Derived Equivalences for RCA

1) Hecke Algebras
2) RCA & cat. $\mathcal{O}$
3) KZ functor & Rouquier's conj.
4) Outline of proof
5) Something perverse

1) W - Weyl gp, $I \subseteq S$
simple refl   reflections

$W \curvearrowright \underline{h}$ (refl. rep). $\underline{h}^{reg} \subset \underline{h}$
$\quad = \{x \in \underline{h} \mid W_x = \{1\}\}$

Braid gp: $B_W = \pi_1 \left(\underline{h}^{reg} / W \right)$

$B_W = \langle T_i \mid i \in I \rangle / \text{reflns}$. Parameter $q: S/W \longrightarrow \mathbb{C}^*$

$H_q(W) = \mathbb{C} B_W / \left((T_i+1)(T_i - q_i) = 0,\ i \in I \right)$

flat deformation of $\mathbb{C}W$

In general, $H_q(W)$-mod has infinite homological dimension (a.k.a singular)

2) RCA: $H_c \subset \mathcal{D}(\underline{h}^{reg}) \rtimes W$, $c: S/W \longrightarrow \mathbb{C}$

gen. by $\underline{h}^*$, W, Dunkl operators D_a, $a \in \underline{h}$

$$D_a = \partial_a + \sum_{s \in S} c(s) \frac{\alpha_s(a)}{a} (s - id) \qquad \alpha_s - \text{root}$$

Triangular decomp. $H_c = S(\underline{h}^*) \otimes \mathbb{C}W \otimes S(\underline{h})$

$\rightsquigarrow$ cat $\mathcal{O}_c = \{ M \in H_c\text{-mod} \mid \text{f. gen} / S(\underline{h}^*), \text{loc. nilp. for } \underline{h} \}$

Verma Modules: $\rightsquigarrow \Delta_c(\lambda) = H_c \otimes_{S\underline{h} \rtimes W} \lambda \qquad (\lambda \rightsquigarrow c_\lambda)$

$+$ order $<^c$: $\quad \lambda <^c \mu \iff c_\lambda - c_\mu \in \mathbb{Q}_{>0}$

$\mathcal{O}_c$ is a highest wt. category

0) $\mathcal{O}_c \cong A\text{-mod}$. A f. dim assoc. alg.

Notation: $L_c(\lambda)$ = irred. quotient of $\Delta_c(\lambda)$

$P_c(\lambda)$ = projective cover of $L_c(\lambda)$

1) $\mathrm{Hom}(\Delta_c(\lambda), \Delta_c(\mu)) \neq 0 \Rightarrow \lambda \leq \mu$

2) $\mathrm{End}\, \Delta_c(\lambda) = \mathbb{C}$

3) $P_c(\lambda) \twoheadrightarrow \Delta_c(\lambda)$ & ker has a filtration with quotients $\Delta_c(\mu)$, $\mu > \lambda$.

highest weight cat. $\Longrightarrow$ homological dimension is finite (a.k.a smooth)

3 $\quad \delta = \prod_{s \in S} \alpha_s^2 \rightsquigarrow H_c[\delta^{-1}] \xrightarrow{\sim} D(\underline{h}^{reg}) \rtimes W$

$$\begin{tikzcd}
M \arrow[r, mapsto] & M[\delta^{-1}] \arrow[r, mapsto] & \text{fibre with monodromy rep.} \\
\in \mathcal{O}_c \arrow[r, mapsto] \arrow[dr] & \text{Loc. syst}^W(\underline{h}^{reg}) \arrow[r] & B_W\text{-mod} \\
& H_q(W)\text{-mod} \arrow[ur, hook] &
\end{tikzcd}$$

$q = e^{2\pi i c}$

Thm (GGOR)

$$\begin{tikzcd}
& \mathcal{O}_c \arrow[dl] \arrow[dr, "KZ"] & \\
\mathcal{O}_c / \mathcal{O}_c^{tor} \arrow[rr, "\sim"] & & H_q(W)\text{-mod}
\end{tikzcd}$$

torsion over $S(\underline{h}^*)$

induced functor is an equivalence

2) KZ is fully faithful on $\mathcal{O}_c$-proj. (& on $\mathcal{O}_c$-tilt)
So KZ is a "resolution of singularities." (whatever that is!)

$$c' \in c + \mathbb{Z}^{S/W} \rightsquigarrow q$$

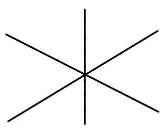
Q: Are $\mathcal{O}_c, \mathcal{O}_{c'}$ derived equivalent

Conj. (Rouquier) Yes
Gordon, I (2011) proof for types A, B

Thm (I L 2014)
conjecture is true in general

(4) Q: When is $\mathcal{O}_c$ & $\mathcal{O}_{c'}$ equivalent as abelian categories?

Orders: $<^c$ on $\operatorname{Irr} W$ defined by linear fns.
Fix $c \in \mathbb{Z}^{S/W} \rightsquigarrow$ rational cones where orders are same

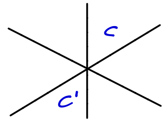


Thm: (Rouquier 05, I.L 2013/2014)
If $\leq^c$ refines $\leq^{c'}$ $(c' \in c + \mathbb{Z}^{S/W})$

$$\Rightarrow \begin{tikzcd} \mathcal{O}_c \arrow[rr, "\sim"] \arrow[dr, "KZ"'] & & \mathcal{O}_{c'} \arrow[dl, "KZ"] \\ & H_q(W)\text{-mod} & \end{tikzcd}$$

Q: $c' = c - \psi$ lie in opposite cones $(\psi \in \mathbb{Z}^{S/W})$

opposite cones



A: $\mathcal{O}_c$ & $\mathcal{O}_{c'}$ are Ringel dual

$$\mathcal{O}_c^{\Delta} \xrightarrow{\sim} \mathcal{O}_{c-\psi}^{\nabla}$$

st. filt. modules   const. filt

$$\begin{tikzcd} R: D^b(\mathcal{O}_c) \arrow[rr, "\sim"] \arrow[dr, "KZ"'] & & D^b(\mathcal{O}_{c-\psi}) \arrow[dl, "KZ"] \\ & D^b(\mathcal{H}_q(W)\text{-mod}) & \end{tikzcd}$$

Hint: $KZ(P_c(\lambda)) = KZ(T_{c-\psi}(\lambda))$

Simply laced case: 2-cones

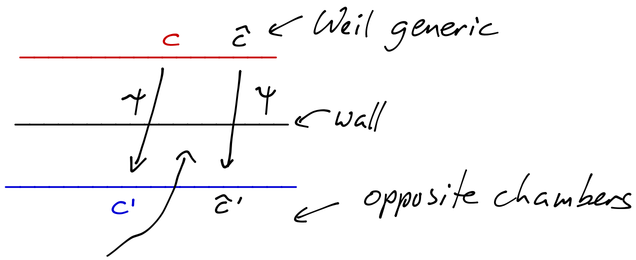
Alternative realization (as $\overset{L}{\otimes}$ with HC bimodule)

Harish-Chandra

Def: A $H_{c'}$-H_c bimodule B is H if ad actions of $S(\underline{h}^*)^W$, $S(\underline{h}^*)^W$ on B are locally nilp.

Claim:
$\exists!$ simple HC bimodule $B_c(\psi)$ s.t. $B_c(\psi)[\delta^{-1}] = D(\underline{h}^{reg}) \rtimes W$

Proposition: $R \cong B_c(\psi) \otimes^L_{HC} - : D^b(\mathcal{O}_c) \longrightarrow D^b(\mathcal{O}_{c'})$



equiv. $\mathcal{B}_{\hat{c}}(\Psi) \otimes^L \cdot$

$\exists$ generic (Zariski) flat family of HC $H_{\hat{c}'}$-$H_{\hat{c}}$-bimodule whose Weil generic fibre is $\mathcal{B}_c(\Psi)$.

Still denote Zariski generic fibre by $\mathcal{B}_{\hat{c}}(\Psi)$. Localization is still $D(\mathfrak{h}^{reg}) \rtimes W$

Thm: $\hat{c}$ is Zariski generic

$$\mathcal{B}_{\hat{c}}(\Psi) \otimes^L_{HC} - : D^b(\mathcal{O}_{\hat{c}}) \longrightarrow D^b(\mathcal{O}_{\hat{c}'})$$

is an equivalence.

Pf Equiv. for Weil generic $\hat{c} \Rightarrow R\,\mathrm{End}_{H_{\hat{c}'}}(\mathcal{B}_{\hat{c}}(\Psi)) = H_{\hat{c}}$

holds for Weil generic c

$\Rightarrow$ also holds for Zariski generic c.

$\Rightarrow$ equiv. for Zariski gen $\hat{c}$

$\mathcal{B}_{\hat{c}}(\Psi) \otimes^L \Delta_c(\lambda)$ is an object with class $= [\nabla_{c-\Psi}(\lambda)]$.

(KZ twist of $\Psi = 1$)

Have equiv. for Zariski generic c on the line

$\rightarrow$ transfer to arbitrary c using Thm 2. $\square$

5) Perverse Equivalences

c, c' are separated by single wall

$B_c(\psi) \otimes^L_{H_c} - = \varphi_c : D^b(\mathcal{O}_c) \longrightarrow D^b(\mathcal{O}_{c'})$ is perverse

Need filtrations by Serre subcategories: $\mathcal{C}_i$

$\mathcal{O}_c = \mathcal{C}' = \mathcal{C}'_0 \supset \mathcal{C}'_1 \supset \mathcal{C}'_2 \supset \dots \supset \mathcal{C}'_n \supset \mathcal{C}'_{n+1} = \{0\}$

$n = \dim \underline{h}$

Sim for $\mathcal{C}^2 = \mathcal{O}_{c'}$ where $\mathcal{C}'_j = \{ M \mid J'_j \cdot M = 0 \}$

2-sided ideal

$H_c = J'_{n+1,c} \supset J'_{n,c} \supset \dots \supset J'_{1,c} \supset J'_{0,c} = \{0\}$

Weil generic c: $J'_{j,c} = \bigcap_{M \in K\text{-}\dim M \leq n-j} \operatorname{Ann} M$

filtration by dim of support.

Zariski Generic c flat degeneration.

Thm 3 (I.L)

$\varphi_c : D^b(\mathcal{O}_c) \longrightarrow D^b(\mathcal{O}_{c'})$ is perverse

1) $D^b_{\mathcal{C}'_j}(\mathcal{C}') \xrightarrow{\sim} D^b_{\mathcal{C}^2_j}(\mathcal{C}^2)$

2) $M \in \mathcal{C}'_j \implies H_k(\varphi_c M) = 0 \quad k < j$

$\in \mathcal{C}^2_{j+1}$ when $k > j$

3) $\mathcal{C}'_j / \mathcal{C}'_{j+1} \xrightarrow{\sim} \mathcal{C}^2_j / \mathcal{C}^2_{j+1}$

$H_j(\varphi_c)$