S. Riche   Koszul duality for modular perverse sheaves.

- joint with P. Achar.

Setting: G conn. red. group $/\mathbb{C}$. $\mathcal{B} = G/B = \bigsqcup_{w \in W} BwB/B$, W = Weyl group.

$\mathbb{F}$ finite field of char $= \ell > 0$.

Goal: Understand $\mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$ Bruhat-constructible perverse sheaves on $\mathcal{B}$ with coefficients in $\mathbb{F}$.

For $\mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{C})$ this category has a graded cover which is Koszul (Beilinson-Ginzburg-Soergel).

The multiplicities $[\Delta_w : IC_v]$ are known - can be computed in terms of Kazhdan-Lusztig polynomials

- this category has a representation-theoretic interpretation: regular block of category $\mathcal{O}$ (= Beilinson Bernstein & Soergel)

Question: What survives for $\mathbb{F}$?

First define main characters in the talk:

§1 Definitions:

$\mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$ is a quasi-hereditary category: $\forall w \in W$ $j_w : BwB/B \hookrightarrow \mathcal{B}$ inclusion.

simple: $IC_w := j_{w!*} \mathbb{F}_{BwB/B}[\ell(w)]$

standards: $\Delta_w = j_{w!} \mathbb{F}_{BwB/B}[\ell(w)]$; costandards $\nabla_w = j_{w*} \mathbb{F}_{BwB/B}[\ell(w)]$.

A tilting object is an object with both a Δ and ∇ filtration.

Indecomposable tiltings parametrized by W: T_w is supported on $\overline{BwB/B}$ and $T_w|_{BwB/B}$ is $\cong \mathbb{F}_{BwB/B}[\ell(w)]$

Parity sheaves: $D^b_{(B)}(\mathcal{B}, \mathbb{F})$. Bruhat-constructible derived category $\mathcal{F} \in D^b_{(B)}(\mathcal{B}, \mathbb{F})$ is called a parity complex if $\mathcal{F} \cong \mathcal{F}_0 \oplus \mathcal{F}_1$ with $\mathcal{H}^i(\mathcal{F}_j) = \mathcal{H}^i(\mathbb{D}(\mathcal{F}_j)) = 0$ if $i \not\equiv j \bmod 2$.

An indecomposable parity complexes are parametrized by $W \times \mathbb{Z}$. For all $w \in W$ there is a unique indecomposable parity complex $\mathcal{E}$ supported on $\overline{BwB/B}$ and such that $\mathcal{E}_w|_{BwB/B} \cong \mathbb{F}_{BwB/B}[-\ell(w)]$. Any indecomposable parity complex is isomorphic to some $\mathcal{E}_w[i]$.

§2. Koszul duality.

Mixed derived category : $D^{mix}_{(B)}(\mathcal{B}, \mathbb{F}) = K^b \operatorname{Parity}_{(B)}(\mathcal{B}, \mathbb{F})$.

Homological shift $[1]$, but also have a shift $\{1\}$ the homological shift in $\operatorname{Parity}_{(B)}(\mathcal{B}, \mathbb{F})$.

Tate twist is then $\langle 1 \rangle = \{-1\}[1]$.

<u>Proposition</u>. There exists a "perverse" t-structure on $D^{mix}_{(B)}(\mathcal{B}, \mathbb{F})$ whose heart $\operatorname{Perv}^{mix}_{(B)}(\mathcal{B}, \mathbb{F})$ is a graded quasi-hereditary : have objects IC^{mix}_w, Δ^{mix}_w, ∇^{mix}_w, T^{mix}_w

$\check{G}$ the Langlands dual group (over $\mathbb{C}$) $\check{T} \subset \check{B} \subset \check{G}$ max. torus and Borel.

$\rightsquigarrow$ category $\operatorname{Perv}_{(\check{B})}(\check{\mathcal{B}}, \mathbb{F}) \subset D^b_{(\check{B})}(\check{\mathcal{B}}, \mathbb{F})$ objects $\check{IC}_w$, $\check{\Delta}_w$, $\check{\nabla}_w$, $\check{T}_w$, $\check{\mathcal{E}}_w$ for $w \in W$

and mixed analogues

<u>Theorem</u>: There exists a diagram :

$$\begin{array}{ccc} D^{mix}_{(B)}(\mathcal{B}, \mathbb{F}) & \xrightarrow[\kappa]{\sim} & D^{mix}_{(\check{B})}(\check{\mathcal{B}}, \mathbb{F}) \\ \text{For} \downarrow & & \downarrow \text{For} \\ D^b_{(B)}(\mathcal{B}, \mathbb{F}) & & D^b_{(\check{B})}(\check{\mathcal{B}}, \mathbb{F}) \end{array}$$

such that : 1) κ is an equivalence, $\kappa \circ \langle 1 \rangle = \langle -1 \rangle \circ [1] \circ \kappa$

$\kappa(\Delta^{mix}_w) = \check{\Delta}^{mix}_{w^{-1}}$, $\kappa(\nabla^{mix}_w) \xrightarrow{\sim} \check{\nabla}^{mix}_{w^{-1}}$, $\kappa(T^{mix}_w) = \check{\mathcal{E}}_{w^{-1}}$, $\kappa(\mathcal{E}_w) = \check{T}^{mix}_{w^{-1}}$

2) For is t-exact, $\text{For} \circ \langle 1 \rangle \cong \text{For}$ and $\bigoplus_i \operatorname{Hom}(\mathcal{F}, \mathcal{G}\langle i \rangle) \xrightarrow{\sim} \operatorname{Hom}(\text{For}\,\mathcal{F}, \text{For}\,\mathcal{G})$

and $\text{For}(\Delta^{mix}_w) = \Delta_w$, $\text{For}(\nabla^{mix}_w) = \nabla_w$, $\text{For}(IC^{mix}_w) = IC_w$ $\text{For}(T^{mix}_w) = T_w$

$\text{For}(\mathcal{E}_w) = \mathcal{E}_w$.

§3. Applications

<u>Theorem</u>: For $v, w \in W$ $[\Delta_w : IC_v] = [\nabla_w : IC_v] = [T_{vw_0} : \nabla_{ww_0}] = \dim \mathbb{H}^\bullet(\check{B}w\check{B}/\check{B}, j^{\vee *}_{ww_0} \check{\mathcal{E}}_{vw_0})$

$$ch(\mathcal{F}) = \sum_{w \in W} \dim \mathbb{H}^i(j^*_w \mathcal{F}) v^{-i-\ell(w)} H_w \in H_W.$$

ℓ-canonical basis : ${}^\ell H_w = ch(\mathcal{E}_w)$

Meaning of this theorem : the combinatorics of $\operatorname{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$ is encoded in the ℓ-canonical basis of the dual group.

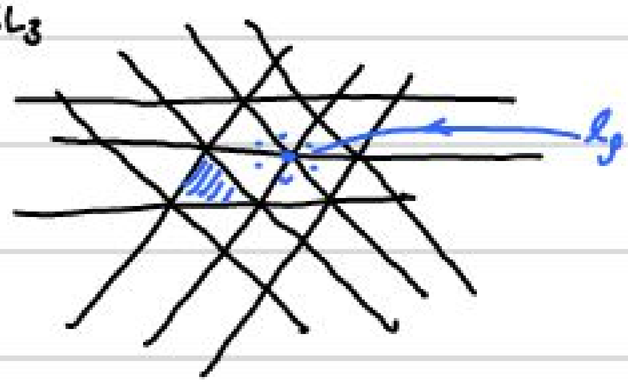
§ Modular category $\mathcal{O}$.

$G_{\mathbb{F}}$ split connected semisimple $\mathbb{F}$-group, simply connected and with root system of G.

Definition: (Soergel)

$\mathcal{O}_{\mathbb{F}} = \mathcal{A}/\mathcal{N}$. $\mathcal{A}$ = Serre subcategory of category of f.d. reps of $G_{\mathbb{F}}$ generated by $L(\lambda)$ with $\lambda \uparrow \ell_\rho$. $\mathcal{N} \subset \mathcal{A}$ subcategory generated by $L(\lambda)$ with $\lambda \notin (\ell-1)\rho + W\rho$

e.g. $G = SL_3$



Soergel: $\mathcal{O}_{\mathbb{F}}$ is a quasi-hereditary category with simple objects L_x $x \in W$, standard objects M_x, costandards N_x

<u>Theorem</u>: There exists an equivalence of q.h. categories

$$\mathcal{O}_{\mathbb{F}} \simeq \mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$$

$$L_w \longleftrightarrow IC_{w^{-1}w_0}$$

$$M_w \longleftrightarrow \Delta_{w^{-1}w_0}$$

$$N_w \longleftrightarrow \nabla_{w^{-1}w_0}$$

§ Further Questions:

$\mathrm{Perv}^{mix}_{(B)}(\mathcal{B}, \mathbb{F})$ is some "graded version" of $\mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$.

Q1: Is this category Koszul? No! In fact $\mathrm{Perv}^{mix}(\mathcal{B}, \mathbb{F})$ is Koszul $\iff$ $\mathcal{E}_w(\mathbb{F}) \cong IC_w(\mathbb{F})$ for all $w \in W$. Known to be false for large ℓ (Williamson).

Q2: Is this category positively graded? Equivalent to the fact that $\mathcal{E}_w(\mathbb{F})$ is perverse for all $w \in W$.