Poisson-de Rham Homology for Cones & Special Polyns

Jt w/ Etingof, jt w/ Proudfoot

I. Weights of Poission Homology:

Suppose X is a complex Poission variety. Assume X is a cone (contracting $\mathbb{C}^*$-action)
Assume $\{-,-\}$ is homogeneous

Then $\mathcal{O}_X$ is non-negatively graded, $(\mathcal{O}_X)_0 = \mathbb{C}$.
Moreover $HP_0(\mathcal{O}_X) = \mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\} = (\mathcal{O}_X)_{H(X)}$ $\leftarrow$ H Vect(X)
is nonnegatively graded $\Longrightarrow$ $h(HP_0(\mathcal{O}_X), t)$ power series

Examples: ① $X = \mathbb{C}^2/\Gamma$ $\quad \Gamma \leq SL_2(\mathbb{C})$ finite

$$h(HP_0(\mathcal{O}_X), t) = \sum_i t^{(2d_i - 2)}$$

$h \equiv$ Hilbert series : $V = \bigoplus_i V_i$
$h = \sum_i (\dim V_i)\, t^i$

d_i = degrees of fund. inv's. (polyn gens.) of $(\operatorname{Sym} \mathfrak{g}^*)^{\mathfrak{g}}$, $\mathfrak{g}$ = McKay corr. simple Lie alg.

② Generalizations: $X \subseteq \mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x,y,z]$ quasi-homogeneous w/ isolated singularity
$= \{Q = 0\}$

$$HP_0(\mathcal{O}_X) = \mathcal{O}_X / (\partial_x Q, \partial_y Q, \partial_z Q)$$

graded

in fact $\{\mathcal{O}_X, \mathcal{O}_X\} = \{\partial_x Q, \partial_y Q, \partial_z Q)$

③ $S^n X$, X as in ② (or ①)

Thm [ES]

$$HP_0(\mathcal{O}_{S^n X}) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda=(\lambda_1,\dots,\lambda_k)}} \mathrm{Sym}^{\lambda_1} H \otimes \dots \otimes \mathrm{Sym}^{\lambda_k} H \,[-d n_\lambda]$$

where $H = HP_0(\mathcal{O}_X) = \mathcal{O}_X / (\partial_x Q, \partial_y Q, \partial_z Q)$

$n_\lambda = \sum_{i \geq 1} (i-1)\lambda_i \qquad d = |Q|$ ← degree of Q

Equivalently: $\bigsqcup S^l X \qquad \mathcal{O}_{\sqcup S^l X} = \mathrm{Sym}\, \mathcal{O}_X$

graded dual

$$HP_0(\mathrm{Sym}(\mathcal{O}_X))^* \cong \mathrm{Sym}(\mathbb{C}[t] \otimes H^*)$$

$|t| = |Q| = d$

weight

This statement implies (with some work)

$$\rho_* \Omega \cong M_{S^n X} \qquad X = \mathbb{C}^2/\Gamma$$

④ Hypertoric Case: $X = T^*V /\!\!/ (\mathbb{C}^*)^k$ ↙ $\{-,-\}$ has degree -2

$(\mathbb{C}^*)^k \curvearrowright V \Rightarrow$ Hamiltonian on T^*V

$$\mu : T^*V \longrightarrow (\mathrm{Lie}(\mathbb{C}^*)^k)^*$$

$$X := \mu^{-1}(0) /\!\!/ (\mathbb{C}^*)^k$$

eg Quiver variety with dim vector $(1,1,\dots,1)$

Theorem (Proudfoot) $HP_0(\mathcal{O}_X) \cong IH^*(X^!)$

$X^! \equiv$ Gale dual hypertoric variety to X.

(BLPW)

Conjecture: (Proudfoot) $X, X^!$ symplectic dual Poisson cones ($\{-,-\}$, deg -2)

$h^{br}_{A^\vee} \equiv$ h-polyn of the broken circuit complex of $A^\vee$.

$$HP_0(\mathcal{O}_X) \cong IH^*(X^!)$$

Cor: $h(HP_0(\mathcal{O}_X), t) = h^{br}_{A^\vee}(t^2) = t^{2\,\mathrm{rk}.A^\vee} \underbrace{T_{A^\vee}}_{\text{Tutte}}(t^{-2}, 0)$

hyperplane arrang. of $X^!$

II. Poisson-de Rham

Observation: $HP^{DR}_*(X)$ has an additional weight grading

Why?

$$HP^{DR}_*(X) = \pi_* M_X,$$

vector fields on V that restrict to $H(X)$ on X

$$\pi : X \longrightarrow pt, \quad M_X = \frac{D_X}{H(X) D_X} = \frac{D_V}{(I_X + \widetilde{H(X)}\, D_V)}$$

$H(X)$ is homogeneous $\Rightarrow$ M_X is weakly $\mathbb{C}^*$-equiv.

$\Rightarrow$ $\pi_* M_X$ weakly $\mathbb{C}^*$-equiv. on pt $=$ graded v.sp.

$$HP^{DR}_i(X) = H_i(M_X \overset{L}{\underset{D_V}{\otimes}} \mathcal{O}_V)$$

graded v.sp $\forall i$

Examples: ①–③ : $S^n(\mathbb{C}^2/\Gamma)$

Same answer $HP_0 \rightsquigarrow HP_*^{DR}$ additional grading

$H \rightsquigarrow HP_*^{DR}(\mathbb{C}^2/\Gamma) = H_0(\mathcal{O}_X) \oplus \mathbb{C}[-2]$ (wt 0)

Theorem: (Proudfoot–S) X hypertoric

$$h(HP_*^{DR}(X); x, y) = \Phi_A(x^2+1, y^{-2}+1, y^2, \dots, y^2)$$

Denham defined using combinatorial Laplacian of associated matroid.

$$= y^{-2\,\mathrm{rk}\,A} \sum_{F \subseteq A} y^{2|F|}\, T_{A^F}(x^2, 0)\, T_{A_F}(0, y^{-2})$$

coloop-free flats of A $\leftrightarrow$ sympectic leaves of X

restrict to F

localization

Cor: [ES] $HP_*^{DR}(X) \cong H^{\dim X - *}(X)$

$(y=1) \leftarrow$ specialisation

How to prove: Slices to sympl. leaves are hypertoric varieties A_F, leaves A^F.

symplectic leaves are hypertoric varieties as well.

(5) Nilpotent Cones, Slodowy Slices

$X = Nil(\mathfrak{g})$, $\mathfrak{g}$ simple complex Lie alg.

Conj (Lusztig)

generalized Kostka polys

$$h(HP_*^{DR}(X), x, y) = \sum_{\chi \in Irr(W)} K_{\mathfrak{g},\chi}(x^2)\, K_{\mathfrak{g},\chi}(y^{-2})$$

where $K_{\mathfrak{g},\chi}(t) := \sum t^i \dim \mathrm{Hom}_W(\chi, H^{2\dim G/B - 2i}(G/B))$

$$W \circlearrowright H^*(G/B) \cong \mathrm{Sym}\, \mathfrak{h}^* / (\mathrm{Sym}\, \mathfrak{h}^*)^W_+$$

$\deg(\mathfrak{h}^*) = 2$ $\quad \mathfrak{h} \subseteq \mathfrak{g}$ Cartan.

Corollary to conjecture:

$$h(HP_0(W_e^{0,cl}), y) = y^{\dim G\cdot e} \sum_{\substack{\chi \text{ s.t.} \\ \mathbb{O}_\chi = G\cdot e}} K_{\mathfrak{g},\chi}(y^{-2})$$

$\leftarrow$ under Springer corr.

Hotta-Kashiwara: $\rho_* \Omega_{T^* G/B} \cong \bigoplus IC(N_\chi) \otimes \chi$

N_χ irred. loc. sys. on $\mathbb{O}_\chi$

$\chi \in Irr(W)$

Refined conj: $h(\chi, y) = K_{\mathfrak{g},\chi}(y^{-2})$

Lusztig conj proved for χ = refl. rep ($\mathbb{O}_\chi$ = subreg) + other char. on subreg non ADE

with G_2 case $\Rightarrow$ conj. holds $\mathrm{rk}\,\mathfrak{g} \le 2$

III Reduction of HP^{DR} to HP_0

Thm (PS): If $\tilde{X} \longrightarrow X$ conical symplectic res. AND $\forall$ sympl. leaves $S \subseteq X$, with X_s = Slice S at $s \in S$, $HP_0(\mathcal{O}_{X_s}) \cong H^{\dim X_s}(\tilde{X}_s)$,

then $\rho_* \Omega \cong M_X$

Thm (PS): In this situation,

$$M_X \cong \bigoplus_{S \subseteq X} IC(L_S)$$

canonical

$n = -\deg\{-,-\}$

L_S = loc. sys. on leaf S

fibre at $s \in S$. $HP_0(X_s)[n \dim S / 2]$

$L_S = H^0 i^* M(X) \quad i: S \hookrightarrow X$

As ordinary (non-equiv.) local systems.

$$L_S \cong H^{\dim X_s}(\rho^{-1}(s))$$

non-can.

Closing thought: ES conj.

$\rho_* \Omega \cong M(X)$ for quiver varieties would follow from $\dim HP_0(X_s) = \dim H^{\dim X_s}(\tilde{X}_s)$ for all S.

quiver varieties