Local Hodge Theory of Soergel Bimodules

Jantzen conjectures:

$\underline{g} \supseteq \underline{b}$, $\underline{h} = \underline{b}/[\underline{b},\underline{b}]$

Given $\lambda \in \underline{h}^*$ highest wt. $\longrightarrow$ $\Delta(\lambda)$ Verma module

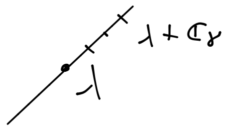
v_λ h.wt vector

$h \cdot v_\lambda = \lambda(h) v_\lambda$

$\gamma \in \underline{h}^*$ deformation direction $\rightsquigarrow$ $\Delta_A(\lambda)$ deformed Verma module

$A = \mathbb{C}[t]$, $(\underline{g}, A)$-bimodule

$h \cdot v_\lambda = (\lambda(h) + \gamma(h) t) v_\lambda$



$(-,-) : \Delta_A(\lambda) \times \Delta_A(\lambda) \longrightarrow A$

$J^\bullet$ Jantzen filtration

$J^0 \supset J^1 \supset \dots$

$J^1 = \mathrm{rad}(-,-) \otimes_A \mathbb{C}$

Jantzen conj. (1978) Certain canonical morphisms are strict for $J^\bullet$.

Deodhar's Question: Does $J^\bullet$ depend on γ? if γ is non-deg.

Gabber-Joseph (1982): JC $\Longrightarrow$ $\sum [\mathrm{gr}^i_J \Delta(\lambda) : L(\mu)] v^i$ is a KL polyn $\Rightarrow$ KL conj.

Barbasch (1984?): JC $\Rightarrow$ $J^\bullet$ is the socle filtration

Beilinson-Bernstein (1982-1990): JC is true if γ is dominant regular integral

(also works for $(\underline{g}, K)$-mod

($J^\bullet$ "is" the weight filt. weight = monodromy)

Soergel (2007) + Kübel (2012): new proof of JC using "local hard Lefschetz".

<u>W</u> (2014): Alg. proof of hard Lefschetz in the language Soergel bim (works for any Coxeter system (W,S))

$\Longrightarrow$ alg. proof of JC.

<u>Cor:</u> Answer to Deodhar's Question is Yes!

True in multiplicity 1 case. False for $\underline{sl}_4(\mathbb{C})$

Classical Hodge Theory: $H^* = H^{*+\dim_{\mathbb{C}} X}(X, \mathbb{R})$ where $X \equiv$ smooth projective var.

$\langle , \rangle$ Poincaré pairing: $H^d \times H^d \longrightarrow \mathbb{R}$

$H^*_{\mathbb{C}} \cong \bigoplus H^{p,q}$ (below only (p,p)-type occurs)

$L : H^* \longrightarrow H^{*+2}$ Lefschetz operator

Hard Lefschetz: $H^*_{\mathbb{C}} = \bigoplus_{d\geq 0} \mathbb{C}[L]/[L^{d+1}] \otimes_{\mathbb{C}} P^{-d}$ $\quad P^{-d} = \ker L^{d+1} \cap H^{-d}$

Hodge Riemann: formula for signature of $\langle x, L^d \bar{y} \rangle$ on $H^{-d}_{\mathbb{C}}$ in terms of Hodge numbers

Local Hodge Theory: $\mathbb{C}^* \circlearrowright \mathbb{C}^n \supseteq X$ with positive weights, X $\mathbb{C}^*$-stable closed subvariety

$A = H^*_{\mathbb{C}^*}(pt, \mathbb{R}) = \mathbb{R}[z]$

We have an s.e.s of graded $A = H^*_{\mathbb{C}^*}(pt)$-modules

$$0 \longrightarrow \underbrace{IH^*_{\mathbb{C}}(X;\mathbb{R})}_{= H^*_{\mathbb{C}^*}(i^! IC_x)}^{M^! ::} \longrightarrow \underbrace{IH^*_{\mathbb{C}^*}(X;\mathbb{R})}_{= IH^*_{\mathbb{C}^*}(i^* IC_x)}^{:: M,\ \langle -,- \rangle} \longrightarrow \underbrace{IH^*_{\mathbb{C}^*}(\dot{X};\mathbb{R})}_{= IH^{*+1}(\dot{X}/\mathbb{C}^*;\mathbb{R})} \longrightarrow 0$$

almost free $\mathbb{C}^*$-action

IC condition $\Longrightarrow$ $M^!$ (resp M) are generated in degrees > 0 (resp. < 0)

$\Longrightarrow$ j^* is a projective cover

$L = z$ satisfies hard Lefschetz on $M/M^!$

Simplest Example: $X = \mathbb{C}^n$

$$0 \longrightarrow A[-n] \longrightarrow A[n] \longrightarrow \underbrace{A/(z^n)[n]}_{= IH^{*+1}(\mathbb{P}^{n-1})} \longrightarrow 0$$

& $\mathbb{C}^n \setminus \{0\} / \mathbb{C}^* = \mathbb{P}^{n-1}$

Poincaré Pairing:

$M^! \times M \longrightarrow A$ non-deg.

$A[z^{-1}] \otimes_A M^! \overset{\sim}{\hookrightarrow} A[z^{-1}] \otimes_A M \rightsquigarrow \langle -,- \rangle$ is a $A[z^{-1}]$-valued form on M.

$M, M^!$ dual lattices $\subset A[z^{-1}] \otimes_A M$.

Everything is encoded in $(M, \langle -,- \rangle)$

orthogonal decomp.

Primitive Decomposition: $M = \overset{\perp}{\bigoplus} M \otimes_{\mathbb{R}} P^{-d}$

Schubert Varieties:

$X \supset \{xB\} \in G/B$

T-stable affine nhd

T-fixed pt.

Deodhar's Q: Does $IH^*(\dot{X}/\mathbb{C}^*; \mathbb{R})$ satisfy hard Lefschetz if $\mathbb{C}^* \subset T$ is regular? ie $X^{\mathbb{C}^*} = \{xB\}$

Key Idea in Proof:

$$\begin{tikzcd} \mathbb{P}^1 \simeq xP_s/B \arrow[r, hook, "i'"] \arrow[d] & G/B \arrow[d] \\ \{xP_s/P_s\} \arrow[r, hook, "i"] & G/P_s \end{tikzcd}$$

$$i^! \pi_* IC_w \longrightarrow i^* \pi_* IC_w$$

$ws > w$ would like to understand

$(i')^! IC_w \longrightarrow (i')^* IC_w$ morphism of complexes on $\mathbb{P}^1$

$\rightsquigarrow V/\mathbb{R}$ real v. sp. together with two symmetric forms

$(-,-)^0$ positive def.
$(-,-)^\infty$ negative def.

hard Lefschetz for global sections $\Longleftrightarrow (-,-)^0 + (-,-)^\infty$ is non-deg.

$HR \Longleftrightarrow (-,-)^0 + (-,-)^\infty$ is PD! ← positive definite

de Cataldo-Migliorini trick:
"weak Lefschetz" + HR for smaller w
$\Rightarrow a(-,-)^0 + (-,-)^\infty$ is non-deg. $\forall a \geq 1$
$\Rightarrow (-,-)^0 + (-,-)^\infty$ is PD!