# Modular Rep Theory & Geometric Satake

## ① Rep. Theory

$G \rightsquigarrow$ conn. red. alg. gp. $/ k$, $k = \mathbb{C}$ or $\overline{\mathbb{F}_q}$

$X^+ =$ dom. wts for $G$.

irred: $L$

① $L_\lambda$, $\lambda \in X^+$

② Understand structure: $k = \mathbb{C}$, wt mult., Weyl Char. Formula

$k = \overline{\mathbb{F}_q}$, hard ??

other indecomposables ?? (interesting

$\lambda \in X^+$

Weyl modules $\quad M_\lambda \quad$ std $\qquad \mathrm{Ext}^i(M_\lambda, N_\mu) = 0$

$\updownarrow$ WCF works $\qquad \downarrow \qquad$ tilting $T_\lambda$

$L_\lambda \quad$ self-dual $\qquad$ (have Weyl + dual-Weyl

dual-Weyl modules $\qquad \downarrow \qquad \Rightarrow \mathrm{Ext}^i(T_\lambda, T_\mu) = 0$

$N_\lambda \quad$ costandard

## ② Geometry:

$G^\vee =$ Langlands dual to $G / \mathbb{C}$.

$$Gr = G^\vee(\mathbb{C}((t))) \big/ G^\vee(\mathbb{C}[[t]])$$

Big $\infty$-dim'l.

* ind-projective variety

${}^iGr$, $\quad {}^0Gr \subset {}^1Gr \subset \ldots \qquad \bigcup_i {}^iGr = Gr$

f.dim proj. var. for each $i$

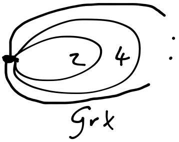
$G^\vee(\mathcal{O})$-stable

$G^\vee(\mathcal{O}) \curvearrowright Gr$

$G^\vee(\mathcal{O}) \backslash Gr \longleftrightarrow$ dominant coweights for $G^\vee \longleftrightarrow$ dom. wts for $G$

$Gr_\lambda \longleftrightarrow \lambda \in X^+$

**Example:** $G^\vee = SL_2\mathbb{C}$, $G = PGL_2\mathbb{C}$, $X^+ = 2\mathbb{N}$

$Gr_{SL_2}$ 

:

$H^\bullet(\overline{Gr_\lambda}, \mathbb{C}) \circlearrowleft PGL_2(\mathbb{C})$ — irred. of h.wt $\lambda$

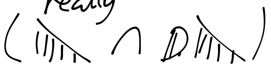
replacement better for spaces with singularities $H^\bullet \longleftrightarrow IH^\bullet$

$\overline{Gr_\lambda}$, singular projective

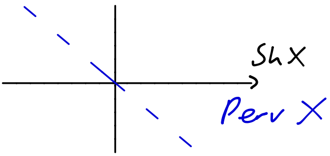
$\mathrm{Perv}_{G^\vee(\mathcal{O})}(Gr, k) \subset D^b_{(G^\vee(\mathcal{O}))}(Gr, k)$ all have f. dim support.

$\quad\quad \|$

$\varinjlim \mathrm{Perv}_{G^\vee(\mathcal{O})}(\dot{G}r, k)$

really 

order by dim support.




all perv. sh. here are f. length objects

$j_\lambda : Gr_\lambda \longrightarrow Gr$

taking ${}^pH$ changes stalks

$${}^pH^0(j_! \underline{k}) \longrightarrow {}^pH^0(j_* \underline{k})$$

$$\quad \| \quad\quad\quad\quad\quad\quad\quad\quad \|$$

$$I_!(\lambda, k) \twoheadrightarrow \underset{\text{simple}}{IC(\lambda, k)} \to I_*(\lambda, k)$$

**Thm:** (Lusztig, Ginzburg $k = \mathbb{C}$; Mirković-Vilonen any field)
(Geometric Satake)

$$(\mathrm{Rep}(G, k), \otimes) \xrightarrow[\sim]{S} (\mathrm{Perv}_{G^\vee(\mathcal{O})}(Gr, k), *)$$ convolution

[MV] $S(M_\lambda \twoheadrightarrow L_\lambda \hookrightarrow N_\lambda) = I_!(\lambda, k) \twoheadrightarrow IC(\lambda, k) \to I_*(\lambda, k)$

Tilting module question: geometric description, sheaf theoretic

$$S(T_\lambda) = ??$$

what do tilting modules corresp. to under $S$?

MV proved: $I_!(\lambda, \mathbb{Z}) = IC(\lambda, \mathbb{Z})$

known. $I_!(\lambda, \mathbb{Z}) \overset{L}{\otimes}_{\mathbb{Z}} k \cong I_!(\lambda, k)$

MV conj: The stalks of $I_!(\lambda, \mathbb{Z}) = IC(\lambda, \mathbb{Z})$ have no torsion.

(Juteau 2008) MV not true

Counterexamples of p-torsion

bad primes (either 2, 3, 5)

Modify:

<u>Thm: (Achar - R)</u>

MV-modified is true: As long as $p$ is JMW prime, then the stalks of $I(\lambda, \mathbb{Z})$ have no $p$-torsion.

known JMW primes:

| | |
|---|---|
| $A_n$ | any |
| $B_n$ | $p > n$ |
| $C_n$ $D_n$ | $p > 2$ |
| $E_6, F_4, G_2$ | $p > 3$ |
| $E_7$ | $p > 19$ |
| $E_8$ | $p > 31$ |

bound is not known to be tight.

Remark: Existence of $p$-torsion $/\mathbb{Z}$ is really a question about parity-vanishing properties $/k$ char $k = p$.

For a point, compute: $(\mathbb{Z}/p\mathbb{Z} \overset{L}{\otimes}_{\mathbb{Z}} k)$

char. $p$

MV p-conj. is implied by vanishing of stalks of $I_!(\lambda, k)$ in odd (or even) degrees.

**Thm [Achar-R]** For $k$ char $p$. JMW $I_!(\lambda, k)$ are "$\star$-parity".

## ④ JMW primes + Parity Sheaves

Decomp. thm doesn't work in modular case

Soergel... inspired theory of "parity sheaves" due to JMW

Defn: $X/\mathbb{C}$, $D^b_c(X, k)$

$\mathcal{F}^\bullet \in D^b_c(X, k)$ is $\star$ even (or $\star$-odd) parity if all stalks are concentrated in even or odd degrees.

!-even (!-odd) $\mathbb{D}$-condition: $\mathbb{D}\mathcal{F}^\bullet$ is $\star$-even (or $\star$-odd)

even = $\star$-even + !-even

odd = " "

parity = even $\oplus$ odd

**Thm [JMW]** If (indecomposable) parity sheaves exist, then they are unique (up to shift in $D^b(X)$). $P$ say

i.e $X_s \subset X$

$X_s$ open is supp $P$, $P|_{X_s} = \underline{k}[\dim X_s]$

Parity sheaves may not be perverse & may not exist

Question: What does it mean (rep thy) if all indecomp. parity sheaves are perverse?

Thm: [JMW] (Answer to Tilting Question)
$S(T_\lambda)$ is a parity sheaf; proved for all primes in JMW table.

Defn: JMW prime is $S(\text{tilting}) = \text{parity}$

Stalks: encoded in derived category. need "derived version" of geometric Satake.

$k = \mathbb{C}$, Arkhipov-Bezrukavnikov-Ginzburg: $D^b(Gr) = D(\quad)$

Thm: (Achar-R. + main ideas + construction of Achar-Riche)

$$D^{mix}_{(G^\vee(\mathcal{O}))}(Gr, k) \cong D^b_{perf.} Coh^{G \times \mathbb{G}_m}(\mathcal{N}/k)$$ nilpotent cone for $G$.

parity $\longleftrightarrow$ tilt(pcoh)

if char. $k$ is JMW, + conditions on $G$.

Implied by: Thm [Achar-R.] (inspired by thm of Ginzburg ~95)

$$\mathrm{Hom}^i(S(T_\lambda), S(T_\mu)) \cong \mathrm{Hom}(T_\lambda \otimes \mathcal{O}_{\mathcal{N}}, T_\mu \otimes \mathcal{O}_{\mathcal{N}}\langle i \rangle)$$

Future work (in progress) (Achar-R., Mautner-Riche)

Relate:

$$D^{mix}_{(I)}(Gr, k) \leftarrow\dashrightarrow D^b(Coh^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}, k))$$