Laura Rider: Modular Representation theory & geometric Satake

w/ P. Achar.

1) Representation Theory

G - conn. red. alg. group. X^+ = dom. weights for G. $k = \mathbb{C}, \overline{\mathbb{F}_q}$.

Irreducibles i) $L_\lambda : \lambda \in X^+$

ii) understand structure : $k = \mathbb{C}$: weight multiplicities, given by Weyl's character formula

$k = \overline{\mathbb{F}_q}$ hard! Weyl modules M_λ (standards) dual Weyl modules N_λ, irreducibles L_λ

(self-dual, $M_\lambda \leftrightarrow N_\lambda$ interchanged by duality). $\mathrm{Ext}^i(M_\lambda, N_\mu) = 0 \quad i > 0$.

Tilting modules : have a Weyl and dual Weyl filtration $\Rightarrow$ no Exts between these.

2) Geometry: $G^\vee$ the Langlands dual group, over $\mathbb{C}$. $Gr = G^\vee(\mathbb{C}((t)))/G^\vee(\underbrace{\mathbb{C}[[t]]}_{\mathcal{O}})$

This is an ind-variety which is projective: Gr is filtered by an increasing union of fin. dim. projective variety: $Gr = \bigcup_i {}^i Gr$ say. $G^\vee(\mathcal{O})$ acts on Gr with orbits given by dominant coweights for $G^\vee$ = dominant weights for G, X^+.

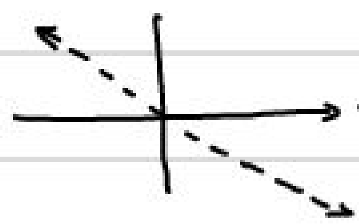
Ex. $G = SL_2$, $G^\vee = PGL_2$, $X^+ = 2\mathbb{N}$. Gr_{SL_2} 

$\overline{Gr_\lambda}$ is singular projective. $H^\cdot(\overline{Gr_\lambda}, \mathbb{C})$ is actually an irreducible PGL_2 representation. In general use IH, get L_λ from $IH^\cdot(Gr_\lambda)$.

$\mathrm{Perv}_{G^\vee(\mathcal{O})}(Gr, k) \subset D^b_{G^\vee(\mathcal{O})}(Gr, k)$

$\quad \| $

$\varinjlim \mathrm{Perv}_{G^\vee(\mathcal{O})}({}^i Gr, k)$

[Aside: Perv: 
Sh X

(really $\mathbin{\text{\\\\}} \cap \mathbb{D}(\mathbin{\text{\\\\}})$)

order by dim of support]

Perverse sheaves are finite length (even here on ind-variety). $j_\lambda : Gr_\lambda \hookrightarrow Gr$.

$$\begin{array}{ccc} {}^pH^0(j_! \underline{k}) & \longrightarrow & {}^pH^0(j_* \underline{k}) \\ \| & & \| \\ I_!(\lambda, k) & & I_*(\lambda, k) \\ & \searrow \quad \nearrow & \\ & IC(\lambda, k) = j_{!*}(\ ) & \end{array}$$

(${}^pH^0$ changes stalks.)

<u>Theorem</u>: (Lusztig, Ginzburg, Mirković-Vilonen) $(\mathrm{Perv}_{G(\mathcal{O})}(Gr, k), *)$ is a Tannakian category equivalent to $(\mathrm{Rep}(G, k), \otimes)$ ("Geometric Satake") Let S be the equivalence

[MV] $S(M_\lambda \to L_\lambda \to N_\lambda) = I_!(\lambda, k) \twoheadrightarrow IC(\lambda, k) \hookrightarrow I_*(\lambda, k)$.

QT : $\delta(T(\lambda)) = ?$ $\quad T(\lambda)$ a tilting object.

↑ would like a sheaf-theoretic description.

MV proved that $I_!(\lambda,\mathbb{Z}) \xrightarrow{\sim} IC(\lambda,\mathbb{Z})$. They knew $I_!(\lambda,\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} k \simeq I_!(\lambda,k)$.

<u>MV conjecture</u>: The stalks of $I_!(\lambda,\mathbb{Z}) = IC(\lambda,\mathbb{Z})$ have no torsion

Juteau: '08 MV not true: Found p-torsion where p is a "bad" prime (at worst 2, 3 or 5).

Proposed a modification:

<u>Theorem</u>: (Achar-R.) MV-modified is true: As long as p is a JMW prime, the stalks of $I_!(\lambda,\mathbb{Z})$ have no p-torsion.

Known JMW primes:

		primes	
sharp	A_n	any	
sharp	B_n	$p > n$	bound not known to be tight
sharp	C_n, D_n	$p > 2$	bound not known to be tight
sharp	E_6, F_4, G_2	$p > 3$	bound not known to be tight
	E_7	$p > 19$	bound not known to be tight
	E_8	$p > 31$	bound not known to be tight

<u>Remark</u>: The existence of p-torsion over $\mathbb{Z}$ is really a question about parity-vanishing properties / k, characteristic $k = p$. (work over a point: $\mathbb{Z}/p\mathbb{Z} \underset{\mathbb{Z}}{\otimes} k$)

Thus MV p-conjecture is implied by vanishing of stalks of $I_!(\lambda,k)$ in odd (or even) degrees.

<u>Theorem</u> (Achar-R.) For $p = \mathrm{char}(k)$ JMW $I_!(\lambda,k)$ are "$*$-parity."

JMW primes and parity sheaves:

Decomposition theorem doesn't work in positive characteristic, but Soergel inspired the theory of parity sheaves developed by Juteau-Mautner-Williamson.

<u>Definition</u>: $X/\mathbb{C}$. $D^b_c(X,k)$. $\mathcal{F} \in D^b_c(X,k)$ is $*$-even (or $*$-odd) if all stalks are concentrated in even (or odd) degrees and !-even (!-odd) is $\mathbb{D}$-condition. Then "even" is $*$ and ! even while "odd" is $*$- and !- odd. (Work with a fixed stratification $X = \bigsqcup X_s$.)

<u>Theorem</u>: [JMW] If (indecomposable) parity sheaves exist they are unique (up to shift)

i.e. if $X_s \subset X$ is open in $\mathrm{supp}(\mathcal{F})$, $\mathcal{P}|_{X_s} = \underline{k}[\dim X_s]$ and this extends uniquely.

These may not be perverse.

<u>Question</u>: What does it mean for rep theory if all parity sheaves are perverse.

<u>Theorem</u>: [JMW] (Answer to Q1). $S(T_\lambda)$ is a parity sheaf for all primes JMW table.

<u>Definition</u>: A JMW prime is one for which $S(\text{tilting}) = \text{parity}$.

Main idea: stalks encoded in derived setting: need a derived geometric Satake.

If $k = \mathbb{C}$ Arkhipov - Bezrukavnikov - Ginzburg $D^b(Gr) = D(\quad)$

<u>Theorem</u>: [Achar-R. + ideas of Achar-Riche] $D^{mix}_{(\mathcal{E}(O))}(Gr, k) \simeq D^b_{perf} Coh^{G \times \mathbb{G}_m}(\underbrace{\mathcal{N}}_{\text{nilp. cone for } G}/k)$.

if char k is JMW, conditions on G. [Match up parity $\leftrightarrow$ tilting (p-coh)

Implied by Theorem of Achar-Riche inspired by Ginzburg '95 : $\mathrm{Hom}^i(S(T_\lambda), S(T_\mu))$

$\cong \mathrm{Hom}(T_\lambda \otimes \mathcal{O}_\mathcal{N}, T_\mu \otimes \mathcal{O}_\mathcal{N}\langle i \rangle)$

Future work in progress: Achar-R. and independently Mautner-Riche.

Relate : $\underset{\substack{\uparrow \\ \text{constructible w.r.t. } I\text{-orbits}}}{D^{mix}_{(I)}}(Gr, k) \leftarrow\text{-}\text{-}\to D^b Coh^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}/k)$.