Koszul Duality for Modular Perverse Sheaves

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G complex reductive gp. $T \subseteq B \subseteq G$
max torus   Borel

$\mathcal{B} = G/B = \bigsqcup_{w \in W} BwB/B$ Bruhat decomp. $W = N_G(T)/T$

$\mathbb{F}$ = f. field char ℓ. ← assume ℓ "good" prime

Goal: Understand category $\mathrm{Perv}_{(\mathcal{B})}(\mathcal{B}, \mathbb{F})$
Bruhat-const. perverse sheaves with coeffs in $\mathbb{F}$.

For $\mathrm{Perv}_{(\mathcal{B})}(\mathcal{B}, \mathbb{C})$ – has a "graded cover" which is Koszul ← Beilinson–Ginzburg–Soergel.

- the multiplicities $[\Delta_w : IC_v]$ can be computed in terms of KL-polys.
- Rep. Thy interpretation: equivalent to a regular block of cat $\mathcal{O}$ (Beilinson-Bernstein + Soergel)

Q: How much of this survives for $\mathbb{F}$.

1. Definitions

$\mathrm{Perv}_{(\mathcal{B})}(\mathcal{B}, \mathbb{F})$ is a quasihereditary category.
$\forall w \in W$ $j_w : BwB/B \hookrightarrow \mathcal{B}$ inclusion

simple : $IC_w := j_{w!*} \mathbb{F}_{BwB/B}[\ell(w)]$

standard: $\Delta_w = j_{w!} \underline{\mathbb{F}}_{BwB/B}[\ell(w)]$

- costandard: $\nabla_w = j_{w*} \underline{\mathbb{F}}_{BwB/B}[\ell(w)]$

An object $\mathcal{F} \in Perv_{(B)}(\mathcal{B}, \mathbb{F})$ is called tilting if it admits both a Δ-filtration & a ∇-filtration.

Indecomp. Tilting Ojects are param. by W.
$\forall w \in W$ $\exists!$ indec. tilting object $\mathcal{T}_w$ supported on $\overline{BwB/B}$
& s.t $\mathcal{T}_w|_{BwB/B} \cong \underline{\mathbb{F}}_{BwB/B}[\ell(w)]$

Parity Sheaves: $D^b_{(B)}(\mathcal{B}, \mathbb{F})$ Bruhat-const. derived cat.
$\mathcal{F} \in D^b_{(B)}(\mathcal{B}, \mathbb{F})$ is called parity complex if

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$$

with $H^i(\mathcal{F}_j) \cong H^i(\mathbb{D}(\mathcal{F}_j)) = 0$ if $i \not\equiv j \mod 2$

Indec. Parity complexes are param. by $W \times \mathbb{Z}$.

$\forall w \in W$ $\exists!$ indec. parity complex $\mathcal{E}_w$ supported on $\overline{BwB/B}$
& s.t $\mathcal{E}_w|_{BwB/B} \cong \underline{\mathbb{F}}_{BwB/B}(\ell(w)]$

Any indec. parity complex is isom. to some $\mathcal{E}_w[i]$
$Parity_{(B)}(\mathcal{B}, \mathbb{F})$ = cat. of Parity complex $\leftarrow$ just an additive sub. cat of derived cat.

2. Koszul Duality

Mixed Derived Category: $D^{mix}_{(B)}(\mathcal{B}, \mathbb{F}) = K^b Parity_{(B)}(\mathcal{B}, \mathbb{F})$
homotop cat.

Shifts: $[1]$ homological shift in $K^b Parity_{(B)}(\mathcal{B}, \mathbb{F})$
$\{1\}$ homological shift in $Parity_{(B)}(\mathcal{B}, \mathbb{F})$
$\langle 1 \rangle = \{-1\}[1]$ $\leftarrow$ Tate twist

Prop: There exist a "perverse" t-structure on $D^{mix}_{(B)}(B,\mathbb{F})$. whose heart, $\mathrm{Perv}^{mix}_{(B)}(B,\mathbb{F})$ is graded quasihereditary.

$\rightsquigarrow$ have objects IC^{mix}_w, Δ^{mix}_w, ∇^{mix}_w, T^{mix}_w

$G^\vee$ Langlands dual gp (over $\mathbb{C}$) : $\check{T} \subset \check{B} \subset \check{G}$ max torus & Borel.

$\rightsquigarrow$ categories $\mathrm{Perv}_{(\check{B})}(\check{B},\mathbb{F}) \subset D^b_{(\check{B})}(\check{B},\mathbb{F})$

objects $\check{IC}_w$, $\check{\Delta}_w$, $\check{\nabla}_w$, $\check{T}_w$, $\check{\mathcal{E}}_w$ for $w \in W$ + mixed structures

Thm: $\exists$ a diagram

$$\begin{tikzcd}
D^{mix}_{(B)}(B,\mathbb{F}) \arrow[r, "\mathcal{K}", "\sim"'] \arrow[d, "\mathrm{For}"] & D^{mix}_{(\check{B})}(\check{B},\mathbb{F}) \arrow[d, "\mathrm{For}"] \\
D^b_{(B)}(B,\mathbb{F}) & D^b_{(\check{B})}(\check{B},\mathbb{F})
\end{tikzcd}$$

s.t

1) $\mathcal{K}$ is an equivalence, $\mathcal{K} \circ \langle 1\rangle = \langle -1\rangle[1] \circ \mathcal{K}$
   $\mathcal{K}(\Delta^{mix}_w) = (\check{\Delta}^{mix}_{w^{-1}})$
   $\mathcal{K}(\nabla^{mix}_w) \simeq \check{\nabla}^{mix}_{w^{-1}}$
   $\mathcal{K}(T^{mix}_w) = \check{\mathcal{E}}_{w^{-1}}$, $\mathcal{K}(\mathcal{E}_w) = \check{T}^{mix}_{w^{-1}}$

2) For is t-exact, $\mathrm{For} \circ \langle 1\rangle \cong \mathrm{For}$
   &
   $$\bigoplus_i \mathrm{Hom}(\mathcal{F}, \mathcal{G}\langle i\rangle) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{For}\,\mathcal{F}, \mathrm{For}\,\mathcal{G})$$
&
   $\mathrm{For}(\Delta^{mix}_w) = \Delta_w$, $\mathrm{For}(\nabla^{mix}_w) = \nabla_w$
   $\mathrm{For}(IC^{mix}_w) = IC_w$, $\mathrm{For}(T^{mix}_w) = T_w$
   $\mathrm{For}(\mathcal{E}_w) = \mathcal{E}_w$

3. Applications

Multiplicities:

Thm: For $v, w \in W$,

$$[\Delta_w : IC_v] = [\nabla_w : IC_v] = (\mathcal{T}_{vw_0} : \nabla_{vw_0}) =$$

$$\dim H^*\left(\check{B}w_0w^{-1}\check{B}/\check{B}, \check{j}^*_{w_0w^{-1}} \check{\Xi}_{w_0v^{-1}}\right) \longleftarrow \text{total cohomology}$$

Interpretation: $\mathcal{H}_W$ = Hecke alg. of W

$\mathcal{F} \in D^b_{(B)}(\mathcal{B}, \mathbb{F})$

$$ch(\mathcal{F}) = \sum_v \dim H^i(j_w^* \mathcal{F})\, v^{-i-\ell(w)} H_w \in \mathcal{H}_W$$

ℓ - Canonical Basis: ${}^\ell\underline{H}_w = ch(\Xi_w)$

Meaning of Thm: the combinatorics of $\mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$ is encoded in the ℓ-canonical basis of the dual gp.

* Modular Category $\mathcal{O}$:

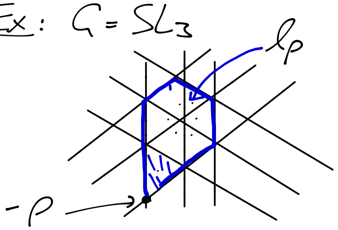
$G_{\mathbb{F}}$ split connected semisimple $\mathbb{F}$-group. simply connected & with the same root system as G.

Defn (Soergel)

$\mathcal{O}_{\mathbb{F}} := \mathcal{A}/\mathcal{N}$ $\qquad$ $\mathcal{A}$ = Serre subcat. of the cat. of f. dim. rep. of $G_{\mathbb{F}}$ gen. by $L(\lambda)$ $\lambda \uparrow \ell\rho$

$\mathcal{N} \subseteq \mathcal{A}$ sub cat. gen by $L(\lambda)$ with $\lambda \notin (\ell-1)\rho + W\rho$

Ex: $G = SL_3$



Soergel: $\mathcal{O}_{\mathbb{F}}$ is a q.hereditary cat. with simple objects L_x $x \in W$, standard objects M_x, costandard objects N_x

Thm: $\exists$ an equivalance of q.h. cats

$$\mathcal{O}_{\mathbb{F}} \longrightarrow \mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$$
$$L_w \longleftrightarrow IC_{w^{-1}w_0}$$
$$M_w \longleftrightarrow \Delta_{w^{-1}w_0}$$
$$N_w \longleftrightarrow \nabla_{w^{-1}w_0}$$

4. Further Questions

$\mathrm{Perv}^{mix}_{(B)}(\mathcal{B}, \mathbb{F})$ "graded cover of $\mathrm{Perv}_{(B)}(\mathcal{B}, \mathbb{F})$

Q1. Is this category Koszul?

- No in general: in fact $\mathrm{Perv}^{mix}_{(B)}(\mathcal{B}, \mathbb{F})$ is Koszul $\iff \mathcal{E}_w(\mathbb{F}) \cong IC_w(\mathbb{F}) \;\; \forall w \in W$

Known to be false even for reasonably large ℓ (Williamson)

Q2: Is this cat positively graded?

Not known

Equivalent to the fact that $\mathcal{E}_w(\mathbb{F})$ is perverse $\forall w \in W$

False in affine type.