K. Kremnizer: Analytic geometry and representation theory.

- joint with Oren Ben-Bassat.

§1 Motivation: $Z \subset X$ a closed subscheme $V = X \setminus Z$. $X = \hat{X}_Z \cup V$ where $\hat{X}_Z$ is the formal completion of X along Z.

The intersection $\hat{X}_Z \cap V = \mathring{\hat{X}}_Z$ $\rightsquigarrow$ algebraically this is completing and localizing.

This is a rigid analytic space. Ben-Bassat and Temkin show that one can glue e.g. vector bundles along this intersection. (Unlike in algebraic geometry, where e.g. the punctured formal disc has no points!)

Also interesting to study $G/\mathbb{Q}_p$, $G/\mathbb{C}((t))$, Beilinson-Bernstein, Satake.

The "usual" approach in algebraic geometry: work via locally ringed spaces. This yields a topological space with a sheaf of rings $\rightsquigarrow$ build $QCoh(X)$, $D(X)$.

problems: QCoh debated in the approach - forget extra structure on rings (note what Coh is however is agreed in the rigid analytic case.)

<u>Need</u>: A functor of points approach.

§ Analysis:

Let k be a field with a valuation (Arch. or non-Arch.)

<u>Remark</u>: Can start with a ring with a valuation, e.g. $(\mathbb{Z}, |\cdot|)$. This yields analytic geometry over $\operatorname{Spec} \mathbb{Z}^{an}$ (or even using sets with valuation, yielding $\operatorname{Spec} \mathbb{F}_1^{an}$ and $\operatorname{Spec} \mathbb{Z}^{an} \longrightarrow \operatorname{Spec} \mathbb{F}_1^{an}$ exists!)

Also $\operatorname{Spec} \mathbb{Z}^{an}$ is already complete, unlike classically.

Assume k is complete and let Ban_k be the category of k-vector spaces which are complete, and take morphisms to be bounded linear maps.

This category has a closed symmetric monoidal category.

$V \hat{\otimes}_k W$ "projective" tensor products. "Closed" means $\underline{\mathrm{Hom}}_k(V,W)$ = Banach space of all bounded linear maps is the adjoint of this $\hat{\otimes}_k$.

This category has finite limits and colimits, but no infinite products, coproducts.

Quasi-abelian categories.

Let $\mathcal{C}$ be an additive category with kernels and cokernels.

$$\begin{tikzcd} \ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \mathrm{coker} f \\ & \mathrm{coker}(\ker f) \arrow[r] & \ker(\mathrm{coker}(f)) = \mathrm{im}(f) \arrow[u] & \end{tikzcd}$$

$\mathrm{coker}(\ker f) = \mathrm{coim}(f)$

Call a morphism "strict" if $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an $\cong$

Ban_k, Filt Ab, LCV_k, Born_k are all examples of Quasi-abelian:

If f is a strict epi
$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & F' \arrow[d] \\ E \arrow[r, "f"] & F \end{tikzcd}$$
$\Rightarrow f'$ a strict epi

Similarly
$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F \arrow[d] \\ E' \arrow[r, "f'"'] & F' \end{tikzcd}$$
f strict monos $\Rightarrow f'$ is also

(check this for examples!).

A quasi-abelian category has a canonical exact structure: $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ strict s.e.s. $\alpha = \ker(\beta)$, $\beta = \mathrm{coker}(\alpha)$

We can define $D(C) = K(C)/$ strict exact. $D(C)$ has a natural t-structure $C \to LH(C)$

<u>Note</u>: i) In the case C has enough projectives, $D^-(C) = K^-(P)$ (and $D^+(C) = K^+(I)$)

ii) In the case C is a closed symmetric monoidal category is quasi-abelian with enough projectives such that projectives are flat and the tensor product of two projectives is projective, then $Ch^{\leq 0}(C)$ has a symmetric monoidal structure.

Ban_k has a subcategory $\mathrm{Ban}_k^{\leq 1}$ has same objects but only non-expanding morphisms. $\mathrm{Ban}_k^{\leq 1}$ has limits and colimits. (quasi-abelian in non-Archimedean case.) $\prod_I k$ is injective in Ban_k and $\coprod_I k$ is injective in Ban_k

How to (co)complete Ban_k? Take $Ind(Ban_k)$: this is complete, cocomplete and is still closed symmetric monoidal.

<u>Note</u>: being a cocomplete q.abelian category with enough projectives means that $LH(Ind(Ban_k))$ is the category of modules over a ring (with many objects...)

$$\begin{tikzcd} CBorn_k \arrow[dr] & \\ LCV_k \arrow[r] & Ind(Ban_k) \\ Pro(Ban_k) \arrow[ur] & \end{tikzcd}$$

Relative algebraic geometry: Grothendieck / Deligne etc.: $(C, \otimes, I)$ closed symmetric monoidal category with limits and colimits.

Example: 1) $(Ab, \otimes_{\mathbb{Z}})$ 2) $(Ind(Ban_k), \hat{\otimes}_k)$

3) $(\underline{Sets}, \times)$

<u>Definition</u>: $Comm(C)$ is the category of comm. monoids in C: $A \otimes A \to A$, $I \to A$.

e.g. $Comm(Ab)$ = commutative rings etc.

Note: $Ind(Comm(Ban_k)) \subsetneq Comm(Ind(Ban_k))$.

$$\begin{tikzcd} & & Comm(Ind(Ban_k)) \\ & Comm(ProBan_k) \arrow[ur, hook] & \\ Pro(Comm(Ban_k)) \arrow[ur] & & \end{tikzcd}$$

Important example: Take V in ob $Ban_k^{\leq 1}$, take $V = k_{r_1} \oplus .. \oplus k_{r_n}$. $S(V) = \bigoplus_n^{\leq 1} S^n(V)$

these are exactly the affinoid algebras of rigid analytic geometry

<u>Definition</u>: $Aff(C) = Comm(C)^{op}$. (spec A associated to A as usual.)

$QCoh(Spec(A)) := A\text{-mod}$ (in $Ind(Ban_k)$).

$M \otimes_A N = coeq(M \otimes A \otimes N \rightrightarrows M \otimes N)$.

Given $f: A \to B$ can define flat, faithfully flat, Zariski immersion, smooth, étale, ...

$B \otimes_A^{\mathbb{L}} B \xrightarrow{\sim} B$ Zariski open immersion classical, but this is different in our setting

"homological Zariski immersion":  not a flat map (because of disjoint discs)

<u>Definition</u>: $PSt(C) := Fun(Aff(C)^{op}, \infty\text{-groupoids})$

Choose a Grothendieck topology τ on $\mathrm{Aff}(C)$, to get $\mathrm{Sch}_\tau(C) \subseteq \mathrm{PSt}(C)$

can also do derived analytic geometry.

If k is non-archimedean and τ topology + homological Zariski morphisms then

rigid analytic spaces $\subseteq \mathrm{Sch}_\tau(\mathrm{Ind}(\mathrm{Ban}_k))$. The locale of this gives the

"points" in (some) classical rigid analytic space.

Can use this to reconstruct Tannakianly the loop group.

Can also construct $\mathcal{D}$-modules. $\mathcal{D}^\infty = R\mathcal{H}om^{top}(\mathcal{O}_X, \mathcal{O}_X)$ — this is a kind of Riemann-Hilbert

topological homs

Should also be a localization theorem — need to compute the global sections algebra!

References: quasi-abelian J.P. Schneiders.