D-modules on Poisson Varieties, Poisson Homology & Symplectic Resolution

X - affine $/\mathbb{C}$, $\mathfrak{g} \equiv$ Lie alg. $\quad \alpha : \mathfrak{g} \longrightarrow \mathrm{Vect}(X) = \mathrm{Der}(\mathcal{O}_X)$

doesn't have to be f.dim

$(\mathcal{O}_X)_{\mathfrak{g}} = \mathcal{O}_X / \mathfrak{g}\mathcal{O}_X$

Question: When is $(\mathcal{O}_X)_{\mathfrak{g}}$ f.dim?

$X_i = \{ x \in X \mid \dim \alpha_x(\mathfrak{g}) = i \}$ where $\alpha_x : \mathfrak{g} \longrightarrow T_x X$, $x \in X$.

locally closed subvar. $\dim X_{i,j} \geq i$

Def: X has finitely many $\mathfrak{g}$-leaves if $\dim X_i = i$.

Prop: In this case, each connected comp. X_{ij} of X_i is smooth & $\alpha_x(\mathfrak{g}) = T_x X_i \quad \forall x \in X_i$

$X_{i,j}$ are called $\mathfrak{g}$-leaves

Thm: If X has f.many $\mathfrak{g}$-leaves, then $\mathcal{O}_X / \mathfrak{g}\mathcal{O}_X$ is f.dim.

Pf (Sketch)

Define a right $\mathcal{D}$-module $M(\alpha) = \mathcal{D}_X / \mathfrak{g} \cdot \mathcal{D}_X$

$\mathrm{Hom}(\mathcal{D}_X, M) = \Gamma(M)$

$= \left(\tilde{\mathfrak{g}} \cdot \mathcal{D}_V + I_X \mathcal{D}_V \right) \backslash \mathcal{D}_V$

$X \subseteq V$ smooth, $\mathcal{D}_X = I_X \mathcal{D}_V \backslash \mathcal{D}_V$

ideal of X in $\mathcal{O}_V$.

undarived direct image

Lemma 1: $(\mathcal{O}_X)_{\mathfrak{g}} = \pi_\circ M(\alpha)$ where $\pi : X \longrightarrow pt$

Pf:

$$M(\alpha) = M(\alpha) \otimes_{\mathcal{D}_V} \mathcal{O}_V = \widehat{\mathfrak{g}}\, \mathcal{D}_V + I_V \mathcal{D}_V \backslash \mathcal{D}_V \otimes_{\mathcal{D}_V} \mathcal{O}_V$$

$$= \mathfrak{g}\cdot\mathcal{O}_X \backslash \mathcal{O}_X = (\mathcal{O}_X)_{\mathfrak{g}}$$

Lemma 2: If X has f. many $\mathfrak{g}$-leaves, then $M(\alpha)$ is holonomic

Pf

$SS(M(x)) \subset T^*V$ is contained in the union of conormal bundles of the $\mathfrak{g}$-leaves (so Lagrangian).

Pf of Thm: Direct image maps holonomic $\mathcal{D}$-modules to holonomic ones $\square$

Special Case: X - Poisson variety, $\mathfrak{g} = \mathcal{O}_X$

$(\mathcal{O}_X)_{\mathfrak{g}} = \mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\} = HP_0(\mathcal{O}_X) = HP_0(X)$ zeroth Poisson homology

$\mathfrak{g}$-leaves $\equiv$ symplectic leaves

Cor: If X has f. many symplectic leaves, then $HP_0(X)$ is f. dim.

$$M_X = \frac{\mathcal{D}_X}{HVect(X)\cdot\mathcal{D}_X}$$

Generalize: take X not nec. affine, take π_* instead of $\pi_\circ$.

Defn: The Poisson-DeRham homology of X is

$$HP_i^{DR}(X) := H^{-i}(\pi_* M_X).$$

Cor: If X has f. many symplectic leaves, then HP_*^{DR} are f. dim.

Example: X symplectic variety of dim $n=2r$. Then $M_X = \Omega$, $HP_i^{DR}(X) = H_{DR}^{n-i}(X,\mathbb{C})$

$-n \leq i \leq n$

Example: V – f.dim symp. v.sp. $G \underset{\text{finite}}{\subseteq} Sp(V)$

take $X = V/\!\!/G$ Poisson, symplectic leaves are strata corresponding to stabilizers.

$\mathcal{O}_{V/\!\!/G} / \{\mathcal{O}_{V/\!\!/G}, \mathcal{O}_{V/\!\!/G}\}$ is f.dim.

conjecture by Alev-Farkas, proved by Berest, E., Ginzburg

$$\mathrm{Hom}(M_{V/G}, \delta_0) := HP_0(V/G)^*$$

Applications to Rep. Theory: A = $\mathbb{Z}_+$-filtered non-comm. alg.

$grA = A_0$. non comm $\Rightarrow$ A_0 has a non-zero Poisson bracket (of some deg. d)

Thm: If $X = \mathrm{Spec}(A_0)$ has f. many symplectic leaves, then A has f. many irred f.dim reps. ($\leq \dim HP_0(X)$)

Pf. By Wedderburn, the char. of irred. f.dim. reps are linearly indep. in $(A/[A,A])^*$ so $\#\text{rep} \leq \dim(A/[A,A])$

but $A_0/\{A_0,A_0\} \twoheadrightarrow A/[A,A]$ so $\#\text{rep} \leq \dim A_0/\{A_0,A_0\}$

Cor: Spherical symplectic ref. alg. has f. many f.dim reps.

Question: Can we compute M_X?

Conj. 1: Let $p: \tilde{X} \longrightarrow X$ be a symplectic resolution. Then

$$M_X = p_* \Omega_{\tilde{X}}$$

Remark ①: p is semismall so R.H.S is a $\mathcal{D}$-module in deg 0.

② Conj. implies that if $\tilde{X}$ exists, then M_X is semisimple.

Suppose $\tilde{X} \longrightarrow X$ is equiv. under a contracting $\mathbb{C}^*$-action then G.K showed that $\exists\, A(c)$, a quantization of $A_0 = \mathcal{O}_X$
Ginzburg-Kaledin
with $c \in H^2(\tilde{X}, \mathbb{C})$. $HH_0(A(c)) \cong H^{top}(X, \mathbb{C})$ for generic c

$$\text{Conj.} \Rightarrow \quad HP_0(A_0) \twoheadrightarrow HH_0(A(c))$$

$$HP_0(A_0) = A_0/\{A_0, A_0\} \cong A(c)/[A(c), A(c)] = HH_0(A(c))$$

So $HH_0 = H^{\dim X}(\tilde{X})$ for all c.

Examples

① Y - sympl. surface, $\mathrm{Hilb}_n(Y) \longrightarrow S^n Y$

More generally, if Y is a Poisson surface with ADE-sing. $\tilde{Y} \longrightarrow Y$ sympl. res. then
$$\mathrm{Hilb}_n(\tilde{Y}) \longrightarrow S^n Y$$

② Springer res: $T^*(G/B) \longrightarrow \mathcal{N}_G$ & slices (Slodowy slices corresp. to nilp. elts. $e \in \mathfrak{g}$)

③ Quiver Varieties.

④ Hypertoric Varieties

In the case of Springer resolution: $X = \mathcal{N}_G \subseteq \mathfrak{g}$

$$M_X = \frac{\mathcal{D}_{\mathfrak{g}}}{\mathfrak{g} \cdot \mathcal{D}_{\mathfrak{g}} + (z - \varepsilon(z))\mathcal{D}_{\mathfrak{g}}} \qquad z \in (S\mathfrak{g}^*)^{\mathfrak{g}} \qquad \varepsilon = \text{augmentation}$$

$\mathfrak{g}$-invariants of sym. alg on $\mathfrak{g}^*$

Hotta-Kash, 1984: $M_X \cong p_* \Omega_{T^* G/B}$

Springer $\mathcal{D}$-module.

Cor: Let $e \in \mathfrak{g}$ be a nilp. elt, W_e^0 classical W-alg $W_e^0 = \mathcal{O}_{S_e}$. Then $HP_0(W_e^0) \cong H^{top}(\rho^{-1}(e))$

Slodowy Slice

Springer fibres

And $\forall$ central char. c, $HH_0(W_e(c)) \cong H^{top}(\rho^{-1}(e))$

let X be a complete intersection 2-surface in $\mathbb{C}^n$ with isolated sing.

$f_1 = \dots = f_{n-2} = 0$. $\{x_i, x_j\} = \det\left(\frac{\partial f_k}{\partial x_\ell}\right)_{\ell \neq i,j}$

Thm: $HH_0(\mathcal{O}_X) \cong H^2(X, \mathbb{C}) \oplus \bigoplus_{s \in \mathrm{Sing}} \mathbb{C}^{\mu_s}$ ← Milnor no. of S.

$t = (t_1, \dots, t_{n-2})$ $\quad X_t = \{f_1 = t_1, \dots, f_{n-2} = t_{n-2}\}$

Thm: $\Leftrightarrow$ $HP_0(\mathcal{O}_{X_t})$ is independent of t.

C smooth curve of degree d in $\mathbb{P}^2$ given by $P(x,y,z) = 0$

X cone of C, $\{x,y\} = \frac{\partial P}{\partial z}$ & cyclic permutations

$g = \frac{(d-1)(d-2)}{2}$, $\mu = (d-1)^3$, $\mu = \dim\left(\mathbb{C}[x,y,z]/(P_x, P_y, P_z)\right)$

What is M_X? One copy of $IC = j_{!*}\Omega$
some copies of δ_0.

Max ext on Bottom is $j_!\Omega$:

$$0 \to \underset{\substack{\parallel \\ \delta \otimes H^1(C,\mathbb{C})}}{\delta^{2g}} \to j_!\Omega \to IC_X \to 0$$

Thm: $M_X = M_{X,ind} \oplus \delta^{\mu-g}$ $\qquad H^0(C, \Omega^1) \otimes \delta$

$$0 \to j_!\Omega \to M_{X,ind} \to \delta^g \to 0$$