Analytic Geometry & Rep Theory

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① Motivation:

$$Z \subseteq X \supseteq X \setminus Z$$

scheme

$$X = \hat{X}_Z \cup X \setminus Z$$

(think of as an open cover)

formal completion

define $\mathring{\hat{X}}_Z := \hat{X}_Z \cap X \setminus Z$

rigid analytic space (not a scheme)

"Usual approach" to algebraic geometry: go through locally ringed spaces $(X, \mathcal{O}_X) \longrightarrow$ want to define $QCoh(X)$, $Coh(X)$, $D(X)$

this approach forgets structure on $(X, \mathcal{O}_X)$

Need a functor of points approach

Analysis

Let k be a field with a valuation.

Remark: One can start with a ring with a valuation:

$(\mathbb{Z}, |\cdot|_\infty) \longrightarrow$ get analytic geometry over $\mathbb{Z}$;

$$\operatorname{Spec} \mathbb{Z}^{an} \longrightarrow \operatorname{Spec} \mathbb{F}_1^{an}$$

We get Ban_k a category of complete k-v.sp w.r.t norm. Morphisms are bounded linear maps. This cat has a closed sym. monoidal structure.

$V \hat{\otimes}_k W$ proj. tensor product

$\underline{Hom}_k(V, W)$ = Banach space of all bounded linear maps

This cat. has finite limits & colimits, but no infinite products or coproducts

Quasi-Abelian Categories:

Let C be an additive category with kernels & cokernels

$$\begin{tikzcd} \ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \operatorname{coker} f \\ & \operatorname{coim} f = \operatorname{coker}(\ker f) \arrow[r] & \ker(\operatorname{coker} f) = \operatorname{im} f \arrow[u, <-] & \end{tikzcd}$$

1. Call f strict if the map $\operatorname{coim} f \longrightarrow \operatorname{im} f$ is an isomorphism.
   Ban_k, IAG, LCV_k, $Born_k$,

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d] & F' \arrow[d] \\ E \arrow[r, "f"] & F \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "f"] \arrow[d] & F \arrow[d] \\ E' \arrow[r, "f'"] & F' \end{tikzcd} \qquad f \text{ strict}$$

f strict epi $\Rightarrow$ f' strict epi ; f strict mono $\Rightarrow$ f' strict mono

A quasi-hereditary abelian category has a canonical structure

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

strict s.es $\alpha = \ker \beta$ $\beta = \operatorname{coker} \alpha$

We can define $D(C) = K(C) \big/ \begin{smallmatrix}\text{strict}\\ \text{exact}\end{smallmatrix}$

left heart

$D(C)$ has a natural t-structure $C \longrightarrow LH(C)$

Note: ① In case C has enough projectives.
$D^-(C) \cong K^-(I)$, $(D^+(C) \cong K^+(I))$

② In case $(C \otimes I)$ is quasi-hereditary with enough projectives s.t proj. are flat & the tensor product of two proj. is proj. $Ch^{\leq 0}(C)$ has a sym. monoidal structure

Ban_k has a subcat $Ban_k^{\leq 1}$, same objects but only non-expanding morphisms.

$Ban_k^{\leq 1}$ has limits & colimits

$\prod_I k$ is injective in Ban_k, $\coprod_I$ is proj. in Ban_k

How to (co)complete Ban_k?

$Ind(Ban_k)$ is complete is still closed, symm. monoidal.

Note: Being complete q-abelian category with enough proj. means that $LH(Ind(Ban_k))$ is a category of modules over a ring (with many objects)

Bornological ?

"all functional analysis sits in in this cat"

$$\begin{tikzcd}
CBorn_k \arrow[dr] & \\
LCV_k \arrow[r] & Ind(Ban_k) \\
Pro(Ban_k) \arrow[ur] &
\end{tikzcd}$$

Relative Algebraic Geometry

Let $(C, \otimes I)$ closed sym. monoidal cat. with limits & colimits.

Ex: ① $(Ab, \otimes_{\mathbb{Z}})$ ② $(Ind(Ban_k),$ ③ $(Set, \times)$

Defn: $Comm(C)$ is the cat. of comm monoids in C

$$A \otimes A \to A \quad , \quad I \to A$$

Ex. $Comm(Ab) = Ring^{comm}$. $Comm(Ind(Ban_k))$

$Comm(Set) =$ comm. monoids

Remark

$$\mathrm{Comm}(\mathrm{Proj}_{\cdot}(\mathrm{Ban}_k)) \longrightarrow \mathrm{Comm}(\mathrm{Ind}(\mathrm{Ban}_k))$$

$$\mathrm{Ind}(\mathrm{Comm}(\mathrm{Ban}_k))$$

$$\mathrm{Comm}(\mathrm{Proj}_{\cdot}(\mathrm{Ban}_k)) \supseteq \mathrm{Proj}_{\cdot}(\mathrm{Comm}(\mathrm{Ban}_k))$$

Take $V \in \mathrm{CBan}_k^{\leq 1}$, $V = k_{r_1} \oplus \dots \oplus k_{r_n}$

$S(V) = \bigoplus^{\leq 1} S^n(V)$: these are exactly the affine alg. of rigid analytic geometry

Def: $\mathrm{Aff}(C) = \mathrm{Comm}(C)^{op}$

$A \longmapsto \mathrm{Spec}\, A$

$\mathrm{QCoh}(\mathrm{Spec} A) := A\text{-mod}$

$M \otimes_A N := \mathrm{cseq}(M \otimes A \otimes N \rightrightarrows M \otimes N)$

$f: A \to B$, can define: flatness, f. flat, Zariski immersion

$B \otimes_A B \xrightarrow{\sim} B$ hom. Zariski

Def: $\mathrm{PSt}(C) := \mathrm{Fun}(\mathrm{Aff}(C)^{op})$

choose a Grothendieck top. on $\mathrm{Aff}(C)$ τ

$\mathrm{Sch}_\tau(C) \subseteq \mathrm{DSt}(C)$

If k is non arch. & τ top. of hom Zariski, then

Rigid analytic spaces $\subseteq \mathrm{Sch}_\tau(\mathrm{Ind}(\mathrm{Ban}_k))$

$X \subseteq \mathrm{Sch}_\tau$