1) Highest weight category.

k field.

A finite dim k-alg.

$\mathcal{C} = \mathrm{mod}(A)$ = finitely generated A-modules

Def $\mathcal{C}$ is a "highest weight category" w.r.t a poset

$\Delta(\mathcal{C}) = \{\Delta_\lambda\}_{\lambda\in\Lambda}$ = standard objects, w/ partial order $<$, if

- $\mathrm{End}(\Delta_\lambda) = k \quad \forall\lambda$
- $\exists P_\lambda$ indecomp proj object s.t. $P_\lambda \overset{\varphi}{\twoheadrightarrow} \Delta_\lambda$ and $\ker\varphi$ filtered by Δ_μ with $\mu>\lambda$
- $\mathrm{Hom}(\Delta_\mu, \Delta_\lambda) \neq 0 \Rightarrow \mu \leq \lambda$

Then $\Delta(\mathcal{C}) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{C})$, $\quad \Delta_\lambda \longmapsto L_\lambda = \mathrm{top}(\Delta_\lambda)$

$\xrightarrow{\sim}$ {Indecomposable projective modules} $\quad P_\lambda$

$\xrightarrow{\sim}$ { " injective modules} $\quad I_\lambda$

$\xrightarrow{\sim}$ { " tilting modules} $\quad T_\lambda$

$\simeq$ { costandard objects} $\quad \nabla_\lambda$

NB Highest weight condition $\Rightarrow$ $\mathrm{gl.dim}(A) < +\infty$.

Brauer reciprocity $\quad (P_\lambda : \Delta_\mu) = [\nabla_\mu : L_\lambda]$

Eg. (Parabolic) BGG-category $\mathcal{O}$, $\quad \Delta(\mathcal{O})$ = Verma modules,

$\nabla(\mathcal{O})$ = dual Verma modules.

Def: $T = \bigoplus_\lambda T_\lambda$ = characteristic tilting module.

$D(A) := \mathrm{End}_A(T)^{op}$ = " Ringel dual of A"

$D(\mathcal{C}) = \mathrm{mod}(D(A))$ a highest weight category w.r.t. Λ^{op}

We have an equivalence of triangulated categories (Ringel duality)

$$D: \quad D^b(\mathcal{C}) \xrightarrow{\sim} D^b(D(\mathcal{C})), \qquad M \mapsto D(M) = D(M) = \mathrm{RHom}_A(M,T)^*$$

and D maps
$$\Delta(\mathcal{C}) \longrightarrow \nabla(D(\mathcal{C}))$$
$$P(\mathcal{C}) \longrightarrow T(D(\mathcal{C}))$$
$$T(\mathcal{C}) \longrightarrow I(D(\mathcal{C}))$$

$I \subset \Lambda$ ideal
$(a<b,\ b\in I \Rightarrow a\in I)$

$\mathcal{C}[I] \subset \mathcal{C}$ Serre subcat gen by L_λ, $\lambda\in I$.

fully faithful on extensions.

$\mathcal{C}(J) = \dfrac{\mathcal{C}}{\mathcal{C}[I]}$ if $J = \Lambda\setminus I$.

$D(\mathcal{C}[I]) = D(\mathcal{C})(I^{op})$

2) Standard Koszul property.

$\mathcal{C} = \mathrm{mod}(A)$. highest weight category.

$\bar{A}$ = positively graded algebra isom to A as algebra.

i.e. $\bar{A} = \bigoplus_{i\geq 0} \bar{A}^i$, and $\bar{A}^0$ semisimple.

<u>Def</u> Let $\bar{A}^! = \mathrm{Ext}_{\bar{A}}(\bar{A}_0, \bar{A}_0)^{op}$ f.d. algebra. "Koszul dual of A"

<u>Def</u> (Agoston-Dlab-Lukacs) $\bar{A}$ is "standard Koszul" if all Δ_λ's have a "linear projective resolution".
i.e.

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to \Delta_\lambda \to 0$$

with P_i generated by degree i-component. i.e. $P_i = \bar{A}\cdot P_i^i$

and if all ∇_λ's has a "linear injective resolution"

<u>Thm</u> (ADL)

$\bar{A}$ standard Koszul $\Leftrightarrow$ $\begin{cases} \bullet\ \bar{A} \text{ Koszul.} \\ \bullet\ \mathrm{mod}(\bar{A}^!) \text{ is a highest weight category (w/ opposit order)} \end{cases}$

In this case, $E: D^b(\bar{A}) \xrightarrow{\sim} D^b(\bar{A}^!)$, $\quad M \longmapsto E(M) = \mathrm{RHom}_A(\bar{A}^0, -)$

takes (graded version of)

∇, I, L to Δ, L, P

Def (Mazorchuk) $\bar{A}$ is <u>balanced</u> if $\bar{A}$ is standard Koszul and the natural grading on $D(\bar{A})$ is positive.

Thm (M). $\bar{A}$ is balanced $\Rightarrow$ $\bar{A}$, $D(\bar{A})$, $\bar{A}^!$, $D(\bar{A})^!$ balanced.

and E commute with D.

E.g. (Parabolic) BGG category $\mathcal{O}$ is balanced.

3) Affine parabolic category $\mathcal{O}$.

$\mathfrak{g}$ simple Lie alg / $\mathbb{C}$.

$\hat{\mathfrak{g}}$ affine Kac-Moody alg ass. with $\mathfrak{g}$ $\supset \hat{\mathfrak{b}} \supset \hat{\mathfrak{t}}$

$\hat{W}$ affine Weyl group

μ, ν proper subsets of simple affine roots.

N Dual Coxeter number.

e positive integer.

$\nu \rightsquigarrow$ parabolic subalg of $\hat{\mathfrak{g}}$

$\rightsquigarrow \mathcal{O}^{\nu}_{\pm e}$ = parabolic (of type ν) affine category $\mathcal{O}$ of $\hat{\mathfrak{g}}$ at level $\pm e - N$

Linkage principle : $\mathcal{O}^{\nu}_{\pm e} = \bigoplus_{\mu} \mathcal{O}^{\nu}_{\pm e, \mu}$

$\mathcal{O}^{\nu}_{\pm e,\mu}$ = subcat gen. by simple objects with highest weight is a $\hat{W}$-dot-orbit of an dominant / antidominant weight whose stabilizer in $\hat{W}$ is a parabolic subgp of type μ.

Thm (SVV) $\mathcal{O}^{\nu}_{\pm e,\mu}$ admits graded lift which is balanced.

such that $\left(\bar{\mathcal{O}}^{\nu}_{e,\mu}\right)^! \simeq \bar{\mathcal{O}}^{\mu}_{-e,\nu}$ as highest weight categories.

(Soergel for nongraded version) $D(\bar{\mathcal{O}}^{\nu}_{e,\mu}) \simeq \bar{\mathcal{O}}^{\nu}_{-e,\mu}$ (with appropriate truncations)

4) Type A case and relation with Howe duality.

Fix $N, \ell > 0$ and $\nu = (\nu_1, \cdots, \nu_\ell) \in \mathbb{N}^\ell$, $\sum_p \nu_p = N$.

$\rightsquigarrow$ the classical affine weight $\sum_p \nu_p \varepsilon_p$ of level 0 for $\widehat{\mathfrak{sl}}_\ell$

$\widehat{\mathfrak{sl}}_e$ action on $\mathbb{C}^e[z^{\pm 1}]$ of level zero $\rightsquigarrow$ $\widehat{\mathfrak{sl}}_e \circlearrowright \Lambda^\nu(\mathbb{C}^e[z, z^{-1}]) = \bigotimes_p \Lambda^{\nu_p}(\mathbb{C}^e[z^{\pm 1}])$

For $\mu \in \mathbb{Z}^e$, $\Lambda^\nu(\mathbb{C}^e[z^{\pm 1}])_\mu$ = weight μ subspace for $\widehat{\mathfrak{sl}}_e$-action.

<u>Howe duality</u>: For $\nu \in \mathbb{N}^\ell$, $\mu \in \mathbb{N}^e$, $\sum \nu_p = \sum \mu_p = N$

$$
\begin{array}{ccc}
 & \Lambda^N(\mathbb{C}^e \otimes \mathbb{C}^\ell[z^{\pm 1}])_{\mu,\nu} & = (\mu,\nu) \text{ weight space for } \widehat{\mathfrak{sl}}_e \times \widehat{\mathfrak{sl}}_\ell \\
 /\!\!/ & & \backslash\!\!\backslash \\
\Lambda^\nu(\mathbb{C}^e[z^{\pm 1}])_\mu & \xleftrightarrow[\text{LR}]{\sim} & \Lambda^\mu(\mathbb{C}^\ell[z^{\pm 1}])_\nu
\end{array}
$$

(explicite on bases given by monomial wedge products).

Consider $\mathfrak{g} = \mathfrak{gl}_N$, $\nu \rightsquigarrow$ parabolic subalg $\begin{pmatrix} \mathfrak{gl}_{\nu_1} & & * \\ & \mathfrak{gl}_{\nu_2} & \\ 0 & & \ddots \end{pmatrix} \subset \mathfrak{g}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \subset \hat{\mathfrak{g}}$

$\mathcal{O}^\nu_{\pm e,\mu}$ parabolic affine category $\mathcal{O}$ ass. with the block corr. to weight

$\hat{\mu} = [(1^{\mu_1} 2^{\mu_2} \cdots e^{\mu_e}) - e\omega_0] - \hat{\rho}$

Thm (SVV)

$$
\begin{array}{ccc}
[\mathcal{O}^\nu_{\mu,-e}] & \xrightarrow[\sim]{\text{D}\circ\text{E}} & [\mathcal{O}^\mu_{\nu,-e}] \\
\theta \downarrow \wr & \circlearrowleft & \theta \downarrow \wr \\
\Lambda^\nu(\mathbb{C}^e[z^{\pm 1}])_\mu & \xrightarrow[\sim]{\text{LR}} & \Lambda^\mu(\mathbb{C}^\ell[z^{\pm 1}])_\nu
\end{array}
$$

where θ = isom of vector spaces sending $[M(\lambda - \rho)]$ to

$(v_{\lambda_1} \wedge v_{\lambda_2} \wedge \cdots \wedge v_{\lambda_{\nu_1}}) \otimes (v_{\lambda_{\nu_1+1}} \wedge \cdots)$ . . .

Further, θ sends tilting $\longrightarrow$ canonical bases, simple $\longrightarrow$ dual can. bases.

PB: Establish similar results for category $\mathcal{O}$ of Rational Cherednik algebras.

$\rightsquigarrow$ categorification of level-rank duality. (Λ^N replaced by $\Lambda^{\frac{\infty}{2}}$)

5) Rational Cherednik algebras

W complex reflection group, $\mathfrak{h}$ reflection representation

$S \subset W$ pseudo-reflections

$c : S/W \longrightarrow \mathbb{C}$

$H(W)_R$ = Rational Cherednik alg over $\mathbb{C}$ ass. with $W, \mathfrak{h}, c$

= subalg of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}]) \rtimes W$ gen. by $\mathbb{C}[\mathfrak{h}]$, W

and Dunkl-Opdam operators $\partial_y + \sum_{s \in S} c(s)(y, \alpha_s)\alpha_s^{-1}(s-1)$, $\quad y \in \mathfrak{h}$, $\alpha_s \in \mathfrak{h}^*$

Triangular decomposition $\quad H(W) = \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}]$

Standard modules $\quad \forall E \in \mathrm{Irr}_{\mathbb{C}}(W)$, $\quad \Delta(E) = \mathrm{Ind}_{W \ltimes \mathbb{C}[\mathfrak{h}^*]}^{H(W)}(E)$

$\mathcal{O}(W) \subset \mathrm{mod}(H(W))$ f.g. $H(W)$-modules locally nilpotent for $\underset{\mathbb{C}[\mathfrak{h}^*]}{\overset{\mathfrak{h}}{\cap}}$-action

Thm (GGOR) $\mathcal{O}(W)$ is highest weight

Cyclotomic case:

Fix $\Gamma \subset \mathbb{C}^\times$ ℓ-roots of 1.

$d > 0$, $\quad G(\ell, 1, d) = \mathfrak{S}_d \ltimes \Gamma^d$ complex reflection group $\overset{\text{def}}{=} \Gamma_d \curvearrowright \mathbb{C}^d$

$$S = \bigsqcup_{\gamma \in \Gamma \setminus \{1\}} \{\gamma_i = (1, \cdots, \gamma, 1 \cdots 1) \mid 1 \le i \le d\} \sqcup \{s_{ij}\gamma_i\gamma_j^{-1} \mid i \neq j\}$$

c determinated by ℓ-parameters $(\kappa, s_p)_{1 \le p \le \ell}$ s.t. $\sum_p s_p = 0$

$$\left(c(s_{ij}) = \frac{1}{\kappa}, \quad c(\gamma_i) = \sum_{p=1}^{\ell} (\gamma^{-p} - 1)\left(\frac{s_p - s_{p+1}}{\kappa} - \frac{1}{\ell} \right) \right)$$

$\mathrm{Irr}_c(\Gamma_d) \simeq \{ \ell\text{-partitions } \lambda = (\lambda^{(1)}, \cdots, \lambda^{(\ell)}) \text{ s.t. } \sum_p |\lambda^{(p)}| = d \}$

$\mathcal{O}(\Gamma_d)^s_\kappa$ category $\mathcal{O}$ with standard objects $\Delta(\lambda)^s_\kappa$

Quotient functor $KZ : \mathcal{O}(\Gamma_d)^s_\kappa \longrightarrow \mathrm{mod}(\mathcal{H}^s_\kappa(\Gamma_d))$

$\mathcal{H}^s_\kappa(\Gamma_d) =$ Hecke alg ass. with Γ_d $= \dfrac{\mathbb{C}\langle T_0, T_1, \cdots, T_{d-1} \rangle}{\text{braid rel}^n \text{ for } \overset{0}{\bullet}\!=\!\overset{1}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\bullet \cdots \overset{d-1}{\bullet}}$

for $q = \exp\left(\frac{2\pi i}{\kappa}\right)$ $\qquad (T_i + 1)(T_i - q) = 0$

$Q_p = q^{s_p}$ $\qquad \prod_{p=1}^{\ell} (T_0 - Q_p) = 0$

NB : If s is "dominant" then $\mathcal{O}(\Gamma_d)^s_\kappa \simeq \mathrm{mod}$ (cyclotomic Schur algebra)

6. Main Thm

Fix e, ℓ, N positive integers. $\mathfrak{g} = \mathfrak{gl}_N$

integral weight of $\mathfrak{g}$ $\simeq \mathbb{Z}^N$

$\rho = (0, -1, -2, \cdots, -N+1)$

$\rho_\nu = (\nu_1, \nu_1 - 1, \cdots, 1, \nu_2, \cdots, 1, \cdots)$

$$P^\nu_d = \{ \lambda \in P^\ell_d \mid \ell(\lambda_p) \le \nu_p \} \hookrightarrow \mathbb{Z}^N$$

$$\lambda \longmapsto \lambda + \rho_\nu - \rho$$

$\lambda \in \mathbb{Z}^N$ weight of $\mathfrak{g}$ $\rightsquigarrow$ $\hat{\lambda} = \lambda + \dfrac{\langle \lambda, 2\rho + \lambda \rangle}{2e} \delta + (-e-N) \omega_0$

Let $A^{\nu}_{-e}(d) \subset \mathcal{O}^{\nu}_{-e}$ Serre subcategory gen. by. $L(\widehat{\lambda+\rho_\nu-\rho})$ for $\lambda \in \mathcal{P}^{\nu}_d$

It is a highest weight category.

<u>NB</u>: If $l=1$. $\rho_\nu=\rho$. A^{ν}_{-e} = Schur category (affine Version of polynomial rep^n of $\mathfrak{g}$)

Conj (Varagnolo-Vasserot, '08) Assume $d \leq \nu_p$ for all p.

Let $s=(-\nu_l, \dots, -\nu_1)$.

Then $A^{\nu}_{-e}(d) \simeq \mathcal{O}(\Gamma_d)^{s}_{-e}$ as highest weight categories

$M(\widehat{\lambda+\rho_\nu-\rho}) \longmapsto \Delta(\lambda^{*})$ for $\lambda \in \mathcal{P}_d$, where $\lambda^{*}=(\lambda^t_l, \dots, \lambda^t_1)$

Proved by Rouquier-S-Varagnolo-Vasserot.

Related to work of Losev

Recall $\mathrm{Fock}^s = \bigoplus_{d\geq 0}\bigoplus_{\lambda \in \mathcal{P}^l_d} \mathbb{C}|\lambda\rangle$ $\circlearrowleft$ $\widehat{\mathfrak{sl}}_e$-module of highest weight $\lambda(s)=\sum_p \omega_{s_p}$

- Uglov: canonical bases for Fock_s
- $\Lambda^{\frac{\infty}{2}} = \text{"}\lim\text{"}\, \Lambda^N(\mathbb{C}^e \otimes \mathbb{C}^l \otimes \mathbb{C}[z^{\pm 1}])$ $\circlearrowleft (\widehat{\mathfrak{sl}}_{el})_1$ ← level one.

$$\cup$$
$$(\widehat{\mathfrak{sl}}_e)_l \times (\widehat{\mathfrak{sl}}_l)_e$$

- Level-Rank duality $\mathrm{Fock}^{\nu}_{\mu} = \Lambda^{\frac{\infty}{2}}_{\mu,\nu} = \mathrm{Fock}^{\mu}_{\nu}$

weight μ subspace for $\widehat{\mathfrak{sl}}_e$-action

weight space for $(\widehat{\mathfrak{sl}}_e)_l \times (\widehat{\mathfrak{sl}}_l)_e$-action

weight ν-subspace for $\widehat{\mathfrak{sl}}_l$-action.

Let $\theta: [\bigoplus_d \mathcal{O}(\Gamma_d)^s_\kappa] \xrightarrow{\sim} \mathrm{Fock}^{\nu}$ $\nu = s^{*}$

$\Delta_\lambda \longmapsto |\lambda^{*}\rangle.$

Cor 1 (Rouquier conj) θ sends $[T(\lambda)]$ to canonical bases
and $[L(\lambda)]$ to dual can. bases.

Proof: Decomposition matrix of Fock^{ν} given by affine parabolic KL-polynomials

$=$ decomposition matrix of $\mathcal{A}^{\nu}_{-e}$

$\bigoplus_d \mathcal{O}(T_d)^s_{-e}$ decomposes into blocks $\bigoplus_{\mu} \mathcal{O}^s_{-e,\mu}$ s.t. $\theta : [\mathcal{O}^s_{\kappa,\mu}] \xrightarrow{\sim} \mathrm{Fock}^{\nu}_{\mu}$

Cor 2 (Chuang-Miyachi conj) $\mathcal{O}(T)^{\nu}_{-e,\mu}$ is balanced (in particular standard Koszul)

and $D^b(\mathcal{O}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{-e,\mu}) \xrightarrow[D\circ E]{\sim} D^b(\mathcal{O}(\mathbb{Z}/e\mathbb{Z})^{\mu}_{-\ell,\nu})$

s.t. the induced map on the Grothendieck gp coincides with the isom

$\mathrm{Fock}^{\nu}_{\mu} \underset{LR}{\simeq} \mathrm{Fock}^{\mu}_{\nu}$ via identification by θ.

Proof: $\underset{\text{highest weight subcat.}}{\mathcal{A}^{\nu}_{-e} \subset \mathcal{O}(\hat{\mathfrak{g}})^{\nu}_{-e}}$ is standard Koszul and balanced.