A conceptual approach to the Generalized Springer Correspondence

I. $\mathcal{P}(\mathcal{N})^G$ = cat. of G eq. perverse sheaves on $\mathcal{N}$

described in terms of Reps of $W(L)$

II. $D(\mathfrak{g})^G$ = derived cat of G equiv. D-modules on $\mathfrak{g}$.

G - conn. red. / $\mathbb{C}$

$\mathcal{N} \subseteq \mathfrak{g} = \mathrm{Lie}(G)$

$$\mathcal{P}(\mathcal{N})^G \subseteq D(\mathcal{N})^G \subseteq D(\mathfrak{g})^G$$

I. Generalised Springer Correspondence

$W(L) = N_G(L)/L$ $\quad$ L = Levi of G

$$\mathcal{P}(G)^G = \bigoplus_{\substack{\text{cuspidal}\\ \text{datum}}} \mathrm{Rep}(W(L))$$

$$\mathrm{Rep}(W) \hookrightarrow \mathrm{Rep}(W(L))$$

W Weyl gp. (when $L = T$ max torus

① Define functors: $\quad L \twoheadleftarrow P \subset G$

$\quad \mathcal{N} \subseteq \mathfrak{l} \quad \mathfrak{p} \quad \mathfrak{g} \supseteq \mathcal{N}_G$

$$\begin{tikzcd}
\mathcal{P}(\mathcal{N}_G)^G \arrow[r, bend left, "\mathrm{Res}^G_L"] \arrow[r, bend right, "\mathrm{Ind}^G_L"'] & \mathcal{P}(\mathcal{N}_L)^L
\end{tikzcd}$$

eg:
$$\begin{tikzcd}
\mathcal{P}(\mathcal{N}_G)^G \arrow[r, bend left] \arrow[r, bend right] & \mathcal{P}(\mathcal{N}_H)^H \cong \mathrm{Vect} \arrow[l, bend left=60]
\end{tikzcd}$$

Springer sheaf

$$\underline{\mathrm{Spr}} \longleftarrow\!\shortmid \; \mathbb{C}$$

$= (\tilde{\mathcal{N}} \longrightarrow \mathcal{N})_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$

② **Defn:** An object $\mathcal{F} \in P(\mathcal{N})^G$ is called cuspidal if $\mathrm{Res}^G_L(\mathcal{F}) = 0$ for all proper Levi L.

Defn: $P(\mathcal{N})^G_{[L]} = \langle \mathrm{Ind}(P(\mathcal{N}_L)^L_{cusp.}) \rangle$

③ **Mackey Thm:** Let Q, P parabolics, $Q \downarrow M$, $P \downarrow L$ Levis

$$\mathrm{Res}^G_M \mathrm{Ind}^G_L(\mathcal{F}) \cong \bigoplus_{\substack{w \in Q\backslash G/P \\ \parallel \\ W_Q \backslash W / W_P}} \mathrm{Ind}^M_{{}^wL \cap M} \mathrm{Res}^{{}^wL}_{{}^wL \cap M}(w_* f)$$

Cor: ⓐ

$$\begin{tikzcd} P(\mathcal{N})^G_{[L]} \arrow[r, bend right] & P(\mathcal{N}_L)^L_{cusp} \arrow[l, bend right] \end{tikzcd}$$

ⓑ $$\mathrm{Res}^G_L \mathrm{Ind}^G_L \Big|_{P(\mathcal{N}_L)^L_{cusp}} (\mathcal{F}) = \bigoplus_{w \in W(L)} w_*(\mathcal{F})$$

$$\cong N_G(L)/L = N_W(W_P)/W_P \subseteq W_P \backslash W / W_P$$

④ Claim: $$P(\mathcal{N}_G)^G = \bigoplus_{\substack{\text{conj cl's} \\ \text{of Levis}}} P(\mathcal{N}_G)^G_{[L]}$$

Need to check:

ⓐ $P(\mathcal{N}_G)^G_{[L]}$ generate ✓

ⓑ $P(\mathcal{N}_G)^G_{[L]} \perp P(\mathcal{N}_G)^G_{[M]}$ if M is not conj. to L.

ⓒ $P(\mathcal{N}_G)^G_{[L]} = P(\mathcal{N}_G)^G_{[L]}$ if L is conj. to M.

ⓐ, ⓑ, ⓒ all proved by Mackey

Barr-Beck Monodicity Thm

If $\mathcal{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{D}$ an adjunction

$RL \in End(\mathcal{D})$ is a monad $\quad (RL)(RL) \longrightarrow RL$

$\qquad\qquad\qquad\qquad\qquad\qquad 1_{\mathcal{D}} \longrightarrow RL$

$\Rightarrow \mathcal{C} \xrightarrow{\sim} \mathcal{D}^{RL} \leftarrow$ RL-modules in $\mathcal{D}$, ie $RL(d) \longrightarrow d$

assuming R is conservative, $R(c)=0 \Rightarrow c=0$

Prop: There is an equivalence of monads

$$Res^G_L Ind^G_L \circlearrowright P(\mathcal{N}_L)^L_{cusp} \circlearrowleft W(L)$$

$$\mathcal{F} \longmapsto \bigoplus_{w \in W(L)} w_*(\mathcal{F})$$

Cor: $P(\mathcal{N}_G)^G_{[L]} \simeq \left(P(\mathcal{N}_L)^L_{cusp}\right)^{W(L)}$

$$P(\mathcal{N}_L)^L_{cusp} = \left(\langle \mathcal{F}_1 \rangle \oplus \cdots \oplus \langle \mathcal{F}_n \rangle\right)^{\circlearrowleft W(L)} = Rep(W(L))$$

II $\quad D(\mathfrak{g})^G$ = derived (strongly) equiv. cat of D-modules on G.

$$\boxed{\text{Thm (G): } D(\mathfrak{g})^G = \bigoplus^{\perp} \left(D(\mathfrak{l})^L_{cusp}\right)^{W(L)}}$$

Remarks:

① c.f McGerty-Nevins, but split

Note: $D(\mathfrak{g})^G \rightsquigarrow D(\mathfrak{gl}_n \times V)^{GL_n}$ $\qquad (D_L \otimes S(\mathfrak{h}[-2]))\#W$

e.g $G = PGL_2$, $\quad D(\mathfrak{g})^G = D(\mathfrak{h})^{N_G(H)} \rightsquigarrow (D\#W)$-mod

smash product

$$② \quad D(\mathcal{N})^G = \bigoplus \ldots$$

$$\cup\!\!\backslash \quad D(pt)^{N_G(H)} = S(\mathfrak{h}[-2]) \# W$$

$$③ \quad \left(D(\mathfrak{l})^L_{cusp}\right)^{W(L)} = \left(\bigoplus D(\mathfrak{z}(\mathfrak{l}))^{Z(L)} \boxtimes f\right)^{W(L)}$$

Hamiltonian Reduction

$$D(\underline{\mathfrak{h}})^{N_G(H)} \hookrightarrow D(\underline{\mathfrak{g}})^G$$

$$\cup\!\!\!\!\!\!\shortmid$$

$$D_{\mathfrak{h}} \rightsquigarrow D_{\underline{\mathfrak{g}}} \big/ D_{\underline{\mathfrak{g}}}\text{-}ad(\underline{\mathfrak{g}}) = M$$

ideal gen. by adjoint action on $\underline{\mathfrak{g}}$.

$$Hom(M,-) = \text{Quantum Hamiltonian Reduction Functor}$$

$$End(M) \text{``}=\text{''} (M)^G \xrightarrow[\text{Levison Stafford}]{} (D_{\underline{\mathfrak{h}}})^W$$

$$End(D_{\underline{\mathfrak{h}}}) = (D_{\underline{\mathfrak{h}}})^W$$