G. Williamson: Global and local Hodge theory of Soergel bimodules.

Global story will have to wait...

Jantzen conjectures: $\mathfrak{g} \supset \mathfrak{b}$, $\mathfrak{h} = \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$. $\lambda \in \mathfrak{h}^*$ a highest weight, has $\Delta(\lambda)$ a Verma module, v_λ a h.w. vector.

Given a direction $\gamma \in \mathfrak{h}^*$ can deform $\Delta_A(\lambda)$ a $(\mathfrak{g}, A)$-bimodule: $h \cdot v_\lambda = (\lambda(h) + \gamma(h)t) v_\lambda$

$$A = \mathbb{C}[t]$$

$(-,-) : \Delta_A(\lambda) \times \Delta_A(\lambda) \to A$ contravariant form. Can form the vanishing filtration $J^0 \supset J^1 \supset \dots$

$$\operatorname{rad}(-,-) \otimes_A \mathbb{C}.$$

Jantzen conjecture ('78) Certain canonical morphisms are strict for $J^\cdot$ (using $\gamma = \rho$).

Deodhar: Does the filtration depend on γ? (for γ non-degenerate)

Gabber-Joseph ('82): J.C. $\Rightarrow \sum [gr^i_J \Delta(\lambda) : L(\mu)] v^i$ is a KL poly $\Rightarrow$ KL conjecture

Barbasch ('84?) J.C. $\Rightarrow$ $J^\cdot$ is the socle filtration

Beilinson-Bernstein ('82-'90): JC is true if γ is dominant integral ($J^\cdot$ "is" the weight filtration

(also works for $(\mathfrak{g}, K)$-mod) weight = monodromy.)

Soergel ('07) & Kübel ('12): new proof of JC using "local hard Lefschetz" ("weight = degree")

(roughly, Koszul dual to BB.)

W. ('14) Algebraic proof of hard Lefschetz in the language of Soergel bimodules (works for (W,S))

$\Rightarrow$ algebraic proof of J.C.

If V is a Weyl module $/\mathbb{Z}$. $V \otimes_{\mathbb{Z}} \mathbb{F}_p$ - suggests deforming / studying is familiar.

Corollary: Answer to Deodhar question is yes! True in multiplicity 1 case

False for $\mathfrak{sl}_4$.

Classical Hodge Theory: Define $H^* = H^{*+\dim_{\mathbb{C}} X}(X, \mathbb{R})$, $X/\mathbb{C}$ smooth projective.

$\langle -,- \rangle$ Poincaré pairing $H^d \times H^{-d} \to \mathbb{R}$.

$$H^*_{\mathbb{C}} \cong \bigoplus H^{p,q} \quad \text{(below only } (p,p)\text{-type).}$$

$L : H^* \to H^{*+2}$ Lefschetz operator.

Hard Lefschetz: $H^*_{\mathbb{C}} = \bigoplus_{d \geq 0} \mathbb{C}[\ell]/(\ell^{d+1}) \otimes P^{-d}$ $\qquad P^{-d} = \ker L^{d+1} \subset H^{-d}$.

Hodge-Riemann relations: formula for signature of $\langle x, L^d y \rangle$ on $H^{-d}_{\mathbb{C}}$ in terms of Hodge numbers.

Local Hodge theory: $\mathbb{C}^* \curvearrowright \mathbb{C}^n$ positive weights $\qquad A = H^*_{\mathbb{C}^*}(pt, \mathbb{R}) = \mathbb{R}[z]$.

$$\begin{array}{l} \mathbb{C}^n \\ \cup \\ X \end{array}$$

X $\mathbb{C}^*$-stable closed subvariety $\qquad \dot{X} = X \setminus \{0\}$

$$\begin{tikzcd}
0 \arrow[r] & \overset{M^!}{IH^*_{\mathbb{C}^*,c}(X;\mathbb{R})} \arrow[r] \arrow[d, equal] & \overset{M}{IH^*_{\mathbb{C}^*}(X;\mathbb{R})} \arrow[r, "j^*"] \arrow[d, equal] & IH^*_{\mathbb{C}^*}(\dot{X};\mathbb{R}) \arrow[r] \arrow[d, equal] & 0 \\
& IH^*_{\mathbb{C}^*}(i^! IC_X) & IH^*_{\mathbb{C}^*}(i^* IC_X) & IH^{*+1}(\dot{X}/\mathbb{C}^*, \mathbb{R}) &
\end{tikzcd}$$

all groups pure $\Rightarrow$ a l.e.s. splits into two s.e.s.

The IC conditions $\Rightarrow$ $M^!$ (resp M) are generated in degrees > 0 (resp. < 0) $\Rightarrow$ j^* is a proj cover.

Simplest example: $X = \mathbb{C}^n$: $0 \to A[-n] \to A[n] \to A/(z^n) \to 0$

$$A/(z^n) = IH^{*+1}(\mathbb{P}^{n-1})$$

Pairing: $M^! \times M \to A$ is nondegenerate. $A[z^{-1}] \otimes_A M^! \xrightarrow{\sim} A[z^{-1}] \otimes M$

Thus we can view the pairing as an $A[z^{-1}]$-valued form on M, and $M, M^!$ are dual lattices in $A[z^{-1}] \otimes_A M$. Thus everything is encoded in $(M, \langle \cdot, \cdot \rangle)$

Primitive decomposition: $M = \overset{\perp}{\bigoplus} A \otimes_{\mathbb{R}} P^{-d}$.

For Schubert varieties, $\{xB\} \in G/B$, a T-fixed point, and pick $X \ni xB$ a T-stable affine neighbourhood.

<u>Deodhar's Question</u>: Does $IH^*(\dot{X}/\mathbb{C}^*, \mathbb{R})$ satisfy hard Lefschetz if $\mathbb{C}^* \subset T$ is regular, i.e. $X^{\mathbb{C}^*} = pt$.

Key idea in proof: $\qquad ws > w$, IC_w

$$\begin{tikzcd}
& G/B \arrow[d, "\pi"] \\
\{xP_s\} \arrow[r, hook, "i"] & G/P_s
\end{tikzcd} \qquad i^! \pi_* IC_w \to i^* \pi_* IC_w$$

would like to understand

$$\begin{tikzcd}
i' : \mathbb{P}^1 \arrow[r, "i'"] \arrow[d] & G/B \arrow[d] \\
\{xP_s\} \arrow[r, hook, "i"] & G/P_s
\end{tikzcd}$$

$(i')^! IC_w \to (i')^* IC_w$ morphism of complexes on $\mathbb{P}^1$

$V/\mathbb{R}$ real vector space with two symmetric forms: $(-,-)^0$ pos. def.

$(-,-)^\infty$ neg. def.

Hard Lefschetz for global sections $\Leftrightarrow$ $(-,-)^0 + (-,-)^\infty$ is non-degenerate

H.R. $:\Leftrightarrow (-,-)^0 + (-,-)^\infty$ is PD !

de Cataldo & Migliorini trick: "weak Lefschetz" + HR for smaller ω

$\Rightarrow a(-,-)^0 + (-,-)^\infty$ is non-deg. $\forall\, a \geq 1$.

$\Rightarrow (-,-)^0 + (-,-)^\infty$ is PD !