D. Jordan : Quantization of character varieties via 4D TFTs.

joint w/ D. Ben-Zvi and A. Brochier.

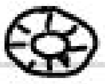
I Character varieties $Ch_G(S)$.

II Its Quantization $Ch_{U_q(\mathfrak{g})}(S)$

III Rep theory for $\otimes$-categories

IV Computing $Ch_{U_q(\mathfrak{g})}(S)$.

I. <u>Definition</u>: The character variety $Ch_G(S)$ is $Ch_G(S) = \{\rho : \pi_1(S) \to G\}$ / conj. action of G.

e.g. i) S =  = Annulus. $Ch_G(S) = G/G$ adjoint quotient.

ii) S =  or  $Ch_G(S) = \frac{G \times G}{G}$

iii) S =  $\Rightarrow$ $Ch_G(S) = Comm_n(G) = \{(A,B) \mid ABA^{-1}B^{-1} = Id\}$.

$G \times G \xrightarrow{\mu} G$ $\cong \mu^{-1}(id)/G$.

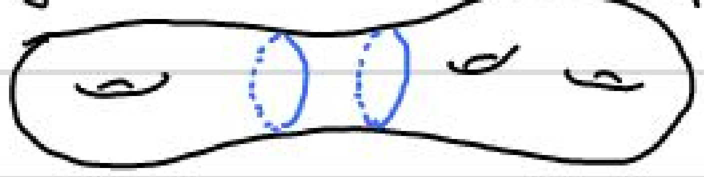
iv) S =  $= \cdot/G = BG$ $QCoh(BG) = Rep\, G$

I. $Ch_G(S)$ carries a canonical Poisson bracket (Goldman Atiyah-Bott) (G red. fix Killing form)

II. $Ch_G(S) = Map(S, BG)$ topological setting: BG algebraic, S simplicial

<u>Observation</u>: $Ch_G(S \sqcup_{P \times I} T)$

S T



$$QCoh(Ch_G(S \sqcup_{P\times I} T)) = QCoh_G(S) \boxtimes_{QCh(P\times I)} QCoh_G(T)$$

Goal: Quantize $Ch_G(S)$ uniformizing G, S.

<u>Theorem</u>/Construction: (B.Z. Brochier. J.) There exists a (2+2)-D TFT $Ch_{U_q(\mathfrak{g})}$ such that $Ch_{U_q(\mathfrak{g})}(S)$ is a quantization of $Ch_G(S)$. When S is punctured, we produce an algebra $a_S \in U_q(\mathfrak{g})$-mod and describe it explicitly.

$D^2 =$  $\longmapsto$ Rep $U_q(\mathfrak{g})$

Known topological / algebraic constructions

$\mathrm{Ann} =$  $\longmapsto$ $\mathcal{O}_q(G/G)$-mod

$V_1 \otimes \ldots \otimes V_n \circlearrowleft B_n(D^2)$.

$\mathcal{O}_q(\mathfrak{g})$-mod$_{U_q(\mathfrak{g})}$

$\uparrow$

braid dual, refl. eq. alg.;

equivariantized q.;

locally finite subalgebra...

$M \otimes V_1 \otimes \ldots \otimes V_n \circlearrowleft PB_n(\mathrm{Ann})$

$\Rightarrow$ Reconstruction.

$S =$  $\longmapsto \mathcal{D}_q^{str}(G/G)$-mod

$= \mathcal{D}_q(G)$-mod$_{U_q(\mathfrak{g})}$

Heisenberg double, quantum diff ops

$B_n(S) := \pi_1(\mathrm{Conf}_n(S))$

$M \otimes V_1 \otimes \ldots \otimes V_n \circlearrowleft PB_n(S)$

Brochier–J.: Reconstruction.

$\mathcal{D}_q(G) \circlearrowleft \widetilde{SL_2}(\mathbb{Z}) = \mathrm{Map}(T^2 \setminus D^2)$.

$\widetilde{SL_2}(\mathbb{Z})$ restore F.T.

$\circlearrowright$

$\mathcal{D}_q(G) \rightsquigarrow \mathcal{O}(G\times G) \circlearrowleft \widetilde{SL_2}(\mathbb{Z})$ (because it is a character variety.)

$\Big\downarrow q = e^{\hbar}$

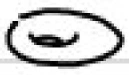
"$\rightsquigarrow$" = degeneration.

$\mathcal{D}(G) \longrightarrow \mathcal{D}(\mathfrak{g})$

$\uparrow$ no Fourier transform.

$\circlearrowleft$ Fourier transform $\rightsquigarrow$ e.g. Springer theory

> Related to DAHA: $q^k = 1$ [Varagnolo–Vasserot]
>
> $eH_{q,t=1}e$ for GL_N. — $q = e^{\hbar}$ [J].
>
> $eH_{q,t\neq 1}e$ for GL_2: joint with M. Balagovic
> $q \in \mathbb{C}$. Note also "Sam_q".

$T^2 =$  $\rightsquigarrow \mathcal{D}_q^{str}(G/G)$-mod

$= \mathcal{D}_q(G) /\!\!/_{\mu_q} U_q(\mathfrak{g})$ "quantum Hamiltonian reduction."

III Factorization homology: (Beilinson–Drinfeld, Lurie, Francis)

$A =$ E_n-algebra in $\mathcal{C}^{\otimes}$ a closed symmetric monoidal $(\infty,1)$-cat

$\mathcal{A}$ – braid monoidal tensor category $\in E_2$-alg $(\mathrm{Cat}_k^{\boxtimes})$ ← Deligne, Kelly tensor product

Factorization homology: $Z_A(S) \in \mathcal{C}^{\otimes} = \mathrm{Cat}_k^{\boxtimes}$

E_2-alg $=$ E_1-alg in E_1-alg

E_1-algebras in $\underline{\mathrm{Cat}}$ $\ni$ tensor categories (rigid)

Note $E_2 = (m_{ver}, m_{hor}) \cong (\otimes, c)$

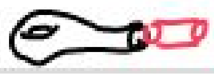
$\nearrow$ braiding

$Z_A(S) = \underset{\longleftarrow}{\mathrm{Colim}}\ A^{\boxtimes n}$

$\coprod_{i=1}^{n} U_i \to S$

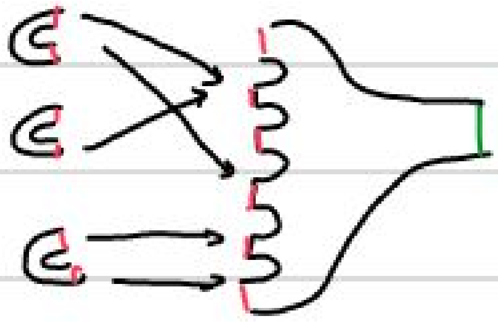
In particular, A = braided

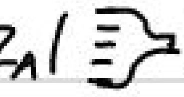
$Z_A(P \times I) = E_1$-algebra

$Z_A(S)$  is a module category.

Theorem: (Francis) $Z_A(S \sqcup_{P\times I} T) \cong Z_A(S) \boxtimes_{Z_A(P\times I)} Z_A(T)$

S a punctured surface has a handle & comb decomposition.



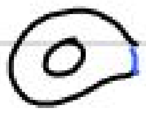
$Z_A($  $) = Z_A($  $) \boxtimes_{Z_A(-)} Z_A($  $)$

Technique: (Beck monadicity) Ostrik: $\mathcal{M}$-right $\mathcal{A}$-mod, $m \in \mathcal{M}$ cyclic generator. $\exists$ an algebra $\mathrm{End}(m) \in \mathcal{A}$-alg. $\mathcal{M} \simeq \mathrm{End}(m)$-mod.

BZ.B.J.: $F: \mathcal{A} \to \mathcal{B}$, have base change $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B} = F(\underline{\mathrm{End}}(m))\text{-mod}_{\mathcal{B}}$.

$Z_A($  $) = \mathcal{O}_{A \otimes A}\text{-mod}_{A \otimes A} \leftarrow$ FRT $\quad \bigoplus_X X^* \otimes X$

$\qquad \updownarrow T^{2n}$

 $\qquad \bigoplus_X X^* \otimes X \circlearrowleft$ Majid braided monoidal

Pivotal $A\otimes$-cat $\rightsquigarrow$ Frobenius algebra, $A \xrightarrow{tr} k$.

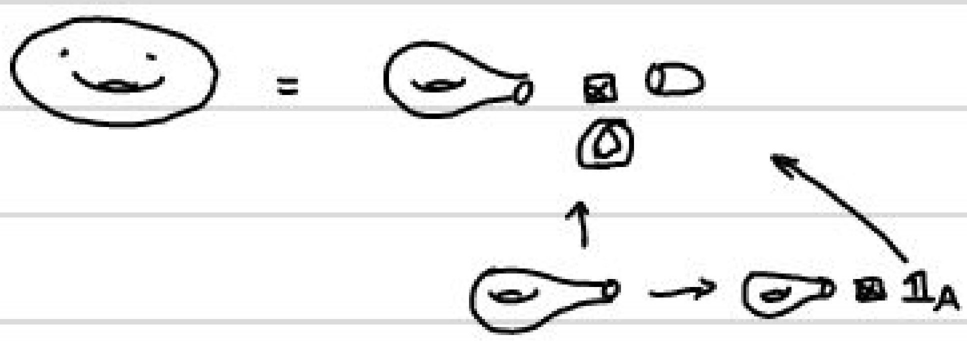
$\mathrm{Hom}(\mathbb{1}, -): \mathcal{A} \to \mathrm{Vect}_k \quad$ pivotal $\Rightarrow (\mathbb{1}, X \otimes Y) = (\mathbb{1}, Y \otimes X^{**})$

$(\mathbb{1}, Y \otimes X)$

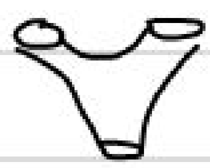
Frobenius algebra: $A = \mathrm{Mat}_n(\mathbb{C})$ with module $\mathbb{C}^n$. Let $\underset{m}{\overset{\shortparallel}{e_i}} \in \mathbb{C}^n$

Game: embed $\mathbb{C}^n \hookrightarrow \mathrm{Mat}_n(\mathbb{C})$ regular rep.

$tr_m(a) = \langle e_i, e_i \cdot a \rangle \in k \qquad tr_m^\vee(1) = E_i^i, \quad \mathbb{C}^n \cong E_i^i \mathrm{Mat}_n(\mathbb{C})$.



$Z_A($  $) = \mathcal{O}_q(G)\text{-mod}_A$

 $\qquad \mathcal{M} \otimes^L_{\mathcal{O}_q(G)} \varepsilon_A$

$\uparrow$

Hamiltonian reduction.