W. Soergel: Uniqueness of grading on category $\mathcal{O}$.

joint with M. Rottmaier.

<u>Theorem</u>: 1) There exists a $\mathbb{Z}$-graded cover of category $\mathcal{O}$ compatible with the action of the centre.

2) Any two such covers are cover-equivalent.

$\mathfrak{g}/\mathbb{C}$ semisimple complex Lie algebra $\supset \mathfrak{b}$ Borel. $\mathcal{O}^{\infty} = \{M \in \mathfrak{g}\text{-mod} \mid$ fin. gen. over $\mathfrak{g}$, locally finite $\mathfrak{b}$ (semisimple $/\mathfrak{h}$) $\}$

<u>Definition</u>: A $\mathbb{Z}$-graded cover of an Artinian category $\mathcal{A}$ is a triple $(\tilde{\mathcal{A}}, v, \varepsilon)$ with

$\tilde{\mathcal{A}} = (\mathcal{A}, [\,])$ artinian with autoequivalence

$v \downarrow$

$\mathcal{A}$ $\quad v$ an exact functor $\varepsilon : v[1] \xrightarrow{\sim} v$

such that 1) $\bigoplus_{i \in \mathbb{Z}} \tilde{\mathcal{A}}(M, N[i]) \xrightarrow{\sim} \mathcal{A}(v(M), v(N))$

2) For $N \in \tilde{\mathcal{A}}$, $M \in \mathcal{A}$, $\exists\ P \in \tilde{\mathcal{A}}$ and $vP \twoheadrightarrow M$, $M \twoheadrightarrow vN$ such that $P \to N$.

A a left Artinian ring, $\tilde{A}\text{-Modf}^{\mathbb{Z}} \to A\text{-Modf}$ where we give A a $\mathbb{Z}$-grading.

<u>Definition</u>: A graded cover $\tilde{\mathcal{O}}$ of $\mathcal{O}$ is compatible with the action of the centre $Z \subset \mathcal{U}(\mathfrak{g})$ if and only if $\tilde{M} \in \tilde{\mathcal{O}}$ and $\chi \in \operatorname{Max} Z$ and $\chi^n(v\tilde{M}) = 0$, then $Z_{\hat{\chi}^n} \to \operatorname{End}_{\mathfrak{g}} v\tilde{M}$ is homogeneous for the natural grading on $Z_{\hat{\chi}^n}$.

$Z \to S^W$, $S = S(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}^*)$. $\lambda \in \mathfrak{h}^* = \operatorname{Max} S \to \operatorname{Max} Z$

$\lambda \mapsto \chi$

$$Z_\chi^\wedge \xrightarrow{\sim} (S_\lambda^\wedge)^{W_\lambda}$$
$$\begin{tikzcd} & (S_\lambda^\wedge)^{W_\lambda} \\ & & (S^{W_\lambda})_\mu^\wedge \arrow[ul, "\sim"'] \end{tikzcd} \qquad \mu = \lambda \cap S^{W_\lambda}$$

$Z/\chi^n \xleftrightarrow{\sim} S^{W_\lambda}/\mu^n \rightsquigarrow$ natural grading on Z/χ^n.

Definition: Let $\mathcal{A}$ be an artinian category. $(\tilde{\mathcal{A}}, \tilde{v}, \varepsilon)$, $(\hat{\mathcal{A}}, \hat{v}, \varepsilon)$ graded covers. A cover equivalence is a triple (F, ε, π) such that:

$$\begin{tikzcd} \tilde{\mathcal{A}} \arrow[rr, "F"] \arrow[dr, "\tilde{v}"'] & \arrow[d, phantom, "\overset{\pi}{\Leftarrow}"] & \hat{\mathcal{A}} \arrow[dl, "\hat{v}"] \\ & \mathcal{A} & \end{tikzcd} \qquad \varepsilon : F[1] \Rightarrow [1]F$$

$$\begin{tikzcd} \hat{v}[1]F \arrow[r, Rightarrow, "\varepsilon"] \arrow[d, Rightarrow, "\hat{\varepsilon}"'] & \hat{v}F[1] \arrow[r, Rightarrow, "\pi"] & \tilde{v}[1] \arrow[d, Rightarrow] \\ \hat{v}F \arrow[rr, Rightarrow, "\pi"] & & \tilde{v} \end{tikzcd}$$

Let A be left Artinian.

1) Any graded cover of A-Modf is equivalent to some $\tilde{A}$-Modf$^{\mathbb{Z}}$

2) $\tilde{A}$-Modf$^{\mathbb{Z}}$ and $\hat{A}$-Modf$^{\mathbb{Z}}$ are cover equivalent iff $\exists$ grading ${}^\sim A^\wedge$ on the group A making it $\tilde{A}$-$\hat{A}$-bimodule

eg. $A = \operatorname{End}_k V$, $\dim_k V < \infty$. $\tilde{V}$ $\mathbb{Z}$-grading $\rightsquigarrow \hat{A} = \operatorname{End}_k \tilde{V}$. $\hat{V}$ a $\mathbb{Z}$-grading, then ${}^\sim A^\vee = \operatorname{Hom}_k(\hat{V}, \tilde{V})$

Definition: Let A be a ring. An A-module Q is called bicentralizing iff

$$A \xrightarrow{\sim} \operatorname{End}_{\operatorname{End}_A Q} Q$$

Example: A-Modf $\cong$ Block of category $\mathcal{O}$.

$Q \longleftrightarrow$ anti-dominant projective-injective.

$$\begin{tikzcd} \operatorname{Hom}_A(-, Q) : A\text{-Modf}^{op} \arrow[r] \arrow[d, dash, "\wr"] & \operatorname{End}_A(Q)\text{-Modf} \\ \mathcal{O}^{op} \arrow[u] \arrow[r] & C\text{-Mod} \end{tikzcd}$$
the coinvariant algebra fully faithful on projectives.

$A \longmapsto Q$

Thus in this case $A \xrightarrow{\sim} (\operatorname{End}_A A)^{op} \xrightarrow{\sim} \operatorname{End}_{\operatorname{End}_A Q} Q$. Thus Q is bicentralizing!

Proposition: Let A be left Artinian, Q bicentralizing. Any two $\mathbb{Z}$-gradings on A compatible with the same $\mathbb{Z}$-grading compatible with the same $\mathbb{Z}$-grading on $\mathrm{End}_A Q$ are cover equivalent.

Proof: $\tilde{A} \xrightarrow{\sim} \mathrm{End}_{\underset{\uparrow \text{grading}}{\mathrm{End}_A Q}} \tilde{Q}$ and $\hat{A} \xrightarrow{\sim} \mathrm{End}_{\mathrm{End}_A Q} \hat{Q}$, and so ${}^{\sim}A^{\wedge} := \mathrm{Hom}(\tilde{Q}, \hat{Q})$ gives the bimodule required above to establish cover equivalence. $\square$

joint work with Wendt:

$\mathrm{Der}^b(\underset{\uparrow \text{principal block.}}{\tilde{\mathcal{O}}_0}) \cong \mathrm{DMT}_{(B)}(G/B)$ derived mixed Tate motives, smooth along Bruhat cells.

$$\begin{array}{lcl} X/k & \xrightarrow[\text{Ayoub}]{\text{Cisinski-Deglise}} & D\mathcal{B}(X,\mathbb{Q}) \supset D\mathcal{B}_c(X,\mathbb{Q}) \\ \downarrow & & \text{triangulated}, \otimes \quad (f_!, f^!), (f^*, f_*) \\ Y/k & & \end{array}$$

$\mathcal{F} \xrightarrow{\sim} \mu_* \mu^* \mathcal{F}$ if $\mu : \mathbb{A}^1 \times X \to X$ (homotopy property).

$\underline{pt}$ = final object. $\mathrm{fin}_X : X \to pt$

Have $\mathrm{fin}_X^* (\underline{pt}) = \underline{X}$. $(\mathrm{fin}_! \underline{\mathbb{A}}^1)[2] = \underline{pt}(-1)$ Tate object, invertible

$D\mathcal{B}(\underline{pt}, \underline{pt}(i)[j])$ = some Adams eigenspace of K-groups of k. If $k = \mathbb{F}_q$, then the higher K groups are all torsion $\Rightarrow$ only $K_0 \otimes \mathbb{Q}$ survives.

$\mathrm{Der}^b(\mathbb{Q}\text{-Modf}^{\mathbb{Z}}) \overset{ff}{\hookrightarrow} D\mathcal{B}_c(pt)$

$\mathbb{Q}[i](j) \longmapsto \underline{pt}[i](j)$

$\{\mathcal{F} \in D\mathcal{B}(G/B) : j_w^* \mathcal{F} \in \bigoplus \underline{BwB/B}(i)[k] \cdots\} = \mathrm{DMT}_{(B)}(G/B)$

$\mathrm{Hot}^b(\mathrm{DMT}_{(B)}(G/B)_{w=0})$ weight zero.

$\downarrow \wr$ $\qquad \mathrm{DMT}_{(B)}(G/B)$

$\mathrm{Hot}^b(C\text{-SMod}^{\mathbb{Z}})$