Elliptic Character Sheaves jt with David Nadler

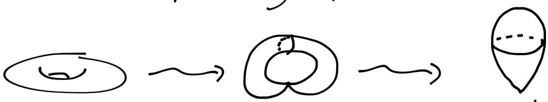
Study family of categories associated to G-reductive over $\mathbb{C}$ & E elliptic curve

geometric Langlands programme

A-side	GLP	B-side
$\mathcal{D}_N(\mathrm{Bun}_G E)$	$\xleftrightarrow[\text{genus } 1]{}$	$\mathrm{Coh}_N(\mathrm{Ch}_G E)$
		$[G^\vee, G^\vee]$ ← comm. up to conjugacy

These two cats. have a family of

- degenerations



Ell. $\rightsquigarrow$ Trig. $\rightsquigarrow$ Rational

- deformations twisted, mixed $\longleftrightarrow$ quantum, q-commuting
  [Jordan] [Helm]

Motivations

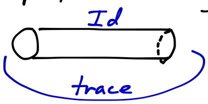
- Avatar of affine character sheaves / have for generalised affine Springer [Lusztig]

[Ginzburg, Schiffman-Vasserot, Lusztig, Bezrukavnikov
Kazhdan-Varshovsky]

- Characters of categorcal rep, traces of Hecke fns

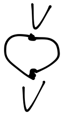
- $\mathrm{Ch}_{Ell} \supset \langle \text{Springer} \rangle$ = reps of Hecke algebra

Caricature from Top. field Theory: $\mathrm{Tr}(\mathrm{Id})$

$E = T^2 =$  $=$ 

e.g $CS_k(T^2)$
= Verlinde algebra ← loop group
= characters of LG-reps

V. f.dim v.sp $\qquad G \circlearrowright V \qquad G$-finite

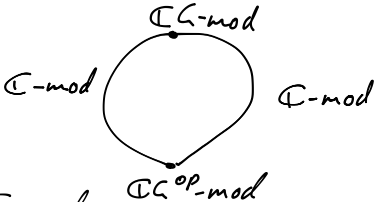


$$\begin{tikzcd}
\mathbb{C} \arrow[r, "Id", "\text{coeval.}"'] \arrow[rr, bend left, "tr(Id) = \dim V"] \arrow[drr, bend right, "\chi_V"'] & \text{End}\, V \arrow[r, "tr", "ev."'] & \mathbb{C} \\
 & \text{“} V \otimes V^* \arrow[d] & \\
 & \mathbb{C}[G] \arrow[r] & \mathbb{C}[\frac{G}{G}]
\end{tikzcd}$$

$\mathbb{C}\,\frac{G}{G} = \text{Cocentre} = HH_* = \underline{\text{trace}}$ of $\mathbb{C}G$

$$\mathbb{C}G \xrightarrow[ab-ba]{tr} \mathbb{C}G \otimes_{\mathbb{C}G \otimes \mathbb{C}G^{op}} \mathbb{C}G = HH_* \mathbb{C}G = Tr(Id_{\mathbb{C}G\text{-mod}})$$

Characters live in Hocsheild Homology



$A = \mathbb{C}G$

$$\begin{tikzcd}
\mathbb{C}\text{-mod} \arrow[r, "A \otimes -"] \arrow[rr, bend right, "A \otimes_{A \otimes A^{op}} A^{op}"'] & A\text{-mod-}A \arrow[r, "- \otimes_{A \otimes A^{op}} A"] & \mathbb{C}\text{-mod}
\end{tikzcd}$$

$G = G(\mathbb{F}_q) \supset B(\mathbb{F}_q) = B$

$\mathbb{C}G\text{-mod} \supseteq \langle \mathbb{C}[G/B] \rangle = H\text{-mod} \qquad H = \text{End}_G\, \mathbb{C}[G/B]$

unipotent principal series

$$\begin{tikzcd}
\mathbb{C}[\frac{G}{G}] & Tr(H) \arrow[l] \\
\chi_{\mathbb{C}[G/B]} & 1 \in H \arrow[l, mapsto] \arrow[u, "Tr"']
\end{tikzcd}$$

$G/\mathbb{C}$ - red. $\supset B$ - Borel

Defn: A G-category is a $(\mathcal{D}(G), *)$-module

e.g $\mathcal{D}(G/B) \cong U\mathfrak{g}\text{-mod}$

Theorem: (BZ-N) $\mathrm{End}_G \mathcal{D}(G/B) = \mathcal{D}(B\backslash G/B) = \mathcal{H}$ ↙ Hecke alg.

↖ cat Soergel Bimodules

$\mathcal{D}(G)\text{-mod} \supseteq \langle \mathcal{D}(G/B) \rangle = \mathcal{H}\text{-mod}$

↙ $\forall$ is principal series

work with Gunningham & Oran: $\mathcal{D}(G) \xleftrightarrow{\sim} \mathcal{D}(N\backslash G/N)$

What are characters of G-categories? or $\mathcal{H}$-mod?

$HH_*(\mathcal{H}, *) = \mathrm{Tr}(\mathcal{H}) = \mathcal{H} \otimes_{\mathcal{H}\otimes\mathcal{H}^{op}} \mathcal{H}$ ← category

$\supseteq HH_*(\mathcal{H})\text{-mod} = \langle \mathrm{Tr}(1) \rangle$

Theorem (BZ-N, 09)

$$\begin{tikzcd}
\mathcal{D}_N\left(\frac{G}{G}\right) = \left\{\begin{array}{c}\text{unipotent}\\ \text{character}\\ \text{sheaves}\end{array}\right\} \cong \mathrm{Tr}(\mathcal{H}) \cong Z(\mathcal{H}) \\
\text{Springer sheaf} \in \mathcal{D}_N\left(\frac{G}{G}\right) \qquad \mathrm{Tr}(1) \in \mathrm{Tr}(\mathcal{H}) \qquad \mathcal{H} \arrow[u]
\end{tikzcd}$$

$\mathrm{Tr}(1) \mapsto$ Springer sheaf

$$\begin{tikzcd}
 & \mathcal{D}(G/B) \arrow[dl] \arrow[dr] & \\
\mathcal{D}(G/G) & & \mathcal{D}(B\backslash G/B)
\end{tikzcd}$$

$$\begin{tikzcd}
 & G/B \cong B/B \arrow[dl] \arrow[dr] & \\
G/G & & B\backslash G/B
\end{tikzcd}$$

Affine version: Want to study loop gp. categories

$D(LG/I)$ 

$\mathfrak{g}\text{-mod} \longleftarrow D(\mathrm{Bun}_G(X, \tilde{x}))$

$D(LG/I) \circlearrowleft \mathcal{H}_{aff} = D(I \backslash LG / I)$

e.g. $D(\mathrm{Bun}_G(X, x))$

Defn: Elliptic character sheaves (unipotent) $= \mathrm{Tr}(\mathcal{H}_{aff})$

$\mathrm{Tr}(1)^{\omega} =$ elliptic Springer sheaf

$\mathcal{H}_{aff} = \mathrm{Coh}_{G^\vee} St$

$$G^\vee \backslash T^*\mathcal{B}^\vee \times_{\mathfrak{g}^\vee} T^*\mathcal{B}^\vee = \frac{\mathfrak{b}^\vee}{B^\vee} \times_{\frac{\mathfrak{g}^\vee}{G^\vee}} \frac{\mathfrak{b}^\vee}{B^\vee}$$

Thm [BZ-N, Preygel 07]

$$\mathcal{H}_{aff} = \mathrm{End}_{\mathfrak{g}^\vee/G^\vee}\left(\mathrm{Coh}(\mathfrak{b}^\vee/B^\vee)\right)$$

Theorem [BZ-N Preygel]

$$\mathrm{Tr}(\mathcal{H}_{aff}) \simeq \mathrm{Coh}_N(L[G^\vee/G^\vee])$$

Want to calculate $\mathrm{Tr}(\mathcal{H}_{aff}) \overset{???}{\simeq} D_N\left[\frac{LG}{LG}\right] \sim D_N(\mathrm{Bun}_G E)$

deformation of LG

$\frac{LG}{LG_q} \longleftrightarrow \mathrm{Bun}_G E, \quad E = \mathbb{C}^*/q^{\mathbb{Z}}$

take $q \to 0$