S. Arkhipov:

G red. alg. group. $T \subset B \subset G$. I the simple roots. P_i almost minimal parabolics.
S_{P_i}

Lie algebras $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$.

Landweber & Sjaamar.

Harada et al: X a real compact manifold. H compact simple Lie group. $H \supset T$ a max torus. $K_T(X) \supset D_i$ $i \in I$, Demazure operators.

$D_i^2 = D_i$, D_is satisfy braid relations of type W.

$K_H(X) = \bigcap_{i \in I} \ker(D_i - 1)$ over $\mathbb{Q}$. $K_H(X) = K_T(X)^W$

<u>Remarks</u>: One can rephrase the action of D_is into an action of W.

<u>Definition</u>: Demazure descent data on a triangulated category C. $D_i : C \to C$ are comonad, $c_i : D_i \to D_i \circ D_i$ c_is isomorphisms, $\varepsilon_i : D_i \to \mathrm{Id}$.

D_is satisfy braid relations in W up to (unspecified) isomorphisms

Natural source of DDD on $\mathcal{C}$:

<u>Definition</u>: $QCHecke(G,B) := QCoh^{B \times B}(G)$ (derived category)

convolution $*: QCHecke(G,B) \times QCHecke(G,B) \longrightarrow QCHecke(G,B)$

$$\begin{tikzcd} & B \backslash \frac{G \times G}{B_\Delta} / B \arrow[dl, "\Delta"'] \arrow[dr, "m"] & \\ B\backslash G/B \times B\backslash G/B & & B\backslash G/B \end{tikzcd}$$

<u>Theorem</u>: (EPS) Consider $\mathcal{O}_{P_i} \in QCHecke(G,B)$. $\mathcal{O}_{P_i}$ is a coalgebra in $(QCHecke(G,B), *)$

1) $\mathcal{O}_{P_i} \xrightarrow{\sim} \mathcal{O}_{P_i} * \mathcal{O}_{P_i}$

2) $w \in W$, $w = s_{i_1} \cdots s_{i_p}$ reduced expression

$$\Rightarrow \mathcal{O}_{P_{i_1}} * \cdots * \mathcal{O}_{P_{i_p}} \cong \mathcal{O}_{\overline{BwB}}$$

Main idea: $\mathcal{O}_{P_{i_1}} * \cdots * \mathcal{O}_{P_{i_p}} = \Gamma(\mathcal{O}_{BS_{i_1,\dots,i_p}}) = \mathcal{O}_{\overline{BwB}}$ $BS_{i_1,\dots,i_p}$ the Bott-Samelson variety.

<u>Remark</u>: 2) $\Leftrightarrow$ braid relations

Corollary: Let C be a triangulated category with a monoid action of $QCHecke(G,B)$

$\Rightarrow$ the functors $D_i = M \to \mathcal{O}_{P_i} * M$ form a DDD on C.

Main idea: $X \circlearrowleft G$. $C = QCoh^B(X)$ $\exists$ a natural monoidal action of

$$QCHecke(G,B) \times QCoh^B(X) \to QCoh^B(X)$$

$$\begin{tikzcd} & B\backslash \frac{G\times X}{B} \arrow[dl] \arrow[dr] & \\ B\backslash G/B \times B\backslash X & & B\backslash X \end{tikzcd}$$

Example of example: $X = pt$

$C = QCoh^B(pt) = Rep(B) = \mathcal{O}_B$-comod

$Rep(B) \underset{Ind_i}{\overset{Res_i}{\rightleftarrows}} Rep(P_i)$ $\qquad D_i = Res\, Ind.$

Remark: Would like to get a categorical action of the braid group action on $QCoh^B(X)$

e.g. on $QCoh^B(X)$, $Rep(B)$. But there is none!

Question: Where does the braid group act?

Interpretation of Bezrukavnikov–Riche:

$QCoh^B(\mathfrak{b}) \underset{f_{i*}}{\overset{f_i^*}{\rightleftarrows}} QCoh^{P_i}(\mathfrak{p}_i)$. $\qquad f_i : \mathfrak{b} \hookrightarrow \mathfrak{p}_i$. $\qquad$ Consider $S_i = f_i^* f_{i*}$

Theorem: (Bezrukavnikov–Riche) $T_i = Cone(S_i \to Id)$, then the functors T_i satisfy braid group relations (up to isomorphism).

Remark: Consider the graded Hopf algebra Ω_B. Then by Koszul duality,

$QCoh^B(\mathfrak{b}) \simeq \Omega_B$-comod $= Rep(LB)$

$\downarrow \uparrow$

$QCoh^{P_i}(\mathfrak{p}_i) \simeq \Omega_{P_i}$-comod $= Rep(LP_i)$

"If DDD acts on C then braid group acts on the double of C"

"Theorem" X a G-variety, then $\exists$ a braid group action on "$QCoh^{LB}(LX)$"

Odd: Consider $H \times X \to X$ $\Rightarrow$ H acts on T^*X with comoment map.

Have $Sym_{\mathcal{O}_X}(\mathfrak{h} \otimes \mathcal{O}_X \to TX)$ $\qquad (\mathfrak{h} = Lie\, H)$.

$$\overset{122}{\Lambda(\mathfrak{h}) \otimes Sym(TX)} = \Lambda(\mathfrak{h}) \otimes \mathcal{O}_{T^*X}$$

H acts on $\Lambda(\mathfrak{h}) \otimes \mathcal{O}_{T^*X}$

<u>Definition</u>: Strongly equivariant quasi-coherent sheaves on T^*X: $(\Lambda(\mathfrak{h}) \otimes \mathcal{O}_{T^*X}\text{-DG-mod})^H$.

<u>Notation</u>: $QCoh^H_{strong}(T^*X)$

<u>Even</u>: Consider $\omega : T^*X \times \mathfrak{h} \to \mathbb{C}$ - moment map.

$$\omega \in \underbrace{\mathcal{O}_{T^*X} \otimes Sym(\mathfrak{h}^*)}_{\text{degree } 2.} = A$$

Get a curved differential graded algebra with curvature in A^2: d differential satisfying $d(xy) = (dx)y + (-1)^{|x|}x(dy)$, $d^2 = [c, x]$. In our case $d=0$, A commutative, but a curved module is $M = \oplus M^n$, $A^n \times M^n \to M^{n+m}$, with $d : M^n \to M^{n+1}$ with $d^2(m) = c.m$

<u>Notation</u>: $MF_{\mathbb{Z}}(T^*X \times \mathfrak{h}, \omega) = CDG\text{-mod}(A)$.

<u>Remark</u>: Polishchuk-Vaintrub define equivariant matrix factorisations and their derived category. $DMF^H_{\mathbb{Z}}(T^*X \times \mathfrak{h}, \omega)$ (also Positselski)

<u>Theorem</u>: (Konstrup, A.) Even and odd match:

$$QCoh^H_{str}(T^*X) \cong DMF^H_{\mathbb{Z}}(T^*X \times \mathfrak{h}, \omega). \qquad \text{(Proof: Koszul duality)}$$

<u>Remark</u>: Suppose H acts freely on X and $X \to X/H = Y$ is an H-principal bundle. Then both sides are equivalent to $D(QCoh(T^*Y))$

<u>Theorem</u> (Konstrup, A) Suppose $G \curvearrowright X$. There exists a categorical braid group action:

$$DMF^B_{\mathbb{Z}}(T^*X \times \mathfrak{b}, \omega)$$

<u>Construction</u>: Have $T^*X \times \mathfrak{b} \xhookrightarrow{f} T^*X \times \mathfrak{p}_i$ induces $DMF^B_{\mathbb{Z}}(T^*X \times \mathfrak{b}, \omega) \underset{f_*}{\overset{f^*}{\rightleftarrows}} DMF^{P_i}_{\mathbb{Z}}(T^*X \times \mathfrak{p}_i, \omega)$

Take composition.

Note: $X = pt$ recovers Bezrukavnikov-Riche.

<u>Final</u> <u>remark</u>: $QCHecke(G, B) = QCoh(B \backslash G / B)$ - two monoidal structures $\otimes$, $*$.

Ben-Zvi-Nadler: the derived double of $(QCHecke, \otimes)$ is $QCoh((\underset{\substack{\uparrow \\ \text{Steinberg.}}}{{}^LB \backslash {}^LG / {}^LB}), \otimes, *)$

& Francis

"hardcore" advanced stuff!

$$C_1 \underset{G}{\overset{F}{\rightleftarrows}} C_2$$

F - monoidal, G satisfying projection formula (left & right).

$G(F(X) \otimes Y) \cong X \otimes G(Y) \Rightarrow \exists$ an action of $\hat{F} : D(C_1) \to D(C_2)$.

$$(Rep(B), \otimes) \underset{Ind}{\overset{Res}{\rightleftarrows}} (Rep(P_i), \otimes)$$

Res monoidal, Ind satisfies projection formula

$$(QCoh(B), *) \underset{i_*}{\overset{i^*}{\leftrightarrows}} (QCoh(P_i), *)$$

i_* is monoidal, i^* satisfies projection formula

$QCoh^B(B) \leftrightarrows QCoh^{P_i}(P_i)$ $\quad S_i = \hat{i^*} \hat{Ind}$