K. McGerty : Morse decomposition for $\mathcal{D}$-modules on stacks

— joint with T. Nevins

§1 Background

§2 Hamiltonian reduction & KN stratifications

§3 Recollement as categorical Morse decomposition.

§4 Applications.

§1 Background:

Many interesting algebras "A" are filtered so that their associated graded is the algebra of functions on some natural variety Y.

eg $\mathcal{U}(\mathfrak{g})_\chi$ — prin. quotient of $\mathcal{U}(\mathfrak{g})$ has $gr\,\mathcal{U}(\mathfrak{g}) = \mathbb{C}[\mathcal{N}]$ the nilcone

$eH_c e$ — $gr = Sym^n(\mathbb{C}^2)$ (H_c = RCA of S_n).

Y is usually singular, but has a nice resolution $X \xrightarrow{f} Y$ where X is no longer affine

Studying representations of A it is then natural to try and microlocalize them over Y to sheaves of modules on Y, a non-commutative f^* : A-mod $\to \mathcal{A}_c$-mod (c a def. parameter.)

Localisation (or microlocalisation) theory then asks when this is an equivalence.

§ Hamiltonian Reduction.

One method for constructing examples of these kinds of resolutions is via quotients:

Suppose W is a smooth affine variety $/\mathbb{C}$ and that $G \circlearrowright X$, G a conn. red. group.

Then we want to consider $T^*(W/G)$.

Since $G \circlearrowright W$ we get a Hamiltonian action on T^*W, so there is a moment map

$\mu : T^*W \to \mathfrak{g}^*$ $(\mathfrak{g} = Lie(G))$.

The correct cotangent bundle for W/G is then $\mu^{-1}(0)/G$.

— a possibly nasty space! How to understand "$\mu^{-1}(0)/G$" :

1) Affine quotient : set $Y = Spec(\mathbb{C}[\mu^{-1}(0)]^G)$

2) GIT : Take character χ of G and consider the GIT quotient

$X = \mathrm{Proj}\left(\bigoplus_{\ell \geq 0} \mathbb{C}[\mu^{-1}(0)]^{G,\chi^\ell}\right) = \mu^{-1}(0)/\!/_\chi G = \mu^{-1}(0)^{ss}/G$ (G acting freely on $\mu^{-1}(0)^{ss} \Rightarrow \mu^{-1}(0)^s/G$ a symplectic manifold)

3) Work with equivariant geometry of $\mu^{-1}(0)$

i.e. consider the whole stack!

$Y = \mathrm{Sym}^n(\mathbb{C}^2)$

$\boxed{\text{EXAMPLE}}$: $W = \begin{array}{c} \mathbb{C} \circ \\ \downarrow \\ V \circ \circlearrowleft \end{array} = \mathfrak{gl}(V) \oplus V \supseteq GL(V)$. $\times \begin{array}{c} \circ \\ i \downarrow\uparrow j \\ X \circlearrowleft \circ \circlearrowright Y \end{array}$. $\mu(X, Y, i, j) = [X, Y] + i \circ j$. $X = \mathrm{Hilb}^n$,

Noncommutative versions: Quantizing X and Y. $T^*W \rightsquigarrow \mathcal{D}_W$ (or $\mathcal{E}_W$ microdifferential operators)

The moment map has a quantum analogue: $\mu^*: \mathfrak{g} \to \mathcal{D}_W$. Can twist by $c \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$

Comm.	Noncomm.
$\mathbb{C}[\mu^{-1}(0)]$	$M_c := \mathcal{D}(W)/\mu_c^*(\mathfrak{g}).\mathcal{D}(W)$
$X = \mathrm{Spec}(\mathbb{C}[\mu^{-1}(0)]^G)$	$U_c = \mathrm{End}(M_c)^{op} \cong M_c^G$.
$Y = \mu^{-1}(0)^{ss}/G$	$((G,c)\text{-equiv } \mathcal{D}_W\text{-mod}) / ((G,c)\text{-equiv. } \mathcal{D}_W\text{-mod})^{un}$ $\;=\;$ "$\mathcal{E}_X$-mod"

where $(\;)^{un}$ are those modules with support in $(T^*W)^{un} = T^*W \setminus (T^*W)^{ss}$.

Note i) $\mathcal{M}$ in $(\mathcal{D}_W, G, c)$-mod $\Rightarrow SS(\mathcal{M}) \subseteq \mu^{-1}(0)$. Any $\mathcal{M}$ in $\mathcal{E}_X$-mod has SS on X etc

ii) If G acts freely on $\mu^{-1}(0)^{ss}$ then there is a sheaf of rings $\mathcal{A}_X^c$ on X and $\mathcal{E}_X^c$-mod <u>is</u> just $\mathcal{A}_X^c$-mod.

Have functors (f^*, f_*) (derived) quantizing commutative push-forward and pull-back, and $Rf_* f^* \xrightarrow{\sim} \mathrm{Id}$.

<u>Theorem</u> (McG. Nevins) If Lf^* is cohomologically bounded then (Lf^*, Rf_*) is an adjoint pair of equivalences of bounded and unbounded derived categories.

In particular, if U_c has fin. global dimension, then we have a derived equivalence.

<u>Remark</u>: Quantizations of line bundles on X give many equivalences of $D^b(\mathcal{E}_X\text{-mod})$ so this yields many interesting derived equivalences (c.f. Losev, Gordon-Losev.)

Thus we see that (in good cases) the quotient of $D((\mathcal{D}, G, c)\text{-mod})$ which is given by

$(T^*W)^{ss}$ (or $\mu^{-1}(0)^{ss}$) is exactly $D^b(\mathcal{U}_c\text{-mod})$.

<u>Question</u>: Can we understand the rest of $\boxed{D_c(T^*W/\!/G) = D((\mathcal{D}_W, G, c)\text{-mod})}$?

<u>Idea</u>: Decompose it by refining the G.I.T. information.

§ Recollement & Kirwan-Ness stratification

Kirwan-Ness stratification: The Hilbert-Mumford criterion shows instability is detected by the action of 1-psgs ("numerical criterion")

Kempf-Rousseau: noticed a point has an "optimal" destabilizing 1-psg which is (almost) unique. Fix a max. torus $T \subset G$ and X_*^+ chamber in $X_* = X_*(T)$.

Then let $S_\lambda = \{ x \in T^*W :$ the optimal 1-psg for x is conjugate to $\lambda \in X_*^+ \}$

<u>Theorem</u>: (Kirwan, Ness) The S_λ are locally closed subsets T^*W which can be ordered to ensure $\overline{S_\lambda} \subseteq \bigcup_{\mu \geq \lambda} S_\mu$, the latter closed, and $(T^*W)^{un} = \bigsqcup_\lambda S_\lambda$

<u>Example</u>: In $T^*(\mathfrak{gl}_n \times \mathbb{C}^n)$ the strata are $S_k = \{ (X, Y, i, j) : \dim(\mathbb{C}\langle X, Y\rangle i) = k \}$.

<u>Remark</u> i) These are the unstable manifolds for the $\|\mu\|^2$ (w.r.t. a Kähler metric) by a theorem of Kirwan. This Morse function is equivariantly perfect, which Kirwan uses to study symplectic quotients

ii) In our case: T^*W, the S_λ are <u>coisotropic</u>

§ Recollement: Consider subset Z which is a closed union of S_λs. Then if $U = T^*W \setminus Z$ we have $U \overset{j}{\hookrightarrow} T^*W \overset{i}{\hookleftarrow} Z$

We may define $D_c(T^*W/\!/G)_Z = \{ M \in D_c(T^*W/\!/G), SS(M) \subseteq Z \}$, and $D_c(U/\!/G)$ the quotient category $D_c(T^*W/\!/G) / D_c(U/\!/G)$ (as a dg category.)

<u>Theorem</u>: (McG. Nevins) The unbounded derived categories above have a recollement:

$$\begin{tikzcd} D_c(T^*W/\!/G)_Z \arrow[r, "i_*"] & D_c(T^*W/\!/G) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D_c(U/\!/G) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

- along with exact triangles

$$i_* i^! \to \mathrm{Id} \to j_* j^* \qquad j_! j^* \to \mathrm{Id} \to i_* i^*$$

- $j_!, j_*, i_*$ are full embeddings.
- These functors preserve <u>coherent</u> and compact objects.

<u>Remark</u>: i) In fact $Z_1 \subset Z_2$ are closed unions of strata the natural refinement also holds.

ii) A version of this when $Z = (T^*W)^{ss}$ in some cases is stated in BPW.

iii) Nadler - recollement for Fukaya category of branes on a Weinstein manifold.

§ <u>Applications</u>:

1) Compact objects: work in progress: a new proof that $D(\mathrm{Bun}_G)$ is compactly generated. (established recently by Drinfeld-Gaitsgory).

2) HH_* is decomposed by these recollement:

$$\begin{tikzcd} HH_*(\mathcal{D}(T^*W/\!/G)) \arrow[r, two heads] & HH_*(\mathcal{E}_X\text{-mod}) \cong H^*(X) \\ H^*(\mathcal{I}_G(T^*W)) \arrow[u, "\cong" {sloped}] \arrow[ur, two heads] & \end{tikzcd}$$

Hyper-Kähler Kirwan surjectivity wants $H^*((T^*W)/G)$ to surject. (Not always true.)