Unicity of the Grading of Category $\mathcal{O}$

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Theorem:

1) $\exists$ a $\mathbb{Z}$-graded cover of category $\mathcal{O}$ compatible with the action of the centre.
2) Any two such graded covers are cover equivalent.

$\mathfrak{g}/\mathbb{C}$ semisimple complex $\supseteq \mathfrak{b}$ Borel

$$\mathcal{O} = \{ M \in \mathfrak{g}\text{-mod} \mid \text{f.gen by}/\mathfrak{g}, \text{ locally finite}/\mathfrak{b}, \text{ s.s over } \mathfrak{h} \}$$

Defn: A graded cover of an artinian category $\mathcal{A}$ is a triple $(\tilde{\mathcal{A}}, v, \varepsilon)$ with

$\tilde{\mathcal{A}} = (\tilde{\mathcal{A}}, [1])$ abelian category with auto shift by 1

$v \downarrow$ exact functor

$\mathcal{A}$

& $\varepsilon : v[1] \Rightarrow v$

such that

1) $\displaystyle\bigoplus_{i \in \mathbb{Z}} \tilde{\mathcal{A}}(M, N[i]) \xrightarrow{\sim} \mathcal{A}(vM, vN)$

2) For $N \in \tilde{\mathcal{A}}$, $M \in \mathcal{A}$ & $M \twoheadrightarrow vN$
$\exists$ $P \in \tilde{\mathcal{A}}$ & $vP \twoheadrightarrow M$ s.t the composition
$P \longrightarrow N$ comes from a morphism in $\tilde{\mathcal{A}}$

Ex.

$A \equiv$ left artinian ring, $\tilde{A}\text{-Modf}^{\mathbb{Z}}$ finitely gen. $\mathbb{Z}$-graded $\tilde{A}$-modules

$\downarrow v$

$\tilde{A}$ the same with $\mathbb{Z}$-grading $A\text{-Modf}^{\mathbb{Z}}$

Q: What does compatible with the action of the centre mean?

Defn: A graded cover $\tilde{\mathcal{O}}$ (of $\mathcal{O}$) is compatible with the action of $Z \subseteq U(\mathfrak{g})$ iff

$$\tilde{M} \in \tilde{\mathcal{O}} \ \& \ \chi \in \text{Max ideal in } Z \ \& \ \chi^n(v\tilde{M}) = 0 \text{ for some } n$$

then $Z/\chi^n \longrightarrow \mathrm{End}_{\mathfrak{g}}\, v\tilde{M}$ is homogeneous for the natural grading on Z/χ^n

What is this natural grading?

$$Z \xrightarrow{\sim} S^W, \quad S = S(\mathfrak{f}) = \mathcal{O}(\mathfrak{f}^*)$$

$$\lambda \in \mathfrak{h}^* = \mathrm{Max}\, S \longrightarrow \mathrm{Max}\, Z$$

$$\lambda \longmapsto \chi \qquad \mu = \lambda \cap S^W$$

$$Z^\wedge_\chi \xrightarrow{\sim} (S^\wedge_\lambda)^{W_\lambda} \cong (S^{W_\lambda})^\wedge_\mu$$

$$Z/\chi^n \xleftrightarrow{\sim} S^{W_\lambda}/\mu^n \rightsquigarrow \text{leads to natural grading on } Z/\chi^n$$

Q: Cover equivalent?

Defn: Let $\mathcal{A}$ artinian. Two graded covers $(\tilde{\mathcal{A}}, \tilde{v}, \tilde{\varepsilon})$ & $(\hat{\mathcal{A}}, \hat{v}, \hat{\varepsilon})$ graded covers.

A cover equivalence is a triple (F, ε, π) s.t

$$\begin{tikzcd} \tilde{\mathcal{A}} \arrow[rr] \arrow[dr, "\tilde{v}"'] & \arrow[d, phantom, "\overset{\pi}{\Leftarrow}"] & \hat{\mathcal{A}} \arrow[dl, "\hat{v}"] \\ & \mathcal{A} & \end{tikzcd}$$

& $\varepsilon : F[1] \overset{\simeq}{\Rightarrow} [1]F$ s.t

$$\begin{tikzcd} \hat{v}[1]F \arrow[r, Rightarrow, "\varepsilon"] \arrow[d, Rightarrow, "\hat{\varepsilon}"'] & \hat{v}F[1] \arrow[r, Rightarrow, "\pi"] & \tilde{v}[1] \arrow[d, Rightarrow] \\ \tilde{v}F \arrow[rr, Rightarrow] & & \tilde{v} \end{tikzcd}$$

$A \equiv$ left artinian

1) Any graded cover of $A\text{-Modf}^{\mathbb{Z}}$ is equivalent to some $\tilde{A}\text{-Modf}^{\mathbb{Z}}$

2) $\tilde{A}\text{-Modf}^{\mathbb{Z}}$ & $\hat{A}\text{-modf}^{\mathbb{Z}}$ are cover equivalent iff $\exists$ grading $\tilde{A}^{\wedge}$ on the group A making it a $\tilde{A}$-$\hat{A}$-bimodule.

eg $A = \mathrm{End}_k V$, $\dim_k V < \infty$

$\tilde{V}$ $\mathbb{Z}$-grading $\rightsquigarrow \tilde{A} = \mathrm{End}_k \tilde{V}$

$\hat{V}$-$\mathbb{Z}$-grading $\rightsquigarrow \hat{A} = \mathrm{End}_k \hat{V}$

$\Rightarrow {}^{\sim}A^{\wedge} = \mathrm{Hom}_k(\hat{V}, \tilde{V})$

Defn: A a ring. An A-module Q is bicentralizing iff

$$A \xrightarrow{\sim} \mathrm{End}_{\mathrm{End}_A Q} Q$$

Example: $A\text{-Modf} \cong$ Block of $\mathcal{O}$

$Q \longleftarrow$ anti-dominant projective

Consider the functor

$$\mathrm{Hom}_A(-,Q) : A\text{-Modf}^{\mathrm{opp}} \longrightarrow \mathrm{End}_A Q\text{-modf}$$

This is fully faithful on projectives.

A is a projective A-module $\Longrightarrow$

$$A \xrightarrow{\sim} (\mathrm{End}_A A)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{\mathrm{End}_A Q} Q$$

Prop: Let A be left artinian, Q bicentralizing. Any two $\mathbb{Z}$-gradings on A compatible with the same $\mathbb{Z}$-grading on $\mathrm{End}_A Q$ are cover equivalent.

Pf (Idea)

graded iso.

$$\tilde{A} \xrightarrow{\sim} \mathrm{End}_{\mathrm{End}_A Q} \tilde{Q}$$

so $\exists$ a grading since $\hat{A}$ is graded

$$\hat{A} \xrightarrow{\sim} \mathrm{End}_{\mathrm{End}_A Q} \hat{Q}$$

$$\Longrightarrow \tilde{A}^{\wedge} = \mathrm{Hom}_{\mathrm{End}_A Q}(\tilde{Q}, \hat{Q})$$

joint with Wendt

$$\mathrm{Der}^b(\widehat{\mathcal{O}}_0) \cong \underset{(\mathcal{B})}{\mathrm{DMT}}(G/B)$$ derived mixed Tate motives

$\mathcal{B} \equiv$ Beilinson

$$X/k \rightsquigarrow D\mathcal{B}(X,\mathbb{Q}) \supset D\mathcal{B}_c(X,\mathbb{Q})$$

$$\begin{array}{c} X/k \\ f\downarrow \\ Y/k \end{array}$$

triangulated, $\otimes$

get 6-functor formalism. $(f_!, f^!)$, (f^*, f_*)

$\mathcal{F} \in D\mathcal{B}(X,\mathbb{Q})$ $\quad \mathcal{F} \xrightarrow{\sim} w_* w^* \mathcal{F}$ $\quad w: \mathbb{A}^1 \times X \to X$

homotopy property $\qquad$ projection map.

$\mathrm{fin}_X^*(\underline{pt}) = \underline{X}$ $\qquad \mathrm{fin}_X : X \to pt$

$(\mathrm{fin}_! \, \underline{\mathbb{A}^1})[2] = \underline{pt}(-1)$ Tate object, invertible

$D\mathcal{B}(\underline{pt}, \underline{pt}(i)[j]) =$ some Adams eigenspace of the K-groups of k.

If $|k| < \infty \Rightarrow$ higher K-groups are torsion
$\Rightarrow$ only $K_0 \otimes_{\mathbb{Z}} \mathbb{Q}$

fully faithful embedding

Consider $\mathrm{Der}^b(\mathbb{Q}\text{-Modf}^{\mathbb{Z}}) \hookrightarrow D\mathcal{B}_c(pt)$

$$\mathbb{Q}[i](j) \longmapsto \underline{pt}[i](j)$$

possibly with multiplicities

$$\underset{(\mathcal{B})}{\mathrm{DMT}}(G/B) := \{\mathcal{F} \in D\mathcal{B}(G/B) \mid j_w^* \mathcal{F} \in \bigoplus \underline{BwB/B}(i)[k]\}$$

$C=$ coinvariant alg.

$$\mathrm{Hot}^b(\mathrm{DMT}_{(\mathcal{B})}(G/B); wt=0)$$

$$\mathrm{Hot}^b(C\text{-SMod}^{\mathbb{Z}}) \qquad \mathrm{DMT}_{(\mathcal{B})}(G/B)$$

S= special modules