Schedler: Poisson deRham homology of cones and special polynomials.

I Weights of Poisson homology

II Poisson deRham

III Reduction from II to I

IV Geometric conjecture.

joint with Etingof, Proudfoot.

I. X complex Poisson. Assume X is a cone with contracting $\mathbb{C}^*$-action. Then $\mathcal{O}_X$ is $\mathbb{Z}_{\geq 0}$ graded, $(\mathcal{O}_X)_0 = \mathbb{C}$. Moreover $HP_0(\mathcal{O}_X) = \mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\} = (\mathcal{O}_X)_{H(X)}$ Ham. vector fields.
is non-negatively graded $\Rightarrow$ $h(HP_0(\mathcal{O}_X), t)$ Power series or polynomial if fin. dim.

Examples: $X = \mathbb{C}^2/\Gamma$, $\Gamma < SL_2(\mathbb{C})$ finite. $h(HP_0(\mathcal{O}_X), t) = \sum_i t^{2d_i - 2}$, d_i - degrees of invariant polynomials of $(\mathrm{Sym}\, \mathfrak{g})^{\mathfrak{g}}$, $\mathfrak{g}$ = McKay simple Lie algebra.

2) Generalizations: $X \subseteq \mathbb{C}^3$, quasihomogeneous, $\{Q = 0\} \subset \mathrm{Spec}\{\mathbb{C}[x,y,z]\}$ and

$$HP_0(X) = \mathcal{O}_X / (\partial_x Q, \partial_y Q, \partial_z Q)$$

in fact $\{\mathcal{O}_X, \mathcal{O}_X\} = (\partial_x Q, \partial_y Q, \partial_z Q)$.

Observed by [AL] following Greuel.

3) $S^n X$, where X is as in 2) (or 1)

Theorem: $HP^0(\mathcal{O}_{S^n X}) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda = (\lambda_1, \ldots, \lambda_k)}} \mathrm{Sym}^{\lambda_1} H \otimes \ldots \otimes \mathrm{Sym}^{\lambda_k} H\, [-d\, n_\lambda]$ where $H = HP_0(\mathcal{O}_X) = \mathcal{O}_X / (\partial_i Q)$, $n_\lambda = \sum_{i=1}^{k} (i-1)\lambda_i$

Equivalently, $\bigsqcup S^n X$, $\mathcal{O}_{\bigsqcup S^n X} = \mathrm{Sym}\, \mathcal{O}_X$, $HP_0(\mathrm{Sym}(\mathcal{O}_X))^* \cong \mathrm{Sym}(\mathbb{C}[t] \otimes H^*)$ $|t| = |Q| = d$
graded dual. bigraded.

This statement implies with some work, that $\rho_* \Omega \cong M_{S^n X}$, $X = \mathbb{C}^2/\Gamma$.

4) Hypertoric case: $X = T^*V /\!/\!/\!/ (\mathbb{C}^*)^k$. $(\mathbb{C}^*)^k \circlearrowright V$ inducing a Hamiltonian action on T^*V & have a moment map $\mu: T^*V \to (\mathrm{Lie}\, \mathbb{C}^*)^k$ and $X = \mu^{-1}(0) /\!/ (\mathbb{C}^*)^k$; Poisson with bracket deg -2

e.g. quiver variety with dim vectors all $= 1$.

$\underline{\text{Theorem}}$: (Proudfoot) $HP_0(\mathcal{O}_X) \cong IH^*(X^!)$, where $X^!$ is the Gale dual hypertoric variety to X. (studied by BLPW).

graded.

$\underline{\text{Conjecture}}$: (Proudfoot). If $X, X^!$ are symplectic dual cones and $\{\cdot,\cdot\}$ has degree -2, then

$$HP_0(\mathcal{O}_X) \cong IH^*(X^!)$$

$\underline{\text{Corollary}}$ (of Thm): $h^{br}_{A^\vee}(t^2) = t^{2rkA^\vee} T_{A^\vee}(t^{-2}; 0)$

hyperplane arrangement of $X^!$

Tutte

II Poisson-deRham.

Observation: $HP^{dR}_*(X)$ has an additional grading by weight.

Why? $HP^{dR}_*(X) = \pi_* M_X$, $\pi: X \to pt$, $M_X = \underset{\uparrow}{H(X)}\mathcal{D}_X \backslash \mathcal{D}_X$. $H(X)$ is homogeneous $\Rightarrow M_X$ is weakly $\mathbb{C}^*$-equivariant.

Ham Vector fields

Thus $\pi_* M_X$ is weakly equivariant, i.e. graded.

In other words $HP^{dR}_i(X) = H^{-i}(M_X \otimes^{L}_{D_V} \mathcal{O}_V)$ graded vector space

$\underline{\text{Examples}}$: $S^n(\mathbb{C}^2/\Gamma)$. Same as before but now with HP_*, $H \rightsquigarrow HP^{dR}_*(\mathbb{C}^2/\Gamma)$. $HP_0(\mathcal{O}_X) \otimes \mathbb{C}[\hbar]$

For example 4.

$\underline{\text{Theorem}}$: (Proudfoot-S.) $h(HP^{dR}_*(X; x, y)) = \underset{\sim}{\Phi_A}(x^{-2}+1, y^{-2}+1, y^2, \dots, y^2)$

Denham - defined using combinatorial Laplacian of associated matroid.

$$= y^{-2rkA} \sum_{F \leq A} y^{2|F|} T_{A^F}(x^2, 0) T_{A_F}(0, y^{-2})$$

coloop free flats $\leftrightarrow$ symp. leaves

restrict to F

localization.

How to prove: slices to symplectic leaves in hypertoric are hypertoric.

5) Nilpotent cones, symplectic leaves:

$\underline{\text{Conjecture}}$: (Lusztig) $h(HP^{dR}_*(X), x, y) = \sum_{\chi \in \hat{W}} K_{\mathfrak{g},\chi}(x^2) K_{\mathfrak{g},\chi}(y^{-2})$

where $K_{\mathfrak{g},\chi}(t)$ is the generalized Kostka polynomials and $K_{\mathfrak{D},\chi}(t) = \sum t^i \operatorname{Hom}_W(\chi, H^{2\dim(G/B)-2i}(G/B))$

Corollary & conjecture: $h(HP_0(W_e^{0,d})) = y^{\dim G\cdot e} \sum_{\chi : \theta_\chi = G\cdot e} K_{\mathfrak{g},\chi}(y^{-2})$

Springer: $\operatorname{Irrep}(W) \hookrightarrow$ (G-orbits on $\mathcal{N}$, local system on orbit)

Hotta-Kashiwara: $\rho_* \Omega_{T^*G/B} \cong \bigoplus_\chi IC(N_\chi) \otimes \chi$

Refined conjecture: $h(\chi; t) = K_{\mathfrak{g},\chi}(y^{-2})$.

III Reduction of HP^{dR} to HP_0.

Theorem: $\tilde{X} \to X$ conical symplectic and $\forall$ $S \subset X$ slices through symplectic leaves and $HP_0(\theta_X) \cong H^{\dim X_S}(\tilde{S}) \Rightarrow \rho_* \Omega \cong M_X$

Theorem: $M_X \cong \bigoplus IC(L_S)$ defined in ES $L_S = i^* M_X$, $i: S \hookrightarrow X$

As ordinary non-equivariant local systems, $L_S \cong H^{\dim X_S}(\rho^{-1}(S))$

non canonical

Closing: ES conjecture $\rho_* \Omega \cong M_X$ for quiver varieties would follow from $\dim HP_0(X_S)$

conjecture: $\dim HP_0(X_S) = \dim H^{\dim X_S}(\tilde{X}_S)$.

The Lusztig conjecture: χ = reflection rep θ_χ and other characters on subregular on ADE.