1. Losev: Derived equivalences for RCAs.

1) Hecke algebras

2) RCA & category $\mathcal{O}$

3) KZ functor & Rouquier's conjecture

4) Outline of proof

5) Something perverse.

§1 W - Weyl group $\qquad W \curvearrowright \mathfrak{h}$ reflection rep.

$\cup$
S = refl $\supset I$ = simple refl $\mathfrak{h}^{reg} \subset \mathfrak{h}$, $\mathfrak{h}^{reg} = \{x \in \mathfrak{h} : W_x = \{1\}\}$. Braid group $B_W = \pi_1(\mathfrak{h}^{reg}/W)$.

$B_W = \langle T_i : i \in I \rangle /$ relations. Parameter $q : S/W \to \mathbb{C}^*$. $H_q(W) = B_W / ((T_i+1)(T_i-q) = 0 \quad i \in I)$

In special points $H_q(W)$-mod has inf. hom. dim. (a.k.a singular.)

2). RCA: $H_c \subset \mathcal{D}(\mathfrak{h}^{reg}) \# W$, $c : S/W \to \mathbb{C}$, generated by $\mathfrak{h}^*, W$, Dunkl operators D_a, $a \in \mathfrak{h}$.

$D_a = \partial_a + \sum_{s \in S} c(s) \frac{\alpha_s(a)}{\alpha_s}(s - id)$ (α_s a root)

Triangular decomp.: $H_c = S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \rightsquigarrow$ category $\mathcal{O}_c = \{M \in H_c\text{-mod} \mid$ fin. gen. $S(\mathfrak{h}^*)$

loc. nilp. for $\mathfrak{h}\}$.

Verma modules $\lambda \in \mathrm{Irr} W \rightsquigarrow \Delta_c(\lambda) = H_c \otimes_{S\mathfrak{h} \# W} \lambda$ and order $<^c$ ($\lambda \rightsquigarrow c_\lambda$) where

$\lambda <^c \mu \Leftrightarrow c_\lambda - c_\mu \in \mathbb{Q}_{>0}$. $\mathcal{O}_c$ is highest weight with standard objects $\Delta_c(\lambda)$ and order

$<^c$: 0) $\mathcal{O}_c \simeq A$-mod where A a fin. dim. algebra

Notation: $L_c(\lambda)$ = irred. quotient of $\Delta_c(\lambda)$. $P_c(\lambda)$ the proj. cover of $L_c(\lambda)$.

1) $\mathrm{Hom}(\Delta_c(\lambda), \Delta_c(\mu)) \neq 0 \Rightarrow \lambda \leq \mu$.

2) $\mathrm{End}(\Delta_c(\lambda)) = \mathbb{C}$

3) $P_c(\lambda) \twoheadrightarrow \Delta_c(\lambda)$ and ker has filtration with quotients $\Delta_c(\mu)$, $\mu > \lambda$.

Note: Highest weight $\Rightarrow$ hom. dim. $< \infty$ (smooth).

4) $\delta = \prod_{s \in S} \alpha_s^2 \rightsquigarrow H_c[\delta^{-1}] \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}^{reg}) \# W$ and $M \to M[\delta^{-1}]$ giving $\mathcal{O}_c \to \mathrm{Loc}^W(\mathfrak{h}^{reg})$

giving a functor to B_W-mod (take monodromy rep. of a fibre)

This factors through $\mathcal{H}_q(W)$-mod, $q = \exp(2\pi i c)$.

Theorem: (GGOR) 1)
$$\begin{tikzcd} \mathcal{O}_c \arrow[d] \arrow[dr, "KZ"] & \\ \mathcal{O}_c/\mathcal{O}_c^{tor} \arrow[r, "\sim"] & H_q(W)\text{-mod} \end{tikzcd}$$

$\mathcal{O}_c^{tor}$ - modules which are torsion over $S(\mathfrak{h}^*)$.

2) KZ is fully faithful on $\mathcal{O}_c$-proj (and on $\mathcal{O}_c$-tilting), so KZ is "a resolution of singularities". (whatever that means.)

$c' \in c + \mathbb{Z}^{S/W} \rightsquigarrow q$ thus we have many $\mathcal{O}_c$s resolving $H_W(q)$-mod.

Conjecture: (Rouquier '05) These $\mathcal{O}_c$s are all derived equivalent.

Gordon-L. ('11) proof for type A, B.

Theorem: (L. '14) Conjecture is true in general.

§4. Question: When is $\mathcal{O}_c, \mathcal{O}_{c'}$ are equivalent as abelian categories?

Orders $<^c$ on Irr W defined by linear functions. Fix $c + \mathbb{Z}^{S/W} \rightsquigarrow$ rational cones where orders are same.

Theorem: (Rouquier '05, Losev '13/'14 in full generality): If $\leq^c$ refines $\leq^{c'}$ ($c' \in c + \mathbb{Z}^{S/W}$)

$$\Rightarrow \begin{tikzcd} \mathcal{O}_c \arrow[r, "\sim"] \arrow[dr, "KZ"'] & \mathcal{O}_{c'} \arrow[d, "KZ"] \\ & H_q(W)\text{-mod} \end{tikzcd}$$

b) $c' = c - \psi$ lie in opposite cones ($\psi \in \mathbb{Z}^{S/W}$)? Answer: $\mathcal{O}_c$ and $\mathcal{O}_{c'}$ are Ringel dual.

$\mathcal{O}_c^\Delta$ - standard-filtered objects, $\mathcal{O}_{c-\psi}^\nabla$ = costandard-filtered objects

Ringel dual $\Leftrightarrow$ $\mathcal{O}_c^\Delta \xrightarrow{\sim} \mathcal{O}_{c-\psi}^\nabla$: defines a derived equivalence:

$$\begin{tikzcd} R: D^b(\mathcal{O}_c) \arrow[r, "\sim"] \arrow[dr, "KZ"'] & D^b(\mathcal{O}_{c-\psi}) \arrow[d, "KZ"] \\ & D^b(H_q(W)) \end{tikzcd} \quad : KZ(P_c(\lambda)) = KZ(T_{c-\psi}(\lambda))$$

Simply-laced case have 2 cones.

Alternative realization (as $\otimes^L$ of HC bimodules):

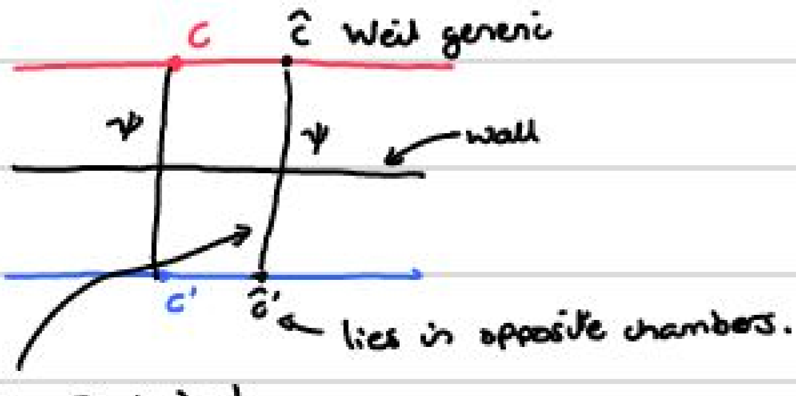
Definition: A $H_{c'}$-H_c-bimodule $\mathcal{B}$ is HC if ad actions of $S(\mathfrak{h}^*)^W$, $S(\mathfrak{h})^W$ on $\mathcal{B}$ are locally nilpotent.

Claim: $\exists !$ simple H.C. bimodule $\mathcal{B}_c(\psi)$ such that $\mathcal{B}_c(\psi)[\delta^{-1}] = \mathcal{D}(\mathfrak{h}^{reg}) \# W$.

(subquotient of shift bimodule).

Proposition: $R \simeq \mathcal{B}_c(\psi) \otimes^L_{H_c} \cdot \quad D^b(\mathcal{O}_c) \longrightarrow D^b(\mathcal{O}_{c'})$

c)



equiv. $\mathcal{B}_{\hat{c}}(\psi) \otimes^L$.

Claim: $\exists$ a generically (Zariski) flat family of HC $H_{\hat{c}'}$-$H_{\hat{c}}$-bimodules whose Weil generic fiber is $\mathcal{B}_{\hat{c}}(\psi)$. Still denote Zariski generic fiber by $\mathcal{B}_{\hat{c}}(\psi)$. Its localization is still $\mathcal{D}(\mathfrak{h}^{reg}) \# W$

Theorem: If $\hat{c}$ is Zariski generic, $\mathcal{B}_{\hat{c}}(\psi) \otimes^L_{H_{\hat{c}}} : D^b(\mathcal{O}_{\hat{c}}) \to D^b(\mathcal{O}_{\hat{c}'})$.

Proof: We know it is an equivalence for gen $\hat{c} \Rightarrow \mathrm{REnd}_{H_{\hat{c}}}(\mathcal{B}_{\hat{c}}(\psi)) = H_{\hat{c}} \Rightarrow$ same statement for Zariski generic $c \Rightarrow$ equiv. for Zarisk gen. $\hat{c}$ (not immediate – also use that $\mathcal{B}_{\hat{c}}(\psi) \otimes^L \Delta_c(\lambda)$ is an object with class $[\nabla_{c-\psi}(\lambda)]$ ($\kappa\mathbb{Z}$ twist of $\psi = 1$).

Thus have equivalence for Zariski generic c on the line. Transfer to arbitrary c using Theorem 2.

§5. Perverse equivalences: c, c' are separated by a single wall: $\varphi_c : D^b(\mathcal{O}_c) \longrightarrow D^b(\mathcal{O}_{c'})$

$\mathcal{B}_c(\psi) \otimes^L_{H_c} \cdot$ is perverse.

Let $\mathcal{C}^1 = \mathcal{O}_c$ filtered $\mathcal{C}^1_0 \supseteq \mathcal{C}^1_1 \supseteq \mathcal{C}^1_2 \supseteq \dots \supseteq \mathcal{C}^1_n \supset \mathcal{C}^1_{n+1} = \{0\}$. $n = \dim \mathfrak{h}$.

Similarly for $\mathcal{C}^2 = \mathcal{O}_{c'}$, where $\mathcal{C}^1_j = \{M : J^1_j M = 0\}$ where J^1_j is a 2-sided ideal

where $H_c = J^1_{n+1,c} \supseteq J^1_{n,c} \supseteq \dots \supset J^1_{1,c} \supset J^1_{0,c} = \{0\}$

For Weil generic c $\quad J^1_{j,c} = \bigcap_{M,\, GK\text{-}\dim M \leq n-j} \mathrm{Ann}(M)$ (filtration by dim of support.)

Zariski generic c – do flat degeneration.

Theorem 3: (I.L) $\varphi_c : D^b(\mathcal{O}_c) \longrightarrow D^b(\mathcal{O}_{c'})$ is perverse: 1) $D^b_{\mathcal{C}^1_j}(\mathcal{C}^1) \xrightarrow{\sim} D^b_{\mathcal{C}^2_j}(\mathcal{C}^2)$

2) $M \in \mathcal{C}^1_j \Rightarrow H_k(\varphi_c M) = 0 \ \ k < j$, $\in \mathcal{C}^2_{j+1}$, $k > j$, 3) $\mathcal{C}^1_j / \mathcal{C}^1_{j+1} \xrightarrow{\sim} \mathcal{C}^2_j / \mathcal{C}^2_{j+1}$ $\quad H_j(\varphi_c)$