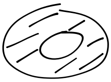
Quantization of Character Varieties via 4D TFT

(j.t D. Ben-Zvi, A. Brochier)

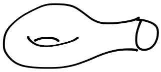
G alg gp. (soon reductive) $\quad S$ = surface

Defn: The character variety $Ch_G(S)$ is

$$Ch_G(S) = \{\rho : \pi_1(S) \longrightarrow G\} \Big/ \text{conj. action of } G$$

e.g $S =$  $=$ Ann $\quad Ch_G(S) = \frac{G}{G}$ adjoint action

$\pi_1(S) = \mathbb{Z}$

$S =$ , , $\quad Ch_G(S) = \frac{G \times G}{G}$ simultaneous conj.

$\pi_1(S) = Free_2$

$S =$  $\Rightarrow Ch_G(S) = Comm_n(G)$

$= \{(A,B) \mid ABA^{-1}B^{-1} = id\}$

$G \times G \xrightarrow{\mu} G$

$= \mu^{-1}(id)/G$

$S =$ , $\quad \pi_1 = *$, $\quad ch_G($$) = {}^{\cdot}/G = BG.$

$QCoh(BG) = Rep\,G$

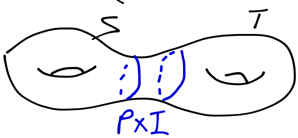
I) $Ch_G(S)$ carries a canonical Poisson bracket

(G-red with a choice of Killing form K on $\mathfrak{g}$)

II) $Ch_G(S) = Maps(S, BG)$

Observation:

$$QCoh\Big(Ch_G(S) = \big(S \underset{P\times I}{\sqcup} T\big)\Big) = QCoh(Ch_G(S)) \underset{QCoh(Ch_G(P\times I))}{\boxtimes} QCoh(Ch_G(T))$$



Goal: Quantize $Ch_G(S)$ uniformly in G, S

Thm (Cons± (BZBJ)): There exists a $(2+\varepsilon)$-Dim TFT $Ch_{U(q)}(S)$ is a quantisation of $Ch_G(S)$. When S is punctured, we produce an algebra $\underline{a}_S \in U_q(\mathfrak{g})\text{-mod}$ & describe it explicitly as follows:

known top./alg. constructions

$D^2 = \longmapsto \operatorname{Rep} U_q(\mathfrak{g})$

$V_1, \dots, V_n$

$V_1 \otimes \dots \otimes V_n \circlearrowleft$ Braid gp $B_n(\ )$

$Ann = \longmapsto \mathcal{O}_q(\frac{G}{G})\text{-mod} \ni M$

also called: $\mathcal{O}_q(G)\text{-mod}_{U_q(\mathfrak{g})}$ braided dual; or refl. eq. alg.; or locally f. sub. alg

$M \otimes V_1 \otimes \dots \otimes V_n \circlearrowleft PB_n(\ )$

$T^2 \setminus D^2 = \longmapsto \mathcal{D}^{wk}_q(\frac{G}{G})\text{-mod}$

$= \mathcal{D}_q(G)\text{-mod}_{U_q(\mathfrak{g})}$

Heisenberg double

quantum diff. ops

$B_n(S) = \pi_1(\operatorname{Config}(S))$

$M \otimes V_1 \otimes \dots \otimes V_n \circlearrowleft PB_n(\ )$

(w. A. Brochier) Reconstruction

$\mathcal{D}_q(G) \circlearrowleft \widetilde{SL_2}(\mathbb{Z})$

mapping class gp. $\longrightarrow$ "$Mapp(T^2 \setminus D^2)$

Fourier trans. $\rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{array}{ccccc} \widetilde{SL_2}(\mathbb{Z}) \circlearrowright \mathcal{D}_q(G) & \xrightarrow{q=1} & \mathcal{O}(G \times G) \circlearrowleft \widetilde{SL_2}(\mathbb{Z}) \\ q = e^{\hbar} \swarrow \quad \searrow & & \\ \mathcal{D}(G) \longrightarrow \mathcal{D}(\mathfrak{g}) & & \end{array}$$

$\Longrightarrow$ Quantum Springer Theory?

$\mathcal{D}(G)$: ~~Fourier trans.~~ don't have

$\mathcal{D}(\mathfrak{g})$: $\circlearrowright$ Fourier trans.

$T^2 =$  $\rightsquigarrow \mathcal{D}_q^{str}\left(\frac{G}{G}\right)\text{-mod} = \left(\mathcal{D}_q(G) /\!\!/_{\mu_q} U_q(\mathfrak{g})\right)$

Related to Double Affine Hecke Alg.

$q^k = 1$: $e\mathbb{H}_{q,t}\, e$ for GL_N [V-V]

$q = e^{\hbar}$: $e\mathbb{H}_{q,t}\, e$ for GL_N

generic q: $e\mathbb{H}_{q,t=1}\, e$ for GL_2

III Factorization Homology:

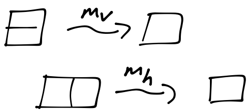
$A = E_n$- algebra in $\mathcal{C}^{\boxtimes}$ = closed symm. monoidal $(\infty,1)$-cat

A = braided tensor category $\subseteq E_2\text{-alg}(\underline{Cat}_k^{\boxtimes})$ Deligne-Kelly tensor product

Factorization homology: $Z_A(S) \in \mathcal{C}^{\otimes} = \underline{Cat}_k^{\boxtimes}$

E_1-algebras in $\underline{Cat}_k$-?: tensor categories (rigid)

E_2-alg: E_1-alg in E_1-alg



$(m_h, m_v) \cong (\otimes, br)$ braiding

$$Z_A(S) = \underset{\coprod_{i=1}^{n} U_i \to S}{\mathrm{Colim}}\ A^{\boxtimes n} \qquad U_i \cong \mathbb{R}^2$$

In particular, A = braided, $Z_A(P \times I) = E_1$-algebra

$Z_A(S)$  is a module category

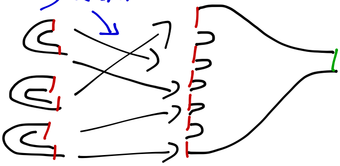
Thm (Francis) Excision

$$Z_A(S \sqcup T) \cong Z_A(S) \boxtimes_{Z_A(P\times I)} Z_A(T)$$

$Z_A($$) = A.$

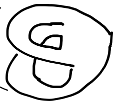
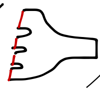
let S = punctured surface has a handle & comb. decom.

attaching data



torus missing a disc

where $n = \#$ handles

$Z_A($$) \simeq Z_A($$) \simeq Z_A($$) \boxtimes_{A^{2n}} Z_A($$)$

$T : A^{2n} \xrightarrow{G} A$ is a $\otimes$-functor

Technique: (Beck monodicity, Ostrik)

Ostrik : M-right A-module $m \in M$ cyclic generator.

$\exists$ an algebra

$\underline{End}(m) \in A$-alg.

$M \cong \underline{End}(m)$-mod

BZBJ: $F : A \longrightarrow B$, base change

$M \boxtimes_A B = F(\underline{End}(m))$-mod$_B$

$Z_A($$) = \mathcal{O}_{A\boxtimes A}$-mod$_{A\boxtimes A}$ $\leftarrow$ FRT

sum over iso. classes of simp. $\to X$ in A (when A is s.s)

$$\bigoplus X^* \boxtimes X$$

$$\Big\} T^{2n}$$

$$\bigoplus \bar{X} \otimes X \circlearrowleft$$

tensor category

pivotal A : $\otimes$-cat $\xrightarrow[K_0]{}$ Frobenius alg. $\qquad A \xrightarrow{tr} k$

decats

$\mathrm{Hom}(\mathbb{1}, -) : A \longrightarrow \mathrm{vec}_k$

pivotal $\Longrightarrow (\mathbb{1}, X \otimes Y) = (\mathbb{1}, Y \otimes X^{**}) = (\mathbb{1}, Y \otimes X)$

Example: Favourite Frobenius Algebra

$A = \mathrm{Mat}_n(\mathbb{C})$, choose $m = e_i \in \mathbb{C}^n$.

Game: embed $\mathbb{C}^n \hookrightarrow \mathrm{Mat}_n(\mathbb{C}^n)$

have map $tr_m(\cdot) : M_n(\mathbb{C}) \longrightarrow k$

$tr_m(a) = (e_i, e_i a) = a^i_i$

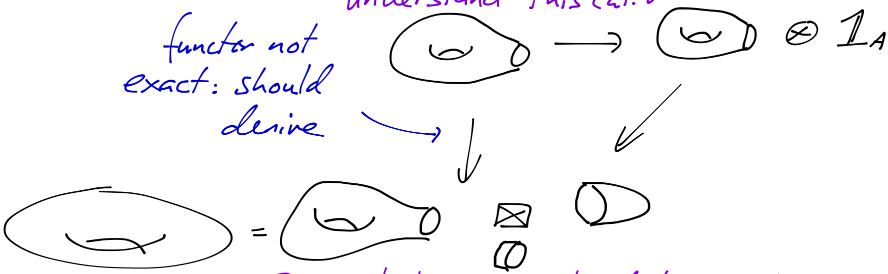
adjoint w.r.t trace map

$tr_m^*(\mathbb{1}) = E^i_i$

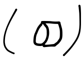
$\mathbb{C}^n \cong E^i_i \,\mathrm{Mat}_n \mathbb{C}$

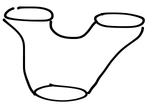
Homs give an inner product on the category

Hamiltonian Reduction for Closed Surfaces



? want to understand this category ?

$Z_A($$) = \mathcal{O}_q(G)\text{-mod}_A$



$M \overset{L}{\underset{\mathcal{O}_q(G)}{\otimes}} \Sigma_A \leftarrow$ Hamiltonian reduction