jt with S. Griffeth, A. G

V f.dim$/\mathbb{C}$, $G \leq_{\text{finite}} GL(V)$

$R = \mathrm{Refl}(G) = \{s \in G \mid \operatorname{codim} V^s = 1\}$

G complex reflection gp : $G = \langle R \rangle$

For each $s \in R$, choose $\alpha_s \in V^*$ s.t $V^s = \ker \alpha_s$

Parameter c : $c : R \longrightarrow \mathbb{C}$, $s \longmapsto c_s$ s.t $\forall g \in G$, $c_{g^{-1}sg} = c_s$

$\rightsquigarrow$ Rational Cherednik algebra (Etingof - Ginzburg

$H_c(G,V)$ subalg. of $\mathrm{End}(\mathbb{C}[V])$ gen. by $\mathbb{C}[V]$, G & Dunkel operators

$S(V^*)$

for $y \in V$ define $T_y(f) = \partial_y f - \sum_{s \in R} c_s \langle \alpha_s, y \rangle \dfrac{f - {}^s f}{\alpha_s}$

for $f \in \mathbb{C}[V]$

Remark: If $c = 0$, we get $H_0(G,V) = \mathcal{D}(V) \rtimes G$

Fact: The T_y's commute

PBW Thm for $H_c(G,V)$

$$\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}V^* \xrightarrow{\sim} H_c(G,V)$$

triangle decomposition like $U(\mathfrak{g}) = S(\mathfrak{n}^-) \otimes S(\mathfrak{h}) \otimes S(\mathfrak{n}^+)$

as $\mathbb{C}$-v.sp.

$\rightsquigarrow$ category, standard modules (Ginzburg, Gray, Opdam Rouquier)

For $E \in \mathrm{Irr}\, \mathbb{C}G$-mod, $\Delta_c(E) = \mathrm{Ind}_{\mathbb{C}[V^*] \rtimes G}^{H_c(G,V)} E$

$$\downarrow$$

unique irr quotient $\longrightarrow$ $L_c(E) = \Delta_c(E) / \mathrm{Rad}\, \Delta_c(E)$

$\mathcal{O}_c = \mathcal{O}_c(G,V)$
$= \{ M \in H_c(G,V)\text{-mod} \mid \text{f. gen over } \mathbb{C}[V], \text{ loc. nilp. for } V \subseteq \mathbb{C}[V^*] \}$

Basic Question: When is $L_c(E)$ finite dimensional?

Support: $M \in \mathcal{O}_c$, since M is f.gen. we get a G-equiv. sheaf on V.

$\rightsquigarrow$ Can consider its support

$L_c(E)$ is f.dim $\iff$ the support is reduced to a point.

Grading: $H_c(G,V)$ is graded with $\deg V^* = 1$, $\deg G = 0$, $\deg V^* = 1$
$\rightsquigarrow$ internal grading induced by deformed Euler elements

$$eu_c = \sum_i x_i y_i + \sum_{s \in R} c_s (1-s)$$

$\in Z(\mathbb{C}G)$
acts by scalar c_E on E.

$M \in \mathcal{O}_c$, $M^d := \{ m \in M \mid (eu - d)^N m = 0, \ N >> 0 \}$

$$\mathcal{O}_c^{>z} = \{ M \mid M^d \neq 0 \Rightarrow d \in z + \mathbb{Z}_{>0} \}$$

$$0 \to \mathrm{Rad}\,\Delta_c(E) \longrightarrow \Delta_c(E) \longrightarrow L_c(E) \longrightarrow 0$$

$\mathrm{Rad}\,\Delta_c(E) \in \mathcal{O}_E^{>c_E}$

Stratum Braid Group

$0 \neq v \in V$, $S = \mathbb{C}^* v$, $G_s = \mathrm{Stab}_G(v)$, $N_s = \mathrm{Stab}_G(S)$
$= G_s \rtimes \langle h \rangle$

B_s group gen. by G_s & T with relation $TgT^{-1} = hgh^{-1}$.

For $F \subset \mathbb{C}N_s$-module irreducible, $\tau \in \mathbb{C}$, define
$F_\tau \in \mathbb{C}B_s$-module (irreducible) s.t
$F_\tau|_{G_s} = F$ & T acts by $e^{2\pi i \tau} h$

Thm (GGJL)

If $\mathbb{C}^* v \not\subset \mathrm{Supp}\, L_c(E)$, then for all $F \in \mathrm{Irr}\, N_s$ s.t
$\langle \mathrm{Rep}^G_{N_s} E, F \rangle \neq 0$, $\exists\, E' \in \mathrm{Irr}\, \mathbb{C}G$, $\exists\, F' \in \mathrm{Irr}\, \mathbb{C}N_s$ s.t

1) $E' >_c E$ i.e $c_{E'} - c_E \in \mathbb{Z}_{>0}$
2) $\langle \mathrm{Res}^G_N E', F' \rangle \neq 0$
3) $F_{\frac{c_E - c_F}{n_s}} \simeq F'_{\frac{c_{E'} - c_{F'}}{n_s}}$ n_s = order of h.

Cor: If $L_c(E)$ is f.dim then
$[\mathrm{Res}^G_{G_E} E] \in K_0(\mathbb{C}G_s)$
is a linear combination of $[\mathrm{Res}^G_{G_s} E']$ for $E' >_c E$.

Spherical Case: G Weyl gp c const: Varagando-Vasserot
= has denominatr "elliptic numbers".

G Coxeter group, c arbitrary, Etingof's criterion
(non-crystallographic) $\frac{P_W(q)}{P_{W_I}(q)}$ $q = e^{2\pi i c}$