Elliptic Character sheaves.

- joint with David Nadler.

Study a family of categories: associated to G reductive (conn.) $/\mathbb{C}$ and E an elliptic curve.

A-side: B-side

$$\mathcal{D}_N(\mathrm{Bun}_G E) \longleftarrow - - - - - \longrightarrow \mathrm{Coh}(\mathrm{Ch}_{G^\vee} E)$$

$\uparrow$ nilpotent sing. support

$\mathrm{Coh}(\mathrm{Ch}_{G^\vee} E) = [G^\vee, G^\vee]$ (pairs of elements which commute up to conjugation.)

$\uparrow$ derived equivalence?

This is Geometric Langlands for genus 1. (Betti and deRham on B-side: distinction is irrelevant because of nilpotent support condition)

Collaborators: B-side: T. Preygel. } + Nadler (globally :-))
A-side: P. Li $\longleftarrow$ elliptic Springer theory.

degenerations:  $[G^\vee, G^\vee] \rightsquigarrow [G^\vee, \mathfrak{g}^\vee] \rightsquigarrow [\mathfrak{g}^\vee, \mathfrak{g}^\vee]$

Elliptic Trig Rational

Sam_K

deformations: twisted, mixed (Nevins, Neitzke)

quantum: Jordan & Brochier & Nevins & Jordan.

mixed: q-commuting $sus^{-1} = u^q$.

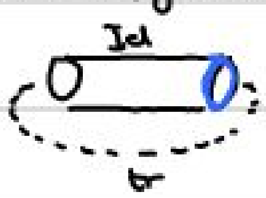
- joint with O. Hehn.

Motivation: • affine character sheaves: Ginzburg, Schiffmann-Vasserot, Lusztig, Bezrukavnikov - Kazhdan-Varshavsky.

$\uparrow$ generalized affine Springer (Lusztig).

• these elliptic character sheaves should be characters of categorical representations, traces of Hecke functors.

• $\mathrm{Ch}_{Ell} \supset \langle \text{Springer} \rangle$ = reps of Hecke algebras.

Caricature: $E = T^2 =$  $=$  $\mathrm{Tr}(\mathrm{Id}_{\mathcal{O}})$

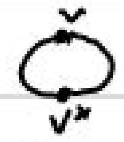
characters of LG-reps.
$\downarrow$
e.g. $CS_K(T^2)$ = Verlinde algebra

Suppose V is a finite dimensional vector space. $\mathrm{End}(V) = V \otimes V^*$. Have natural maps:

$$\mathbb{C} \xrightarrow[\text{coev}]{\text{id}} \underset{V \otimes V^*}{\overset{\mathrm{End}(V)}{\wr\wr}} \xrightarrow[\text{ev}]{\text{tr}} \mathbb{C}.$$

} having these map is a "finiteness" condition.

$\mathrm{tr}(\mathrm{id}) = \mathrm{ev}(\mathrm{coev}) = \dim$:

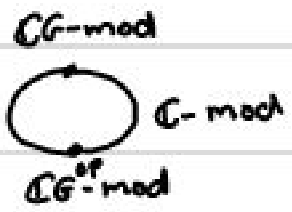


Suppose $G \circlearrowright V$ finite. Refine: $V \otimes V^* \to \mathbb{C}[G]$ (matrix coefficient)

$$\begin{tikzcd} & V \otimes V^* \arrow[r] \arrow[dr, "\text{ch}"] & \mathbb{C}[G] \arrow[d] & \\ \mathbb{C} \arrow[ur, "\text{Id}"] \arrow[rr, bend right, "\chi_V"'] & & \mathbb{C}[G/G] \arrow[r] & \mathbb{C} \end{tikzcd}$$

character. evaluate at 1.

Ex: $\mathbb{C}^{G/G}$ = cocenter = HH_* = trace of $\mathbb{C}G$: universal trace: $\mathbb{C}G \to \mathbb{C}G \overset{L}{\underset{\mathbb{C}G \otimes \mathbb{C}G^{op}}{\otimes}} \mathbb{C}G = HH_*(\mathbb{C}G)$.
$ab - ba$ $\quad\quad = \mathrm{Tr}(\mathrm{Id}_{\mathbb{C}G\text{-mod}})$



(previous picture one level up)

$$\begin{tikzcd} \mathbb{C}\text{-mod} \arrow[r, "A\otimes"] \arrow[rr, bend right, "A \otimes^L_{A \otimes A^{op}} A"'] & A\text{-mod-}A \arrow[r, "\otimes_{A \otimes A^{op}} A"] & \mathbb{C}\text{-mod}. \end{tikzcd}$$

Ex Suppose $G = G(\mathbb{F}_q) \supseteq B(\mathbb{F}_q) = B$.

$\mathbb{C}G\text{-mod} \supset \langle \mathbb{C}[G/B] \rangle = H\text{-mod}$ where $H = \mathrm{End}_G(\mathbb{C}[G/B])$.
↑
unipotent principal series

$$\begin{array}{ccc} \mathbb{C}[G/G] & \longleftarrow & \mathrm{Tr}(H) \\ \Psi & & \uparrow \mathrm{Tr} \\ \chi_{\mathbb{C}[G/B]}. & & 1 \in H \end{array}$$

<u>Definition</u>: A G-category is a $(\mathcal{D}(G), *)$-modules e.g. $\mathcal{D}(G/B)$ is a module category for $(\mathcal{D}(G), *)$ (via Beilinson-Bernstein this is $\cong \mathcal{U}(\mathfrak{g})$-mod.)

<u>Theorem</u>: (BZ. Nadler) $\mathrm{End}_G \mathcal{D}(G/B) = \underset{\mathcal{H}}{\overset{\mathcal{D}(B \backslash G/B)}{\shortparallel}}$ dg Soergel bimodules.

$\mathcal{D}(G)\text{-mod} \supseteq \langle \mathcal{D}(G/B) \rangle = \mathcal{H}\text{-mod}$ (by Lurie Bar-Beck)

<u>Remark</u>: w/ S. Gunningham and Oran: Everything is a principal series "module"

$(\mathcal{D}(G), *) \simeq \mathcal{D}(N \backslash G/N) = \mathrm{End}(\mathcal{D}(G/N))$ (equivalence of their module categories.)

What are characters of G-categories, or (by above) or of H-mod?

$HH_*(\mathcal{H}, *) \cong \mathrm{Tr}(\mathcal{H}) = \mathcal{H} \underset{\mathcal{H} \otimes \mathcal{H}^{op}}{\otimes} \mathcal{H}$ ← since we started with the <u>monoidal</u>

category $(\mathcal{H}, *)$, HH_* is a (one)-category. One can also consider $\mathcal{H}$ just as a category, and form $HH_*(\mathcal{H})$. This is then only an algebra, but we can consider its module category. This is naturally a full subcat. of $\mathcal{H} \underset{\mathcal{H}\otimes\mathcal{H}^{op}}{\otimes} \mathcal{H}^{op}$ but not everything. In fact it is $\langle Tr(1) \rangle$

Theorem: $\mathcal{D}_N(\mathcal{O}/G) \supset \left\{ \begin{matrix} \text{unipotent} \\ \text{character sheaves} \end{matrix} \right\} \simeq Tr(\mathcal{H})$.

$\underset{\text{Springer}}{\overset{\Downarrow}{}} \longleftarrow Tr \uparrow 1 \in \mathcal{H}$

gives Groth.-Springer sheaf

In fact $Z(\mathcal{H}) \simeq Tr(\mathcal{H})$ (Calabi-Yau)

$\mathcal{D}(G/G) \longleftarrow \mathcal{D}(G/B) \longleftarrow \mathcal{D}(B\backslash G/B)$ functors coming from $G/G \swarrow G/B \supset B/B \searrow B\backslash G/B$

Affine version: Want to study loop group categories:

$\mathcal{D}(LG/I)$, $\hat{\mathfrak{g}}$-mod

$\mathcal{D}(Bun_G(X, \hat{x}))$ full level structure at a point.

$\mathcal{D}(LG/I)$ carries an action of $\mathcal{H}_{aff} = \mathcal{D}(I\backslash LG/I)$ e.g. $\mathcal{D}(Bun_G(X, x))$ parabolic structure.

Definition: Elliptic character sheaves $= Tr(\mathcal{H}_{aff}) \ni Tr(1) =$ elliptic Springer sheaf.

$\mathcal{H}_{aff} = Coh_{G^\vee}(St)$ $\quad G^\vee\backslash T^*\mathcal{B}^\vee \underset{\mathfrak{g}^\vee}{\times} T^*\mathcal{B}^\vee = \tilde{\mathfrak{b}}^\vee/B^\vee \underset{\mathfrak{g}^\vee/G^\vee}{\times} \tilde{\mathfrak{b}}^\vee/B^\vee$ (of the form $X \times_Y X$ so get a monoidal category.)

Theorem [BZ-N. Preygel] $\mathcal{H}_{aff} = End_{\mathfrak{g}^\vee/G^\vee}(Coh\, \tilde{\mathfrak{b}}^\vee/B^\vee)$.

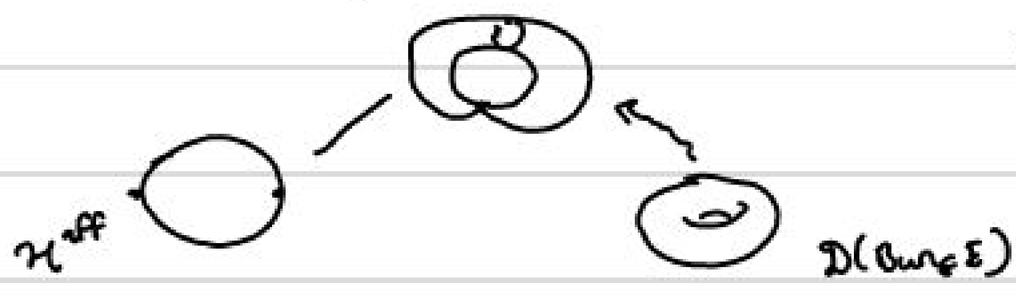
Theorem: [BZ-N. Preygel]: $Tr(\mathcal{H}_{aff}) \simeq Coh_N([G^\vee, G^\vee])$

analogous to finite dimensional case but harder. [Big Steinberg, i.e. $B^\vee/B^\vee \times_{G^\vee/G^\vee} B^\vee/B^\vee$ gives the q-commuting versions.)

(this is different to HH_* of category again!)

$Tr(\mathcal{H}_{aff}) \underset{\simeq}{\overset{??}{\longrightarrow}} \mathcal{D}_N(LG/LG)$ Now LG/LG has a "q" deformation $Bun_G E$ $\quad \mathbb{C}^\vee/q^{\mathbb{Z}} \downarrow E$

Idea: Replace with $\mathcal{D}_N(Bun_G E)$ and take $q \to 0$ nodal curve degeneration:



Thus the goal is to calculate $Tr(\mathcal{H}_{aff})$ in this way by degenerating the elliptic curve.