Braid Group Action on Equivariant Matrix Factorizations

G - alg. gp. $/\!/ \mathbb{C}$, $\quad T \subseteq B \subseteq G \qquad \underline{t} \leq \underline{b} \leq \underline{g}$
W = Weyl, $I = \{$simple roots$\}$.

Horodo,
X-rad compact manifold, $\quad H \times X \longrightarrow X$
H compact simple Lie gp.
$\cup$
T

$$K_T(X) \circlearrowleft D_i \quad i \in I$$

D_i = Demazure operator, $D_i^2 = D_i$ $\leftarrow$ satisfies braid rels of type W.

$$K_H(X) = \bigcap_{i \in I} \ker(D_i - 1)$$

over $\mathbb{Q}$, $\quad K_H(X) = (K_T(X))^W$

Remark: One can rephrase the action of D_i in terms of action of W.

Categorical Picture

Defn: Demazure descent data on a triangulated cat $\mathcal{C}$

$D_i : \mathcal{C} \longrightarrow \mathcal{C}$

$c_i : D_i \longrightarrow D_i \circ D_i$

$\varepsilon_i : D_i \longrightarrow Id$

D_i are
c_i - isomorphisms
D_i's satify braid rels of type W up to iso.

Natural source of DDD on $\mathcal{C}$.

Def: $QCHecke(G,B) := QCoh_{B\times B}(G)$

derived category

convolution: $*: \; QCHecke(G,B)\times QCHecke \longrightarrow QCHecke$

$$B\backslash G/B \times B\backslash G/B \longleftarrow B\backslash \frac{G\times G}{B_\Delta}/B \longrightarrow B\backslash G/B$$

Thm: (CPS)

Consider $\mathcal{O}_{P_i} \in QCHecke(G,B)$,

1) $\mathcal{O}_{P_i}$ is a coalg. in $QCHecke(G,B),*)$ $\mathcal{O}_{P_i} \xrightarrow{\sim} \mathcal{O}_{P_i} * \mathcal{O}_{P_i}$

2) $w\in W$, $w = s_{i_1}\cdots s_{i_k}$ = red. exp.

$$\Longrightarrow \mathcal{O}_{P_{i_1}} * \cdots * \mathcal{O}_{P_{i_k}} \cong \mathcal{O}_{\overline{BwB}}$$

Main Idea:

$$\mathcal{O}_{P_{i_1}} * \cdots * \mathcal{O}_{P_{i_k}} = \Gamma\left(\mathcal{O}_{BS_{i_1\ldots i_k}}\right) = \mathcal{O}_{\overline{BwB}}$$

Remark: 2) $\Longleftrightarrow$ Braid relations on $\mathcal{O}_{P_i}$

Corollary: let $\mathcal{C}$ be a tri. cat with a monoidal action of $QCHecke(G,B)$

$\Longrightarrow$ the fun, $D_i : M \longmapsto \mathcal{O}_{P_i} * M$

form a DDD on $\mathcal{C}$.

Main Example: X - acted by G.

$C = QCoh^B(X)$.

$\exists$ a natural monoidal action

$$QCHecke(G,B) * QCoh^B(X) \longrightarrow QCoh^B(X)$$

$$B\backslash G/B \times B\backslash X \longleftarrow B\backslash \frac{G\times X}{B} \longrightarrow B\backslash X$$

Example of example

$X = pt$

$$\mathcal{C} = QCoh^{B}(pt) = Rep(B) = \mathcal{O}_B\text{- comodules}$$

$$Rep(B) \underset{Ind_i}{\overset{Res_i}{\rightleftarrows}} Rep(P_i)$$

$D_i = Res_i \circ Ind_i$

Remark: Would like to get a categorical braid gp. action from a DDD on $\mathcal{C}$
e.g on $QCHecke(G,B)$ e.g on $Rep(B)$

Question: Where does braid group act?

An interpretation of Bezrukavnikov - Riche construction:
BR

$$QCoh^{B}(\underline{b}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} QCoh^{P_i}(\underline{p}_i)$$

$f_i : \underline{b} \hookrightarrow \underline{p}_i$

Consider $S_i = f_i^* f_{i*}$

Thm: (BR)

Let $T_i = cone(S_i \longrightarrow Id)$. The functors T_i satisfy braid group relations.

Remark: Consider the graded Hopf-alg. Ω_B. Then by Koszul duality,

notation: loops into B

$$QCoh^{B}(\underline{b}) \cong \Omega_B\text{-comodules} =: Rep(LB)$$

$$QCoh^{P_i}(\underline{p}_i) \cong \Omega_{P_i}\text{-comod} =: Rep(LP_i)$$

"Thm" X - a variety with a G-action, $\exists$ a braid gp on "$QCoh^{LB}(LX)$".

Even & odd versions of a category on which braid groups will act:

Odd: $H \circlearrowright X$, $H \circlearrowright T^*X$ H alg. gp.

comoment map $\to$ $\underline{h} \times \mathcal{O}_X \longrightarrow TX$

Consider the complex

$$Sym_{\mathcal{O}_X}(\underline{h} \otimes \mathcal{O}_X \xrightarrow{\mu} TX) \cong \Lambda(\underline{h}) \otimes Sym(TX)$$
$$\cong \Lambda(\underline{h}) \otimes \mathcal{O}_{T^*X}$$

H acts on $\Lambda(\underline{h}) \otimes \mathcal{O}_{T^*X}$.

Defn: Strongly equivariant coh. sheaves on T^*X
$(\Lambda(\underline{h}) \otimes \mathcal{O}_{T^*X}\ DG\text{-mod})^H$

Notation: $QCoh^H_{Strong}(T^*X)$

Even: Consider $w: T^*X \times \underline{h} \longrightarrow \mathbb{C}$
(an incarnation of moment map).

$w \in \mathcal{O}_{T^*X} \otimes Sym^1(\underline{h}^*)$

deg 2.

$A = \mathcal{O}_{T^*X} \otimes Sym(\underline{h}^*)$, deg $\underline{h}^* = 2$

get a curved graded differential algebra, $d=0$

Curvature: $w \in A^2$

differential: $d(xy) = (dx)y + (-1)xdy$

$d^2(x) = [c, x]$ c = curvature element

if $d=0$, alg. is commutative

CDG mod(A)

$$M = \bigoplus M^n \; ; \quad A^m \times M^n \longrightarrow M^{m+n}$$
$$d : M^n \longrightarrow M^{n+1}$$
$$d^2(m) = c \cdot m$$

Notation: $MF_{\mathbb{Z}}(T^*X \times \underline{h}, w) = \text{CDG mod}(A)$

Remark: Polishchuk - Vaintrob defined equivariant matrix factorization & their derived category
$$DMF^H_{\mathbb{Z}}(T^*X \times \underline{h}, w)$$

Thm: (A) Even = Odd:
$$QCoh^H_{\text{Strong}}(T^*X) \cong DMF^H_{\mathbb{Z}}(T^*H \times \underline{h}, w)$$

Pf: Koszul duality

Remark: Suppose that H acts freely on X
$X \longrightarrow X/H =: Y$ - H-principal bundle. Then both sides of the thm are iso. to $D(QCoh(T^*Y))$

Main Thm: (Kanstamp-A) $G \curvearrowright X$
$\exists$ a categorical braid gp action on $DMF^B_{\mathbb{Z}}(T^*X \times \underline{b}, w)$

Construction: $T^*X \times \underline{b} \xrightarrow{f} T^*X \times \underline{p_i}$

$$DMF^B_{\mathbb{Z}}(T^*X \times \underline{b}, w) \underset{f_*}{\overset{f^*}{\rightleftarrows}} DMF^P_{\mathbb{Z}}(T^*X \times \underline{p_i})$$
$$S_i = f_i^* f_{i*}$$
$$T_i = \text{Cone}(S_i \longrightarrow Id)$$

Claim: T_i gen. the action of the braid gp.

Final Remarks:

$$\text{QCHecke}(G,B) = \text{QCoh}(B\backslash G/B)$$

two monoidal structures $\otimes, *$

Ben-Zvi,
the derived double of $(\text{QCHecke}, \otimes)$ is $\text{QCoh}(LB\backslash LG/LB)$
$\otimes, *$