S. Gunningham : A conceptual approach to the generalized Springer correspondence.

I $\mathcal{P}(\mathcal{N})^G$ – G-equiv. perverse sheaves on $\mathcal{N}$. Describe in terms of $W(L)$s

II $D(\mathfrak{g})^G$ – derived category of G-equiv. $\mathcal{D}$-modules.

G connected reductive group over $\mathbb{C}$, $\mathfrak{g} = \mathrm{Lie}(G)$.

$$\begin{array}{ccc} \mathcal{P}(\mathcal{N})^G \subseteq & D(\mathcal{N})^G & \text{the derived category.} \\ & \cap\mid & \\ & D(\mathfrak{g})^G. & \end{array}$$

I General Springer correspondence (Lusztig '84).

$$\mathcal{P}(\mathcal{N}) = \bigoplus_{\text{indexing set to come}} \mathrm{Rep}(W(L)) \longleftrightarrow \mathrm{Rep}(W)$$

$W(L) = N(L)/L$, L a Levi of G.

① Define functors : $L \subset P \subset G$, $P \twoheadrightarrow L$. Lie algebras $\mathfrak{l}, \mathfrak{p}, \mathfrak{g}$. $\mathcal{N}_L$ the nilcone of L, $\mathcal{N}_G$ that of G.

$$\mathcal{P}(\mathcal{N}_G)^G \underset{\mathrm{Ind}_L^G}{\overset{\mathrm{Res}_L^G}{\rightleftarrows}} \mathcal{P}(\mathcal{N}_L)^L \quad \text{obtained by } \mathfrak{g} \leftarrow \mathfrak{p} \rightarrow \mathfrak{l} \text{ correspondence.}$$

$\cong \mathfrak{p}/\mathfrak{n}_{\mathfrak{p}}$.

e.g. $L = H$ a max. torus then $\mathcal{P}(\mathcal{N}_G)^G \rightleftarrows \mathcal{P}(\mathcal{N}_H)^H \cong \mathrm{Vect}$. $\mathrm{Ind}(\mathbb{C}) = \mathrm{Spr}$

a point

$\mathrm{Res}(-) = \mathrm{Hom}(\mathrm{Spr}, -)$.

② An object $\mathcal{F} \in \mathcal{P}(\mathcal{N}_G)^G$ is called cuspidal if $\mathrm{Res}_L^G(\mathcal{F}) = 0$ for all proper Levi L.

Definition: $\mathcal{P}(\mathcal{N})^G_{[L]} = \langle \mathrm{Ind}(\mathcal{P}(\mathcal{N}_L)^L_{cusp}) \rangle$.

③ Mackey Theorem :

Let Q, P be two parabolics with Levis M and L

$$\mathrm{Res}^G_M \circ \mathrm{Ind}^G_L(\mathcal{F}) \cong \bigoplus_{\substack{w \in Q\backslash G/P \\ \shortparallel \\ W_Q \backslash W / W_P}} \mathrm{Ind}^M_{{}^wL \cap M} \mathrm{Res}^{{}^wL}_{{}^wL \cap M}(w_* \mathcal{F})$$

Corollary : a) $\mathcal{P}(\mathcal{N})^G_{[L]} \rightleftarrows \mathcal{P}(\mathcal{N}_L)^L_{cusp}$

b) $\mathrm{Res}^G_L \, \mathrm{Ind}^G_L |_{\mathcal{P}(\mathcal{N}_L)^L_{cusp}} (\mathcal{F}) = \bigoplus_{w \in N(L)/L} w_*(\mathcal{F})$.

$$N(L)/L = W(L) = N_W(W_P)/W_P \subseteq W_P \backslash W / W_P$$

4) Claim: $\mathcal{P}(\mathcal{N}_G)^G = \bigoplus \mathcal{P}(\mathcal{N}_G)^G_{[L]}$

$\uparrow$ conjugacy classes of Levis in G.

What needs to be checked?

a) $\mathcal{P}(\mathcal{N}_G)^G_{[L]}$ generate (standard: Restrict any object to a minimal parabolic)

b) $\mathcal{P}(\mathcal{N}_G)^G_{[L]} \perp \mathcal{P}(\mathcal{N}_G)^G_{[M]}$ if M is not conjugate to L (use Mackey and stability under Verdier duality).

c) $\mathcal{P}(\mathcal{N}_G)^G_{[L]} = \mathcal{P}(\mathcal{N}_G)^G_{[M]}$ if L is conjugate to M.

Barr-Beck monadicity theorem.

If $\mathcal{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{D}$ an adjunction $RL \in \mathrm{End}(\mathcal{D})$ is a monad: $(RL)\circ(RL) \to RL$, $1_{\mathcal{D}} \to RL$

$\Rightarrow \mathcal{C} \xrightarrow{\sim} \mathcal{D}^{RL}$ (RL-modules in $\mathcal{D}$) i.e. $RL(d) \to d$.

(assuming R is conservative, ie. $R(c) = 0 \Rightarrow c = 0$.

P<u>roposition</u>: There is an equivalence of monads

$$\mathrm{Res}^G_L \circ \mathrm{Ind}^G_L \circlearrowright \mathcal{P}(\mathcal{N}_L)^L_{cusp} \supseteq W(L) \qquad F \to \bigoplus_{w \in N(L)/L} w_*(F)$$

<u>Corollary</u>: $\mathcal{P}(\mathcal{N}_G)^G_{[L]} \simeq (\mathcal{P}(\mathcal{N}_L)^L_{cusp})^{W(L)}$

and $\mathcal{P}(\mathcal{N}_L)^L_{cusp}\ (\langle \mathcal{F}_1 \rangle \oplus \dots \oplus \langle \mathcal{F}_n \rangle)^{\supseteq W(L)} \cong \bigoplus \mathrm{Rep}(W(L))$

II $\mathcal{D}(\mathfrak{g})^G$ derived equivariant category of $\mathcal{D}$-modules on G.

<u>Theorem</u>: (G.) $\mathcal{D}(\mathfrak{g})^G = \overset{\perp}{\bigoplus} (\mathcal{D}(\mathfrak{l})^L_{cusp})^{W(L)}$.

<u>Remarks</u>: Note: McGerty-Nevins had a recollement: this is similar, but split!

<u>Note</u>: $\mathcal{D}(\mathfrak{gl}_n)^{GL_n} \rightsquigarrow \mathcal{D}(\mathfrak{gl}_n \times V)^{GL_n, c}$

e.g. PGL_2: $\mathcal{D}(\mathfrak{g})^G = \mathcal{D}(\mathfrak{h})^{N_G(\mathfrak{h})} = (\mathcal{D}_{\mathfrak{h}} \otimes S(\mathfrak{h}[-2])) \# W.$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\downarrow \heartsuit$

$\quad\quad\quad\quad\quad\quad\quad\quad(\mathcal{D}_{\mathfrak{h}} \# W)$-mod.

for SL_2 there is then also an additional cuspidal object, which is orthogonal to this category.

2) $D(\mathcal{N})^G = \bigoplus \ldots$

$$\begin{array}{c} \cup \\ D(\mathfrak{h})^{N_G(H)} \end{array} = (S(\mathfrak{h}[-2]) \# W)\text{-mod}$$ (c.f. L. Rider's derived Springer correspondence.)

3) $D(\mathfrak{l})^L_{\text{cusp}} = \bigoplus_{\substack{\text{irred} \\ \text{cuspidals} \\ \text{on } \mathcal{N}_L}} D(\mathfrak{z}(\mathfrak{l}))^{Z(L)} \boxtimes f_i$

Hom reduction: $D(\mathfrak{h})^{N_G(H)} \hookrightarrow D(\mathfrak{g})^G$

$$\begin{array}{c} \cup \\ \mathcal{D}_{\mathfrak{h}} \end{array} \longleftarrow \mathcal{D}_{\mathfrak{g}} / \mathcal{D}_{\mathfrak{g}} \cdot ad(\mathfrak{g}) = M.$$

$\text{Hom}(M, -) = QHR$ (quantum Hamiltonian reduction - everything is very derived here tho!)

$\circlearrowleft$

$\text{End}(M) \text{"="} M^G$.

$\searrow$ Levasseur - Stafford

$\mathcal{D}_{\mathfrak{h}}^W$

} the above equivalence of categories recovers this: $\text{End}(\mathcal{D}_{\mathfrak{h}}) = (\mathcal{D}_{\mathfrak{h}})^W$.

c.f. Gan - Ginzburg, Etingof, ...

Bellamy - Ginzburg: work on mirabolic sheaves.