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joint with Travkin & Dobrovolska

Q a quiver with vertex set I, k a field. $d \in \mathbb{Z}^I_{\geq 0}$. Consider $Rep_d(k) = Rep_d(Q, k)$, d-dim. reps of Q in k-vector spaces.

Q fixed throughout. $Ind_d(k)$ = indecomposable reps, $Abs_d(k)$ = absolutely indecomposable reps ie. $Ind_d(\bar{k}) \cap Rep_d(k)$.

Let $k = \mathbb{F}_q$ and $a_q = \{$isoclasses of abs indecomp. reps of dim d over $\mathbb{F}_q\}$

<u>Proposition</u>: (Kac): $a_q \in \mathbb{Z}[q]$ (a_q is a polynomial with integer coefficients)

<u>Conjecture</u>: (Kac '80) This poly. has non-negative coefficients.

If g.c.d.$(d_i; i \in I) = 1$ then Crawley-Boevey-van den Bergh. General case: Hausel-Letellier-Rodriguez-Villegas ('12). Both proofs use Nakajima quiver varieties.

More recently Davison '13 gave a proof using CoHa.

Approach of Hausel: use Hua Lemma ('00):

$$t^d = \prod_{i \in I} t_i^{d_i}$$

$$a_q(d) = \sum_m a^m(d) q^m. \qquad \prod_{d \geq 1} \prod_{m,n \geq 0} (1 - q^{m-n} t^d)^{a^m(d)} = \text{some alternating sum}.$$

Idea: Relate to Poincare polynomial of Nakajima quiver varieties: HLRV show that

(parts) $\quad RHS(Hua) = \sum q^i \dim H^{2i}$ (Nakajima variety, d', w' of a quiver Q')

Let $\mathcal{C}$ be an abelian category over a finite field $k = \mathbb{F}_q$ and $Vect_k$ the category of fin. dim. vector spaces. I a finite set.

Assumption: i) Homological dim. of $\mathcal{C}$ is 1.

ii) $\forall i$ we are given an exact functor $F_i : \mathcal{C} \to Vect_k$ such that $F = \bigoplus F_i$ is a faithful functor $\mathcal{C} \to \bigoplus_I Vect_k$

Given a finite extension K of k, $\mathcal{C}_K = K \otimes \mathcal{C}$. Define $\mathrm{Obj}_{d,k}$: objects A in $\mathcal{C}$ with such that $\dim F(A) = d$ ($d \in \mathbb{Z}^I_{\geq 0}$.)

$a_q(d) = |$ absolutely indecomposable isoclasses of objects in $\mathrm{Obj}_{d,k}|$

Examples: 1) $\mathcal{C} = \mathrm{Rep}(Q; \mathbb{F}_q)$. F_i = vector space at vertex i.

"2)" (almost an example). X a smooth projective curve $/\mathbb{F}_q$. $\mathcal{C} = \mathrm{Coh}(X)$ an abelian category. Let $S \subset X(\mathbb{F}_q)$. For each $s \in S$, F_s : sheaf $\mapsto$ fibre at s.

Indecomposable objects : really about counting indecomposable vector bundles - difference is torsion sheaves, which are easy to analyse (do not involve the geometry of curve).

Using enough points F_s is faithful once you fix Chern class (there are only finitely many isoclasses of such.)

$\{\rho \in \mathcal{C} : F_i(\rho) \overset{\varphi_i}{\simeq} k^{d_i}\}$ - no automorphisms. Isoclasses of such pairs are then an algebraic variety. Let

$$Y_d = \text{isoclasses of } (\rho, \varphi_i, i \in I)$$

The group $G_d = \prod GL_{d_i} \circlearrowright Y_d$.

$|$ Isoclasses of $\mathcal{C}(k)| = |G_d(k)\text{-orbits in } Y_d(k)|$

However the indecomposable locus is just a constructible set.

$T^*Y_d \longrightarrow \mathfrak{g}_d := \mathrm{Lie}\, G_d$ the moment map.

Y_d is actually a smooth stack = homological dim. 1 $\Rightarrow T^*Y_d$ is smooth.

Pick $\mathcal{O} \subset \mathfrak{g}_d$ sufficiently generic semisimple regular conjugacy class

$\mathcal{M}_\theta := \mu^{-1}(\mathcal{O}) /\!/ G_d$ Ham. reduction.

Theorem: (D.T.G.) $\#\{\text{abs. ind. objects of } \mathcal{C}(k)\} = \mathrm{STr}(\mathrm{Frob}_k, H^*(\mathcal{M}_\theta))$.

Case where $\mathcal{C} = \mathrm{Rep}_k Q$: μ is usual moment map. For generic θ, $\mu^{-1}(\theta)$ is smooth

It is known that $\mathcal{M}_\theta$ has no odd homology and eigenvalues of Fr are powers of q.

$S \subset X(\mathbb{F}_q)$.

Y = vector bundles with complete flags at each $s \in S$. T^*Y = Higgs bundle (V, φ)

φ has a simple pole at each $s \in S$ and V a parabolic bundle on Y.

$\mu^{-1}(\mathcal{O})$ prescribes the residues of φ at $s \in S$:

$M_{\mathcal{O}} = \{(V, \varphi) : res_s(\varphi) \in \mathcal{O}_s\}$ where $\mathcal{O} = \{\mathcal{O}_s : s \in S\}$.

Then the theorem counts indecomposable Higgs bundles.

Loc_d = ℓ-adic local system on $X \setminus S$ such that monodromy at $s \in S$ belongs to $\mathcal{O}_s$

- not an algebraic variety!

$Loc_d^{Fr} = \mathcal{L}$ s.t. $\mathcal{L} \cong Fr^*\mathcal{L}$. Drinfeld counted this for $d=2$. Deligne extends to GL_d

using Langlands for GL_d.

Deligne gets the same answer! This shows the number of Fr-fixed local systems

is the same for a Higgs bundle count.