P. Etingof: $\mathcal{D}$-modules on Poisson varieties, Poisson homology, and symplectic resolutions

X affine $/\mathbb{C}$, $\mathfrak{g}$ Lie algebra. $\alpha: \mathfrak{g} \to \mathrm{Vect}(X) = \mathrm{Der}(\mathcal{O}_X)$. $\mathcal{O}_X/\mathfrak{g}\mathcal{O}_X = (\mathcal{O}_X)_{\mathfrak{g}}$ coinvariants.

When is $(\mathcal{O}_X)_{\mathfrak{g}}$ fin. dim.?

Let $X_i = \{x \in X : \dim \alpha_x(\mathfrak{g}) = i\}$, a locally closed subvariety, $\dim X_{ij} \geq i$ (X_{ij} the components of X_i.

Def: We say that X has finitely many $\mathfrak{g}$-leaves if $\dim X_i = i$. In this case the connected components of X_i, X_{ij}, is smooth and $\alpha_x(\mathfrak{g}) = T_x X_i$ $\forall x \in X_i$.

The X_{ij} are called the $\mathfrak{g}$-leaves

Theorem: If X has finitely many $\mathfrak{g}$-leaves then $\mathcal{O}_X/\mathfrak{g}\mathcal{O}_X$ is fin. dim

Proof: (Sketch) Define a right $\mathcal{D}$-module $M(\alpha) = \mathfrak{g}\mathcal{D}_X \backslash \mathcal{D}_X$. Any X has a canonical $\mathcal{D}_X$ such that $\mathrm{Hom}(\mathcal{D}_X, M) = \Gamma(M)$: X singular, $X \subset V$ smooth, $\mathcal{D}_X = \underset{\substack{\uparrow \\ \text{ideal of } X \text{ in } \mathcal{O}_V.}}{I_X} \cdot \mathcal{D}_V \backslash \mathcal{D}_V$

$\tilde{\mathfrak{g}} = \{v \in \mathrm{Vect}(X) \mid v \text{ tangent to } X, v|_X \in \alpha_X(\mathfrak{g})\}$.

Lemma 1. $(\mathcal{O}_X)_{\mathfrak{g}} = \pi_0 M(\alpha)$ where π_0 is underived push-forward, $\pi: X \to pt$.

Proof: $\pi_0 M(\alpha) = M(\alpha) \otimes_{\mathcal{D}_V} \mathcal{O}_V = (\tilde{\mathfrak{g}}\mathcal{D}_V + I_X \mathcal{D}_V) \backslash \mathcal{D}_V \otimes_{\mathcal{D}_V} \mathcal{O}_V$

Lemma 2: If X has finitely many $\mathfrak{g}$-leaves then $M(\alpha)$ is holonomic

Proof: $SS(M(\alpha)) \subseteq$ conormals to the $\mathfrak{g}$-leaves, and so Lagrangian.

Special case: X a Poisson variety, $\mathfrak{g} = \mathcal{O}_X$ acting on itself. Coinvariants are $\mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_X\} = HP_0(\mathcal{O}_X)$ (or write $HP_0(X)$). In this case $\mathfrak{g}$-leaves are symplectic leaves.

Corollary: If X has finitely many symplectic leaves, $HP_0(X)$ is fin. dim

For any Poisson X, have a canonical $\mathcal{D}$-module $M_X = H\mathrm{Vect}(X).\mathcal{D}_X \backslash \mathcal{D}_X$.

Generalize: X not necessarily affine, take π_* instead of π_0.

Define: The Poisson-deRham homology of X is

$$HP_i^{dR}(X) := H^{-i}(\pi_* M_X)$$

If X has fin. many symplectic leaves, then $HP_*^{dR}(X)$ is fin. dimensional.

e.g. X symplectic of dim $n = 2r$, then $M_X = \Omega$, $HP_i^{dR}(X) = H^{n-i}_{dR}(X, \mathbb{C})$.

e.g. V fin. dim. vector space, $G < Sp(V)$ finite. Then V/G is Poisson and symplectic leaves are given by strata given by conjugacy classes of stabilizers $\Rightarrow \mathcal{O}_{V/G} / \{\mathcal{O}_{V/G}, \mathcal{O}_{V/G}\}$ is fin. dim.

(conjectured by Alev-Farkas, first proved by Berest-Etingof-Ginzburg.)

Note $\mathrm{Hom}(M_{V/G}, \delta_0) = HP_0(V/G)^*$ so computing multiplicity of M_X along leaves is essentially the problem "what is $\dim HP_0$". Hard!

Application to rep. theory: If A is a $\mathbb{Z}_+$-filtered algebra, $\mathrm{gr} A = A_0$ is commutative, then A_0 is Poisson bracket $\{\cdot,\cdot\}$ of degree $-d$ (some $d > 0$ - if A noncommutative $\{\cdot,\cdot\}$ must be nonzero.)

<u>Theorem</u>: If $X = \mathrm{Spec}(A_0)$ has fin. many symp. leaves then A has fin. many irreducible fin. dim. representations ($\leq \dim HP_0(X)$).

<u>Proof</u>: By Wedderburn, characters of irred. fin. dim. reps are linearly independent in $(A/[A,A])^*$ so $\#$reps $\leq \dim A/[A,A]$, but $A_0/\{A_0,A_0\} \twoheadrightarrow A/[A,A]$.

<u>Question</u>: Can we compute M_X?

<u>Conjecture</u>: Let $\rho : \tilde{X} \to X$ be a symplectic resolution. Conjecture that $M_X = \rho_* \Omega_{\tilde{X}}$.

Remark: 1) ρ is semismall so $\rho_* \Omega_{\tilde{X}}$ is a $\mathcal{D}$-module.

2). Conjecture implies that in this case M_X is semisimple.

Suppose $\rho : \tilde{X} \to X$ is equivariant w.r.t. a $\mathbb{C}^*$-action with positive weight on $\omega_{\tilde{X}}$. Ginzburg and Kaledin show $\exists$ $A(c)$ a quantization of A_0 with $c \in H^2(\tilde{X}, \mathbb{C})$, $HH_0(A(c)) \cong H^{\dim(X)}(\tilde{X}, \mathbb{C})$.

$$\left.\begin{array}{c} \text{The conjecture implies } HP_0(A_0) \twoheadrightarrow HH_0(A(c)) \\ \| \\ A_0/\{A_0,A_0\} \cong A(c)/[A(c),A(c)] \end{array}\right\} \text{so } HH_0 = H^{\dim(X)}(\tilde{X}) \text{ for all such } c.$$

Examples: 1) Y a symplectic surface. $\mathrm{Hilb}^n(Y) \xrightarrow{\rho} S^nY$.

More generally Y an ADE singularity, $\tilde{Y} \to Y$ symplectic resolution, then $\mathrm{Hilb}^n(\tilde{Y}) \to S^nY$.

2). Springer resolutions: $T^*G/B \xrightarrow{\rho} \mathcal{N}_G$ and Slodowy slices for nilpotent elements.

3). Quiver varieties.

4) Hypertoric varieties.

Known: in case 1) Etingof-Schedler.

2) $\mathcal{N} \hookrightarrow \mathfrak{g}$ and $M_X = ((z-\varepsilon(z))\mathcal{D}_\mathfrak{g} + \mathfrak{g}.\mathcal{D}_\mathfrak{g}) \setminus \mathcal{D}_\mathfrak{g}$. Hotta-Kashiwara, 1984:

$z \in S(\mathfrak{g}^*)^\mathfrak{g}$, ε = aug. $M_X \cong \rho_* \Omega_{T^*G/B}$.

Corollary: Let $e \in \mathfrak{g}$ be a nilpotent element, W_e^0 the classical W-algebra, $(W_e^0 = \mathcal{O}_{S_e})$, then

$HP_0(W_e^0) \cong H^{\mathrm{top}}(\rho^{-1}(e))$. Moreover $\forall$ central characters c, $HH_0(W_e(c))^0 \cong H^{\mathrm{top}}(\rho^{-1}(e))$.

Springer fibre

Case 3) is open, and case 4) is known by Proudfoot-Schedler.

Let X be a complete intersection in $\mathbb{C}^n$ of dim 2. $X = \{f_1 = \dots = f_{n-2} = 0\}$.

$\{x_i, x_j\} = \det(\partial f_k / \partial x_\ell)_{\ell \neq i,j}$

<u>Theorem</u>: If X has isolated singularities, $HP_0(\mathcal{O}_X) \cong H^2(X,\mathbb{C}) \oplus \bigoplus_{s \in \mathrm{Sing}X} \mathbb{C}^{\mu_s}$

μ_s is Milnor number of s.

Note Theorem $\Leftrightarrow$ $\dim HP(\mathcal{O}_{X_t})$ is independent of t, X_t the versal deformation of X given by $\{f_i = t_i : 1 \leq i \leq n-2\} = X_{\underline{t}}$.

C a smooth curve of degree d in $\mathbb{P}^2$ given by $P(x,y,z) = 0$. X = cone on C, $\{x,y\} = \frac{\partial P}{\partial z}$.

and cyclic permutations. $g = \frac{(d-1)(d-2)}{2}$, $\mu = (d-1)^2$, $\mu = \dim(\mathbb{C}[x,y,z]/(P_x, P_y, P_z))$

What is M_X? Copies of $IC(j_{!*}\Omega)$ and δ_0 are only composition factors.

Max extension is $j_!\Omega_X$. $0 \to \delta_0^{2g} \to j_!\Omega \to IC_X \to 0$

<u>Theorem</u>: $M_X = M_{X,\mathrm{ind}} \oplus \delta_0^{\mu-g}$, and $0 \to j_!\Omega \to M_{X,\mathrm{ind}} \to \underset{H^0(C,\Omega^1)\otimes\delta_0}{\overset{\shortparallel}{\delta_0^g}} \to 0$