P. Shan.

§1 Highest weight categories.

$\Bbbk$ a field alg. closed, A fin.-dim $/\Bbbk$. $\mathcal{C} = \mathrm{mod}(A)$.

<u>Definition</u>: $\mathcal{C}$ is a h.w. category if we have $\Delta(\mathcal{C}) = \{\Delta_\lambda\}_{\lambda \in \Lambda}$ such that

- $\mathrm{Hom}(\Delta_\lambda, \Delta_\mu)$
- $P_\lambda \twoheadrightarrow \Delta_\lambda$ $\quad \Delta(\mathcal{C}) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{C})$, $\Delta_\lambda \mapsto L_\lambda$.

$\simeq \{\text{ind. proj. mod in } \mathcal{C}\}$ $\quad P_\lambda$

$\simeq$ " inj . . . $\quad I_\lambda$

$\simeq \nabla(\mathcal{C})$ costandards

$\simeq \{\text{ind. tiltings}\}$ $\quad T_\lambda$

$T = \bigoplus_{\lambda \in \Lambda} T_\lambda$, characteristic tilting. $D(A) = \mathrm{End}_A(T)^{op}$ 'Ringel dual of A', $D(\mathcal{C}) = \mathrm{mod}(D(A))$

highest weight on Λ^{op}. $\quad \mathbb{D} = R\mathrm{Hom}_A(-, T)^* : D^b(\mathcal{C}) \to D^b(D(\mathcal{C}))$

$$\Delta(\mathcal{C}) \to \nabla(D(\mathcal{C}))$$

$$P \to T$$

$$T \to I$$

$I \subset \Lambda$ an ideal if $a < b$, $b \in I \Rightarrow a \in I$. $\mathcal{C}[I]$ Serre subcategory gen. by L_λ, $\lambda \in I$, a h.w. subcat. of $\mathcal{C}$. If $M, N \in \mathcal{C}[I]$ $\mathrm{Ext}^i_{\mathcal{C}[I]}(M,N) = \mathrm{Ext}^i_{\mathcal{C}}(M,N)$.

If J is a coideal (so $I = \Lambda \setminus J$ is an ideal). $\mathcal{C}[J] = \mathcal{C}/\mathcal{C}[\Lambda \setminus J]$. $D(\mathcal{C}[J]) = D(\mathcal{C})[J^{op}]$

2) Standard Koszul: $\mathcal{C} = \mathrm{mod}(A)$. Assume $\bar{A}$ is a positive grading on A ($\bar{A}_0$ semisimple)

Define $\bar{A}^! = \mathrm{Ext}_{\bar{A}}(A^0, A^0)^{op}$

$$\uparrow$$
$$\cong \bar{A}/\bar{A}_{>0}.$$

<u>Definition</u>: Say $\bar{A}$ is standard Koszul if any Δ_λ has a linear projective resolution:

$$0 \to P_d \to \dots \to P_1 \to P_0 \to \Delta_\lambda \to 0$$

$P_i = \bar{A} \cdot P_i^i$ and ∇_λ has a linear injective resolution.

<u>Theorem</u>: $\bar{A}$ is standard Koszul $\Leftrightarrow$ $\bar{A}$ is Koszul and $\mathrm{mod}\, \bar{A}^!$ is highest weight on Λ^{op}.

$E = R\mathrm{Hom}_{\bar{A}}(\bar{A}^0, -) : D^b_{gr}(\bar{A}) \longrightarrow D^b_{gr}(\bar{A}^!)$

$\nabla, J, I \mapsto \Delta, L, P$

Definition: Say $\bar{A}$ is a balanced algebra if it is standard Koszul and $D(\bar{A})$ is positively grading (using natural grading on $T_\lambda 0$).

Theorem (Mazorchuk) If $\bar{A}$ is balanced, then $D(\bar{A}), \bar{A}^!, D(\bar{A}^!)$ are balanced and Σ and E and D commute.

3) Affine parabolic category $\mathcal{O}$.

Fix $\mathfrak{g}$ a simple Lie algebra over $\mathbb{C}$. $\hat{\mathfrak{g}}$ an affine Kac-Moody algebra $\supset \hat{\mathfrak{b}} \supset \hat{\mathfrak{t}}$. $\hat{W}$ the affine Weyl group. Let μ, ν be two proper subsets of simple affine roots. N = dual Coxeter number, e a positive integer.

$\nu \rightsquigarrow$ parabolic subalgebra of $\hat{\mathfrak{g}}$. $\mathcal{O}^\nu_{\pm e}$ parabolic affine category $\mathcal{O}$ of $\hat{\mathfrak{g}}$ at level $\pm e - N$

Linkage principle $\mathcal{O}^\nu_{\pm e} = \bigoplus_\mu \mathcal{O}^\nu_{\pm e,\mu}$ where $\mathcal{O}^\nu_{\pm e,\mu}$ is subcategory generated by $L(\lambda)$ such that λ in $\hat{W}$-dot orbit of dom/anti-dom. weight whose stabilizer is a parabolic subgroup of type μ

Theorem: (S-Varagnolo-Vasserot)

$\mathcal{O}^\nu_{\pm e,\mu}$ admits a graded lift which is balanced such that $(\bar{\mathcal{O}}^\nu_{e,\mu})^! = \bar{\mathcal{O}}^\mu_{-e,\nu}$, $D(\bar{\mathcal{O}}^\nu_{e,\mu}) = \bar{\mathcal{O}}^\nu_{-e,\mu}$.

h.w. structure: $\mathcal{O}^\nu_{-e,\mu}$ has $\hat{W}/W_\mu$ Bruhat order
$\mathcal{O}^\nu_{e,\mu}$ " " opposite order

} make sense of $\int$ in truncated categories.

§4 Type A case and Howe duality.

$\nu = (\nu_1, \nu_2, \dots, \nu_\ell) \in \mathbb{Z}^\ell_{>0}$, $\sum \nu_p = N$. $\mathfrak{p}_\nu := \begin{pmatrix} \mathfrak{gl}_{\nu_1} & & * \\ 0 & \ddots & \\ & & \mathfrak{gl}_{\nu_\ell} \end{pmatrix}$ $\mathfrak{p}_\nu \subseteq \hat{\mathfrak{g}}$.

$\hat{\mathfrak{sl}}_e$ on $\mathbb{C}^e[z^{\pm 1}]$ of level 0. Extends to an action so $\hat{\mathfrak{sl}}_e$ on $\Lambda^\nu(\mathbb{C}^e[z^{\pm1}]) = \bigotimes_p \Lambda^{\nu_p}(\mathbb{C}^e[z^{\pm1}])$

$\mu \in \mathbb{Z}^e \rightsquigarrow$ weight for $\hat{\mathfrak{sl}}_e$ of level 0.

$$\Lambda^\nu(\mathbb{C}^e[z^{\pm1}])_\mu \overset{\text{Howe}}{=} \Lambda^\mu(\mathbb{C}^\ell[z^{\pm1}])_\nu$$

$$\| \qquad\qquad$$

$$\Lambda^N(\mathbb{C}^e \otimes \mathbb{C}^\ell[z^{\pm1}])_{\mu,\nu} \longleftarrow \text{acted on by } \hat{\mathfrak{sl}}_e \times \hat{\mathfrak{sl}}_\ell$$

Theorem:
$$\begin{array}{ccc} [\mathcal{O}^{\nu}_{\mu,-e}] & \xrightarrow[\sim]{\text{DoE}} & [\mathcal{O}^{\mu}_{\nu,-e}] \\ \theta \downarrow & & \downarrow \theta \\ \Lambda^{\nu}(\mathbb{C}^{e}[z^{\pm1}])_{\mu} & \xrightarrow{\text{Howe}} & \Lambda^{\mu}(\mathbb{C}^{\ell}[z^{\pm1}])_{\nu} \end{array}$$

$\theta : [\mathcal{O}^{\nu}_{\mu,-e}] \xrightarrow{\sim} \Lambda^{\nu}(\mathbb{C}^{e}[z^{\pm1}])_{\mu}$, $[\Delta\lambda] \mapsto$ monomial wedges , $[L\lambda] \longrightarrow$ canonical bases.

§5 Rational Cherednik algebras.

Fix $\Gamma \subseteq \mathbb{C}^{\times}$ the subgroup of ℓ-th roots of unity. Let $\Gamma_d = \mathfrak{S}_d \ltimes \Gamma^{d}$ acting naturally on $\mathbb{C}^{d}$.

$$S = \bigsqcup_{\gamma \in \Gamma \setminus \{1\}} \{\gamma_i = (1,\dots,1,\gamma,1,\dots,1) : 1 \le i \le d\} \sqcup \{ s_{ij}\gamma_i\gamma_j^{-1} : i \neq j, \gamma \in \Gamma\}.$$

Fix $\kappa > 2$ an integer. $s = (s_1,\dots,s_\ell) \in \mathbb{Z}^{\ell}$, $\sum_p s_p = 0$. $c(s_{ij}\gamma_i\gamma_j^{-1}) = 1/\kappa$ and set
$$c(\gamma_i) = \sum_{p=1}^{\ell} (\gamma^{-p} - 1)\left(\tfrac{1}{\kappa}(s_p - s_{p+1}) - 1/\ell\right)$$
$\rightsquigarrow H(\Gamma_d)^{s}_{\kappa}$ and $\mathcal{O}(\Gamma_d)^{s}_{\kappa}$.

$\operatorname{Irr}(\Gamma_d) = \{\ell\text{-multipartitions of } d\} = \mathcal{P}^{\ell}_{d}$, $\lambda = (\lambda^{(1)},\dots,\lambda^{(\ell)})$, $\sum |\lambda^{(p)}| = d$.

KZ : $\mathcal{O}(\Gamma_d)^{s}_{\kappa} \longrightarrow \operatorname{mod}(\mathcal{H}(\Gamma_d)^{s}_{\kappa})$. $\mathcal{H}(\Gamma_d)^{s}_{\kappa} = B(\overset{0}{\circ}=\overset{1}{\circ}-\circ-\dots-\circ) / (T_j+1)(T_j-q)$ $1 \le i \le d$, $\prod (T_0 - Q_p) = 0$

<u>Remark</u>: (Rouquier) If s is dominant $\mathcal{O}(\Gamma_d)^{s}_{\kappa} \simeq \operatorname{mod}(\text{cyclotomic Schur algebra})$. ($\kappa > 2$ + condition on s).

Note $\bigoplus_d [\mathcal{O}(\Gamma_d)^{s}_{\kappa}] \simeq \text{Fock}^{s}_{\kappa} \circlearrowleft \widehat{\mathfrak{sl}}_{\kappa}$. $[\Delta\lambda] \longmapsto \mathbb{C}|\lambda\rangle$.

§6 Main Theorem:

Fix e, ℓ, N positive integers. $\mathfrak{g} = \mathfrak{gl}_N$. ν an ℓ-composition of N. Integral weight of $\mathfrak{g} \simeq \mathbb{Z}^{N}$.

$\mathcal{P}^{\nu}_{d} = \{\lambda \in \mathcal{P}^{\ell}_{d} \mid \ell(\lambda^{(p)}) \le \nu_p\} \hookrightarrow \mathbb{Z}^{N}$.

$$\lambda \longmapsto \lambda + \rho_\nu - \rho \qquad \rho = (0,-1,-2,\dots,-N+1) \ , \ \rho_\nu = (\nu_1, \nu_1 - 1, \dots, \nu_2, \nu_2 - 1, \dots)$$

$\mathcal{P}^{\nu}_{d}$ actually maps to $\mathcal{P}^{\nu}$, the ν-dominant weights.

To any $\lambda \in \mathcal{P}^{\nu}$ define $\hat{\lambda} = \lambda + (-e-N)\omega_0 + \frac{\langle \lambda, 2\rho + \lambda \rangle}{2e}\delta$, a weight for $\hat{\mathfrak{g}}$.

Let $A^{\nu}_{e}(d)$ = Serre subcategory generated by $L(\widehat{\lambda + \rho_\nu - \rho})$, $\lambda \in \mathcal{P}^{\nu}_{d}$.

<u>Lemma</u>: $A^{\nu}_{-e}(d)$ is highest weight.

Conjecture (Varagnolo-Vasserot) If $d \le \nu_p$ $\forall p$, $1 \le p \le \ell$. Take $s = (-\nu_\ell, -\nu_{\ell-1}, \dots, -\nu_1)$

Then there exists an equivalence of h.w. categories:

$$A^{\nu}_{-e}(d) \xrightarrow{\sim} \mathcal{O}(\Gamma_d)^s \qquad \lambda \in P^{\ell}_d$$

$$M(\lambda+\hat{\rho}_\nu-\rho) \mapsto \Delta(\lambda^*)^e \qquad \lambda^* = (\lambda^t_\ell, \lambda^t_{\ell-1}, \dots)$$

- proved by Rouquier-S.Varagnolo-Vasserot / Losev.

The Fock space $\mathrm{Fock}^s \circlearrowleft \hat{sl}_e$ with highest weight $\lambda_s = \sum_{p=1}^{\ell} \omega_{s_p}$.

Uglov: canonical basis of Fock^s.

$$\Lambda^{\infty/2} = \lim \Lambda^N(\mathbb{C}^e \otimes \mathbb{C}^\ell[z^{\pm 1}]) \circlearrowleft (\hat{sl}_{e\ell})_1$$

$$\hat{sl}_e \curvearrowright \mathrm{Fock}^{\nu}_{\mu} = \Lambda^{\infty/2}_{\mu,\nu} = \mathrm{Fock}^{\mu}_{\nu} \circlearrowleft \hat{sl}_\ell \qquad (\hat{sl}_e)_\ell \times (\hat{sl}_\ell)_e \subset (\hat{sl}_{e\ell})_1$$

Corollary: (Conjectured by Rouquier) $[\mathcal{O}(\Gamma_d)]^s \to \mathrm{Fock}^\nu$

simple $\to$ dual can. basis.

tilting $\to$ canonical basis.

Corollary: (Conjecture of Chuang-Miyachi)

$$\begin{array}{ccc}
D^b(\bar{\mathcal{O}}(\mathbb{Z}/e\mathbb{Z})^{\nu}_{-e,\mu}) & \xleftrightarrow{\text{DoE}} & D^b(\bar{\mathcal{O}}(\mathbb{Z}/e\mathbb{Z})^{\mu}_{-e,\nu}) \\
\downarrow \theta & & \downarrow \theta_\mu \\
\mathrm{Fock}^{\nu}_{\mu} & \longrightarrow & \mathrm{Fock}^{\mu}_{\nu}
\end{array}$$