Quiver Hecke algebras

Cargese, 6/2014.

— x —

O. DUDAS

Overview.

Why quiver Hecke algebras?

(= KLR algebras, for Khovanov – Lauda – Rouquier algebras
≠ algebras associated to quivers.)

* active research area
* involves modern rep'n theory (categorification & "geometry")

The example of $\mathfrak{sl}_2$ (Chuang – Rouquier) will be the core example.
Start by the categorification (2) of the $\mathfrak{sl}_2$-action (1)

Problem ⇝ add structure ⇝ go to a higher level ⇝ solve ⇝ go down

rep'n theory	$\xrightarrow{\text{categorification}}$ higher rep'n theory
group / algebra A	monoidal category $\mathcal{A}$ (not unique in general)
acts on vector space V	(k-linear) category (abelian/triangulated) $\mathcal{V}$
ie $A \longrightarrow \mathrm{End}(V)$	$\mathcal{A} \longrightarrow \mathrm{End}(\mathcal{V})$ (endofunctors of $\mathcal{V}$)
	objects $\longmapsto$ functors
($\mathfrak{g}$ = any Kac-Moody algebra)	$\boxed{\text{maps}} \longmapsto$ natural transform'n

$A = U_q(\mathfrak{g})$

$\boxed{\text{Quiver Hecke algebra}}$

(1) Useful for "decategorification" which allows to prove things (hidden pptes of the weaker structure).
(Ex.: positivity results / integrality
Ex.: KL polynomials, cf Williamson)

(2) Obtain stronger invariants:
cf geometry: knot homology / Khovanov homology, ...

(3) Uncover hidden symmetries,
eg Ennola duality for finite reductive gps, mirror symmetry ...)

(4) Etc.

<u>Ex of $\mathfrak{sl}_2$</u>: Start with the simplest quiver: $\bullet$

<u>$\mathfrak{sl}_2$ categorized action</u>:

- $\mathfrak{sl}_2$: $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = [E,F] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ cf (L2)
- $U_q(\mathfrak{sl}_2)$ (flat deform'n of $U(\mathfrak{sl}_2)$)

Want to find some "nice representations" of $\mathfrak{sl}_2$.

$\boxed{\text{* From geometry}}$:

V = n-dim'l vector space / $\mathbb{F}_q$

- $\rightsquigarrow$ all subspaces on V of dimension d: $Gr(d,V)$
- "induction" & "restriction":

$\rightsquigarrow \{0 \subset U \subset U' \subset V : \dim U = d, \dim U' = d+1\}$

$$\begin{tikzcd} & \{0 \subset U \subset U' \subset V\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Gr(d,V) & & Gr(d+1,V) \end{tikzcd}$$

$\rightsquigarrow \bigsqcup_{d=0}^{n} Gr(d,V) =: X$ & $\rightsquigarrow$ $\mathbb{C}$-valued functions on X

For $\psi : X \to \mathbb{C}$, set:

$$\boxed{(E\cdot\psi)(U') := q^{*} \sum_{\substack{U \subset U' \\ \dim U'/U = 1}} \psi(U) \quad , \quad (F\cdot\psi)(U) = q^{*} \sum_{\substack{U' \supset U \\ \dim U'/U = 1}} \psi(U')}$$

<u>Fact</u> (Beilinson-Lusztig-McPherson?)
This defines an action of $U_q(\mathfrak{sl}_2)$ on the space of functions on X.
(cf ex. session)

<u>How to upgrade this rep'n?</u>

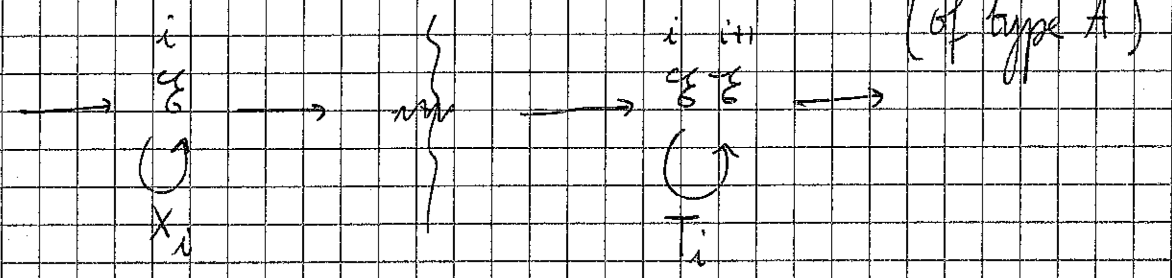
- Replace functions by sheaves (use functions-sheaves dictionary cf (L8) & (L9))

$\rightsquigarrow$ functors $\mathcal{E}$, $\mathcal{F}$.

- Miss extra-structure: endomorphisms of these: $\mathrm{End}(\mathcal{E}^n)$?

$\left.\begin{array}{l} \cdot\ X \in \mathrm{End}(\mathcal{E}) \\ \cdot\ T \in \mathrm{End}(\mathcal{E}^2) \end{array}\right]$ satisfying some Hecke relations (see below)

In the end: $\mathrm{End}(\mathcal{E}^n)$ is an <u>affine nil-Hecke algebra</u> (L4)
(of type A)



$\boxed{\text{* From rep'n theory}}$

In rep'ns of $\left\{\begin{array}{l}\text{symmetric groups,} \\ GL_n \\ \text{cyclotomic/rational Cherednik} \\ \text{(cf Iain Gordon)}\end{array}\right.$ recall i-induction & i-restriction functors

<u>From $\mathfrak{sl}_2$ to $\mathfrak{g}$</u>

<u>Rouquier</u>: "Modulo some Hecke isomorphisms, the generalisation is quite natural!"

One of the goals of the summer school: convince the world it's true, ie it's easy to go from $\mathfrak{sl}_2$ to any KM-algebra.
— how to glue these $\mathfrak{sl}_2$ actions?

Let $\mathfrak{g}$ = semisimple Lie algebra $\rightsquigarrow$ Cartan matrix, quiver $\vec{Q}$.

Starting point: <u>Ringel</u>.

Ringel: Hall algebra of $\mathbb{F}_q\vec{Q}$ (cf (L1)) is isomorphic to $U_q(n^+)$

Product on ~~...~~ the Hall algebra: "counting" some composition series on flags.

Lusztig (L15) geometric interpretation of the condition (in terms of convolution of sheaves)

$\rightsquigarrow$ produce, out of generators $(E_i)_{i=1,\dots,n}$ in $U_q^+(\mathfrak{g})$

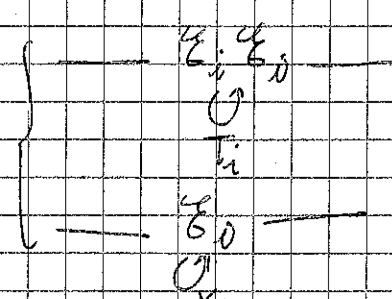
functors $(\mathcal{E}_i)_{1\leq i\leq n}$

Extra-structure: $\mathrm{End}((\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_n)^n) = ?$ [in some derived category]

$\simeq$ quiver Hecke algebra (that we'll meet first by a generators-and-relations def'n)

Contains: * idempotents — projections over $\mathcal{E}_{i_1}\cdots\mathcal{E}_{i_n}$

* $\mathfrak{sl}_2$-categorised action:



(more ...): * endomorphisms $\mathcal{E}_i\mathcal{E}_j$ — $\mathcal{E}_j\mathcal{E}_i$ as soon as have arrow $i\longrightarrow j$ in $\vec{Q}$ (L12)? or (L9)

Main goal: * $\mathrm{Ext}^*((\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_n)^n)$ is a quiver algebra Hecke

* Cor: description of Lusztig's canonical basis in terms of projective indecomposable modules (PIM) of a quiver Hecke algebra.

Representations of quivers

Plan: I Def'ns
II kQ-mod
III Standard resolutions
IV Gabriel's thm.

k = alg'c'ly closed field.

I Def: Quiver $Q = (Q_0, Q_1, s, t)$ (Q_0, Q_1 finite)
w/ Q_0, Q_1 are sets (vertices, arrows), & $s,t: Q_1 \to Q_0$.

Eg: $1 \circlearrowleft \alpha$

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \end{tikzcd}$$

Def. Rep'n of Q = M given by $(V_i)_{i\in Q_0}$ k-vector spaces, $(f_\alpha)_{\alpha\in Q_1}$ maps, $f_\alpha: \mathrm{Hom}(V_{s(\alpha)}, V_{t(\alpha)})$

Def: Morphism $u: M\longrightarrow N$ is a collection $(u_i: V_i\longrightarrow W_i)_{i\in Q_0}$
[where $M = ((V_i)_i, (f_\alpha)_\alpha)$, $N = ((W_i), (g_\alpha))$]
s.th.

$$\begin{tikzcd} V_{s(\alpha)} \arrow[r, "f_\alpha"] \arrow[d, "u_{s(\alpha)}"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{t(\alpha)} \arrow[d, "u_{t(\alpha)}"] \\ W_{s(\alpha)} \arrow[r, "g_\alpha"'] & W_{t(\alpha)} \end{tikzcd}$$

for all $\alpha\in Q_1$.

The composition of morphisms is done component-wise.

For any rep'n M of Q, there is an identity $id_M = \{(id_{V_i})_{i\in Q_0}$

This gives a category $\mathrm{Rep}_k Q$ — of rep'ns of Q over k.

Say M is a finite dimensional rep'n of Q if all V_i's are f.d.

Then, define $\underline{\dim}\, M = (\dim V_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$.

Eventually, will see that for Dynkin quivers, $\underline{\dim}\, M$ characterises M.

Def.: Say Q is of finite orbit type if Q has only finite iso classes of rep'ns of Q for any prescribed dim'n vector.

II Path algebra.

Def.: Path algebra associated to Q (over k) is the associated algebra kQ w/ generators $(e_i)_{i\in Q_0}$ and $(f_\alpha)_{\alpha\in Q_1}$ and relations:

- $e_i^2 = e_i$
- $e_i e_j = \delta_{ij} e_i$
- $e_{t(\alpha)}\alpha = \alpha = \alpha e_{s(\alpha)}$

Easy: basis of kQ given by paths.

Prop. Cat'y of rep'ns of Q is equivalent to the category of left kQ-mods.

Rem: * Unit: $\sum_{i\in Q_0} e_i =: 1$

* Q has no oriented cycles $\iff \dim_k kQ < \infty$.

Def.: Simple modules associated to $i\in Q_0$: $S(i)\in kQ$-mod. defined by * $S(i)_i = k$, $S(i)_j = 0$ if $j\neq i$.

* $f_\alpha = 0 \quad \forall \alpha\in Q_1$.

Prop: If Q has no oriented cycles, then any simple kQ-mod is isomorphic to some $S(i)$ for $\exists i\in Q_0$.

Ex. Let $Q = \circlearrowleft$. Then $kQ \cong k[X]$

For any $\lambda\in\mathbb{C}$, $S(\lambda) = k[X]/(X-\lambda)$ is simple // $k \circlearrowleft f_\alpha = \lambda\,\mathrm{Id}$

$\hookrightarrow$ the decomposition: $kQ = \bigoplus_{i\in Q_0} P(i)$ w/ $P(i) = kQ\, e_i$ $(i\in Q_0)$

Then $P(i)$ are indecomposable projectives

Prop: (i) The vector space $P(i)$ is the linear span of all paths w/ source i

(ii) $P(i) \ncong P(j)$ if $i\neq j$

(iii) $P(i)$ is indecomposable, which is equivalent to saying that e_i is primitive.

(iv) If Q has no oriented cycles, every indecomposable projective module is (up to iso) one of the $P(i)$'s (and only one).

Replace $P(i)$ by $I(i) :=$ linear span of paths w/ sink i

$\longrightarrow$ same proposition w/ "injective" instead of "projective".

III Standard resolution.

Prop: Let $M\in kQ$-mod. Then there is an exact sequence

$$0 \longrightarrow \prod_{\alpha\in Q_1} P(t(\alpha)) \underset{k}{\otimes} e_{s(\alpha)} M \xrightarrow{u} \prod_{i\in Q_0} P(i)\underset{k}{\otimes} e_i M \xrightarrow{v} M \longrightarrow 0$$

w/
$$u(a\otimes m) = a\alpha\otimes m - a\otimes \alpha m \qquad (a\in P(t(\alpha)),\ m\in e_{s(\alpha)}M)$$
$$v(a\otimes m) = am \qquad (a\in P(i),\ m\in e_i M)$$

$\longrightarrow$ "standard resolution" of M = projective resol'n of length at most 1

Together w/ Schanuel's lemma & the short exact sequence
$$0 \to I \to kQ \to kQ/I \to 0$$
(for every left ideal I in kQ), one can show that every submodule of a projective kQ-module is also projective

This ppty defines the class of hereditary algebras.

Let M, N be rep'ns of Q (as above): there is an exact sequence:

$$0 \to \mathrm{Hom}(M,N) \to \prod_{i\in Q_0} \mathrm{Hom}(V_i, W_i) \to \prod_{\alpha\in Q_1} \mathrm{Hom}(V_{s(\alpha)}, W_{t(\alpha)})$$

$$0 \to \mathrm{Hom}(M,N) \to \prod_{i\in Q_0} \mathrm{Hom}(V_i, W_i) \xrightarrow{c_{M,N}} \prod_{\alpha\in Q_1} \mathrm{Hom}(V_{s(\alpha)}, W_{t(\alpha)}) \to \mathrm{Ext}^1(M,N) \to 0$$

$$c_{M,N}((u_i)_{i\in Q_0}) = (f_\alpha u_{s(\alpha)} - u_{t(\alpha)} g_\alpha)_{\alpha\in Q_1}$$

Take dimensions:

Cor: if $\dim M = (m_i)_{i\in Q_0}$ & $\dim N = (n_i)_{i\in Q_0}$, one has:

$$\dim \mathrm{Hom}_{kQ}(M,N) - \dim \mathrm{Ext}^1(M,N) = \sum_{i\in Q_0} m_i n_i - \sum_{\alpha\in Q_1} m_{s(\alpha)} n_{t(\alpha)}$$

Def: This defines the Euler form associated to Q:

$$\langle -,- \rangle_Q : \mathbb{R}^{Q_0} \times \mathbb{R}^{Q_0} \to \mathbb{R}$$
$$(\underline{m}, \underline{n}) \longmapsto \sum_{i\in Q_0} m_i n_i - \sum_{\alpha\in Q_1} m_{s(\alpha)} n_{t(\alpha)}$$

Set $(\underline{m}, \underline{n}) := \langle \underline{n}, \underline{m} \rangle$ (Euler form of "opposite quiver")

The Euler form is not symmetric in general.

Def: Tits form of Q:
$$q_Q(\underline{n}) = \langle \underline{n}, \underline{n} \rangle_Q \qquad (\underline{n} \in \mathbb{R}^{Q_0})$$
— independent on the shape of the quiver.

IV Gabriel's theorem

Def: Representation space of a quiver Q for a fixed dim'n vector $\underline{n} \in \mathbb{N}^{Q_0}$:
$$\mathrm{Rep}(Q, \underline{n}) := \bigoplus_{\alpha: i\to j} \mathrm{Hom}(k^{n_i}, k^{n_j}) = \bigoplus_{\alpha: i\to j} \mathrm{Mat}_{n_j\times n_i}(k)$$

Obvious: $\dim \mathrm{Rep}(Q, \underline{n}) = \sum_{\substack{\alpha\in Q_1 \\ \alpha: i\to j}} n_i n_j$

Action of $GL(\underline{n}) := \prod_{i\in Q_0} GL(n_i, k)$

$g = (g_i)_{i\in Q_0}$ act on $x = (x_\alpha)_\alpha \in \mathrm{Rep}(Q, \underline{n})$ by:
$$g\cdot x = (g_j x_\alpha g_i^{-1})_{\substack{\alpha\in Q_1 \\ \alpha: i\to j}}.$$

Obvious: • $(Q\text{-reps of } \dim = \underline{n})/\simeq \;\longleftrightarrow\; GL(\underline{n})\text{-orbits in } \mathrm{Rep}(Q,\underline{n})$

• $k^* \hookrightarrow GL(\underline{n})$ acts trivially

Hence action of $GL(\underline{n})$ factorises through $PGL(\underline{n}) := GL(\underline{n})/k^* \mathrm{Id}_{\underline{n}}$
& $PGL(\underline{n})$ is irreducible of dim'n $\sum_{i\in Q_0} n_i^2 - 1$.

We are to prove the...

Claim: Q has finite orbit type $\Rightarrow$ Q is of type ADE.

Note: $\dim GL(\underline{n}) - \dim \mathrm{Rep}(Q, \underline{n}) = q_Q(\underline{n})$

hence $\dim PGL(\underline{n}) - \dim \mathrm{Rep}(Q,\underline{n}) = q_Q(\underline{n}) - 1$.

Let $O_1, \dots, O_\ell$ be the orbits of $PGL(\underline{n})$. Recall that:

- O_j is irred'ble $\forall j$
- $\dim G_x = \dim G - \dim O_x \quad \forall x \in X \supset G$.
- $\mathrm{Rep}(Q, \underline{n}) = \bigcup_{i=1}^{\ell} O_i =$
- $\overline{O_x} = O_x \cup$ (orbits of smaller dim'n) $\forall x \supset G$

Then $O_{i_0} \subset \mathrm{Rep}(Q, \underline{n})$ is open ~~$\forall i$~~ since for some i_0
hence $q_Q(\underline{n}) \geq 1 \quad \forall \underline{n} \in \mathbb{N}^{Q_0}$

~~Réciproque~~: As for the converse, let us state:

Th: Let q_Q be positive definite. Then M is indecomposable $\Leftrightarrow$ $q_Q(\dim M) = 1$
and M is uniquely determined by $\dim M$ (up to $\simeq$). Then: {indec}/$\simeq$ is finite.

KM-algebras & their quantum groups

Classical grad course: f dim $\rightsquigarrow$ semi-simple $\rightsquigarrow$ Cartan matrix
$\rightsquigarrow$ natural ∞-dim. Lie algebras!

§1 Semisimple Lie algebras.

$\mathfrak{g}$ semi-simple: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$ $\qquad$ $\mathfrak{g}_\alpha$ = weight spaces of $\mathfrak{h}$ acting by adjoint action

(say, over field of char 0 & alg'ly closed)

Def: $\alpha \in \mathfrak{h}^*$ is a root if $\mathfrak{g}_\alpha \neq (0)$. $\qquad$ | $\mathfrak{h} \simeq \mathfrak{h}^*$ via Killing form κ

$\Delta = \{\text{roots}\}$. $\qquad$ $\rightsquigarrow (\,,\,)$ on $\mathfrak{h}^*$.

Reflection associated to $\alpha\in\Delta$: $s_\alpha : u \mapsto u - \frac{2(\alpha,u)}{(\alpha,\alpha)}\alpha$ $\quad (u \in \mathfrak{h}^*)$.

Weyl group $W = \langle s_\alpha, \alpha\in\Delta\rangle$.

Choose a basis of Δ, say Π

~~Def: coroot $\alpha^\vee \in \mathfrak{h}$ associated to $\alpha\in\Delta$~~

Th.: (i) Δ spans $\mathfrak{h}^*$

(ii) for $\alpha\in\Delta$, one has: $c\alpha\in\Delta \Rightarrow c=\pm 1$

(iii) $W\Delta = \Delta$

(iv) $\forall \alpha,\beta\in\Delta$, $\langle\langle \alpha,\beta\rangle\rangle := 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

Def: Coroot: $\alpha^\vee$ associated to $\alpha\in\Delta$ characterised by $\forall\beta\in\Delta$, $\langle\alpha^\vee,\beta\rangle = \langle\langle\alpha,\beta\rangle\rangle$

Cartan matrix: $A = (\langle\alpha_i^\vee, \alpha_j\rangle)_{i,j=1,\dots,n}$ if $\Pi = \{\alpha_1,\dots,\alpha_n\}$.

ZKB

Th.: Let $A=(a_{ij})$ be the Cartan matrix of $\mathfrak{g}$ then: $\qquad [\alpha_i^\vee =: h_i]$

(i) $a_{ii} = 2 \quad \forall i\in\{1,\dots,n\}$

(ii) $a_{ij} \leq 0 \quad \forall i\neq j$

(iii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$

(iv) $\det A > 0$

(v) Moreover, any A s.th. (i)-(iv) are satisfied, is the Cartan matrix of some $\mathfrak{g}$.

Ex. $\mathfrak{sl}_n(\mathbb{C})$: $D_{ij} = E_{ii} - E_{jj}$ (elem'ry matrices) $\rightsquigarrow h_i = D_{i,i+1}$

$h_1,\dots,h_{n-1}, E_{ij}$ $(i\neq j)$ span $\mathfrak{sl}_n(\mathbb{C})$

set $\varepsilon_j : \mathfrak{h} \to \mathbb{C}$, $\sum_{i=1}^n x_i E_{ii} \mapsto x_j$

then $\Delta = \{\varepsilon_i - \varepsilon_j\}_{i\neq j}$, $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\}_{1\leq i\leq n-1}$ for instance

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

§2 Kac-Moody algebras.

Def: $A \in \mathcal{M}_n(\mathbb{Z})$. A realisation of A is $(\mathfrak{h}, \Pi, \Pi^\vee)$ s.q.:

- $\mathfrak{h}$ = f.d. vector space
- $\Pi = \{\alpha_1,\dots,\alpha_n\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_1^\vee,\dots,\alpha_n^\vee\} \subset \mathfrak{h}$ s.th.

(1) Π & $\Pi^\vee$ are linearly indep.

(2) $\langle\alpha_i^\vee, \alpha_j\rangle = a_{ij}$

(3) $\dim\mathfrak{h} = n + \operatorname{corank} A$.

Fact: Exists & unique up to permutations of rows & columns of A.

Def: Root lattice $Q \subset \mathfrak{h}^*$ is the lattice spanned by Π

Coroot lattice $Q^\vee \subset \mathfrak{h}$ ——— $\Pi^\vee$

$Q_+ \subset Q$ is $Q_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$

Def: $\tilde{\mathfrak{g}}(A)$ is the Lie algebra generated by $(e_i)_{1\leq i\leq n}$, $(f_i)_{1\leq i\leq n}$ & $\mathfrak{h}$ w/ relations:

(R1) $[e_i, f_j] = \delta_{ij}\alpha_i^\vee \quad \forall i,j\in\{1,\dots,n\}$.

(R2) $[h,h'] = 0 \quad \forall h,h'\in\mathfrak{h}$

(R3) $[h,e_i] = \langle h, \alpha_i\rangle e_i \quad \forall h\in\mathfrak{h}, \forall i\in\{1,\dots,n\}$

(R4) $[h,f_i] = -\langle h,\alpha_i\rangle f_i$

Th. $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$

* $\exists$ maximal ideal $\mathfrak{r}$ s.th. $\mathfrak{r} \cap \mathfrak{h} = \{0\}$.

Def. $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$

Fact: $\mathfrak{g}(A) = \underbrace{\left(\bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \mathfrak{g}_{-\alpha}\right)}_{=: \mathfrak{n}_-} \oplus \mathfrak{h} \oplus \underbrace{\left(\bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \mathfrak{g}_{\alpha}\right)}_{=: \mathfrak{n}_+}$

Def. If A is a generalised Cartan matrix, ie if it satisfies conditions (i)-(iii) of Th (ZLB), then $\mathfrak{g}(A)$ is called a Kac-Moody algebra

Fact: $\mathfrak{g}(A) \simeq \mathfrak{g}(A') \iff \exists P \in GL_n(\mathbb{Z}),\ A' = PAP^{-1}$.

Prop: A Kac-Moody algebra is generated by $(e_i)_{1\le i\le n}, (f_i)_{1\le i \le n}$ & $\mathfrak{h}^*$ and satisfies (R1)-(R4) of def. above and:

(R5) $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \qquad (\iff \sum_{s\geq 0} c(s,i,j)\, e_i^s e_j e_i^{1-a_{ij}-s} = 0)$

(R6) $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$

§3 Quantum story

$A = (a_{ij})_{1\le i,j\le n}$ generalised Cartan matrix $\rightsquigarrow \mathfrak{g} = \langle e_i, f_i, h \in \mathfrak{h}\rangle$

Look at the free associative alg. generated by $e_i, f_i, h \in \mathfrak{h}$, and mod out by same relations $\rightsquigarrow$ get $U(\mathfrak{g})$ - same rep'ns

Want $U_q(\mathfrak{g})$ = deformation of $U(\mathfrak{g})$: 2 meanings.

(1) Let q be an indeterminate, and $\mathbb{K} = \mathbb{Q}(q)$

$U_q(\mathfrak{g})$ = associative algebra over $\mathbb{K}$ w/ generators $\begin{cases} E_i, F_i & (1 \le i \le n) \\ K_h & (h \in Q^\vee \subset \mathfrak{h}) \end{cases}$

Relations
- $K_h K_{h'} = K_{h+h'}$, $\forall h, h' \in Q^\vee$
- $K_0 = 1$
- $K_h E_i K_h^{-1} = q^{\alpha_i(h)} E_i$
- $K_h F_i K_h^{-1} = q^{-\alpha_i(h)} F_i$
- $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$
- $\sum_{s \geq 0} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \quad \forall i \neq j$
- idem for F

Morally: $K_h = q^{h}$

A is symmetrisable: there are $d_i \in \mathbb{N}$, s.th. $a_{ij} d_j = d_j a_{ji}$

$[\ldots]$; $q_i = q^{d_i}$

$K_i = K_{d_i h_i}$

$\begin{bmatrix} m \\ k \end{bmatrix}_q \xrightarrow{q \to 1} \binom{m}{k}$

(2) "Specialisation at $q = \varepsilon \in \mathbb{C}^*$". Fix ε = nonzero complex number

Take same generators & relations, interpret things in $\mathbb{C}$?

Start w/ $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Note that most relations already hold over $\mathcal{A}$.

Define $U_{\mathcal{A}}$ = subalgebra of $U_q(\mathfrak{g})$ over $\mathcal{A}$ generated by E_i, F_i, K_i, $\tilde{H}_i = \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ + divided powers

$E_i^{(n)} = \dfrac{E_i^n}{[n]_{q_i}!}$ $\quad$ (& relations: $[E_i, F_i] = \delta_{ij} \tilde{H}_i$, $(q_i - q_i^{-1})\tilde{H}_i = K_i - K_i^{-1}$)

Then this is an integral form of $U_q(\mathfrak{g})$, ie:

$$U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(q) \simeq U_q(\mathfrak{g})$$

Good, since have a $\mathbb{Z}[q, q^{-1}]$-structure on $\mathbb{C}$ $\quad q.x = \varepsilon.x$

Set $U_\varepsilon \mathfrak{g} = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$.

Fact. $U_1 \mathfrak{g} / (K_i - 1) \simeq U(\mathfrak{g})$

$E_i \longmapsto e_i$

$F_i \longmapsto f_i$

$\tilde{H}_i \longmapsto h_i = \alpha_i^\vee$

$K_i \longmapsto 1$.

[Show how some relations degenerate:

$[E_i, F_i] = \delta_{ij} \tilde{H}_i \rightsquigarrow [e_i, f_j] = \delta_{ij} \, q_i^\vee$

$[\tilde{H}_i, E_i] = ?\rightsquigarrow [h_i, e_i] = 2 e_i$

$[\tilde{H}_i, E_i] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} E_i - E_i \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} = q E_i K_i - q^{-1} E_i K_i^{-1} \underset{K_i \to 1}{\overset{q_i \to 1}{\rightsquigarrow}}$

— similar for the other ones.]

Ex: $\mathfrak{sl}_2$: $q=1$: e, f, h, w/ usual relations

PBW: Basis of $U\mathfrak{sl}_2$: $\{e^a h^b f^c : a, b, c \in \mathbb{N}\}$

$q \neq 1$: $U_q(\mathfrak{sl}_2) = \langle E, F, K^{\pm 1} \rangle / KEK^{-1} = q^2 E, KFK^{-1} = q^{-2} F, [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$

PBW - Basis of $U_q \mathfrak{sl}_2$: $(E^a K^b F^c)_{a, c \in \mathbb{N}, b \in \mathbb{Z}}$.

PBW for $U\mathfrak{g}$

Pick a basis of $\mathfrak{g}$ (eg basis of $\mathfrak{h}$ + $e_\alpha \in \mathfrak{g}_\alpha$ + $f_\alpha \in \mathfrak{g}_{-\alpha}$)
$(x_1, \dots, x_N)$

Then a basis of $U(\mathfrak{g})$ is: $(x_1^{a_1} \cdots x_N^{a_N})_{a_1, \dots, a_N \in \mathbb{N}}$.

Cor.:

① $U\mathfrak{n}^+ \otimes U\mathfrak{h} \otimes U\mathfrak{n}^- \xrightarrow{\text{mult}^n} U(\mathfrak{g})$

② Basis for $U\mathfrak{n}^+$ = monomials in root vectors; idem w/ $\mathfrak{n}^-$.

PBW for $U_q \mathfrak{g}$:

Let $U_q(\mathfrak{n}^+) := \langle E_i, i \in \{1, \dots, n\} \rangle$, $U_q(\mathfrak{n}^-) := \langle F_i, 1 \leq i \leq n \rangle$, $U_q(\mathfrak{h}) := \langle K_i^{\pm 1} \rangle$

Then: ① $U_q \mathfrak{n}^+ \otimes U_q \mathfrak{h} \otimes U_q \mathfrak{n}^- \xrightarrow{\text{mult}^n} U_q(\mathfrak{g})$ is an iso of v. spaces

② $U_q(\mathfrak{h}) \cong \mathbb{K}[Q^\vee]$

$U_q(\mathfrak{n}^+) = \langle E_1, \dots, E_n \mid \text{Serre relations (RS)} \rangle$, idem $U_q(\mathfrak{n}^-)$

Some work to understand "root vectors" in the q-version

Category $\mathcal{O}$.

Last talk: $\left[\begin{array}{l} U\mathfrak{g} \simeq U(\mathfrak{n}^-)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n}^+) \quad \text{w/ } \mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta} \mathfrak{g}_{\pm\alpha} \\ U_q\mathfrak{g} \simeq U_q\mathfrak{n}^- \otimes \mathbb{C}[K_i^{\pm 1}]\otimes U_q(\mathfrak{n}^+) \end{array}\right.$

Stick w/ $\mathfrak{g}$ — forget quantum.

Category $\mathcal{O}$ contains – all f dim'l $U\mathfrak{g}$-modules & their projective covers
– Verma modules

x is an abelian cat'y, closed under $\oplus$

x every simple has a projective cover & injective envelope.

Def. of $\mathcal{O}$: A $U\mathfrak{g}$-module M is in $\mathcal{O}$ if:

* M is f generated
* M is locally $\mathfrak{n}^+$ finite: $\forall m\in M$, $\dim(U\mathfrak{n}_+ . m) < \infty$
* $M \simeq \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$ ~~and~~ w/ $M_\lambda = \{m\in M : \forall h\in\mathfrak{h},\ h.m = \lambda(h)m\}$

~~last~~ ~~these~~ conditions imply that $\dim M_\lambda < \infty \quad \forall\lambda$

(cf corresponding cat'y for Cherednik algebras)

Important fact: $\forall\lambda\in\mathfrak{h}^*$, $\exists!$ simple module $L(\lambda)$, $\exists!$ projective module $P(\lambda)$ and $\exists!$ Verma module $M(\lambda)$ s.th.: $\begin{cases} P(\lambda) \twoheadrightarrow L(\lambda) \\ M(\lambda) \twoheadrightarrow L(\lambda)\end{cases}$ (unique simple quotients)

Meaning? Start w/ Verma modules.

Fix $\lambda\in\mathfrak{h}^*$; then $\mathbb{C}$ is an $\mathfrak{h}$-module by: $h.v = \lambda(h)v$ ($v\in\mathbb{C}, h\in\mathfrak{h}$)

Inflate to a $U(\mathfrak{h})\otimes U(\mathfrak{n}_+)$-module by letting $\mathfrak{n}_+$ act by 0.

Induce: $M(\lambda) = \mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}_+)\otimes U(\mathfrak{h})} \mathbb{C}_\lambda$

Variant: $M(\lambda) = U(\mathfrak{g}) \big/ U(\mathfrak{g})\cdot\left(h-\lambda(h), x \;\middle|\; \begin{array}{l} h\in\mathfrak{h} \\ x\in\mathfrak{n}_+\end{array}\right)$

Useful for getting the simples: find the max submodule $N(\lambda)$ of $M(\lambda)$
then $L(\lambda) = M(\lambda)/N(\lambda)$

Properties of Verma modules.

x They sit in between L's & P's in a precise manner: BGG reciprocity

Meaning: $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$ — Jordan-Hölder

Th.: $P(\lambda)$ admits a filtration by submodules s.th. successive subquotients are Verma modules; the number of occurrences of a Verma module is well-defined.

x Characters of Verma modules are easy to describe:
If $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$, set $\mathrm{ch}\, M = \sum_{\nu\in\mathfrak{h}^*} \dim(M_\nu)\, e^\nu$

x Computing multiplicities $[M(\mu) : L(\lambda)]$ is hard — KL theory.

There is a partial order on $\mathfrak{h}^*$: $\mu \geq \lambda \iff \mu - \lambda \in \mathbb{N}\Pi$ — simple roots

Th: ~~[illegible]~~ $[M(\mu) : L(\lambda)] \neq 0 \implies \mu \geq \lambda$

Quantum case:

$U_q\mathfrak{g} = U_q\mathfrak{n}_- \otimes \mathbb{C}[K_i^\pm] \otimes U_q\mathfrak{n}^+$ (PBW theorem)

Category $\mathcal{O}$ is defined as the full subcategory of $U_q(\mathfrak{g})$-modules M s.th.:

1) M is f.g.
2) $U_q(\mathfrak{n}_+)$ acts locally finitely
3) weight spaces for $\mathbb{C}[K_i^\pm]$: for $\lambda = (\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$,
$M_\lambda = \{m\in M : \forall i=1\dots n,\ K_i m = \lambda_i m\}$.

Duality: There is an involution that fixes $\mathfrak{h}$, switches $\mathfrak{n}_+$ & $\mathfrak{n}_-$ and maps $\mathcal{O}$ to its opposite category $\mathcal{O}^{op}$, which is isom. to $\mathcal{O}$.
→ "self-dual".

Highest weight category $\mathcal{C}$ (Cline - Parshall - Scott)

* $\mathcal{C}$ = abelian, locally artinian, contains enough projectives
* There is a poset Λ ("weights") which is "locally finite" (ie $\lambda \leq \mu \implies$ there are finitely many ν's sth $\lambda \leq \nu \leq \mu$)

Conditions:

1) Λ indexes simples $L(\lambda)$ in $\mathcal{C}$
2) Λ indexes "dual Verma modules" $A(\lambda) \supset L(\lambda)$ sth. simple subquotients of $A(\lambda)$ are $L(\mu)$ w/ $\mu < \lambda$ (& $L(\lambda)$ w/ mlp 1)
3) There is an injective envelope $I(\lambda) \hookleftarrow L(\lambda)$ for every λ + conditions.

Ex. 1: kQ = path algebra of $(\overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot} \longrightarrow \cdots \longrightarrow \overset{n}{\cdot})$

Then mod-kQ is a highest weight cat'y

Simples: $S_i = A_i = (0 \to \cdots \to 0 \to \underset{i\text{th vertex}}{\mathbb{C}} \to 0 \to \cdots \to 0)$

Injectives: $I_i = (0 \to \cdots \to 0 \to \mathbb{C} \to \mathbb{C} \to \cdots \to \mathbb{C})$

Ex. $\underline{U(sl_2)}$ & $\underline{U_q(sl_2)}$. e, f, h; $\mathfrak{h}^* \cong \mathbb{C}$, $Q = 2\mathbb{Z}$, wt lattice $\sim \mathbb{Z}$

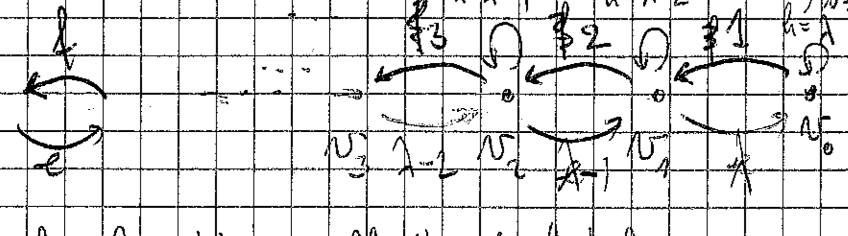
1) char = 0 / generic case.

For $\lambda \in \mathfrak{h}^*$, $M(\lambda)$ is free of rank 1 as a $U(\mathfrak{n}^-)$-module.

Fix $v_0 \in M(\lambda)_\lambda$: highest weight vector

Set $v_i = \frac{f^i v_0}{i!}$ ($i \geq 0$) Then $\times$ $v_i \in M(\lambda)_{\lambda - 2i}$ ✓

$\times (v_i)_{i \geq 0}$ basis of $M(\lambda)$



$$\begin{cases} h.v_i = (\lambda - 2i) v_i \\ e.v_i = (\lambda - i + 1) v_{i-1} \\ f.v_i = (i+1) v_{i+1} \end{cases}$$

Apply f: obtain all the tail below a vector.

~~Apply~~ Only way to have a proper submodule: $e.v_i = 0$ for $\exists i$

Then: $i = \lambda + 1$ ~~[illegible]~~ $\rightsquigarrow v_i \in M_{-\lambda-2}$. Then: $M(\lambda)$: $\frac{L(\lambda)}{L(-\lambda-2)}$

Thus: if $\lambda \geq 0$ (ie $\lambda = 2k$ for $\exists k \in \mathbb{Z}$, $k \geq 0$)

then $M(\lambda)$ has a unique submodule isomorphic to $L(-\lambda-2)$ (generated by $v_{\lambda+1}$) & $M(\lambda)/M(-\lambda-2) \simeq L(\lambda)$

otherwise: $M(\lambda)$ is irreducible.

Cor.: $\{$f. d. simple $sl_2(\mathbb{C})$-modules$\}/_{\simeq} \longleftrightarrow \mathbb{N}$

$L(\lambda) \longleftarrow\!\shortmid \lambda$

Ex.: $L(0)$ = trivial, $L(1)$ = std, $L(2)$ = adjoint

Fact: a) (Weyl's complete reducibility theorem.)

All f. dim. sl_2-modules are semi-simple.

b) Fix k of charact. $\neq 2$, $q \in k \setminus \{0, 1\}$, $q^2 \neq 1$

Recall $U_q(sl_2) = \left\langle E, F, K^{\pm 1} \,\middle|\, KK^{-1} = 1 = K^{-1}K, KEK^{-1} = q^2 E, KFK^{-1} = q^{-2} F, EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \right\rangle$

With the same arg'ts, check that $M(\lambda)$ has a basis: $(m_i)_{i \in \mathbb{N}}$ w/

$K.m_i = \lambda q^{-2i} m_i$

$F.m_i = m_{i+1}$

$E.m_i = \begin{cases} 0 & \text{if } i = 0 \\ \frac{\lambda q^{-i+1} - \lambda^{-1} q^{i-1}}{q - q^{-1}} & \text{other} \end{cases}$

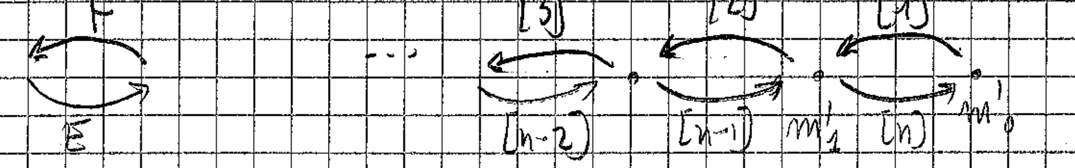
$\lambda = \pm q^a$, $a \in \mathbb{N}$

$\pm [i][a + 1 - i]$

Assume q not a root of unity: for $n \geq 0$,

$L(n, \pm) = M(\pm q^n) / \langle m_i \mid i > n \rangle$.

Let m_i' be the image of $\frac{m_i}{[i]!}$ in this quotient:



At roots of 1, $U_q(\mathfrak{g})$ is f.g. over its center $\rightsquigarrow$ "baby Vermas", etc & loose semi-simplicity.

Eirini Chavli — Tuong-Huy Nguyen

Affine nil-Hecke algebras

Symmetric group:

Prop: $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \mid [\dots] \rangle$.

Def: $w \in \mathfrak{S}_n$, write $w = s_{i_1} \cdots s_{i_k}$; if k is minimal, call it length; $l(w)$.

Ex: · $w = 13212 = 13121 = 31121 = 321$ minimal: $l(w) = 3$.
· $w = 13 = 31$ $l(w) = 2$. — several decomp'ns.

Matsumoto's lemma: let a be reduced decomp'n of w. Then all reduced decomp'n can be deduced from a by braid rel'ns.

Rks: * $l(w) = l(w^{-1})$ $\forall w \in W$.
* $l(w) = 1 \iff w \in S = \{s_1, \dots, s_{n-1}\}$.
* $l(w) = 0 \iff w = e$.

Prop: $R(w) := \{(i,j) : i<j \ \& \ w(i) > w(j)\}$. Then $l(w) = |R(w)|$.
(let $w \in \mathfrak{S}_n$)

$\mathfrak{S}_4$: (1,2) (2,3) (3,4)
(1,3) (2,4)
(1,4)

Cor.: $w_0 = (1,n)(2,n-1)\cdots$ is the unique elt of max. length $\frac{n(n-1)}{2}$.

(Indeed, for any w, $R(w) \subset R(w_0)$.)

Hecke algebra of symmetric group.

$R = \mathbb{Z}[u_1, u_2]$.

$H_n^f = R$-alg generated by $T_1, \dots, T_{n-1}$ & rel'ns $\begin{cases} (T_i - u_1)(T_i - u_2) = 0 \\ \text{braid rel'ns.} \end{cases}$

Fact: $T_w = T_{i_1} \cdots T_{i_{l(w)}}$ is well-defined by Matsumoto.

Th (Iwahori): $(T_w)_{w \in \mathfrak{S}_n}$ is a basis / $\mathbb{Z}[u_1,u_2]$ of H_n^f.

Rk: * $u_1 \mapsto 1$, $u_2 \mapsto -1$ gives $\mathbb{Z}[\mathfrak{S}_n]$. ($T_w$ becomes w)
* $l(w) + l(w') = l(ww') \Rightarrow T_w T_{w'} = T_{ww'}$.

Def: Nil-Hecke algebra: ${}^0H_n^f = \langle T_i \mid T_i^2 = 0, \text{ braid rel'ns} \rangle$.
— specialisation under $u_1 = u_2 = 0$.

Fact: $T_w T_{w'} = \begin{cases} T_{ww'} & \text{if } l(w) + l(w') = l(ww') \\ 0 & \text{otherwise.} \end{cases}$ (exchange condition)
by

Grading of ${}^0H_n^f$ by: $\deg T_i = -2$. / $\deg T_w = -2l(w)$.

$({}^0H_n^f) = \bigoplus_{i \in \mathbb{Z}} ({}^0H_n^f)_i$ with $({}^0H_n^f)_i = \bigoplus_{l(w) = -\frac{i}{2}} \mathbb{Z} T_w$.

Ex: $H = {}^0H_3^f$. $H_i = 0$ unless $i \in \{0, -2, -4, -6\}$.

$$\begin{cases} H_0 = \mathbb{Z} T_e = \mathbb{Z}.1 \\ H_{-2} = \mathbb{Z} T_{s_1} \oplus \mathbb{Z} T_{s_2} \\ H_{-4} = \mathbb{Z} T_{s_1 s_2} \oplus \mathbb{Z} T_{s_2 s_1} \\ H_{-6} = \mathbb{Z} T_{s_1 s_2 s_1} \end{cases}$$

For $x_1 \in H_{-2}$, $x_2 \in H_{-4}$, do we have $x_1 x_2 \in H_{-6}$?

$x_1 = aT_1 + bT_2$, $x_2 = cT_1T_2 + dT_2T_1$ AQT

Def: Affine nil-Hecke algebra: generated over $\mathbb{Z}$ by $\{T_1, \dots, T_{n-1} ; X_1, \dots, X_n\}$

& rel'ns:
$$\begin{cases} T_i^2 = 0, \quad \text{braid rel'ns} \\ X_i X_j = X_j X_i \\ T_i X_j = X_j T_i \text{ if } j-i \notin \{0,1\} \\ T_i X_{i+1} - X_i T_i = 1 \quad \& \quad T_i X_i - X_{i+1} T_i = -1 \end{cases}$$

?

Ex: $n=2$, $\mathfrak{S}_2$: $T_1 = T$, $X_1 = X$, $X_2 = Y$.
$$\begin{cases} T^2 = 0 \\ XY = YX \\ TY - XT = 1 \\ TX - YT = -1 \end{cases}$$
— checker? oui (avec signe)

— pour que $T(X+Y) = (X+Y)T \neq 0$

Demazure op's.

Def: $P_n := \mathbb{Z}[X_1, \dots, X_n]$; $\mathfrak{S}_n$ acts on P_n by permuting the X_i's.

Set $(1 \leq i \leq n-1)$ $\partial_i P := \dfrac{P - s_i(P)}{X_{i+1} - X_i}$ ← well-defined & linear / $P_n^{\mathfrak{S}_n}$.

Indeed: write $P = \sum_{k,l} P_{kl}(X_1, \dots, \hat{X}_i, \hat{X}_{i+1}, \dots, X_n) \cdot X_i^k X_{i+1}^l$

$s_i P = \sum_{k,l} P_{k,l} \cdot X_i^l X_{i+1}^k$

$P - s_i P = \prod (X_i^{k-l} - X_i^{k-l}) P = (X_i - X_{i+1}) \cdot P'$. OK.

$= \sum_{k>l} P_{kl} X_i^{l} X_{i+1}^{l} (X_i^{k-l} - X_{i+1}^{k-l}) + \sum_{k<l} \dots$

Prop: Get an action of ${}^0H_n^f$ on P_n by $T_i P = \partial_i P$ $(\forall i)$

Not: $w \in \mathfrak{S}_n \rightsquigarrow \partial_w$ = image of T_w in $\mathrm{End}(P_n)$

Prop: Get an action of ${}^0H_n$ on P_n by:

$\rho(T_i)(P) = \partial_i(P)$

$\rho(X_i)(P) = X_i P$.

Prop: $\rho : {}^0H_n \to \mathrm{End}\, P_n$ is faithful & ${}^0H_n = P_n \otimes {}^0H_n^f$.

Pf: $a = \sum P_w T_w \in {}^0H_n$ $(a \neq 0)$. Construct $h \in {}^0H_n$ s.th. $\rho(a)(h) \neq 0$

(1) w_1 of min. length s.t. $P_{w_1} \neq 0$

Then, is: $a T_{w_1^{-1} w_0} = \left(\sum P_w T_w\right) T_{w_1^{-1} w_0} = P_{w_1} T_{w_0}$

(since $l(w_1^{-1} w_0) = l(w_0) - l(w_1)$.)

$\rho(a T_{w_1^{-1} w_0})(X_2 X_3^2 \cdots X_n^{n-1}) = \rho(\underset{\substack{\cap \\ \mathbb{Z}[X_1 \dots X_n]}}{P_{w_1}} T_{w_0})(X_2 \cdots X_n^{n-1})$

$= P_{w_1} \partial_{w_0}(X_2 \cdots X_n^{n-1})$

$= P_{w_1} \neq 0$ } check!

Hence $\rho(a) \neq 0$ & $\rho : {}^0H_n \to \mathrm{End}\, P_n$ is faithful ∎

Th (Zelevinsky – Bernstein – Lusztig) $\boxed{Z({}^0H_n) = P_n^{\mathfrak{S}_n}}$

— difficult.

Conseq: get $\tilde{\rho} : {}^0H_n \longrightarrow \mathrm{End}_{Z({}^0H_n)}(P_n) = \mathrm{End}_{P_n^{\mathfrak{S}_n}}(P_n)$

Goal: $\boxed{\tilde{\rho} = \text{isomorphism}}$

(Remains to show that P_n is a free $P_n^{\mathfrak{S}_n}$-module of rank $(n!)^2$.)

— then get descr'n of ${}^0H_n$ as a matrix ring.

Prop: $\tilde{\rho}$ is an isomorphism.

Sketch:

① P_n is a progenerator: ie $\begin{cases} P_n \text{ is a f.g. } {}^0H_n\text{-module} \\ {}^0H_n \text{ is a summand in a multiple of } P_n \text{ (as } {}^0H_n\text{-module)} \end{cases}$

② Then $\tilde{\rho} : {}^0H_n \longrightarrow \mathrm{End}_{P_n^{\mathfrak{S}_n}}(P_n)$ is a split injection (good exercise in commutative algebra)

③ ${}^0H_n$ has rank $(n!)^2$, so that $\tilde{\rho}$ is an iso. ∎

Prop: $b_n := T_{w_0} X_2 X_3^2 \cdots X_n^{n-1}$

Then: (1) $b_n^2 = b_n$

(2) ${}^0H_n = {}^0H_n b_n {}^0H_n$

(3) $P_n \xrightarrow{\sim} {}^0H_n b_n$

$p \longmapsto p b_n$

Implies ①.

"Proof": Claim: T_{w_0} is the only elt in ${}^0H_n$ sth. $\begin{cases} \rho(T_{w_0})(X_2 \cdots X_n^{n-1}) = 1 \\ \rho(T_{w_0})(\partial_w(X_2 \cdots X_n^{n-1})) = 0. \end{cases}$

(Key for the claim/hidden fact: we have a basis of P_n over $P_n^{\mathfrak{S}_n}$: $(\partial_w(X_2 \cdots X_n^{n-1}))_{w \in \mathfrak{S}_n}$)

Proof of (1): $T_{w_0} X_2 \cdots X_n^{n-1} T_{w_0} = T_{w_0}$

Compute the action of LHS on $(\partial_{w_0}(X_2 \cdots X_n^{n-1}))_{w \in \mathfrak{S}_n}$

— for $w \neq 1$, $\rho(T_{w_0} X_2 \cdots X_n^{n-1}) \underbrace{\rho(T_{w_0})(\partial_w(X_2 \cdots X_n^{n-1}))}_{=0} = 0$

— for $w = 1$: similarly easy.

Proof of (2): similar

Proof of (3): Injectivity

If $P b_n = 0$ (for $P \in \mathbb{Z}[X_1, \dots, X_n]$) then $\rho(P b_n)(1) = 0$

But $P = P \rho(T_{w_0})(X_2 \cdots X_n^{n-1}) = \rho(P b_n)(1) = 0$!

ZRB

Relations:

- $e(\underline{i}) e(\underline{j}) = \delta_{\underline{i},\underline{j}} e(\underline{i})$
- $\sum_{\underline{i} \in I^\alpha} e(\underline{i}) = 1$
- $y_r e(\underline{i}) = e(\underline{i}) y_r \qquad y_r y_t = y_t y_r$
- $\psi_r e(\underline{i}) = e(s_r \underline{i}) \psi_r$
- $(\psi_r y_t - y_{s_r(t)} \psi_r) e(\underline{i}) = \delta_{i_r, i_{r+1}} (\delta_{r+1,t} - \delta_{r,t}) e(\underline{i})$
- $\psi_r^2 e(\underline{i}) = q_{i_r, i_{r+1}}(y_r, y_{r+1}) e(\underline{i})$
- $(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) e(\underline{i}) = \delta_{i_r, i_{r+2}} \dfrac{q_{i_r, i_{r+1}}(y_r, y_{r+1}) - q_{i_r, i_{r+1}}(y_{r+2}, y_{r+1})}{y_r - y_{r+2}} e(\underline{i})$
- $\psi_r \psi_t = \psi_t \psi_r$ if $|t - r| \geq 2$.

Quiver Hecke algebras

A few notations

$k=\bar{k}$ alg'cally closed field

Γ = loop-free quiver w/ finite vertex set I

For $i,j\in I$, let $m_{ij} = \#\{\text{directed edges from } i \text{ to } j\}$

Define a symmetric Cartan matrix $C=(c_{ij})_{i,j\in I}$ by

$$c_{ij}=\begin{cases}2 & \text{if } i=j\\ -m_{ij}-m_{ji} & \text{if } i\neq j.\end{cases}$$

Let $\mathfrak{g}$ be the Kac-Moody algebra corresponding to C.

Root datum: P = weight lattice; $(\alpha_i)_{i\in I}$ simple roots; $(\Lambda_i)_{i\in I}$ fund. wts.
$\cup$
Q = root lattice ; P_+, Q_+ = what you guess.

$(\cdot,\cdot): P\times Q \longrightarrow \mathbb{Z}$ bilinear form defined by:

$(\alpha_i,\alpha_j)\longmapsto c_{ij}$

$(\Lambda_i,\alpha_j)\longmapsto \delta_{ij}$ (Kronecker)

For $\alpha=\sum_{i\in I} c_i\alpha_i$ a root, define its height as $\sum_{i\in I}c_i = |\alpha|$

Let $\langle I\rangle$ be the set of all words in the alphabet I

For $\alpha\in Q_+$, $|\alpha|=d$, let $\langle I\rangle_\alpha$ = set of words $\underline{i}=(i_1,\dots,i_d)$ s.th. $\alpha_{i_1}+\dots+\alpha_{i_d}=\alpha$.

$\mathfrak{S}_d$ acts on $\langle I\rangle_\alpha$, it is generated by $s_1,\dots,s_{d-1}$ transpositions
↳ by place permutation.
(s_r exchanges r^{th} & $(r+1)^{th}$ entries)

Now $q_{ij}(u,v)=\begin{cases}0 & \text{if } i=j\\ (v-u)^{m_{ij}}(u-v)^{m_{ji}} & \text{other}\end{cases}$ $\quad(i,j\in I)$

We come to the main definition.

For $\alpha\in Q_+$, $|\alpha|=d$, the quiver Hecke algebra $R_\alpha(\Gamma)$ is the associative algebra generated by

$$\left\{\begin{array}{l} \cdot\ e(\underline{i}) \quad \underline{i}\in\langle I\rangle_\alpha\\ \cdot\ y_1,\dots,y_d\\ \cdot\ \psi_1,\dots,\psi_{d-1}\end{array}\right.$$ subject to the relations on the previous page

similar to the Hecke algebra, chopped up by the idempotents $e(\underline{i})$'s.

Eg 1: $\alpha=n\alpha_i$, then $R_\alpha={}^0H_n$
(indeed, $e(i)=1$)

Eg 2: Γ_1: $1\to 2\to 3\to\cdots\to n-1$ & $\alpha=\alpha_1+\alpha_2$.

$R_\alpha(\Gamma_1)=\langle e(21), e(12), y_1, y_2, \psi_1\rangle$ is isomorphic to the path algebra of:

$$\begin{tikzcd} e(12) \arrow[loop above, "y_1"] \arrow[loop below, "y_2"] \arrow[r, bend left, "\psi_1"] & e(21) \arrow[loop above, "y_1"] \arrow[loop below, "y_2"] \arrow[l, bend left, "\psi_1"] \end{tikzcd}$$

Grading of $R_\alpha(\Gamma)$ over $\mathbb{Z}$:

- $e(\underline{i})$: degree 0
- y_r : degree 2
- $\psi_r e(\underline{i})$: degree $-(\alpha_{i_r},\alpha_{i_{r+1}})$.

Basis: For $w\in\mathfrak{S}_d$, fix a choice of reduced expression, say $w=s_{i_1}\cdots s_{i_k}$, & set $\psi_w=\psi_{i_1}\cdots\psi_{i_k}$.

Th.: $\left(y_1^{m_1}\cdots y_d^{m_d}\,\psi_w\, e(\underline{i}) \;\middle|\; \underline{i}\in\langle I\rangle_\alpha,\ w\in\mathfrak{S}_d,\ \underline{m}\in\mathbb{N}^d\right)$ is a basis

Centre: Pick $\underline{i}$ s.th. $\mathfrak{S}_{\underline{i}} := \mathrm{Stab}_{\mathfrak{S}_d}(\underline{i})$ is a standard parabolic subgroup of $\mathfrak{S}_d$.

Then:

$$z_j = \sum_{w \in \mathfrak{S}_d / \mathfrak{S}_{\underline{i}}} y_{w(j)}\, e(w(\underline{i})) \qquad (j = 1, \dots, d)$$

generate a polynomial ring $k[z_1, \dots, z_d]$.

$$\boxed{\textbf{Th}: \quad Z(R_\alpha(\Gamma)) = k[z_1, \dots, z_d]^{\mathfrak{S}_{\underline{i}}}}$$

Cor: R_α is free of finite rank over its centre, of rank $(n!)^2$.

— mostly follows from what's known of H_n, cf ...

Cyclotomic quiver Hecke algebra.

Γ = any loop-free quiver.

$e \in \mathbb{N}^*$, $\Gamma = \Gamma_e$: • vertices : $I = \mathbb{Z}/e\mathbb{Z}$

• arrows $i \longrightarrow i+1$ $(i \in I)$

$e = 0$: Γ_0 : $\cdots \to \bullet \to \bullet \to \bullet \cdots$ $\quad A_\infty$

$e = 2$: $0 \rightleftarrows 1$ $\quad\quad \Gamma_3$:  $\quad \Gamma_e = \tilde{A}_{e-1}$.

Def: The cyclotomic quiver Hecke algebra associated to

$\begin{cases} \Lambda \in P_+ \text{ of level } l \\ \alpha \in Q_+ \text{ of height } d \end{cases}$

is the k-algebra denoted R_α^Λ generated by

$e(\underline{i})$, y_i $(1 \le i \le d)$, ψ_j $(1 \le j \le d-1)$

subject to the same relations as R_α together w/: $y_1^{\langle \Lambda, \alpha_{i_1} \rangle} e(\underline{i}) = 0$ $\quad \forall \underline{i} \in \langle I \rangle_\alpha$ $\quad (\langle \Lambda, \alpha_{i_1}\rangle \in \mathbb{N})$

So $R_\alpha^\Lambda = R_\alpha / J^\Lambda$ w/ $J^\Lambda = \left\langle y_1^{\langle \Lambda, \alpha_{i_1} \rangle} e(\underline{i}) \;\middle|\; \underline{i} \in \langle I \rangle_\alpha \right\rangle$.

Eg 1: $\begin{cases} \alpha = \alpha_1 + \alpha_2 \\ \Lambda = \Lambda_1 + 2\Lambda_2 \end{cases}$ $\quad$ (Γ of type A ... not too interested in that)

$$y_1, y_2 \circlearrowright e(12) \underset{\psi_1}{\overset{\psi_1}{\rightleftarrows}} e(21) \circlearrowleft y_1, y_2$$

$0 = y_1^{\langle \Lambda, \alpha_1 \rangle} e(12) = y_1 e(12)$ $\qquad 0 = y_1^{\langle \Lambda, \alpha_2 \rangle} e(21) = y_1^2 e(21)$

(Note that $\psi_1^2 = \dots e(12)$ so that y_2 to some power annihilates $e(12)$ — not necess'ly all of them.)

Eg 2: $\begin{cases} \alpha = 2\alpha_1 + \alpha_2 \\ \Lambda = 5\Lambda_1 + 2\Lambda_3 \end{cases}$

$$? \psi_1 \circlearrowright e(211) \underset{\psi_1}{\overset{\psi_1}{\rightleftarrows}} e(121) \underset{\psi_2}{\overset{\psi_2}{\rightleftarrows}} e(112) \circlearrowleft \psi_1 \; ?$$

$0 = y_1^{\langle \Lambda, \alpha_2 \rangle} e(211)$. $\quad 0 = y_1^5 e(121)$.

$0 = e(211)$!

Lemma (Brundan – Kleshchev): The elts $y_k e(\underline{i}) \in R_\alpha^\Lambda$ are nilpotent for all $k \in \{1, \dots, d\}$, $\underline{i} \in \langle I \rangle_\alpha$.

Sk.: Induction on k. Obvious for $k = 1$ by def'n of J^Λ.

• Two cases: $\begin{cases} i_k = i_{k+1} \\ i_k \neq i_{k+1} \end{cases}$

Cor.: R^Λ_α is a finite dimensional algebra.

Indeed, one has a basis of R_α: $y_1^{m_1} \cdots y_d^{m_d} \psi_w e(\underline{i})$
This set spans R^Λ_α and in R^Λ_α, only a finite number survive.

Fix some $q \in k^\times$

Let H_d be the affine Hecke algebra of type A. It's the k-algebra generated by $T_1, \ldots, T_{d-1}, X_1, \ldots, X_d^{\pm 1}$ w/ some relations.

Fix $\Lambda \in P_+$

The cyclotomic Hecke algebra H^Λ_d is:

$$H_d / \left\langle \prod_{i \in I} (X_1 - q^i)^{(\Lambda, \alpha_i)} \right\rangle$$

Rk: There is a degenerate cyclotomic affine Hecke algebra defined by analogous cyclotomic quotient.

Fact: • dim H^Λ_d is finite.

• there exist a set of mutually orthogonal idempotents $e(\underline{i})$ $(i \in I^d)$

Indeed, let M be a f. dim. H^Λ_d-module.
Denote by $M_{\underline{i}}$ the wt space of M corresponding to $\underline{i}$
ie

$$M_{\underline{i}} = \left\{ v \in M \;\middle|\; (X_r - q^{i_r})^N v = 0 \quad \begin{array}{l} \forall r = 1, \ldots, d \\ \forall N \gg 0 \end{array} \right\}$$

$\Rightarrow$ can decompose $M = \bigoplus_{\underline{i} \in I^d} M_{\underline{i}}$

Do this now for H^Λ_d $\rightsquigarrow$ find $e(\underline{i})$, $\underline{i} \in I^d$.

Now, let $\alpha \in Q_+$ & $e_\alpha = \sum_{\underline{i} \in \langle I \rangle_\alpha} e(\underline{i}) \in H^\Lambda_d$.
{central}

Remark that one can show e_α is zero or is a primitive idempt.

Hence $H^\Lambda_\alpha := e_\alpha H^\Lambda_d$ is zero or a block of H^Λ_d.

Let e be the least positive integer sth $1 + q + \cdots + q^{e-1} = 0$
or zero if no such e exists.

Th. (B-K): The algebra H^Λ_α is generated by

$$\{e(\underline{i}),\ i \in \langle I \rangle_\alpha\} \cup \{y_1, \ldots, y_d\} \cup \{\psi_1, \ldots, \psi_{d-1}\}$$

subject to the same relations (ZKB) for $\Gamma = \Gamma_e$
together w/ cyclotomic relations.

Cor.: Grading on the cyclotomic Quiver Hecke algebras.

NB: these are cyclotomic Hecke alg. attached to complex reflection groups. (are they?)

(L6) Khovanov–Lauda approach

Γ quiver w/ vertex set I

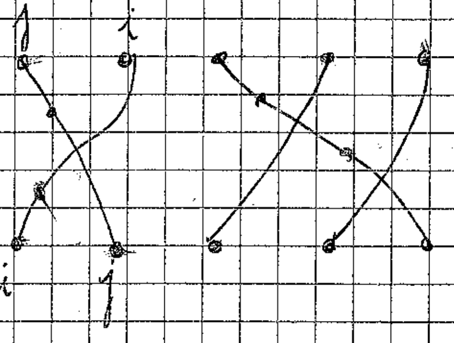
assume the quiver is simply laced for simplicity:

ie in C, $c_{ij} \geq -1$ $\forall i,j$

Co Q_+ weights, fix $\alpha \in Q_+$, $\alpha = \sum_{i \in I} n_i \alpha_i$, $|\alpha| = \sum_i n_i$.

R_α

$\langle I \rangle_\alpha$



$|\alpha|$ dots

$|\alpha|$ dots

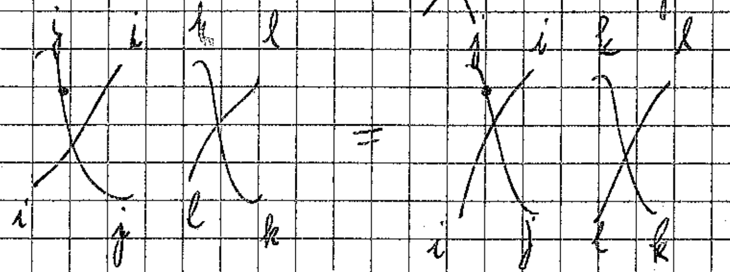
α strands "directed up" (no critical pts)

α ~~points~~ dots on the strands

take such diagrams up to isotopy

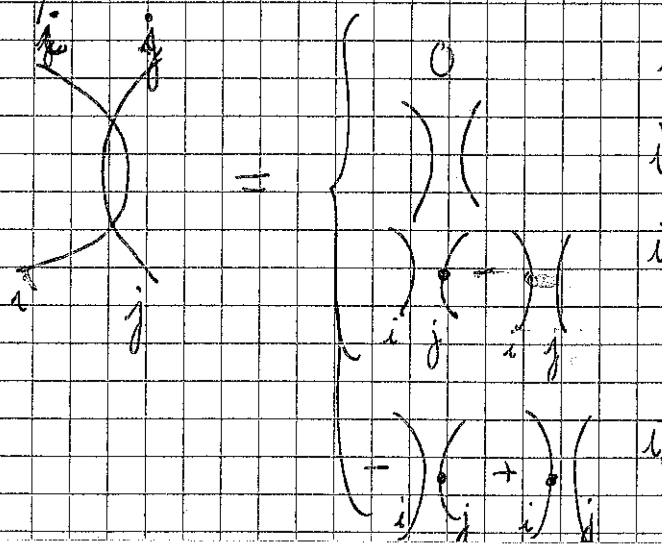
can't move dots across $\times$ crossings of strands.

eg.



R_α = finite linear combinations of such diagrams

w/ local relations:

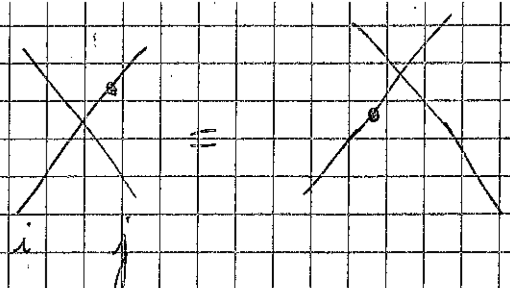


0 if $i = j$

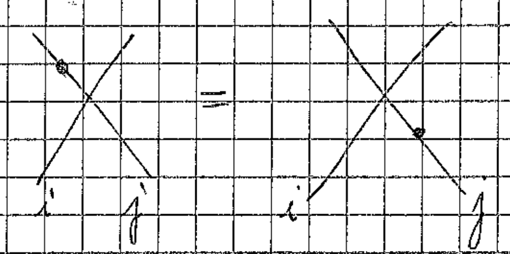
if $c_{ij} = 0$

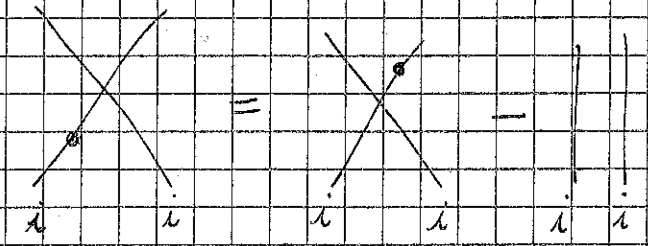
if $i \longrightarrow j$

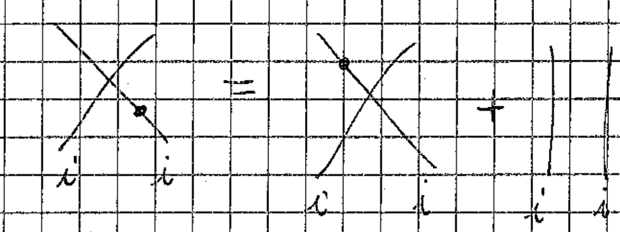
if $j \longrightarrow i$

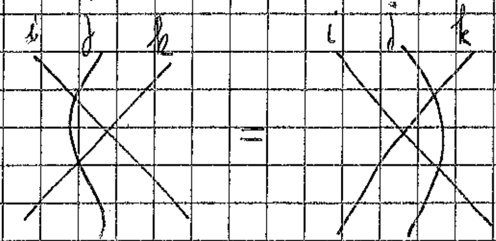


if $i \neq j$

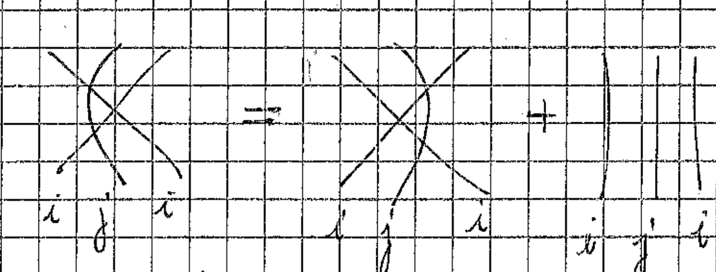




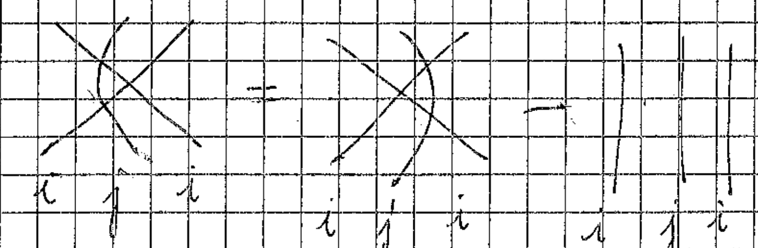




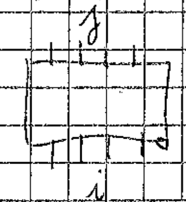
unless ...



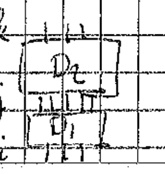
if $i \longrightarrow j$



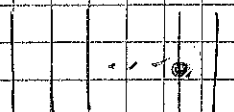
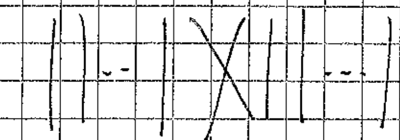
if $j \longrightarrow i$

Let. ${}_j R_{\alpha\, \underline{i}} = \Big\{$  $\Big\}$. So, $R_\alpha = \bigoplus_{\underline{i}, \underline{j} \in \langle I \rangle_\alpha} {}_{\underline{j}} R_{\alpha\, \underline{i}}$

Product: $D_1 D_2$? $D_1 \in {}_{\underline{j}} R_{\alpha\, \underline{i}}$, $D_2 \in {}_{\underline{l}} R_{\alpha\, \underline{k}}$

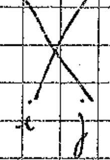


$D_1 D_2 \in {}_{\underline{l}} R_{\alpha\, \underline{i}}$ if $\underline{j} = \underline{k}$, 0 otherw.

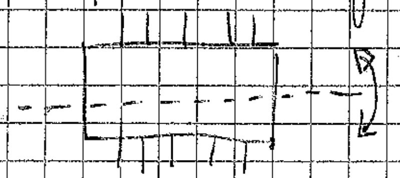
Generators:  & 

single dot — single crossing

Previous talk: y_i — ψ_r

Grading:  : degree 2; deg  $= c_{ij}$

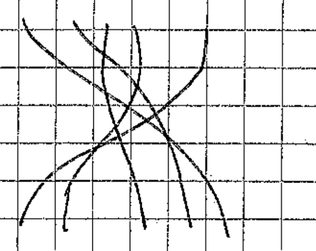
Anti-involution: ψ : reflects a diagram across the x-axis



ψ. $\left(\psi: {}_{\underline{j}}R_{\underline{i}} \to {}_{\underline{i}}R_{\underline{j}}\right)$

Idempotents, $e(\underline{i}) = 1_{\underline{i}}$ $|||||||$ ($\underline{i}$) $\quad \sum_{\underline{i}\in\langle I\rangle_\alpha} 1_{\underline{i}} = 1 \in R_\alpha$

Eg: $\alpha = m\alpha \rightsquigarrow R_\alpha = {}^0H_m$

$e_m =$ 

- w_0
- put $m-1$ dots on the 1^{st} strand
  - $m-2$ ——— 2^{nd}
  - ⋮

$\to$ degree 0 idempotent.

These R_α's talk to each other.

$R(\alpha) \otimes R(\alpha') \longrightarrow R(\alpha+\alpha')$ Which gives

Ind, Res : $R(\alpha)\otimes R(\alpha')$-mod $\rightleftarrows$ $R(\alpha+\alpha')$-mod

Fact: (graded) projective modules are mapped to (graded) projective modules

(obvious for Ind, prove for Res)

$\mathcal{C}$: Category of $\mathbb{Z}$-graded left projective R_α-modules

$K_0(R_\alpha)$ = Grothendieck group

$K_0(R) = \bigoplus_{\alpha\in Q_+} K_0(R_\alpha)$

$$\left(\begin{array}{l} [\mathrm{Ind}] : K_0(R)\otimes K_0(R) \longrightarrow K_0(R) \\ [\mathrm{Res}] : K_0(R) \longrightarrow K_0(R)\otimes K_0(R) \end{array} \right)$$

Projective modules:

For $\underline{i} \in \langle I\rangle_\alpha$, $1_{\underline{i}} = ||| \cdots |$ ($i_1 \cdots i_{|\alpha|}$), $P_\alpha := R_\alpha 1_{\underline{i}} = \{ \cdots \}$

More general than $\langle I\rangle_\alpha$ — divided powers.

$\mathrm{Seqd}(\alpha) := \left\{ i_1^{(n_1)} \cdots i_r^{(n_r)} : n_j \in \mathbb{N},\ \alpha = \sum_{j=1}^{r} n_{i_j}\alpha_{i_j} \right\}$

Eg: $\mathrm{Seqd}(2\alpha_1 + \alpha_2) = \{ 112,\ 1^{(2)}2, \ldots \}$.

For $\underline{i} \in \mathrm{Seqd}(\alpha)$ $\underline{i} = i_1^{(n_1)} \cdots i_r^{(n_r)}$

form $1_{\underline{i}} := \underbrace{e_{i_1, n_1}}_{e_m \text{ above for}} \otimes \cdots \otimes e_{i_r, n_r}$

$P_{\underline{i}} := R_\alpha \psi(1_{\underline{i}}) \underbrace{[-\langle \underline{i}\rangle]}_{\text{some shift}}$ w/ $\langle \underline{i}\rangle = \sum_{k=1}^{r} \binom{n_k}{2}$

where

NB: There is $P_{\underline{i}} \longrightarrow P_{\mathrm{dup}(\underline{i})}$

w/ $\mathrm{dup}(\underline{i}) = (\underbrace{i_1 \cdots i_1}_{n_1} \underbrace{i_2 \cdots i_2}_{n_2} \cdots)$

Induction?

Prop: $\mathrm{Ind}\ P_{\underline{i}} \otimes P_{\underline{j}} = P_{\underline{i}\,\underline{j}}$

Prop: $\left| \begin{array}{ll} P_{\cdots ij \cdots} \simeq P_{\cdots ji \cdots} & \text{if } c_{ij} = 0 \\ P_{\cdots iji \cdots} = P_{\cdots i^{(2)} j \cdots} \oplus P_{\cdots j i^{(2)} \cdots} & \text{if } c_{ij} = -1 \end{array} \right.$

cf Serre rel'ns in $U_q(\mathfrak{n}^+)$

Def: f is a $\mathbb{Q}(q)$-algebra w/ generators θ_i $(i \in I)$
& relations:

$$\begin{cases} \theta_i \theta_j = \theta_j \theta_i & \text{if } c_{ij} = 0 \\ \theta_i^{(2)} \theta_j - \theta_i \theta_j \theta_i + \theta_j \theta_i^{(2)} = 0 & \text{if } c_{ij} = -1 \end{cases}$$

— same rel'ns as $P_{\cdots iji \cdots}$

<u>Relations</u> between f & $K_0(R)$?
$\quad\hookrightarrow$ bigebra

Want to transfer:
- Coproduct
- bilinear form
- anti-involution

Recall $K_0(R)$ comes from <u>graded modules</u> $\to$ q acts by shifting the grading,
which makes $K_0(R)$ into a $\mathbb{Z}[q^{\pm 1}]$-module.

Then $K_0(R)_{\mathbb{Q}(q)} := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q^{\pm 1}]} K_0(R)$

Let
$$\gamma : f \longrightarrow K_0(R)_{\mathbb{Q}(q)}$$
$$\theta_i \longmapsto [P_i] \in K_0(R_{\alpha_i})$$

<u>Well-defined?</u>

Claim: $[n]_q! \, [P_{i^{(n)}}] = [P_{\underbrace{i \cdots i}_{n}}]$

module over $R_{n\alpha_i}$ = nil-Hecke
more precisely ~~...~~ regular rep'n
= $n!$ copies of polynomial ring
($[n]_q!$ is here to take the grading into account)

<u>Coproduct on f</u>:

Def: $f \otimes f$ is an algebra w/ product:
$$(x \otimes y)(z \otimes t) := q^{\cdots} \, xz \otimes yt \quad \text{w/ } \cdots = -(\mathrm{wt}(y), \mathrm{wt}(z))$$
$$(\text{where } (\alpha_i, \alpha_j) = c_{ij})$$

Def. Coproduct = unique algebra morphism s.th.
$$r : f \longrightarrow f \otimes f$$
$$\theta_i \longmapsto \theta_i \otimes 1 + 1 \otimes \theta_i$$

<u>i.e.</u>: $r(\theta_{i_1} \cdots \theta_{i_n}) = (\theta_{i_1} \otimes 1 + 1 \otimes \theta_{i_1}) \cdots (\theta_{i_n} \otimes 1 + 1 \otimes \theta_{i_n})$

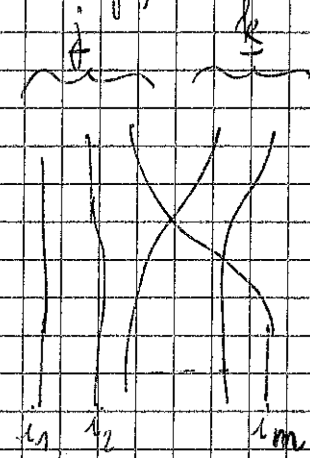
$$= \sum_{\substack{j_1, \ldots, j_r \\ k_1, \ldots, k_m}} q^{\cdots} \, \theta_{j_1} \cdots \theta_{j_r} \otimes \theta_{k_1} \cdots \theta_{k_m}.$$

$j_1, \ldots, j_r$, $k_1, \ldots, k_m$ are complementary subsequences of $i_1, \ldots, i_n$

<u>Claim</u>: $\mathrm{Res}_{\alpha,\beta}(P_{i_1 \cdots i_n}) \simeq \bigoplus_{\underline{j}, \underline{k}} P_{j_1 \cdots j_r} \otimes P_{k_1 \cdots k_m} [\ldots]$ ← some shift

<u>Picture</u>

$P_{i_1 \cdots i_m}$



← restrict ⇒ multiply by image of unit, since $R_\alpha \otimes R_\beta \to R_{\alpha+\beta}$ doesn't map unit to unit.

<u>Involution on f</u>:
$$\bar{\ } : f \longrightarrow f$$
$$\theta_i \longmapsto \theta_i$$
$$q \longmapsto q^{-1}$$

On the level of quiver Hecke algebras: for P projective module over R_α, define $\bar{P} := \underbrace{\mathrm{Hom}(P, R_\alpha)}_{\text{left-module}}{}^{\psi}$ ← twist by anti-autom.

Easy: $\overline{[P_i]} = [P_i]$; $\overline{P[n]} = \bar{P}[-n]$

Bilinear form on f: There is a unique one sth.

(i) $(1, 1) = 1$

(ii) $(\theta_i, \theta_j) = \delta_{ij} \dfrac{1}{1-q^2}$

(iii) $(x, yy') = (r(x), y \otimes y')$

(iv) $(yy', x) = (y \otimes y', r(x))$

Bilinear form on $K_0(R)$:

For P, Q projective modules over R_ν: $\hookrightarrow$ graded dimension:

$([P],[Q]) := \mathrm{grdim}\,(P^\psi \otimes_{R_\nu} Q) \in \mathbb{Q}((q))$

~~Rks~~:

Lemma: (i) $([P_\phi],[P_\phi]) = 1$

(ii) $([P_i],[P_j]) = \delta_{ij} \dfrac{1}{1-q^2}$ since P_i = polynomial ring $k[X]$ ($\deg X = 2$)

(iii) $([P], [Q][Q']) = (\mathrm{Res}\,[P], [Q] \otimes [Q'])$] standard facts (?)

(iv) vice versa.

Cor.: $(\gamma(x), \gamma(y)) = (x, y)$ ie $\gamma : f \longrightarrow K_0(R)_{\mathbb{Q}(q)}$ is an isometry.

Cor.: γ is injective. Indeed, $(\cdot,\cdot)$ is nondegenerate on f.

~~Surjectivity is ~~

Th.: γ is surjective — much harder, but less interesting

$\llcorner$ dealing w/ $K_0(R)$ before $K_0(R) \otimes \mathbb{Q}(q)$.

More precisely: let $f_\mathcal{A}$ be the $\mathcal{A}$-integral form defined by divided powers (over $\mathcal{A} = \mathbb{Z}[q^{\pm 1}]$). Then $\gamma_\mathcal{A} : f_\mathcal{A} \longrightarrow K_0(R)$.

Khovanov Lauda Conj: $f_\mathcal{A}$ has a canonical basis $\mathcal{B}$ (cf L7), hence a $\mathbb{Z}$-basis $\mathcal{B}_\mathbb{Z} = \bigsqcup_{n\in\mathbb{Z}} q^n \mathcal{B}$; then $\mathcal{B}_\mathbb{Z}$ corresponds to indec'ble proj. in $K_0(R)$.

(L7 is the next talk)

(L7) Relation w/ canonical basis.

* Keep the same notations

* $\mathfrak{g}$ = ~~semi~~ simple Lie algebra $/\mathbb{C}$, $U_q\mathfrak{g}$ associated q-gp: $\langle f_i, e_i, k_h \mid {i\in I \atop h\in\mathfrak{h}} \rangle_{\mathbb{C}(q)}$

* Triangular decomp'n: $U_q(\mathfrak{g}) = U_q^- \otimes U_q(\mathfrak{h}) \otimes U_q^+$ w/ $U_q^- = \langle e_i \rangle$, $U_q^+ = \langle f_i \rangle$, $U_q\mathfrak{h} = \langle k_h \rangle$

Motivation: λ dominant weight, $V(\lambda)$ = simple f. dim. rep'n of $U_q\mathfrak{g}$, v_λ h.w. vect.

$\hookrightarrow \varphi : U_q^- \twoheadrightarrow V(\lambda)$, $x \longmapsto x.v_\lambda$; $\mathrm{Ker}\,\varphi = \sum_{i\in I} U_q^- f_i^{\langle\lambda, \alpha_i^\vee\rangle+1}$

Aim: Construct a basis B of U_q^- sth:

$(U_q^-)_\mu$ (for μ = weight) & $U_q^- f_i^k$ are coordinate subspaces (ie spanned by some elts of B)

Then: $\varphi(B) \setminus \{0\}$ will be a basis of $V(\lambda)$ for all λ — «good basis».

Want more: that $\mathrm{Im}\,(f_i)^k \subset V(\lambda)$ is a coordinate subspace $\forall i, k$. This means that $\varphi^{-1}(\mathrm{Im}\, f_i) = f_i^k U_q^- + \ker\varphi$

Lusztig: (1990) | Built a basis B of U_q^- sth $(U_q^-)_\mu$ $(\forall\mu)$ & $U_q^- f_i^k$, $f_i^k U_q^-$ $(\forall k)$ are coordinate spaces

Two constructions: Lusztig's canonical basis, Kashiwara's global basis.

Lusztig's construction of B:

Start w/ a PBW basis of U_q^-, then change it to find B.

PBW basis of U_q^-

Define automorphisms T_i $(i \in I)$ of $U_q\mathfrak{g}$: $\left[\begin{array}{l} r = -\alpha_j(\alpha_i^\vee) \text{ if } j \neq i \\ T_i f_j = \sum_{s=0}^{r} (-1)^s q_i^s f_i^{(s)} f_j f_i^{(r-s)} \end{array}\right.$

$\left[\begin{array}{l} q_i = q^{(\alpha_i,\alpha_i)/2} \\ f_i^{(k)} = \dfrac{f_i^k}{[k]_{q_i}!} \\ [k]_{q_i} = \dfrac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}} \end{array}\right.$ & $T_i(k_h) = k_{s_i(h)}$ & $T_i(f_i) = -k_{\alpha_i}^{-1} e_i$, similar for $T_i(e_i)$

Th. $\boxed{\text{The } (T_i)_i \text{ satisfy the braid relations}}$ — big computation.

Let w_0 = longest elt in W; write $w_0 = s_{i_1} \cdots s_{i_t}$ (reduced) & set $\underline{i} = (i_1, \ldots, i_t)$

The list of positive roots are obtained as follows:

$$\alpha_{i_1},\ s_{i_1}(\alpha_{i_2}),\ s_{i_1}s_{i_2}(\alpha_{i_3}),\ \dots,\ s_{i_1}\cdots s_{i_{t-1}}(\alpha_{i_t}).$$

For all $\underline{c}\in\mathbb{N}^t$, set $F_{\underline{i}}^{\underline{c}} := f_{i_1}^{(c_1)}\, T_{i_1}(f_{i_2}^{(c_2)}) \cdots T_{i_1}\cdots T_{i_{t-1}}(f_{i_t}^{(c_t)})$.

Th: The family $(F_{\underline{i}}^{(\underline{c})})_{\underline{c}\in\mathbb{N}^t}$ is a (PBW) basis of U_q^- over $\mathbb{C}(q)$.

Canonical basis of U_q^-

Let $\mathcal{L}$ = sub-$\mathbb{Z}[q]$-module generated by a PBW-basis

Let $\pi : \mathcal{L} \to \mathcal{L}/q\mathcal{L}$ be the canonical projection.

Then the PBW-basis maps onto a basis $\tilde{B}$ of $\mathcal{L}/q\mathcal{L}$.

Prop: $\mathcal{L}$ and $\tilde{B}$ are independent on the choice of $\underline{i}$ (= reduced expr'n of w_0)

— reduces to rank 2, use braid relations.

Introduce an involution $\bar{\ } : U_q\mathfrak{g} \to U_q(\mathfrak{g})$; $e_i \mapsto e_i$, $f_i\mapsto f_i$, $k_\mu \mapsto k_{-\mu}$, $q\mapsto q^{-1}$

Th. (main result) — "canonical basis"

(i) For all $b\in\tilde{B}$, there is a unique $G(b)\in\mathcal{L}$ s.th. $\overline{G(b)}=G(b)$ & $G(b)-b\in q\mathcal{L}$.

(ii) $B := \{G(b) : b\in\tilde{B}\}$ is a basis of U_q^- & of the $\mathbb{Z}[q,q^{-1}]$-module $\mathcal{L}\cap\bar{\mathcal{L}}$.

(iii) The subspaces $U_q^- f_i^k$, $f_i^k U_q^-$ are coordinate subspaces of U_q^- for B.

Sketch: * Use the link between U_q^- and geometry, i.e. rep'n theory of quiver Γ.

* The canonical basis is constructed from the PBW basis & the transition matrix is unitriangular

Ex: type A_1: $I=\{1\}$, $B=\{f_1^{(k)},\ k\in\mathbb{N}\}$

Ex: type A_2: $B = \{f_1^{(p)}f_2^{(q)}f_1^{(r)} \mid q\geq p+r\}\cup\{f_2^{(p)}f_1^{(q)}f_2^{(r)} \mid q\geq p+r\}$.

In general, it's difficult to compute B.

Kashiwara's construction

× Crystal bases of finite-dim'l $U_q\mathfrak{g}$-modules:

$V(\lambda)$ has a unique crystal basis $(\mathcal{L}(\lambda), B(\lambda))$ (+ crystal operators)

$\mathcal{A} = \{f\in\mathbb{C}(q) : f \text{ regular at } 0\}$ — $\mathcal{L}(\lambda)$: free $\mathcal{A}$-module s.th. $V(\lambda)=\mathbb{C}(q)\otimes\mathcal{L}(\lambda)$ — $B(\lambda)$: basis of $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$

× By inductive limit, one gets $(\mathcal{L}(\infty), B(\infty))$ = crystal basis of U_q^-.

↳ meaning: for $\xi\in Q$, $V(\lambda)_{\lambda+\xi} = (U_q^-)_\xi \Big/ \underbrace{\sum (U_q^-)_{\xi+(1+\lambda(\alpha_i^\vee))\alpha_i}}_{\text{this space is 0 for } \lambda\gg 0}$

× Introduce $(U_q^-)_{\mathbb{Z}}$ = $\mathbb{Z}[q,q^{-1}]$-module generated by $f_i^{(k)}$ (i,k ...)

Th (Kashiwara): $(U_q^-)_{\mathbb{Z}}\cap\mathcal{L}(\infty)\cap\overline{\mathcal{L}(\infty)} \xrightarrow{\sim} \mathcal{L}(\infty)/q\mathcal{L}(\infty)$

Let G be the inverse of this map.

Then $\{G(b) : b\in B(\infty)\}$ satisfies the properties of the ~~global~~ canonical basis.

Lusztig: it is indeed the canonical basis, called "global basis".

Crystal: One can parametrise the canonical basis by using the combinatorics of crystals.

↳ quite explicit.

(L8) Sheaves & their cohomology

Fix an algebraically closed field & X = affine variety $/k$.

$\rightsquigarrow$ $k[X]$ = ring of regular functions.

Study geometry of X through alg. pties of $k[X]$.

Efficient for global pties: eg $X \to Y$ is iso $\iff$ $k[Y] \to k[X]$ iso

Ex: $X = \mathbb{A}^n$, U = open in $\mathbb{A}^n$.

Define $\mathcal{F}(U) = \{ f : U \to \mathbb{A}^1,\ f \text{ continuous} \}$ * wrt Zariski

Then $\mathcal{F}(X) = k[X_1, \dots, X_n]$

Assume $x \in \mathbb{A}^n$, then can $\rightsquigarrow$ nbhds of $x \in V_0 \subset V_1 \subset \cdots \subset V_k$

What happens to $\mathcal{F}(V_i)$?

If $V \subset U \subset \mathbb{A}^n$ are open sets, has "restriction map"

$$\rho_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \qquad \text{w/ } i_{U,V} : V \longrightarrow U$$
$$f \longmapsto f \circ i_{U,V} \qquad \text{is the inclusion.}$$

Nice pties — look for local pties that can be extended globally.

Let $U = \bigcup_{i \in I} U_i$ be an open covering of $U \subset \mathbb{A}^n$ (open). Then:

(i) $s|_{U_i} = t|_{U_i}$ $\forall i \in I \Rightarrow s = t$ ("local determination")

(ii) If $s_i \in \mathcal{F}(U_i)$ $\forall i \in I$ sth $\forall i,j$ $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$
then there exists some $s \in \mathcal{F}(U)$ sth $s|_{U_i} = s_i$ $\forall i \in I$.
"glueing pty"

Presheaves

Let X be a topological space. ~~A presheaf on X~~

Let $\mathcal{U}_X$ be the category of open subsets of X, where

$$\mathrm{Hom}_{\mathcal{U}_X}(V,U) = \begin{cases} \{ i_{V,U} : V \hookrightarrow U \} & \text{if } V \subset U \\ \emptyset & \text{otherwise.} \end{cases}$$

A *presheaf* on X is a contravariant functor $\mathcal{F} : \mathcal{U}_X \to \mathcal{A}$
w/ $\mathcal{A}$ is a "nice" category such as Ab, Sets, k-Vector Spaces ...

Denote by $\rho^{\mathcal{F}}_{U,V} = \mathcal{F}(i_{V,U}) : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

An elt $s \in \mathcal{F}(U)$ is called a section of $\mathcal{F}$ (on U).

* $\mathrm{PreSh}_X(\mathcal{A})$:= cat'y of presheaves on X in $\mathcal{A}$.

Here, a morphism $f \in \mathrm{Hom}_{\mathrm{PreSh}_X}(\mathcal{F}, \mathcal{G})$ is a collection of maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ ($U \in \mathcal{U}_X$) sth $\forall V \subset U$ in $\mathcal{U}_X$,

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \downarrow \rho^{\mathcal{F}}_{U,V} & & \downarrow \rho^{\mathcal{G}}_{U,V} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

Examples:

(1) The constant presheaf $\underline{k}_X^{pre}$ in $\mathrm{PreSh}_X(\mathrm{Vec}_k)$ is given by:
$$\underline{k}_X^{pre}(U) = k \quad \forall U \in \mathcal{U}_X.$$

(2) The skyscraper ~~presheaf~~ sheaf: If $x_0 \in X$ is then define $\mathcal{F}_{x_0} \in \mathrm{PreSh}_X(\mathrm{Vec}_k)$ by
$$\mathcal{F}_{x_0}(U) = \begin{cases} k & \text{if } x_0 \in U \\ \{0\} & \text{otherwise.} \end{cases}$$

Def. Say a presheaf $\mathcal{F}$ on X in $\mathcal{A}$ is a *sheaf* if the nice properties (i) & (ii) hold.

Sheaves form a category — in fact, a full subcategory of $\mathrm{PreSh}_X(\mathcal{A})$
ie $\mathrm{Hom}_{\mathrm{Sh}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{PreSh}_X}(\mathcal{F}, \mathcal{G})$.

Example: * The skyscraper ~~pre~~sheaf is a sheaf.
* The constant presheaf is not (in general) a sheaf.

Sheafification:

We have a forgetful functor $\mathrm{For} : \mathrm{Sh}_X(\mathcal{A}) \longrightarrow \mathrm{PreSh}_X(\mathcal{A})$.

Does it admit a left adjoint?

Def: If $x \in X$ and $\mathcal{F} \in \mathrm{PreSh}_X(\mathcal{A})$, the *stalk* of $\mathcal{F}$ at x is
$\varinjlim \mathcal{F}(U)$, limit being taken over open sets U containing x.

$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U) = \coprod_{U \ni x} \mathcal{F}(U) / \sim$

where for $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$, we say $s \sim t$ if there is an open nbhd $W \subset U \cap V$ containing x sth $s|_W = t|_W$.

If $x \in U$, have a natural map $\mathcal{F}(U) \longrightarrow \mathcal{F}_x$

$s \longmapsto s_x$ ("germ of s at x")

Obs: If $\mathcal{F} \in Sh_X(\mathcal{A})$ then $s \in \mathcal{F}(U)$ is uniquely determined by $(s_x)_{x \in U}$ — ex.: prove it!

If $\mathcal{F} \in PreSh_X(\mathcal{A})$, set $\tilde{\mathcal{F}}(U) = \coprod_{x \in U} \mathcal{F}_x \quad (U \in \mathcal{U}_X)$

(ex.: this is a sheaf)

Say that $s \in \tilde{\mathcal{F}}(U)$ is continuous if there is an open covering $U = \bigcup_{i \in I} U_i$ and $t_i \in \mathcal{F}(U_i)$ $(i \in I)$ sth $s_x = (t_i)_x$ $\forall x \in U_i$ $\forall i \in I$.

Now set: $\mathcal{F}^+(U) = \{ s \in \tilde{\mathcal{F}}(U) \mid s \text{ is continuous} \}$.

Then (ex.) $\mathcal{F}^+ \in Sh_X(\mathcal{A})$ is called sheafification of $\mathcal{A}$.

Lemma: For any $x \in X$, $\mathcal{F}^+_x = \mathcal{F}_x$

Eg. Let $\underline{k}^{pre}_X$ be the ~~ct~~ sheaf on X. Then $\underline{k}_X = (\underline{k}^{pre}_X)^+$ is called the constant sheaf — and $(\underline{k}_X)_x \cong k$ $\forall x \in X$.

Prop: If $\mathcal{A}$ is an abelian cat'y, then $Sh_X(\mathcal{A})$ is iso.

Lemma: A sequence of sheaves $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$ is exact ~~iff~~ {???}

$\iff \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x$ is exact $\forall x \in X$.

Ex.: Prove!

Operations on sheaves.

Let $f : X \longrightarrow Y$ be a C^0 map.

Define • $f_* : Sh_X \longrightarrow Sh_Y$ ("push forward" of $\mathcal{F}$) by

$\forall \mathcal{F} \in Sh_X$, $\left[f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \quad \forall U \in \mathcal{U}_Y \right]$ (is a sheaf)

• $f^* : Sh_Y \longrightarrow Sh_X$ — pullback of

$f^*\mathcal{F} = \left(U \longmapsto \varinjlim_{V \supset f(U)} \mathcal{F}(V) \right)^+$ ← sheafification.

Eg 0: Let $a : X \longrightarrow \{*\}$ — only one nonempty set on $\{*\}$.

$a_*\mathcal{F} = \Gamma(\mathcal{F})$

ie $a_*\mathcal{F}(\{*\}) = \mathcal{F}(X)$

Eg 1: $f : X \to Y$: $f^*\underline{k}_Y = \underline{k}_X$

On stalks: $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$

indeed: $\varinjlim_{U \ni x} f^*\mathcal{F}(U) = \varinjlim_{U \ni x} \varinjlim_{V \supset f(U)} \mathcal{F}(V) = \varinjlim_{V \ni f(x)} \mathcal{F}(V)$

$\boxed{\text{Cor. } f^* \text{ is exact}}$

But f_* is not exact in general.

Define $f_! : Sh_X \longrightarrow Sh_Y$ — proper pushforward — by

$f_!\mathcal{F}(U) = \{ s \in \mathcal{F}(f^{-1}(U)) \mid f|_{Supp\, s} : Supp(s) \longrightarrow U \text{ is proper} \}$.

[Recall • $f : X \to Y$ is proper if, for every compact $K \subset Y$, $f^{-1}(K)$ is compact

• $Supp(s) = \{ x \in U : s_x \neq 0 \}$ for $s \in \mathcal{F}(U)$

Let U = locally closed set in X and $i : U \longrightarrow X$.
(↳ intersection of a closed and an open set)

Then $i_!\mathcal{F}$ is "extension by zeros": $(i_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$

Then $i_!$ is exact — nicer than in general.

Def: $\mathcal{H}om_{Sh_X}(\mathcal{F},\mathcal{G})$ is defined by: $\mathcal{H}om_{Sh_X}(\mathcal{F},\mathcal{G})(U) = Hom_{Sh_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$.

Product: $\mathcal{F} \otimes \mathcal{G} = \left(U \longmapsto \mathcal{F}(U) \otimes \mathcal{G}(U)\right)^+$

Prop: $\mathcal{H}om(\mathcal{F}, f_*\mathcal{G}) \cong \mathcal{H}om(f^*\mathcal{F}, \mathcal{G})$ $\qquad$ $f : \overset{\mathcal{G}}{X} \longrightarrow \overset{\mathcal{F}}{Y}$

$Hom(\mathcal{F}, f_*\mathcal{G}) = f_*\mathcal{H}om(f^*\mathcal{F}, \mathcal{G})$ $\qquad$ [???]

Def: $\mathcal{F} \in Sh_X$ is a local system if it is locally constant, ie
$\forall x \in X, \exists U \in \mathcal{U}_x, x \in U$ sth. $\mathcal{F}_{|U} \cong \underline{k}_U$.

Prop: Local systems on $\underset{(\text{nice})}{X}$ correspond to representations of $\pi_1(X)$
(1:1)

Sk: Say $\mathcal{F}$ is a local system. Fix a base point x_0.
Then $\mathcal{F}_{x_0}$ is a f.d. vector space. For $[\alpha] \in \pi_1(X, x_0)$ (α is a loop $\alpha : [0,1] \to X$ based)
divide $[0,1]$ into intervals on which α lies in a conected open set V_1, ...
Then composing $\mathcal{F}_{x_0} \xrightarrow{\sim} \mathcal{F}_{V_1} \xrightarrow{\sim} \mathcal{F}_{V_2} \longrightarrow \cdots \longrightarrow \mathcal{F}_{V_n} = \mathcal{F}_{x_0}$
defines action of $[\alpha]$ on $\mathcal{F}_{x_0}$.

If $\gamma : \pi_1(X, x_0) \longrightarrow GL(V)$,
set $\mathcal{F}(U) = \{U \longrightarrow V : \text{compatible w/ the rep'n}\}$

Def: A sheaf $\mathcal{F}$ on X is constructible if $\exists$ partition $X = \bigsqcup_{i=1}^{n} X_i$
sth $\mathcal{F}_{|X_i}$ is a local system $\forall i$.

Cohomology of sheaves: $\mathcal{F}$ has an ~~can be~~ injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\cdot$
$H^i(X, \mathcal{F}) := h^i(\Gamma(\mathcal{I}^\cdot))$

(1.9) Derived category of constructible sheaves

I Derived categories

$\mathcal{A}$ Abelian category ⇝ $C^\#(\mathcal{A})$ = complexes in $\mathcal{A}$
w/ $\# \in \{\emptyset; +; -; b\}$
bounded from below, from above, from both sides.

Why? Think about derived functors, eg Tor, Ext, H^i...
(Take cohomology in order that the result doesn't depend on the resolution)
- $C^\# \longrightarrow H^\cdot(C)$: loose info!
- compose derived functors & avoid spectral sequences
- $f^!$ = right adjoint to (left exact) functor $f_!$

$\mathcal{A}$ abelian category ⇝ $C(\mathcal{A})$ = chain complexes + chain maps
— abelian too; but too fine.

Assume $f : X^\cdot \longrightarrow Y^\cdot$ sth. $H^i(f) : H^i(X^\cdot) \longrightarrow H^i(Y^\cdot)$ is $\cong$ $\forall i$.
— then say f is a quasi-isomorphism (qis)
~~qis~~

We are to formally invert qis — localize. Pb: not commutative.
⇒ conditions on the inverted strings — One condition: $as^{-1}bt^{-1}$ can't be written as cu^{-1}? No in general!

Say $f : X^\cdot \to Y^\cdot$ is homotopic to zero — $f \sim 0$ — if
$\exists s^n : X^n \longrightarrow Y^{n-1}$ sth. $f = sd + ds$.
— say s is a chain homotopy. $\qquad$ ⇒ gets homotopic cat'y

Say $f \sim g$ if $(f-g) \sim 0$.

Def: Homotopy cat'y: $K(\mathcal{A})$: objects: chain complexes as in $C(\mathcal{A})$
mph: chain maps up to homotopy.

Let S = {quasi-isomorphisms}.
Roughly: « $D^\#(\mathcal{A}) = S^{-1}K^\#(\mathcal{A})$ »

Think of morphisms in $\mathcal{D}^\#(\mathcal{A})$ as: "$s^{-1}f$" w/ $\begin{cases} f = \text{morph} \\ s = \text{qis} \end{cases}$

$$\left[X \xleftarrow{s} W \xrightarrow{f} Y \right]$$

Price to pay: $\mathcal{D}^\#(\mathcal{A})$ is not abelian (no kernels nor cokernels).

But: $\mathcal{D}^\#(\mathcal{A})$ is triangulated.

↳ stands for "long ex. seq'ce associated to s.e.s".

II Derived categories of sheaves & functors

X = top. space, $\mathbb{C}_X$ = constant sheaf, $\mathcal{A} = \mathbb{C}_X$-mod

Derived cat'y $\mathcal{D}(\mathbb{C}_X\text{-mod}) \overset{\text{short}}{=} \mathcal{D}(\mathbb{C}_X)$

Recall $\mathcal{H}om(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{Sh_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ — left-exact.

$\mathcal{H}^\cdot(\mathcal{F}^\cdot,\mathcal{G}^\cdot): C(\mathcal{A})^{op} \times C(\mathcal{A}) \longrightarrow C(\mathcal{A})$?

Take the double complex: $\mathcal{H}om(\mathcal{F}^{-i},\mathcal{G}^j)$

$$\begin{tikzcd}
\mathcal{H}om(\mathcal{F}^i,\mathcal{G}^{j'}) \arrow[r, "-\partial_{i-1}"] \arrow[d, "\partial_j"] & \mathcal{H}om(\mathcal{F}^{i+1},\mathcal{G}^{j+1}) \arrow[r] \arrow[d, "\partial_j"] & \cdots \\
\mathcal{H}om(\mathcal{F}^i,\mathcal{G}^j) \arrow[r, "+\partial_i"] \arrow[d] & \mathcal{H}om(\mathcal{F}^{i+1},\mathcal{G}^{j+1}) \arrow[r] \arrow[d] & \cdots \\
{} & {} &
\end{tikzcd}$$

$j=i-1$

$j=i$

$j=i+1$

Total complex: $\mathcal{H}om^n(\mathcal{F}^\cdot,\mathcal{G}^\cdot) := \prod_{j=i+n} \mathcal{H}om(\mathcal{F}^i,\mathcal{G}^j)$

Easy: $K^\#(\mathcal{A}) \times C(\mathcal{A}) \longrightarrow K(\mathcal{A})$

More subtle: $X \xleftarrow{s} W \xrightarrow{f} Y$ Want the functor to preserve qis

Trick: $\mathcal{G}^\cdot \simeq \mathcal{J}^\cdot$ injective resol'ns

↳ qis!

On the subcat'y of of injective resol'ns $\mathcal{H}om(\mathcal{F},-)$ does preserve qis

Then get: $\mathcal{H}om(\mathcal{F}^\cdot,\mathcal{J}^\cdot): K(\mathcal{A}) \times \mathcal{D}^\#(\mathcal{A}) \longrightarrow \mathcal{D}^\#(\mathcal{A})$

$\Vert$

$\rightsquigarrow$ $R\mathcal{H}om(\mathcal{F}^\cdot,\mathcal{G}^\cdot): \mathcal{D}^-(\mathcal{A}) \times \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{A})$

Remember f_* = left-exact, & $f_!$ too.

$$\left.\begin{array}{l} Rf_*(\mathcal{F}^\cdot) := f_*(\mathcal{J}^\cdot) \\ Rf_!(\mathcal{F}^\cdot) := f_!(\mathcal{J}^\cdot) \end{array}\right\} \text{ w/ } \mathcal{F}^\cdot \xrightarrow[\sim]{\text{qis}} \mathcal{J}^\cdot \leftarrow \text{injective}$$

Prop: $R\mathcal{F} \circ R\mathcal{G} = R(\mathcal{F} \circ \mathcal{G})$ provided that either $\mathcal{G}$ preserves injectives or ~~$\mathcal{G}$~~ ???

↳ eg

III

$Shv(X)$ = cat'y of sheaves of $\mathbb{C}$-v.spaces on X

$\mathcal{D}^\#(X) = \mathcal{D}^\#(Shv(X))$ w/ $\# \in \{\emptyset, +, -, b\}$.

Recall $f^*: Shv(Y) \longrightarrow Shv(X)$ exact functor.

— so f^* descends to $\mathcal{D}^\#(Y) \longrightarrow \mathcal{D}^\#(X)$

Saw before f^* is ~~R~~ adjoint to f_*; goes to $\mathcal{D}$:

Adjunction: $\mathrm{Hom}_{\mathcal{D}^\#(Y)}(\mathcal{G}^\cdot, Rf_*\mathcal{F}^\cdot) = \mathrm{Hom}_{\mathcal{D}^\#(X)}(f^*\mathcal{G}^\cdot, \mathcal{F}^\cdot)$

Stronger version (this was global):

$$Rf_* R\mathcal{H}om^\cdot(f^*\mathcal{G}^\cdot, \mathcal{F}^\cdot) = R\mathcal{H}om^\cdot(\mathcal{G}^\cdot, Rf_*\mathcal{F}^\cdot).$$

Th (Proper base change theorem)

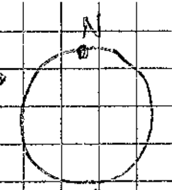
Assume have a cartesian square

$$Z \times_Y X$$

$$\begin{tikzcd}
Z \times_Y X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Z \arrow[r, "g"] & Y
\end{tikzcd}$$

Then: $g^* \circ f_! = f'_! \circ (g')^*$

$g^* \circ Rf_! = Rf'_! \circ (g')^*$

Exo: $X = S^1 \setminus \{N\} \coprod \{(\cdot)\} \longrightarrow$ 

$\downarrow \qquad\qquad \downarrow f$

$Y = [-1,1] \quad \{0\} \overset{g}{\subset} [\ \longrightarrow\ ]\ Y$

Compute $g^* f_!(\underline{\mathbb{C}}_X) = f_! \circ g^*(\underline{\mathbb{C}}_X)$

$g^* f_*(\underline{\mathbb{C}}_X) \neq f_* \circ g^*(\underline{\mathbb{C}}_X)$

Exo: Use PBC thm to compute $R^i f_!(\underline{\mathbb{C}}_X)_z$ $(z \in \mathbb{C})$ w/ $f : \mathbb{C}\setminus\{0\} \hookrightarrow \mathbb{C}$.

Prop: Let $\mathcal{F}^\cdot \in \mathcal{D}^-(X)$, $\mathcal{G}^\cdot \in \mathcal{D}^-(Y)$

$$Rf_! \mathcal{F}^\cdot \overset{L}{\otimes} \mathcal{G}^\cdot = Rf_!(\mathcal{F} \overset{L}{\otimes} f^* \mathcal{G})$$

ex. define $\overset{L}{\otimes}$ (use that $\exists$ enough injectives)

Verdier duality: $f_!$ doesn't have an adjoint at the abelian level.

Th: Suppose $R^i f_! = 0$ for $i \gg 0$ [f is cohomologically finite]

Then there is a functor $f^! : \mathcal{D}^+(Y) \to \mathcal{D}^+(X)$ s.th.:

$$\mathrm{Hom}_{\mathcal{D}^+(Y)}(Rf_! \mathcal{F}^\cdot, \mathcal{G}^\cdot) = \mathrm{Hom}_{\mathcal{D}^+(X)}(\mathcal{F}^\cdot, f^! \mathcal{G}^\cdot).$$

$$R\mathcal{H}om(\qquad) = Rf_* R\mathcal{H}om^\cdot(\ ,\ )$$

— hence $f^!$ is not derived from any functor on $Shv(X)$.

Addendum: if X, Y are complex spaces, if f has smooth fibers of complex dim'n d, then $f^! = f^*[2d]$.

Def: let $a : X \to pt$ be the unique map from X to a point.

The dualizing sheaf on X is: $\omega_X = a^!(\underline{\mathbb{C}}_{pt})$.

If $\mathcal{F} \in \mathcal{D}^b(X)$ define $D\mathcal{F} = R\mathcal{H}om(\mathcal{F}, \omega_X)$.

Properties: (easy to check)

- $f^! \omega_Y = \omega_X$ (cf $f^* \underline{\mathbb{C}}_Y = \underline{\mathbb{C}}_X$)
- if X = cpx mfd, then $\omega_X = \underline{\mathbb{C}}_X[2\dim_{\mathbb{C}} X]$

Two admitted facts:

Fact #1: $D^2 \neq \mathrm{Id}$ on $\mathcal{D}^b(X)$

Fact #2: Local systems are not preserved by $f_*, f_!, f^!$

[Eg to think about: $f : \mathbb{C}\setminus\{0\} \hookrightarrow \mathbb{C}$: look at $f_!$ & f_* ...]

Def: The (bounded) derived category of constructible complexes $\mathcal{D}^b_c(X)$ is the full subcat'y of $\mathcal{D}^b(X)$ consisting of objects $\mathcal{F}^\cdot$ s.th. $\mathcal{H}^i(\mathcal{F}^\cdot)$ is a constructible sheaf $\forall i$.

Upshot of this is...

Th: * The category $\mathcal{D}^b_c(X)$ is closed under our six operations $f_*, f_!, f^*, f^!$, $R\mathcal{H}om^\cdot$ and $\otimes$.

* Moreover, $\mathcal{D}^b_c(X)$ is closed under D and $D^2 = \mathrm{Id}$ on $\mathcal{D}^b_c(X)$.

Summary:

$f : X \to Y$ morphism $\rightsquigarrow$ adjoint pairs (f^*, f_*), $(f_!, f^!)$

$$\mathcal{D}^b_c(X) \underset{f^*, f^!}{\overset{f_*, f_!}{\rightleftarrows}} \mathcal{D}^b_c(Y)$$

f proper (e.g. closed immersion) $\Rightarrow f_* = f_!$

f open immersion $\Rightarrow f^* = f^!$

$\omega_X = p_X^! \underline{\mathbb{C}}_{X^{an}}$ w/ $p : X \to pt$

$D_X = \mathcal{H}om(-, \omega_X)$ duality on $\mathcal{D}^b_c(X)$, $\begin{cases} \mathbb{D}_Y f_! \simeq f_* \mathbb{D}_X \\ \mathbb{D}_Y f^* \simeq f^! \mathbb{D}_X \end{cases}$

X smth $\Rightarrow \omega_X = \mathbb{C}_X[2d_X]$

X smth, $\mathcal{L} \in Loc(X) \Rightarrow D_X \mathcal{L}[d_X] \simeq \mathcal{L}^\vee[d_X]$

Perverse sheaves

Recall

X = ~~ℂ~~ var'y $\Rightarrow$ X^{an} cpx var'y.

Stratif'n: $X = \bigsqcup X_\lambda$, X_λ loc. closed & $\overline{X_\lambda} = \bigsqcup_{\text{some }\mu} X_\mu$

Constructible: $\mathbb{C}_{X^{an}}$-module $\mathcal{F}$ w/ $\mathcal{F}_{|X_\lambda}$ a local system for $\exists$ stratif'n (X_λ) of X ~~constructible~~

I Perverse sheaves

X = alg var'y (not nec'lly smooth, connected nor irred'ble)

Def: $\mathcal{F} \in \mathrm{Perv}(X)$ if

(i) $\forall j \in \mathbb{Z}$, $\dim(\mathrm{Supp}\, H^j \mathcal{F}) \leq -j$

(ii) $\mathbb{D}_X \mathcal{F}$ satisfies (i).

$\rightsquigarrow$ full subcat'y of $\mathcal{D}^b_c(X)$

We'll see it is abelian & there is some "cohomology" on $\mathrm{Perv}(X)$.

Ex: $i_x : \{x\} \hookrightarrow X$ emb. of a point, ~~then~~ & F = $\mathbb{C}$-vector space.
Then skyscraper sheaf $(i_x)_* F$ is a perverse sheaf.

II t-structure on a triangulated category $\mathcal{D}$

1) **Data**: two full subcat'ies $\mathcal{D}^{\leq 0}$, $\mathcal{D}^{\geq 0}$ of $\mathcal{D}$

$\rightsquigarrow$ $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq n}[-n]$, $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq n}[-n]$ $(n \in \mathbb{Z})$

Def: $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ gives a t-structure on $\mathcal{D}$ if

(i) $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ [hence $\mathcal{D}^{\leq -n} \subset \mathcal{D}^{\leq 0}$ $\forall n \geq 0$]

(ii) $\forall X \in \mathcal{D}^{\leq 0}$, $\forall Y \in \mathcal{D}^{\geq 1}$, $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \{0\}$

(iii) $\forall X \in \mathcal{D}$, $\exists$ distinguished triangle $X_0 \to X \to X_1 \xrightarrow{+1}$
s.th. $X_0 \in \mathcal{D}^{\leq 0}$, $X_1 \in \mathcal{D}^{\geq 1}$ — replaces short ex. seq. in ab'n cat'ie

2) **Truncation functors** $\tau^{\leq n}$, $\tau^{\geq n}$ $(n \in \mathbb{Z})$ functors

Prop: For $n \in \mathbb{Z}$, let $\iota^{\leq n} : \mathcal{D}^{\leq n} \to \mathcal{D}$ and $\iota^{\geq n} : \mathcal{D}^{\geq n} \to \mathcal{D}$ be inclusions.
Then $\iota^{\leq n}$ has a left adjoint $[\tau^{\leq n}]$ & $\iota^{\geq n}$ has a right adjoint $[\tau^{\geq n}]$

Pf: easy & natural.

(1) It's enough to construct $\tau^{\leq 0}$ or $\tau^{\geq 1}$ ~~[then $\tau^{\leq n} = \tau^{\leq 0}[-n]$, $\tau^{\geq n} = \tau^{\geq 1}[1-n]$]~~
(Then $\tau^{\leq n} = [-n] \circ \tau^{\leq 0} \circ [n]$ and $\tau^{\geq n} = [1-n] \circ \tau^{\geq 1} \circ [n-1]$)

(2) Set $\tau^{\leq 0}(X) = X_0$. Works since we want:

$(\forall)$ $\mathrm{Hom}_{\mathcal{D}}(Y, X) \cong \mathrm{Hom}_{\mathcal{D}}(Y, \tau^{\leq 0} X)$ for $Y \in \mathcal{D}^{\leq 0}$.

But the defining property of distinguished triangles says:
from $X_1[-1] \to X_0 \to X \xrightarrow{+1}$ ~~X_1~~ is a dist. triangle, ~~get~~ [shift by 1]
so get exact seq'ce $\mathrm{Hom}_{\mathcal{D}}(Y, X_1[-1]) \to \mathrm{Hom}_{\mathcal{D}}(Y, X_0) \to \mathrm{Hom}_{\mathcal{D}}(Y, X) \to \mathrm{Hom}_{\mathcal{D}}(Y, X_1)$

$\| \; 0$ since $Y \in \mathcal{D}^{\leq 0}$, $X_1[-1] \in \mathcal{D}^{\geq 2} \subset \mathcal{D}^{\geq 1}$; $\| \; 0$ since $Y \in \mathcal{D}^{\leq 0}$, $X_1 \in \mathcal{D}^{\geq 1}$.

Hence $(\forall)$. The other statement is similar.

Properties of truncation functors.

Prop: $\forall a, b \in \mathbb{Z}$, $\tau^{\leq a} \circ \tau^{\geq b} \cong \tau^{\geq b} \circ \tau^{\leq a}$.

Pf: long, see literature.

Th: (i) $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is an abelian subcategory of $\mathcal{D}$

(ii) Define $H^n := [n] \circ \tau^{\leq n} \circ \tau^{\geq n} : \mathcal{D} \to \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$
Then, for any distinguished triangle $X \to Y \to Z \xrightarrow{+1}$,
there is a long exact sequence in $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$:
$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots$$

Pf: Long. Main result about t-structures.

3) **t-exactness**

We have two triangulated categories endowed w/ t-structures
$(i = 1, 2)$ $(\mathcal{D}_i, \mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$. Let $\mathcal{C}_i := \mathcal{D}_i^{\leq 0} \cap \mathcal{D}_i^{\geq 0}$.
and a functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ of triangulated categories.

Def: Say F is left t-exact if $F(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$
— right t-exact if $F(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$

Prop: Let $F: \mathcal{D}_1 \to \mathcal{D}_2$ & $G: \mathcal{D}_2 \to \mathcal{D}_1$ be adjoint [(F,G) is an adj. pair]
Then F is right t-exact IFF G is right t-exact.

II Perverse sheaves

1) $\mathcal{D}^b_c(X) \supset \mathrm{Perv}(X)$ defined above.

New version: $\mathrm{Perv}(X) := \underbrace{{}^p\mathcal{D}_c^{\leq 0}(X)}_{\substack{\mathcal{F}\text{'s satisfying} \\ (i)}} \cap \underbrace{{}^p\mathcal{D}_c^{\geq 0}(X)}_{\substack{\mathcal{F}\text{ satisfying} \\ (ii)}}$

$$\begin{array}{ccc} \mathbb{D}_X : {}^p\mathcal{D}_c^{\leq 0}(X) & \xrightarrow{\sim} & {}^p\mathcal{D}_c^{\geq 0}(X) \\ \cup & & \cup \\ \mathrm{Perv}(X) & \xrightarrow{\sim} & \mathrm{Perv}(X) \end{array}$$

$\boxed{\textbf{Th.}\ ({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X)) \text{ is a t-structure on } \mathcal{D}^b_c(X)}$

Cor.: long exact sequences associated to dist. triangles in $\mathcal{D}^b_c(X)$

Pb: ~~no good since~~ usually, $f_*, f_!$ etc. don't map $\mathrm{Perv}(X)$ on $\mathrm{Perv}(Y)$.

2) t-exactness

Recall (f^*, f_*) and $(f_!, f^!)$ are adjoint pairs.

Prop: If $f: X \hookrightarrow Y$ is injective, then
f^* is right t-exact
$f^!$ is left t-exact.

Ex.: $i: Z \to X$ closed immersion $\Rightarrow$ then $i_* = i_!$

Indeed: i^* is right t-exact so i_* is left t-exact
$i^!$ is left —— $i_!$ is right ——

So i_* and $i_!$ are left & right t exact, so that $i_* = i_!$

As a consequence: $i_*: \mathrm{Perv}(Z) \to \mathrm{Perv}(X)$

Ex 2.: Assume X smooth of $\mathbb{C}$-dimension d_X. Then:
Then $\underline{\mathbb{C}}_X[d_X] \in {}^p\mathcal{D}^{\leq 0}(X)$
but also: $\mathbb{D}_X(\underline{\mathbb{C}}_X[d_X]) \in {}^p\mathcal{D}^{\leq 0}(X)$, so that $\underline{\mathbb{C}}_X[d_X] \in \mathrm{Perv}(X)$

IV Intersection cohomology complex.

Some categorical aspects:

a) $\mathrm{Perv}_X : \mathrm{Open}(X)^{op} \to \mathrm{Cat}$ this is a stack ("2-sheaf")
$U \longmapsto \mathrm{Perv}(U)$ good!

b) $\mathrm{Perv}(X)$ is noetherian & artinian. Natural q?:
$\rightsquigarrow$ what are the simple objects?

Answer: they are the "intermediate extensions" $IC(X, \mathcal{L})$ of irreducible local systems $\mathcal{L}$ on smooth locally closed irreducible subvarieties of X. [there might be isomorphisms].

Hey, what is "intermediate extension"?

Let $j: U \to X$ be an open immersion & $i: Z = X \setminus U \to X$.

Claim: Let $\mathcal{F} \in \mathrm{Perv}(U)$, then there is a unique $\tilde{\mathcal{F}} \in \mathrm{Perv}(X)$ s.th.
(a) $j^*\tilde{\mathcal{F}} \simeq \mathcal{F}$
(b) $i^*\tilde{\mathcal{F}} \in {}^p\mathcal{D}_c^{\leq -1}(Z)$ and $i^!\tilde{\mathcal{F}} \in {}^p\mathcal{D}^{\geq 1}(Z)$.

Call: ${}^pj_{!*}\mathcal{F} := \tilde{\mathcal{F}}$ ~~characterises~~ is the intermediate ext'n of $\mathcal{F}$.
$\rightarrow$ "perversification".

Now Assume U smooth irred. loc. closed subvariety, $j: U \to \overline{U}$ (open immersion) & $i: \overline{U} \hookrightarrow X$ (closed imm'n).
If $\mathcal{L} \in \mathrm{loc}(U)$ irreducible, set $IC(X, \mathcal{L}) := {}^pi_* {}^pj_{!*}\mathcal{L}[d_U] \in \mathrm{Perv}(X)$.
$\rightarrow$ these are the simple objects in $\mathrm{Perv}(X)$.

Special case: $IC(X) := IC(X, \underline{\mathbb{C}}_{X^{an}_{reg}})$ for X irreducible.

NB. X smooth $\Rightarrow IC(X) = \mathbb{C}_{X^{an}}[d_X]$

- $\mathbb{D}_X$ commutes w/ [?]
- Get generalised Poincaré duality for singular var'ies.

V. Fixed ~~str~~ Λ.

Fix a stratif'n Λ, ie $X = \coprod_{\lambda \in \Lambda} X_\lambda$ sth X_λ are connected + some additional cond'ns you will learn by yourselves.

↳ Λ-constructible ⇝ $\mathcal{D}^b_\Lambda(X) \subset \mathcal{D}^b_c(X)$

Set $Perv_\Lambda(X) = Perv(X) \cap \mathcal{D}^b_\Lambda(X)$

This is the heart of the t-structure

$\mathcal{D}^{\le 0}_\Lambda(X) = \{\mathcal{F}$ sth $i_\lambda^* \mathcal{F}$ is concentrated in degrees $\le -\dim X_\lambda$ w/ $i_\lambda : X_\lambda \hookrightarrow X \}$

$\mathcal{D}^{\ge 0}_\Lambda(X) = \{ \quad i_\lambda^! \mathcal{F} \ \ldots \ \deg \ge -\dim X_\lambda \}$

What are the simple objects of $Perv_\Lambda(X)$?

Ans: $IC(X, \mathcal{L})$ sth $\mathcal{L} \in Loc(X_\lambda)$ irreducible.

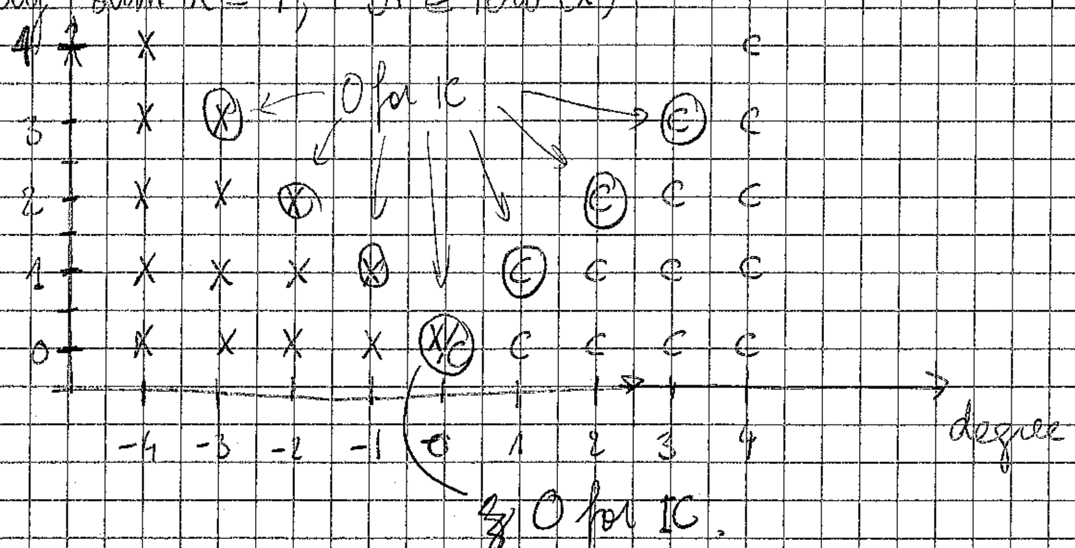
$\|$ unique ext'n of $\mathcal{L}[d_{X_\lambda}]$ to X, supported on $\overline{X_\lambda}$, sth

$\forall \mu \in \Lambda$, $X_\mu \subset \overline{X_\lambda}$, $\mu \neq \lambda$:
$\begin{cases} i_\mu^* IC(X,\mathcal{L}) \text{ is concentrated in degrees } < -\dim X_\mu \\ i_\mu^! IC(X,\mathcal{L}) \text{ ——— } > -\dim X_\mu \end{cases}$ ([illegible])

$\mathcal{F} \in Perv_\Lambda(X) \Rightarrow H^i(\mathcal{F})$ are Λ-constructible, ie $H^i(\mathcal{F})|_{X_\lambda}$ is a local system. This implies $H^i(\mathcal{F})|_{X_\lambda}$ is loc. constant, thus the stalks of $H^i(\mathcal{F})$ are constant on X_λ.

Say $\dim X = 4$, $\mathcal{F} \in Perv(X)$

x = usual coho
c = compactly supported coho.



VI Decomposition theorem.

Important q? : understand $f_* IC(Y) \in \mathcal{D}^b_c(X)$ for $f : Y \to X$ a mph.

Th (BBDG) If $f : Y \to X$ is proper, then $f_* IC(Y)$ is a direct sum of shifts of twisted ICC's.

— very close to be a perverse sheaves (up to shifts)
— false in charact. p.

Plan for next talk.

① Category $P_G(X)$
② Equivariant decomposition thm.

Equivariant perverse sheaves & representation theory.

① The category $P_G(X)$

Motivation: G alg'c gp acting on a variety X (over $\mathbb{C}$)
Let $a: G\times X \to X$ be the action, $p: G\times X\to X$ be the proj'n.
Say $f: X\to\mathbb{C}$ is equivariant if $\forall g\in G, \forall x\in X$, $f(gx)=f(x)$.
Means: $a^*f = p^*f : G\times X \to \mathbb{C}$

Notation Let $\varphi: X\to Y$ be a smth mph of rel. dim'n d
For $\mathcal{F}\in Shv(Y)$, write $\underline{\varphi}^*\mathcal{F} = \varphi^*\mathcal{F}[d] = \varphi^!\mathcal{F}[-d]$
if φ smooth
Then $\underline{\varphi}^*: P(Y) \to P(X)$.

Def: A G-equivariant perverse sheaf on X is a pair $(\mathcal{F}, i)$ w/ $\mathcal{F}\in P(X)$ and $\{i: \underline{a}^*\mathcal{F} \xrightarrow{\sim} \underline{p}^*\mathcal{F}\}\in P(G\times X)$ s.th.:

(1) $(\underline{m\times id})^* i = (\underline{id\times p})^* i \circ (\underline{id\times a})^* i$ ("cocycle cond'n")

(2) $(e\times id)^* i\,[-\dim G] = id_{\mathcal{F}}$.

NB: (2) is in fact useless — would get equivalent cat'y — but makes comput'ns easier.

Def: A mph $(\mathcal{F}, i) \to (\mathcal{G}, j)$ is a mph $\alpha: \mathcal{F}\to\mathcal{G} \in P(X)$ s.th. $j\circ(\underline{a}^*\alpha) = (\underline{p}^*\alpha)\circ i$.

⇝ get a cat'y $P_G(X)$.

Rk: Would get analogous def'n for eg $QCoh(X)$ or $\mathcal{D}$-mod(X) ...

Intuition from simple settings.
1) Suppose $\mathcal{F}\in QCoh(X)$, $\mathcal{F}$ locally free — ie sheaf of sections of a vector bdl $F\xrightarrow{\pi} X$. Then the data of i s.th. $(\mathcal{F}, i)\in QCoh_G(X)$ is equivalent to the data of

$$\begin{tikzcd} G\times F \arrow[r, "\Phi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & F \arrow[d] \\ X \arrow[r] & X \end{tikzcd}$$

inducing $F_x \simeq F_{g\cdot x}$

— work this out.

2) If G is connected, then $P_G(X) \to P(X)$ is fully faithful.
$(\mathcal{F}, i) \mapsto \mathcal{F}$

[means[??]]
— wrong in general: could have 2 diff structures on a $\mathcal{F}$.

3) If G acts freely on X, can form the scheme X/G.
Then $P_G(X) \simeq P(X/G)$.

Properties of $P_G(X)$ [push forward doesn't preserve perversity in gal]

1) Verdier duality: anti-equiv'ce $\mathbb{D}_X: P_G(X)^{op} \xrightarrow{\sim} P_G(X)$

2) Intermediate extensions: $j: Z\hookrightarrow X$ locally closed & G-stable
⇝ $j_{!*}: P_G(Z) \to P_G(X)$

3) External product $\boxtimes: P_G(X)\times P_G(Y) \to P_G(X\times Y)$.

4) Forgetful functor: $P_G(X)\to P(X)$ of ab'n cat'ies respecting the above functors.

But be careful!
× Could defined G-equivariant constructible sheaves but berk.

× $\mathcal{D}_G(X)$ is not the naive thing.
But we do have a forgetful fctor

$$\begin{tikzcd} \mathcal{D}_G(X) \arrow[r] & \mathcal{D}(X) \\ P_G(X) \arrow[u, hook, "\heartsuit"] \arrow[r] & P(X) \arrow[u, hook, "\heartsuit"'] \end{tikzcd}$$

Even for G connected, an extension of 2 equivariant perv. sheaves need not be G-equivariant ~~[illegible]~~.

(2) The equivariant decomposition theorem.

Lemma 1: Suppose $x_0 \in X$, $\mathbb{O} = G \cdot x$, $H = G_x = \mathrm{Stab}_G(x_0)$.
Then $P_G(\mathbb{O}) \simeq \{$ cat'y of f.d. rep'n of $H/H^0 \}$.
Moreover, each object is an G-equivariant local system on $\mathbb{O}$ shifted by $\dim \mathbb{O}$.

Idea of pf: $P_G(\mathbb{O}) \simeq P_G(G/H \cdot pt) \overset{pr}{\simeq} P_H(pt)$
work

Recall $\pi_1(G) \longrightarrow \pi_1(G/H) \longrightarrow \pi_0(H) \cong H/H_0$

Then: A local system on G/H is G-equivariant iff the corresponding $\pi_1(G/H)$-rep'n is pulled back from a rep'n of H/H_0.

NB: $IC(Y, \mathcal{L})$: ($\mathcal{L}$ is a local system on Y, — we pushforward
$IC(Y, \mathcal{L})$ is a local system on X, supported on $\overline{Y}$.

Lemma 2: $IC(Y, \mathcal{L}) = IC(Y', \mathcal{L}') \Longleftrightarrow \exists\, Z \subset Y \cap Y'$ locally closed, smooth, connected, dense in Y and Y' with $\mathcal{L}|_Z \simeq \mathcal{L}'|_Z$.

Thm (Beilinson–Lunts).
Let M be a smooth G-variety, $\mu : M \longrightarrow N$ G-equiv't, projective, w/ N = G-variety w/ finitely many orbits.
Let $\mathcal{C}_M$ be sth. $(\mathcal{C}_M)|_{M_i} = \mathbb{C}_{M_i}[\dim_{\mathbb{C}} M_i]$
Then: $\mu_* \mathcal{C}_M = \bigoplus_{\substack{i \in \mathbb{Z} \\ \varphi = (\mathbb{O}, \chi)}} L_\varphi(i) \otimes IC(\mathbb{O}, \chi)[i]$

$\mathbb{O}$ orbit in N
χ = irred. G-equiv. local system on $\mathbb{O}$
i.e. a $G(x_0)/G(x_0)^0$-rep'n.

Sketch — By the ordinary decomp'n theorem:
$$\mu_* \mathcal{C}_M = \bigoplus_{\substack{i \in \mathbb{Z} \\ \varphi = (Y, \mathcal{L})}} L_\varphi(i) \otimes IC(Y, \mathcal{L})$$
— need to prove that only φ that occur have supp $Y \in \{\mathbb{O}\}$.

Know canonical G-equiv't structure on $\mu_* \mathcal{C}_M$
$\Rightarrow$ each $IC(Y, \mathcal{L})$ on the RHS acquires a G-equiv't struct.

Hence $\mathrm{Supp}\, IC(Y, \mathcal{L}) = \overline{Y}$ is G-stable, so $\overline{Y}$ contains an orbit $\mathbb{O} \subset \overline{Y}$ open in $\overline{Y}$.
Restrict $IC(Y, \mathcal{L})$ to $\mathbb{O} \rightsquigarrow$ get an object in $P_G(\mathbb{O})$
i.e. an irred G-equiv't local system on $\mathbb{O}$, shifted by $\dim \mathbb{O}$.
Since $IC(Y, \mathcal{L})|_{\mathbb{O}} \simeq IC(\mathbb{O}, \chi)|_{\mathbb{O}}$,
deduce that $IC(Y, \mathcal{L}) \simeq IC(\mathbb{O}, \chi)$. ∎

(3) (Improved) Springer correspondence.

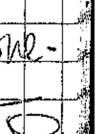
G connected alg gp $/\mathbb{C}$, $T \subset B \subset G$, $N_G(T)/T = W$

Springer, around 1976, found W acts on $H^*(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell)$
w/ $\mathcal{B}_u = \{ B'$ Borel containing $u \}$ w/ u = unipotent elt of G.
— though W doesn't act on $\mathcal{B}_u$. $IC \dots$

Lusztig reformulates this in terms of perv. sh.

Springer corresp. in type A.

$G = GL_n(\mathbb{C})$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $W = \mathfrak{S}_n$, $\mathcal{N} \subset \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ nilp. cone.
G acts on $\mathcal{N}$ w/ finitely many orbits.

Ex.: $n=2$ 

$\mathcal{N}/\!/G \longleftrightarrow P_n = \{ \lambda \vdash n \} \longleftrightarrow \mathrm{Irr}(\mathfrak{S}_n)$
$G\cdot x \longmapsto \lambda = (\lambda_1, \dots, \lambda_\ell)$ = sizes of Jordan blocks of x
$\lambda \longmapsto S^\lambda$ Specht modules

These orbits stratify $\mathcal{N}$ — Orbit closure ordering $\longleftrightarrow$ dominance ordering

Q: What is the explicit W-action?

Define $\pi : G \times^B \mathfrak{b} \longrightarrow \mathfrak{g}$ $\quad (\mathfrak{b} = \mathrm{Lie}\,B,\ \mathfrak{g} = \mathrm{Lie}\,G)$

$(g, x) \longmapsto g.x$ (conjugation) $\quad$ "Grothendieck resolution"

$G \times^B \mathfrak{b} = G \times \mathfrak{b}/\sim$ w/ $(g,x) \sim (gb^{-1}, b.x)$ $\forall b \in B$.

π has two important restrictions:

$\pi_{nil} : \widetilde{\mathcal{N}} := G \times^B \mathfrak{n} \longrightarrow \mathcal{N}(\mathfrak{g})$ $\quad$ Springer resol'n

$\pi_{rs} : \widetilde{\mathfrak{g}}_{rs} = G \times^B \mathfrak{b}_{rs} \longrightarrow \mathfrak{g}_{rs}$ $\quad$ rs = regular semisimple

Miraculous: • π_{nil} is proper & semi-small $\hookrightarrow$ dim of fibers don't drop.

• π_{rs} is a W-covering hence $\mathrm{End}(\pi_* \mathbb{C}_{\widetilde{\mathfrak{g}}_{rs}}) \simeq \mathbb{C}W$

$\mathrm{End}((\pi_{rs})_* \mathbb{C}_{\widetilde{\mathfrak{g}}_{rs}})$

$$\begin{tikzcd}
\widetilde{\mathfrak{g}}_{rs} \arrow[r] \arrow[d, "\pi_{rs}"', "W"] & \widetilde{\mathfrak{g}} \arrow[d, "\pi"] & \widetilde{\mathcal{N}} \arrow[l, hook'] \arrow[d, "\pi_{nil}"] \\
\mathfrak{g}_{rs} \arrow[r] & \mathfrak{g} & \mathcal{N} \arrow[l, hook']
\end{tikzcd}$$

Fact: $R\pi_* \mathbb{C}_{\widetilde{\mathfrak{g}}}$ is the IC extension of the local system $(\pi_{rs})_* \mathbb{C}_{\widetilde{\mathfrak{g}}_{rs}}$

W acts on $R\pi_* \mathbb{C}_{\widetilde{\mathfrak{g}}}$ and therefore on $H^i(\mathcal{B}_x)$ for $x \in \mathfrak{g}$

w/ $\mathcal{B}_x = \{$Borels of $\mathfrak{g}$: $x \in \mathfrak{b}\}$

By the decomposition theorem:

$$(\pi_{nil})_* \mathbb{C}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_{\substack{\mathcal{O} \in \mathcal{N}/G \\ \mathcal{L} \text{ irred loc. syst.}}} IC(\mathcal{O}, \mathcal{L}) \otimes \underbrace{V_{\mathcal{O},\mathcal{L}}}_{\text{vector space to recall mtp.}}$$

$\left[V_{\mathcal{O},\mathcal{L}} \simeq H^{top}(\mathcal{B}_x)^{\rho} \text{ w/ } \rho \in \mathrm{Irr}(C_G(x)/C_G(x)^{\circ}) \right]$

Taking endomorphisms algebras one gets:

$$\mathrm{End}\left((\pi_{nil})_* \mathbb{C}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]\right) \simeq \mathbb{C}W$$

since $\mathrm{Hom}(IC(\mathcal{O}_\lambda), IC(\mathcal{O}_\mu)) = \delta_{\mu\nu}\mathbb{C}$.

Springer correspondence for GL_3.

$W = \mathfrak{S}_3$, $\quad GL_3/B$: recall $H^*(G/B) \simeq \mathbb{C}[\mathfrak{S}_3]$ (w/ doubling degrees)

as $\mathfrak{S}_3$-mod: $V_{triv} \oplus V_\rho^{\oplus 2} \oplus V_\varepsilon$ — reflection rep'n

$\mathcal{N} = \{x \in \mathfrak{gl}_3 : x^3 = 0\}$

G-orbits

$\mathcal{O}_{(3)} = G.\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ (dim 6), $\quad \mathcal{O}_{(2,1)} = G.\begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$ (dim 4), $\quad \mathcal{O}_{(1^3)} = G.\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$ (dim 0)

What IC sheaves occur?

It's enough to describe their stalks.

By base change, it's enough to compute cohomology of fibres.

$\rightsquigarrow$ What are fibres of π_{nil}?

If $x \in \mathcal{O}_{(3)}$, there only one flag stab. by x, so: fibre is a point.

If $x \in \mathcal{O}_{(2,1)}$ $\quad \pi_{nil}^{-1}(x) \overset{ex}{=} \mathbb{P}^1 \vee \mathbb{P}^1$ $\quad$ (bouquet de 2 sphères)

If $x = 0$ $\quad \pi_{nil}^{-1}(x) = G/B$. — corresponds to regular rep'n

$$(\pi_{nil})_* \mathbb{C}_{\mathcal{N}}[6] = IC(\mathfrak{g}_{rs}, \mathcal{L})|_{\mathcal{N}} = \boxed{IC(\mathfrak{g}_{rs}, \mathcal{L}_{tr})|_{\mathcal{N}}} \oplus IC(\mathfrak{g}_{rs}, \mathcal{L}_\rho)|_{\mathcal{N}}^{\oplus 2} \oplus IC(\mathfrak{g}_{rs}, \mathcal{L}_\varepsilon)|_{\mathcal{N}}$$

	-6	-5	-4	-3	-2	-1	0
$\mathcal{O}_{(3)}$	$\mathbb{C}$						
$\mathcal{O}_{(2,1)}$	$\mathbb{C}$		$\mathbb{C}^2$				
$\mathcal{O}_{(1^3)}$	$\mathbb{C}$		$\mathbb{C}^2$		$\mathbb{C}^2$		$\mathbb{C}$

(IC cohomology sheaves are supported on the strata and below)

Equivariant cohomology

G = topological gp acting on X = G-space.
Want $H_G(X)$ so that it is $H(X/G)$ whenever this makes sense.
Idea: replace X by something on which G acts freely and the cohomology of which is "as close to X as possible".

(I) Classifying space

Def: A principal G-bundle is $p: E \to B$, $E = G$-space, for which there is a covering $B = \bigcup U_i$ sth:

1) $\forall j$, $p^{-1}(U_j) \xrightarrow{\varphi_j} U_j \times G$

2)
$$\begin{tikzcd} p^{-1}(U_j) \arrow[rr, "\varphi_j"] \arrow[dr, "p"'] & \circlearrowleft & U_j \times G \arrow[dl, "pr_1"] \\ & B & \end{tikzcd}$$

3) $g\,\varphi_j(u,h) = \varphi_j(u, gh)$ $\forall \dots$

Such a family (φ_j, U_j) is called a trivialisation of p, E =: total space.

Ex: G Lie gp, H closed subgp $\rightsquigarrow$ $G \to G/H$ is a ppal H-bdl.

Def: A universal principal G-bdl is a ppal G-bdl $p: E_G \to B_G$ sth E_G is contractible. Then E_G = classifying space.

Th (Milnor) | Every topological group has a ~~[illegible]~~ universal principal G-bundle.

Won't prove the thm but give constructions in examples.

Eg 0: $G = \{e\}$: $p: pt \to pt$ works.

Ex 1: $G = \mathbb{C}^*$: acts on $\mathbb{C}^n$ by $c\,(x_1, \dots, x_n) = (cx_1, \dots, cx_n)$.
$\mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ is a ppal $\mathbb{C}^*$-bdl.
$\pi_k(\mathbb{C}^n \setminus \{0\}) = 0$ if $k < n$
but $\pi_n(\mathbb{C}^n \setminus \{0\}) \neq 0$. So take some inductive lim

$$\begin{tikzcd} \mathbb{C}^n\setminus\{0\} \arrow[r, hook] \arrow[d] & \mathbb{C}^{n+1}\setminus\{0\} \arrow[d] & \mathbb{C}^\infty\setminus\{0\} := \varinjlim_n \mathbb{C}^n\setminus\{0\} \arrow[d] \\ \mathbb{CP}^{n-1} \arrow[r, hook] & \mathbb{CP}^{n} & \mathbb{CP}^\infty := \varinjlim_n \mathbb{CP}^{n-1} \end{tikzcd}$$

contractible

ppal $\mathbb{C}^*$-bdl

E_G, B_G

Ex 2: $G = S^1$
Put $S^{2n+1} \hookrightarrow \mathbb{CP}^n$ (deformation retraction) & build $\varinjlim S^{n+1}$ $= S^\infty$.

Ex 3: GL_n
Cs $\mathcal{M}^{max}_{N,n} = \{A \in \mathcal{M}_{N,n} : rk\,A = n\}$ $(N \geq n)$ is a ppal GL_n-bdl.

$$\begin{tikzcd} \mathcal{M}^{max}_{N,n} \arrow[d] & A \arrow[d, mapsto] \\ Gr_{n,N} & \ni Im(A) \end{tikzcd}$$

Has $\mathcal{M}_{N,n} \setminus \mathcal{M}^{max}_{N,n} = \{M \in \mathcal{M}_{N,n} : \det A = 0\ \forall A = (N-n)\times(N-n) \text{ minor of } M\}$
— has complex codim. $N-n+1$
— hence it is $(N-n)$-connected.

With the same trick: $\mathcal{M}^{max}_{\infty,n} \to Gr(\infty, n)$ is a universal ppl GL_n-bdl.

Ex 4: H = closed subgp of G = Lie gp
Then $E_G \to E_{G/H}$ is a universal principal H-bundle (UPGL)

Let $E_G \to B_G$ be a UPG-bdl: G acts freely on $X \times E_G$ by $g(x,e) = (gx, ge)$
$\rightsquigarrow$ form $(X \times E_G)/G$.

Def: $H_G(X) := H\big((X \times E_G)/G\big)$ — coho. w/ coeff's ~~in~~ $\mathbb{Q}$.

Prop: This definition is independent on the choice of E_G.

Sk: Cs, if E_G and E'_G are two classifying spaces:
$(E_G \times E'_G \times X)/G \longrightarrow (E_G \times X)/G$ is a E'_G fibration, so that $\pi_k(E'_G) = 0 \to \pi_k\big((E_G \times E'_G \times X)/G\big) \to \pi_k\big((E_G \times X)/G\big) \to \pi_{k+1}(E'_G) = 0$
By Whitehead's thm: $H\big((E_G \times E'_G \times X)/G\big) \simeq H\big((E_G \times X)/G\big)$ $\blacksquare$

Spent some time bothering about limits. That was unnecessary.

Prop: Let G compact Lie group, M mfd of dim $\leq n$ and $E_G^n \to B_G^n$ a ppal G-bdl.
Then: $\forall m \leq n$, $H^m((E_G^n \times M)/G) \cong H^m_G(M)$.

Let $X = pt \circlearrowleft G$.
So $H_G(pt) = H(E_G/G) = H(B_G)$.

Ex. $G = \mathbb{C}^\times$ or S^1: $B_G = \mathbb{P}^\infty$ so that $H_G(pt) = \mathbb{Q}[x]$, $x \in H^2(\mathbb{P}^\infty)$ (degree 2)

Ex. If $G = (\mathbb{C}^\times)^d$ or $(S^1)^d$, then $B_G = (\mathbb{P}^\infty)^d$
so that: $H_G(pt) = H(\underbrace{\mathbb{P}^\infty \times \dots \mathbb{P}^\infty}_{d \text{ copies}}) \overset{\text{Künneth}}{\simeq} \bigotimes_{i=1}^{d} \mathbb{Q}[x_i] \simeq \mathbb{Q}[x_1, \dots, x_d]$ (deg $x_i = 2$)

Ex: If G acts on X trivially:
then $(X \times E_G)/G \simeq X \times B_G$ so that $H_G(X) \overset{\text{Künneth}}{\cong} H(X) \otimes H_G(pt)$.

Ex. If G acts on X freely.
Then $(X \times E_G)/G \xrightarrow[E_G\text{-fibration}]{} X/G$ and as in the proof of proposition, homotopy groups are all the same & $H_G(X) \simeq H(X/G)$. Good

Ex. $H \subset G$ closed subgp. G?
$E_G \to E_G/H$ is a UPH-bdl so get $(G/H \times E_G)/G \to E_G/H$
$G.(1H, v) \longleftarrow v$
$G.(gH, v) \longmapsto Hg^{-1}v$
So $H_G(G/H) \simeq H(EG/H) \simeq H(B_H) = H_H(pt)$ (???)

Basic constructions.

Pull-back: Let $f: X \to Y$ be a continuous map of G-spaces.
Then get $f^-: (X \times E_G)/G \to (Y \times E_G)/G$
$G.(x, v) \longmapsto G.(f(x), v)$
So get: $f^*: H_G(Y) \to H_G(X)$ — denoted f^*.

For $f: X \to pt$, can do pullback: $\boxed{H_G(pt) \to H_G(X)}$
This, together with the ring structure of $H_G(X)$, makes $H_G(X)$ into a $H_G(pt)$-module.

Let X = G-space & $H \subset G$ subgp. → or H-space?
Relation between $H_G(?)$ & $H_H(?)$?
Do action of H on $G \times X$ & quotient $(G \times X)/H$
Then get $((G \times X)/H \times E_G)/G \xleftarrow{\sim} (X \times E_G)/H$ (check, similar to the previous one)
$G.((g, Hx), v) \longmapsto (g^{-1}x, g^{-1}v)H$
$(pt \times E_G)/G \longleftrightarrow (pt \times E_G)/H$

⇝ induction formula: $H_H(X) \simeq H_G((G \times X)/H)$
& this isomorphism is compatible w/ the $H_H(pt)$-module structure

[use natural map $E_G/H \to E_G/G$]

$G_{\mathbb{C}}$ = complex reductive
Let G = compact connected Lie gp & X = a G-space.
$T \supset K$ = max. torus ($\simeq (S^1)^d$) & $W = N_G(K)/K$ & $R = H_K(pt)$
max torus

Prop: (i) W acts on $H_K(X)$ and $H_G(X) \cong H_K(X)^W$.
[if X is a point: $H_K(pt)^W = H(B_K)^W = R^W$.] $G/K \simeq G_{\mathbb{C}}/T$

(ii) $H_K(pt) = H_G(G/K) \to H(G/K)$ is surjective and induces an isomorphism $R/R^W_+ \simeq H(G/K)$. (where $R^W_+ = \dots$)

(iii) $H_K(X) \simeq H_K(pt) \otimes_{R^W} H_G(X)$
X = flag var'y
[if $X = G/K$ flag var'y, then $H_K(X) = R \otimes_{R^W} R$] $H_K(X) \cong H_T(X)$
Ex: Work out $G = SL_2$ case

More equivariant cohomology

Recall $\times$ G = topological group (probably Lie) acting on X.

$\rightsquigarrow$ $E_G \rightsquigarrow H^*_G(X) := H^\cdot((X \times E_G)/G)$
$\downarrow$ (UP G-bdl)
B_G

$\times$ Typical ingredient: go from compact to complex ss.

Lemma: Let K be a compact subgroup of G, assume G acts on X, and that $K \hookrightarrow G$ is a homotopy equivalence. Then: $H^\cdot_K(X) \cong H^\cdot_G(X)$.

Pf: Recall E_G/K is a UP K-bdl, i.e. can take: $E_G =: E_K$

Canonical map: $(X \times E_G)/K \longrightarrow (X \times E_G)/G$

— it is a fibration w/ fiber G/K. By assumption, $\begin{cases} H^0(G/K) = \mathbb{R} \\ H^{\geq 1}(G/K) = 0 \end{cases}$ (since $K \hookrightarrow G$ is an homotopy equivalence) so that AQT. ∎

Examples: 1) G reductive $/\mathbb{C}$ & K = max. compact. Iwasawa decomp'n shows they are homotopy equiv.

2) $T \subset B$ (torus in Borel)

3) $\{1\} \hookrightarrow N$ w/ N unipotent ($\Rightarrow N \simeq \mathbb{A}^{\dim N}$ as a variety)

4) $L \subset P$ (Levi inside parabolic)

Assume G, K act freely on X and the actions commute.

$$H^\cdot_G(X/K) \overset{?}{\longleftrightarrow} H^\cdot_K(X/G)$$

Lemma: There is an isomorphism $H^\cdot_G(X/K) \xrightarrow{\sim} H^\cdot_K(X/G)$.

Pf (formal, not deep): ~~$H^\cdot_G(X/K) = H^\cdot((E_G \times X/K)/G) = H^\cdot((E_G \times X)/K \times G)$~~

Instead of E_G, use $E_G \times E_K$ to compute $H^\cdot_G$; same w/ $E_G \times E_K$ & E_K.

$$H_G(X/K) = H^\cdot((X/K \times E_G \times E_K)/G) = H^\cdot((X \times E_G \times E_K)/K \times G)$$

Since K action on X is free, doesn't matter what action we take here, i.e. diagonal action or only on X.

Obviously can obtain the same descrip'n of $H^\cdot_K(X/G)$.

Applications $H \leqslant G$ closed, $G/H =: X$ (not boring: any orbit is such!)

$$H^\cdot_G(G/H) = H^\cdot_H(G/G) = H^\cdot_H(pt)$$

Ex. $H^\cdot_G(G/B) = H^\cdot_B(pt) = H^\cdot_T(pt)$ — known. [used that $T \overset{\text{hmtp}}{\simeq} B$]

Prop: T torus, $X^*(T) = \{\text{characters of } T\}$. There exists an isomorphism
$$c : \mathrm{Sym}(X^*(T)) \xrightarrow{\sim} H^\cdot_T(pt)$$

— compared to previous ~~result~~ statement, it is coordinate-free.

Sk. Pf (not coord.-free): Fix $\lambda : T \to \mathbb{C}^*$, $E_T \to B_T$ ppal bdl

$\rightsquigarrow$ form $\mathcal{L}_\lambda = (E_T \times \mathbb{C}_\lambda)/T \longrightarrow B_T$ — line bdl on B_T.

Then $c(\lambda) \longmapsto c_1(\mathcal{L}_\lambda) \in H^2(B_T)$. Chern ~~character~~ class.

Both algebras have the same graded dimension (characters of T are in degree 2), hence need only surjectivity.

Follows from the case of $T = \mathbb{C}^*$.

On $\mathbb{P}^n$, the tautological line bdl $\mathcal{O}(-1)$: its first Chern class generates $H^\cdot(\mathbb{P}^n)$ ∎

Ex. $S := \mathrm{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}) = \mathbb{C}[\lambda_1, \dots, \lambda_n]$ n characters that generate $X^*(T)$.

Then $H^\cdot(BT) = S = H^\cdot_G(G/B)$.

Prop. $T \subset G$, G reductive, W = Weyl, $G \circlearrowright X$ (coefficients in a field)

(i) $H^\cdot_G(X) = H^\cdot_T(X)^W$

(ii) $H^\cdot_G(G/B) \twoheadrightarrow H^\cdot(G/B)$

(iii) $H^\cdot_T(X) = H^\cdot_G(X) \otimes_{S^W} S$.

$H^\cdot(G/N) = H^\cdot(G/T)^W$ (Galois) $= \mathbb{Q}$

(ii), (iii): easy. ∎

Pf.: Let $N = N_G(T)$, so: $(E_G \times X)/T \longrightarrow (E_G \times X)/N$ — W-Galois covering.

hence $H((E_G \times X)/N) \simeq H^\cdot_T(X)^W$.

Now: $(E_G \times X)/N \longrightarrow (E_G \times X)/G$ is a fibration w/ fibers G/N.

It is thus enough to prove that $H^\cdot(G/N) = \mathbb{C}$. (or $\mathbb{Q}$)

affine fibration $\begin{cases} G/T \longrightarrow G/N & W\text{-Galois covering} \\ \downarrow \\ G/B \end{cases}$

$\chi(G/B) = |W| \overset{\text{fibr'n}}{\Rightarrow} \chi(G/T) = |W| \overset{\text{Galois}}{\Rightarrow} \chi(G/N) = 1$ $\left\{ H^\cdot(G/N) = H^\cdot(G/T)^W \right.$

Important example: double flag variety $H_G^\bullet(G/B \times G/B^-)$?

(1M) Have $G/B \times G/B \simeq G \times_B (G/B)$

$(gB, hB) \longmapsto [g, g^{-1}hB]B$ (compute $\longleftarrow$)

So: $H^\bullet(G/B \times G/B) \simeq H^\bullet(G \times_B (G/B)) \simeq H_B^\bullet(G/B) \simeq H_T^\bullet(G/B)$

By prop. (iii): $H_T(G/B) \simeq H_G(G/B) \otimes_{S^W} S \simeq H_T(pt) \otimes_{S^W} S \simeq S \otimes_{S^W} S$.

(2M) $H_G(G/B \times G/B) \simeq H_G^\bullet((G\times G)/(B\times B)) = H^\bullet_{B\times B}((G\times G)/G)$

(action of G on the left, $B\times B$ on the right $\Rightarrow$ commute, + free.)

$\simeq H^\bullet_{B\times B}(G) \simeq H^\bullet_B(G/B)$ etc.

Localisation theorem (Borel, Atiyah-Bott, ...)

* T-actions only.

Idea: reduce everything to X^T — fixed point set

Why? $H_T^\bullet(X^T) = \underbrace{H_T(pt)}_{S} \otimes H^\bullet(X^T)$ — easy to compute!

Th. Suppose that

(1) either X is a compact smooth manifold (+ orientation ...)

(2) or X is some variety that can be embedded T-equivariantly into some f. di. representation of T.

Then: $H_T(X) \xrightarrow{i^*} H_T(X^T)$ becomes an isomorphism after inverting finitely many non trivial elts of $H_T^\bullet(pt)$.

Def: A T-space X is called equivariantly formal if (as $H_T(pt)$ module) $H_T(X) \simeq H_T(pt) \otimes H^\bullet(X)$. An equivalent condition is: $H_T^\bullet(X) \longrightarrow H^\bullet(X)$ is surjective.

So nearly all info is contained in the spectral sequence associated to the fibration $X \longrightarrow E_T \times_T X$

$\downarrow$

B_T

Case: If $H^{odd}(X) = 0$ then X is equivariantly formal.

Fact: If X is equivariantly formal, then $H_T(X) \to H_T^\bullet(X^T)$ is injective.

Eg: $T = \mathbb{C}^*$, $X = \mathbb{P}^1$ — action: $\lambda[x_0 : x] = [x_0 : \lambda x]$

$\hookrightarrow$ no odd coho $\Rightarrow$ equiv'ly formal.

so $H_T^\bullet(\mathbb{P}^1) \simeq H_T^\bullet(pt) \otimes H^\bullet(\mathbb{P}^1) \simeq H_T^\bullet(pt) \otimes \mathbb{C}[x]/(x^2)$

$\simeq H_T^\bullet(pt) \oplus H_T^\bullet(pt)\langle -2 \rangle$.

But still don't know the ring structure.

$H^\bullet(\mathbb{P}^1)$ is generated by $c_1(\mathcal{O}(-1))$ — denote by $\mathcal{L} = \mathcal{O}(-1)$.

Facts: If $\mathcal{L}$ is a T-equivariant bdl on X, then one can construct the equivariant Chern class of $\mathcal{L}$, $c_1^T(\mathcal{L}) \in H_T^\bullet(X)$

Here, $\mathcal{O}(-1)$ can be given an equiv. structure $\rightsquigarrow c_1^T(\mathcal{L}) \in H_T^\bullet(\mathbb{P}^1)$

$i^* c_1^T(\mathcal{L}) = c_1^T(\underbrace{\mathcal{L}_{|p}}_{T\text{-module}}) \oplus c_1^T(\underbrace{\mathcal{L}_{|q}}_{\text{idem}})$ in $H_T^\bullet(p) \oplus H_T^\bullet(q)$ — p, q are the two fixed points of $\mathbb{C}^*$ in $\mathbb{P}^1$.

Believe her or do it yourself:

Know: $c_1^T(\mathcal{L}_{|p}) \in H_T^\bullet(p) \simeq S =: \mathbb{C}[t]$

Fact: $i^* c_1^T(\mathcal{L}) = (t, -t)$ in

$H_T^\bullet(pt) \longrightarrow H_T^\bullet(\mathbb{P}^1) \xrightarrow{i^*}$

$t \longmapsto t\phi \longmapsto t \oplus t$

$-t \longmapsto -t \longmapsto -t \oplus t$.

$$c_1^T(\mathcal{L}) - t \xmapsto{i^*} 0 \oplus -2t$$

$$c_1^T(\mathcal{L}) + t \mapsto 2t \oplus 0$$

$$\underbrace{c_1^T(\mathcal{L})^2 - t^2}_{\text{already zero in } H_T^\cdot(\mathbb{P}^1)} \mapsto 0 \quad \text{— dingue}$$

In the end: $H_T^\cdot(\mathbb{P}^1) \simeq \mathbb{C}[x,t]/(x^2-t^2)$

Extend: $\mathbb{C}^* \curvearrowright \mathbb{P}^n$ $\quad \lambda.[x_0 : \cdots : x_n] = [\lambda^{m_0}x_0 : \cdots : \lambda^{m_n}x_n]$

Get $H_{\mathbb{C}^*}^\cdot(\mathbb{P}^n) = \mathbb{C}[x,t] / \prod_{i=0}^{n}(x - m_i t)$

The example of $\mathfrak{sl}_2$

or: geometric construction of the affine nil-Hecke algebra

Picture: $G = GL_n$, $\pi : \mathcal{F}l_n \longrightarrow pt \;/\; GL_n$. $\qquad K = \mathbb{C}$.

affine nil-Hecke algebra: ${}^0H_n \xrightarrow[\text{①}]{\text{mph of algebras}} \mathrm{Ext}_{\mathcal{D}_G^\cdot(pt)}(\pi_*\underline{K}_{\mathcal{F}l_n}, \pi_*\underline{K}_{\mathcal{F}l_n}) \underset{\text{②}}{\simeq} H_G^\cdot(\mathcal{F}l_n \times \mathcal{F}l_n)$.

Remarks: From the last talks, we know that, up $P_n = \mathbb{C}[x_1, \dots, x_n]$

$*$ ${}^0H_n \simeq \mathrm{End}_{P_n^{\mathfrak{S}_n}}(P_n) \simeq H_G^\cdot(\mathcal{F}l_n \times \mathcal{F}l_n)$.

①

Let X be a complex variety, $Y = \mathbb{P}(E) \xrightarrow{\pi} X$ a $\mathbb{P}^1$-fibration for some rank-2 vector bundle E over X

Prop: There exists a morphism of algebras ${}^0H_2 \dashrightarrow \mathrm{Ext}_{\mathcal{D}}(\pi_*\underline{K}_Y, \pi_*\underline{K}_Y)$
(up $\mathcal{D} = \mathcal{D}^b(X)$)

Sketch of proof or example. Pretend $X = \{pt\}$, $Y = \mathbb{P}^1$, $E = \mathbb{C}^2$.

Recall ${}^0H_2 = \langle X, Y, T \mid T^2 = 0,\ XY = YX,\ T(X+Y) = (X+Y)T \rangle$

Construction of T.

(π^*, π_*) adjoint pair $\rightsquigarrow$ canonical map $\underline{K}_{pt} \xrightarrow{\text{can}} \pi_*\underline{K}_{\mathbb{P}^1}$
— corresponding to 1 under $\mathrm{Hom}(\underline{K}_{\mathbb{P}^1}, \underline{K}_{\mathbb{P}^1}) \to \mathrm{Hom}(\pi^*\underline{K}_{pt}, \underline{K}_{\mathbb{P}^1})$

Fact: $\mathcal{F} \in \mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq i} \Rightarrow \mathcal{F} \simeq \mathcal{H}^i\mathcal{F}[-i]$

So we get a map:

$$\pi_*\underline{K}_{\mathbb{P}^1} \xrightarrow{1_{\text{can}}} \mathcal{H}^2(\pi_*\underline{K}_{\mathbb{P}^1})[-2]$$

— indeed, $\pi_*\underline{K}_{\mathbb{P}^1} \longrightarrow \tau_{\geq 2}\underline{K}_{\mathbb{P}^1}$.

There is the trace map $\mathcal{H}^2(\pi_* \underline{K}_{\mathbb{P}^1}) \xrightarrow{tr} \underline{K}_{pt}$.

Define $T : \pi_* \underline{K}_{\mathbb{P}^1} \xrightarrow{can} \mathcal{H}^2(\pi_* \underline{K}_{\mathbb{P}^1})[-2] \xrightarrow{tr} \underline{K}_{\mathbb{P}^1}[-2] \xrightarrow{can} \pi_* \underline{K}_{\mathbb{P}}[-2]$

— thus $T \in \mathrm{Hom}(\pi_* \underline{K}_{\mathbb{P}^1}, \pi_* \underline{K}_{\mathbb{P}^1}[-2])$.

Rk: $T^2 \in \mathrm{Hom}(\pi_* \underline{K}_{\mathbb{P}^1}, \pi_* \underline{K}_{\mathbb{P}^1}[-4])$ so, since $\pi_* \underline{K}_{\mathbb{P}^1} \in \mathcal{D}^{\leq 2}$ and $\pi_* \underline{K}_{\mathbb{P}^1}[-4] \in \mathcal{D}^{\geq 4}$, we get $T^2 = 0$.

Construction of X and Y.

$$\begin{array}{ccc} \mathcal{O}(-1) \subset \mathbb{P}^1 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\pi} & pt \end{array}$$

Fact: $H^n(X, \mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(\underline{K}, \mathcal{F}[+n])$

Let $\alpha = c_1(\mathcal{O}(-1))$, $\beta = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^1, \underline{K}_{\mathbb{P}^1}) \simeq \mathrm{Hom}_{\mathcal{D}^b(\mathbb{P}^1)}(\underline{K}_{\mathbb{P}^1}, \underline{K}_{\mathbb{P}^1}[2])$

Let $X = \pi_*(\alpha)$, $Y = \pi_*(\beta)$

so $X, Y \in \mathrm{Hom}_{\mathcal{D}^b(pt)}(\pi_* \underline{K}_{\mathbb{P}^1}, \pi_* \underline{K}_{\mathbb{P}^1}[2])$

Rk. • Relations are ok (admitted)

• Non-equivariant version: $X = -Y$.

• Equivariant version: equivariant Chern class.

$$\mathrm{Ext}(\pi_* \underline{K}_{\mathbb{P}^1}, \pi_* \underline{K}_{\mathbb{P}^1}) \simeq \mathrm{Mat}_{2\times 2}(\mathbb{K}[X,Y]^{\mathfrak{S}_2})$$

copy of $\mathcal{A}$ ⊕ copy of $\mathcal{A}$ (shifted)

Put things in perspective

$\mathfrak{g}$ semisimple ~~(n~~ $(\mathfrak{g} = \mathfrak{sl}_2)$ | Quiver Hecke algebra (affine nil Hecke algebra)

Def: ${}^0H_n$ = $\mathbb{C}$-alg. generated by $x_1, \dots, x_n, t_1, \dots, t_{n-1}$ + relations.

$t_i^2 = 0$, $t_{i+1} t_i t_{i+1} = t_i t_{i+1} t_i$, $t_i t_j = t_j t_i$ $(|j-i| \geq 2)$

Def: ${}^0H_n$ is the $\mathbb{C}$-algebra generated by $x_1, \dots, x_n, t_1, \dots, t_{n-1}$ subject to:

$$\begin{cases} t_i^2 = 0 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \quad \& \quad t_i t_j = t_j t_i \quad (|i-j| \geq 2) \\ x_i x_j = x_j x_i \\ t_i x_i - x_{i+1} t_i = x_i t_i - t_i x_{i+1} = 1. \end{cases}$$

$\mathcal{B}(pt)$ = strict monoidal $\mathbb{C}$-linear category generated by E (equipped w/ a ⊗, associative & w/ unit) $\quad \mathrm{Ob}\,\mathcal{B}(pt) = \{$finite sums of powers of $E\}$

and morphisms generated by $X \in \mathrm{End}(E)$, $T \in \mathrm{End}(EE, EE)$

subject to relations in $\mathrm{End}(E^2)$:

$$X 1_E \circ T - T \circ 1_E X = 1_{E^2}$$
$$T \circ 1 X_E - X 1_E \circ T = 1_{E^2}$$
$$T^2 = 0$$
$$(T 1_E) \circ (1_E T) \circ (T 1_E) = (1_E T) \circ (T 1_E) \circ (1_E T) \in \mathrm{End}_{E^3}.$$

Thm: ${}^0H_n \longrightarrow \mathrm{End}_{\mathcal{B}(pt)}(E^n)$ is an isom. of algebras

degree		
	1	$\longmapsto 1_{E^n}$
2	x_i	$\longmapsto 1_{E^{i-1}} X 1_{E^{n-i}}$
-2	t_i	$\longmapsto 1_{E^{i-1}} T 1_{E^{n-i-1}}$

Recall from (L4) that ${}^0H_n \simeq \mathrm{End}_{P_n^{\mathfrak{S}_n}}(P_n) \simeq H^\cdot_{GL_n}(Fl_n \times Fl_n)$

as $P_n^{\mathfrak{S}_n}$-modules

full flag variety in $\mathbb{C}^n$

Arabia '86: ${}^0H_n \xrightarrow{\sim} H_{GL_n}(Fl_n \times Fl_n)$ as algebras

this is convolution (à la Borel-Moore)

The isomorphism (2) evoked in the begining was / Ginzburg

$$H^\cdot_{GL_n}(Fl_n \times Fl_n) \xrightarrow{\sim} \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b_{c, GL_n}(pt)}(\pi_{n!}\underline{\mathbb{C}}, \pi_{n!}\underline{\mathbb{C}}[d]) \text{ w/ } \pi_n : Fl_n \to \cdot \ (GL_n\text{-equiv.})$$

Want to explain to what the generators x_i, t_i correspond here.

Restatement: $\mathrm{End}_{\mathcal{B}(pt)}(E^n) \xrightarrow{\sim} \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(\pi_{n!}\mathbb{C}_{Fl_n}, \pi_{n!}\mathbb{C}[d])$

$x \longmapsto x \in \mathrm{Hom}(\pi_{1!}\mathbb{C}_{Fl_n}, \pi_{1!}\mathbb{C}_{Fl_n}[2])$

$T \longmapsto t \in \mathrm{Hom}(\pi_{2!}\mathbb{C}, \pi_{2!}\mathbb{C}[-2])$

$x \in \mathrm{Hom}_{\mathcal{D}}(\pi_{1!}\mathbb{C}_{Fl_1}, \pi_{1!}\mathbb{C}[2])$

$\mathrm{Hom} \quad = \quad = H^2_{\mathbb{C}^*}(Fl(1))$

Natural candidate: c_1 = Chern class of tangent line bdl over $Fl(1)$

$c_1 \in H^2_{\mathbb{C}^*}(Fl(1))$

$x = \pi_{1!}(c_1)$.

$\pi_2 : \mathbb{P}^1 \to pt \quad \rightsquigarrow t \in \mathrm{Hom}_{\mathcal{D}}(\pi_{2!}\mathbb{C}, \pi_{2!}\mathbb{C}[-2])$ as above.

Jacinta claimed: "$yT - Tx = 1$"

Relation between this x & our x.

$\pi_{1!} \circ \pi_{1!}(\mathbb{C}) = \pi_{2!}(\mathbb{C})$

$\begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$ $\rightsquigarrow x$ from Jacinta, $\rightsquigarrow y$ from —

Lusztig's canonical basis.

§1: Moduli spaces ~~Hall algebra~~ associated to quivers.

§2: The Hall category.

§3: Examples: $\cdot$ & $\cdot\!\to\!\cdot$

§4: Relation to quantum groups.

Fix $k = \mathbb{C}$ (any field) & a quiver Q.

§1 Moduli spaces associated to quivers.

As in (L1), cs moduli spaces of Q-rep'ns. Fix $\alpha \in \mathbb{N}^I \subset \mathbb{Z}^I = K_0(\mathrm{Rep}\text{-}Q)$

$E_\alpha := \bigoplus_{h\in Q_1} \mathrm{Hom}(k^{\alpha_{s(h)}}, k^{\alpha_{t(h)}}) =$ "Q-rep'ns on $\bigoplus_{i\in I} k^{\alpha_i}$".

Cs $\mathcal{M}_\alpha := E_\alpha / G_\alpha$ w/ $G_\alpha = \prod_{i\in I} GL(\alpha_i, k)$

— indeed a stack

but we don't care: merely a notation.

Further $E_{\alpha_1,\dots,\alpha_n} := \{0 \subseteq N_1 \subset \dots \subset N_n : \text{flags in Rep-}Q,\ N_i/N_{i-1} \simeq E_{\alpha_i}\}$

$G_{\alpha_1+\dots+\alpha_n} \circlearrowright E_{\alpha_1,\dots,\alpha_n} \xrightarrow[\text{proper}]{q} E_{\alpha_1+\dots+\alpha_n}$

Cs $\mathcal{E}_{\alpha_1,\dots,\alpha_n} := E_{\alpha_1,\dots,\alpha_n} / G_{\alpha_1+\dots+\alpha_n}$

Correspondence:

$$\begin{tikzcd} & \mathcal{E}_{\alpha,\beta} \ni N \subset M \arrow[dl, "p"', "\text{smooth}"] \arrow[dr, "q"] & \\ \mathcal{M}_\alpha \times \mathcal{M}_\beta \ni (M/N, N) & & \mathcal{M}_{\alpha+\beta} \ni M \end{tikzcd}$$

§2 $\mathcal{D}^b(\mathcal{M}_\alpha) := \mathcal{D}^b_{G_\alpha, c}(E_\alpha)$

Induction: $m : \mathcal{D}^b(\mathcal{M}_\alpha \times \mathcal{M}_\beta) \longrightarrow \mathcal{D}^b(\mathcal{M}_{\alpha+\beta})$

$F \longmapsto q_! p^*(F)[\dim p]$

Restriction: $\Delta : \mathcal{D}^b(\mathcal{M}_{\alpha+\beta}) \longrightarrow \mathcal{D}^b(\mathcal{M}_\alpha \times \mathcal{M}_\beta)$

$F \longmapsto p_! q^*(F)[\dim p]$.

$\rightsquigarrow$ convolution: $F * G := m(F \boxtimes G)$

These maps are (co)associative in an obvious categorical way.

We want a subcategory of $\mathcal{D}^b(\mathcal{X}_\alpha)$. Set $\mathbb{1}_\alpha := \underline{k}_{E_\alpha}[\dim E_\alpha] \in \mathcal{D}^b(\mathcal{X}_\alpha)$

<u>Lusztig's sheaves</u>: $L_{\alpha_1,\dots,\alpha_n} := \mathbb{1}_{\alpha_1} * \mathbb{1}_{\alpha_2} * \dots * \mathbb{1}_{\alpha_n} \in \mathcal{D}^b(\mathcal{X}_{\alpha_1+\dots+\alpha_n})$.

Can show: $L_{\alpha_1,\dots,\alpha_n} = q_!\left(\underline{k}_{E_{\alpha_1,\dots,\alpha_n}}\right)\left[\dim E_{\alpha_1,\dots,\alpha_n} + \sum_{i=1}^n \dim E_{\alpha_i}\right]$

Indeed: essentially the associativity of m.

<u>NB</u>: By the decomposition theorem, Lusztig's sheaves are semisimple.

<u>Main def'n</u>: $\mathcal{P}_\alpha := \left\{ F : \text{simple } G_\alpha\text{-equivariant perverse sheaf which is a summand of } L_{\alpha_1+\dots+\alpha_n} \text{ where } \alpha_i = \varepsilon_i \text{ (basis vector)} \right\}$

$\mathcal{H}_\alpha :=$ « the category generated by (shifts of) elts of $\mathcal{P}_\alpha$ under $\oplus$ »

$$\mathcal{P}_Q := \coprod_{\alpha \in \mathbb{N}^I} \mathcal{P}_\alpha, \qquad \mathcal{H}_Q := \coprod_{\alpha \in \mathbb{N}^I} \mathcal{H}_\alpha \quad \text{"Hall category".}$$

<u>Prop</u>: $\mathcal{H}_Q$ is preserved under m and Δ.

meaning: $m : \mathcal{H}_Q \boxtimes \mathcal{H}_Q \longrightarrow \mathcal{H}_Q$

as $\mathbb{N}^I$-graded categories

(cf tensor product of complexes)

<u>Formula</u>: $\Delta(L_{\underline{\alpha}}) = \bigoplus_{\underline{\beta}+\underline{\gamma}=\underline{\alpha}} L_{\underline{\beta}} \boxtimes L_{\underline{\gamma}}[d_{\underline{\beta},\underline{\gamma}}]$. w/ $d_{\beta,\gamma} = -\sum_k (\beta_k, \gamma_k) - \sum_{k<l} \langle \beta_k, \gamma_l \rangle$ (?)

<u>Sk.</u>: Easy for m, more work for Δ.

<u>Prop</u>: $\mathcal{P}_Q$ is closed under Verdier duality.

Indeed, Verdier duality exchanges $!$ & $*$

<u>Geometric pairing</u>. $F, G \in \mathcal{D}^b(\mathcal{X}_\alpha)$

Set: $\{F, G\} := \sum_{j \in \mathbb{Z}} \dim H^j_{G_\alpha}(F \otimes G, E_\alpha)\, v^j \in \mathbb{N}((v))$.

Recall
$$\begin{tikzcd} & E_G \times X \arrow[dl, "f"'] \arrow[dr, "s"] & \\ X & & (E_G \times X)/G \end{tikzcd}$$
F on X, $\tilde{F}$ on $(E_G \times X)/G$: $f^* F \cong s^* \tilde{F}$.

This pairing is bilinear, symmetric

Module over $\mathbb{Z}[v, v^{-1}]$ where v acts by $[1] : \mathcal{H}_Q \longrightarrow \mathcal{H}_Q$, v^{-1} by $[-1]$.

Then $\{F[n], G\} = v^n \{F, G\}$.

This is a "Hopf pairing":

<u>Prop</u>: $\{F * F', G\} = \{F \boxtimes F', \Delta(G)\}$

<u>Sk.</u>: Use the projection formula.

§3 <u>Examples</u>.

3.1 $Q = \bullet$

Dimension vectors: $n \in \mathbb{N}$. $E_n = \{pt\}$ (only 1 rep'n: k^n)

Divide n as sum of basis vectors: only 1 way: $E_{\underbrace{1,1,\dots,1}_n}$.

Hence $E_{1,\dots,1} = \{0 \subseteq W_1 \subseteq \dots \subseteq W_n = k^n : \dim W_i/W_{i-1} = 1\}$

$= Fl_n$ flags variety, dimension $n(n-1)/2$.

and $q : E_{1,\dots,1} \longrightarrow E_n$ is $q : Fl_n \longrightarrow \{pt\}$ — project to a point

Also: $L_{1,\dots,1} = q_!\left(\underline{k}_{E_{1,\dots,1}}\right)\left[\frac{n(n-1)}{2}\right] = \bigoplus_i H^i\left(E_{1,\dots,1}, \underline{k}_{E_{1,\dots,1}}\right)\left[\frac{n(n-1)}{2} - i\right]$

$= (\mathbb{1}_n)^{\oplus [n]_v!} = \bigoplus_j \mathbb{1}_n^{\oplus \{r_j\}}[j]$ w/ $[n]_v! = \sum_j r_j v^j$.

meaning: $\{\ \}$

Thus: $\mathcal{P}_n = \{\mathbb{1}_n\}$, $\mathcal{P}_Q = \{\mathbb{1}_n, n \in \mathbb{N}\}$

$$\mathcal{H}_Q = \coprod_{n \in \mathbb{N}} \mathcal{D}^b_{G_n}(E_n)^{ss}$$

Can check that: $\Delta(\mathbb{1}_n) = \bigoplus_{\beta+\gamma=n} \mathbb{1}_\beta \boxtimes \mathbb{1}_\gamma [d_{\beta,\gamma}]$

~~and $\{\mathbb{1}_n, \mathbb{1}_n\} = \sum_j \dim H^j(E_n)[\dim E_n]$~~ $= \bigoplus_{\beta+\gamma=n} \mathbb{1}_\beta \boxtimes \mathbb{1}_\gamma [-\beta\gamma]$

$= \bigoplus_{i=0}^{n} \mathbb{1}_i \boxtimes \mathbb{1}_{n-i} [-i(n-i)]$

$$\{\mathbb{1}_1, \mathbb{1}_1\} = \sum_j \dim H^j_{k^\times}(E_1)[\dim E_1]\, v^j \qquad E_1 = \{pt\} \Rightarrow \dim E_1 = 0$$

$$= \sum_j \dim H^j_{k^\times}(pt)\, v^j = 1 + v^2 + v^4 + \cdots = \frac{1}{1-v^2}.$$

Check that $\{\mathbb{1}_n, \mathbb{1}_n\} = \prod_{j=1}^{n} \frac{1}{1-v^{2j}}$

Note: every sheaf in $\mathcal{P}_Q$ is self-dual.

$\boxed{3.2}$ $Q = 1 \bullet \longrightarrow \bullet 2$

$S_1 = (k \to 0)$, $S_2 = (0 \to k)$ $\quad$ $\dim S_1 =: \varepsilon_1$, $\dim S_2 =: \varepsilon_2$.

$I = (k \xrightarrow{1} k)$, $\dim I = \varepsilon_1 + \varepsilon_2$.

For $\alpha = d_1\varepsilon_1 + d_2\varepsilon_2$

$G_\alpha = GL(d_1, k) \times GL(d_2, k)$

A rep'n of $\dim = \alpha$ is given by a linear map $f: k^{d_1} \to k^{d_2}$.

Obviously: $\mathrm{rk} f =: r \leqslant \inf(d_1, d_2)$.

Orbits: $\mathcal{O}^\alpha_r$ are determined by $r \in \{0, 1, \dots, \inf(d_1, d_2)\}$.

Rep'n space: $E_\alpha = \bigsqcup_r \mathcal{O}^\alpha_r$; $\overline{\mathcal{O}^\alpha_r} = \bigsqcup_{s \leq r} \mathcal{O}^\alpha_s$.

Lemma: Any local system on $\mathcal{O}^\alpha_r$ is trivial.

Sk: $\mathrm{Stab}_{G_\alpha}(f)$ (for f in $\mathcal{O}^\alpha_r$) is: an elt in $\mathrm{Aut}(I^{\oplus r} \oplus S_1^{d_1-r} \oplus S_2^{d_2-r})$ and this group is connected [Indeed, it is a subgp in $\mathrm{End}(I^\square \oplus S_1^\square \oplus S_2^\square)$, hence it is stable under multiplication by $k^\times$, hence $0 \in \overline{Z}$ for any Z = connected component of Stabilizer, hence stab is connected] ∎

Cor: Any simple G_α-equivariant perverse sheaf on E_α is of the form $IC(\mathcal{O}^\alpha_r)$ for some r.

Hence anything in $\mathcal{P}_Q = \{IC(\dots) \mid \dots\}$ what conditions?

Eg: let $\alpha = \varepsilon_1 + \varepsilon_2$

Two Lusztig sheaves: $L_{\varepsilon_1, \varepsilon_2}$ & $L_{\varepsilon_2, \varepsilon_1}$.

sub rep'n of $\dim = \varepsilon_2$

$$E_{\varepsilon_1, \varepsilon_2} = \left\{ \begin{matrix} 0 \longrightarrow k \\ \uparrow \quad\ \uparrow f \\ k \longrightarrow k \end{matrix} \right\} = \{f : k \to k\} = k \qquad \left(\dim E_{\varepsilon_1+\varepsilon_2} + \dim E_{\varepsilon_1} + \dim E_{\varepsilon_2} = 1 + 0 + 0\right)$$

of a rep'n sth quotient has $\dim = \varepsilon_1$

Thus $q: E_{\varepsilon_1, \varepsilon_2} \xrightarrow{Id} E_{\varepsilon_1+\varepsilon_2} = k$

then $L_{\varepsilon_1, \varepsilon_2} = q_! \underline{k}_k [1] = \mathbb{1}_{\varepsilon_1+\varepsilon_2} = IC(\mathcal{O}_1)$

$$E_{\varepsilon_2, \varepsilon_1} = \left\{ \begin{matrix} k \longrightarrow 0 \\ \mathrm{id}\uparrow \quad \uparrow \\ k \longrightarrow k \end{matrix} \right\} = \{pt\} \quad \& \quad L_{\varepsilon_2, \varepsilon_1} = \underline{k}_{\{0\}} = IC(\mathcal{O}_0).$$

Eg: $\alpha = 2\varepsilon_1 + \varepsilon_2$: cf exercise session.

§4 Grothendieck group of $\mathcal{H}_Q$

Let $K_Q = \bigsqcup_{\alpha \in \mathbb{Z}^I} K^\alpha_Q$ w/ $K^\alpha_Q = K_0(\mathcal{H}_\alpha)$ (F should be $\mathcal{F}_1$, &c.)

For $F \in \mathcal{H}_Q$, set $[F] =: b_F$ in the Grothendieck group.

K_Q becomes a $\mathbb{Z}[v, v^{-1}]$-module by:

$$v^n b_F = b_{F[n]}$$

For $\alpha, \beta \in Q$, has $\begin{cases} m_{\alpha,\beta} : \mathcal{H}^\alpha_Q \boxtimes \mathcal{H}^\beta_Q \longrightarrow \mathcal{H}^{\alpha+\beta}_Q \\ \Delta_{\alpha,\beta} : \mathcal{H}^{\alpha+\beta}_Q \longrightarrow \mathcal{H}^\alpha_Q \boxtimes \mathcal{H}^\beta_Q \end{cases}$

these descend to $\begin{cases} m : K^\alpha_Q \times K^\beta_Q \longrightarrow K^{\alpha+\beta}_Q \\ \Delta : K^{\alpha+\beta}_Q \longrightarrow K_Q \otimes K_Q \end{cases}$ (respecting the grading).

Involution: $\bar{\ } : K_Q \longrightarrow K_Q$, $b_F \longmapsto b_{DF}$, $v \longmapsto v^{-1}$ $\quad$ | Geom pairing: $\{b_F, b_G\} = \{F, G\}$.

Th. [Lusztig] Set $q = v^{-1}$. There is an isomorphism:

$$\Phi : U_q^{\mathbb{Z}}(\mathfrak{n}^-) \xrightarrow{\sim} K_Q$$ (preserving product, coproduct & ~~...~~ geometric pairing)

$$E_i^{(n)} \longmapsto D\,\mathbb{1}_{n\epsilon_i}$$

Moreover, $\left(\Phi^{-1}(b_F)\right)_{F\in\mathcal{P}_Q}$ is the canonical basis.

Data.

Γ = quiver; I = vertices, H = arrows, $s,t : H \to I$

Construct a Kac-Moody algebra $\mathfrak{g}$ & qgp $U_q(\mathfrak{g})$.

$(\alpha_i)_{i\in I}$ simple roots, $K_0(\Gamma) = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ root lattice.

Th. (Ringel) Embedding $\Phi : U_q(\mathfrak{n}^-) \hookrightarrow H_\Gamma$ = Ringel-Hall algebra.

Def: $C_\Gamma := \mathrm{Im}\,\Phi$ — "composition algebra".

Convolution algebra.

Assume have 3 maps of sets
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[d, "q"] \arrow[dr, "r"] & \\ X & X & X \end{tikzcd}$$
sth r has finite fibers.

$\rightsquigarrow \mathcal{F}(X) = \{\text{functions } X \to \mathbb{C}\}$ endowed w/ convolution

$$f * g := r_*\left(p^*f \cdot q^*g\right) \qquad (f,g\in\mathcal{F}(X))$$

└ usual product of functions on Z.

Equivariant subalgebra

Assume a group G acts on X, [~~...~~ and] on Z and that p, q, r are G-equivariant.

$\rightsquigarrow \mathcal{F}_G(X) = \{ f : X \to \mathbb{C} : f(gx) = f(x)\ \forall g\in G, \forall x \in X\}$

Fact: $\mathcal{F}_G(X)$ is a subalgebra of $\mathcal{F}(X)$ — isomorphic to $\mathcal{F}(X/G)$.

Hence:
$$\begin{tikzcd} & Z \circlearrowleft G \arrow[dl, "p"'] \arrow[d, "q"] \arrow[dr, "r"] & \\ X & X & X \circlearrowleft G \end{tikzcd}$$
$\rightsquigarrow$ convolution algebra of functions on X/G.

Now: Fix k = finite field

$X = \mathrm{Rep}_k(\Gamma)$ = rep'ns of Γ over k

$X = \coprod_{\alpha\in Q^+} \mathrm{Rep}_{\alpha,k}(\Gamma)$ w/ $Q^+ = \sum_{i\in I}\mathbb{N}\alpha_i$

$Z =$

$$\mathcal{E} = \mathcal{E}_k(\Gamma) = \{(L \subset L'),\ L, L' \in X\} \circlearrowleft GL_k(\Gamma) \leftarrow \text{some infinite group you guess}$$

$$L \in X \qquad L'/L \in X \qquad L' \in X \circlearrowleft GL_k(\Gamma)$$

⇝ Hall algebra := convolution algebra of functions on $Rep_k(\Gamma)/GL_k(\Gamma)$
H'_Γ.

⇝ twist the product a little : get the $\boxed{\text{Ringel-Hall algebra } H_\Gamma}$.
└ renormalize : $q^{\langle \text{Euler form} \rangle}$ somewhere.

<u>Strange remark / question.</u>

Assume $\mathfrak{g}$ is f. dim. semi simple to simplify.
a) Associate Γ to $\mathfrak{g}$ (non unique way), then $Rep_k(\Gamma)$ to Γ.
Ringel: $U_q(\mathfrak{n}^-)$ is the convolution algebra of GL-equiv't fn's on $Rep_k(\Gamma)$

b) $\mathfrak{g}$ ⇝ G Lie group
Dixmier (?): $U(\mathfrak{g})$ = convolution algebra of G-equiv't generalized functions on G w/ support is concentrated on e.
[there is a non-G-equivariant algebraic version (Archipov)]

<u>Faisceau-fonction correspondence.</u>

Set $q = |k|$, X = variety / $\mathbb{F}_q$, $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} = \{ Rep_{\overline{\mathbb{F}_q}}(\Gamma)$
$\mathrm{Fr} \in Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ ⇝ Frobenius endom. of $\overline{X}$.

<u>Philosophy</u>: "natural" functions on $X = \overline{X}^{Fr}$ should be traces of Weil (perverse) sheaves in $D^b_c(\overline{X})$.

This suggest that H_Γ should be categorified in $D^b_{G_{\overline{\mathbb{F}_q}}(\Gamma), c}(\overline{X})$
($G_{\overline{\mathbb{F}}}(\Gamma)$-equivariant bounded derived cat'y of constructible sheaves on $\overline{X}$ (over $\overline{\mathbb{Q}_\ell}$ for some prime $\ell \nmid q$)

<u>Th.</u> (Lusztig)
1) Let Rep := {f.d. rep'ns of Γ over $\mathbb{C}$} $\circlearrowleft GL(\Gamma)$
Then $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{n}^-) \xrightarrow{\sim} K_0^{gr}(\mathcal{P})$ (Graded Grothendieck group of some subcat'y of $D^b_{GL,c}(\overline{X})$)

2) $\mathcal{P} \xrightarrow{\text{"trace map"}} C_\Gamma \simeq K_0^{gr}(\mathcal{P})$.

<u>Th 1</u>. Presentation by generators & relations of $\mathcal{P}$ — idempotent completion.
$$R : \left(\overline{\mathbb{Q}_\ell} \underset{\mathbb{Z}}{\otimes} \mathcal{B}(\Gamma)\right)^{id} \xrightarrow{\sim} \mathcal{P}$$
(for some explicitely defined monoidal category of $\mathcal{B}(\Gamma)$)

<u>Def.</u>: $\mathcal{B} = \mathcal{B}(\Gamma)$ = strict monoidal category generated by objects F_i ($i \in I$)
and arrows $\begin{cases} x_i : F_i \longrightarrow F_i \\ \tau_{ij} : F_i F_j \longrightarrow F_j F_i \end{cases}$ subject to some relations.

<u>Def</u> of $\mathcal{P}$? ~~Choose~~ For $\mu \in K^+ = Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$,
Set $\mathcal{E}^\mu = \{(L \subset L') : L, L' \in Rep,\ L'/L \in Rep^\mu\}$, so $\mathcal{E} = \coprod_\mu \mathcal{E}^\mu$.

Define $R(F_{i_1}^{(n_1)} \cdots F_{i_n}^{(n_n)})$ as follows.
$F_i^{(n)}$ = divided power of F_i^n in $\mathcal{B}^{id}$: $\left(F_i^{(n)}\right)^{\oplus n!} = F_i^n$

$$Rep \xleftarrow{\pi} Rep^\circ \underset{Rep}{\times} \mathcal{E}_{n_1 i_1} \underset{Rep}{\times} \cdots \underset{Rep}{\times} \mathcal{E}_{n_n i_n} = \{(0 = L_n \subset L_{n-1} \subset \cdots \subset L_1 : L_{k-1}/L_k \in Rep^{n_k i_k}\}$$
$L_1 \leftarrow\!\shortmid (L_k)$

$$F := R(F_{i_1}^{(n_1)} \cdots F_{i_n}^{(n_n)}) := \pi_! \overline{\mathbb{Q}_\ell}[\text{shift}]$$
— this is a semi-simple perverse sheaf in $D^b_{GL}(Rep)$.

<u>Def</u>: Let $\mathcal{P}$ be the smallest full additive subcat'y of $D^b_{GL}(Rep)$ closed under translations and ~~[illegible]~~ containing summands of objects of the form F (above)

Construction of the map in Th. 1.

$x_i \in End(F_i)$ "over" $\mathcal{G}^{\alpha_i}$.

$\tau_{ij} \in Hom(F_iF_j, F_jF_i)$ "over" $\mathcal{G}^{\alpha_i} \times_{Rep} \mathcal{G}^{\alpha_j}$ and $\mathcal{G}^{\alpha_j} \times_{Rep} \mathcal{G}^{\alpha_i}$.

{ Recall, $\mathcal{G}^{\alpha_i} = \{(L \subset L') : L'/L \in Rep^{\alpha_i}\}$ }

Thus have a ~~cano~~ canonical line bundle L'/L over $\mathcal{G}^{\alpha_i}$.

hence a 1st Chern class $c_1(L'/L) \in H^2(\mathcal{G}^{\alpha_i})$

hence $\tilde{x}_i \in Ext^2(F_i, F_i)$ w/ $\left\{ \begin{array}{l} F_i = p_!^{\alpha_i} \overline{\mathbb{Q}_\ell} [\ldots] \\ p^{\alpha} : \mathcal{G}^{\alpha} \longrightarrow Rep \times Rep. \end{array} \right.$

Rem.: $\tilde{x}_i$ gives $x_i \in Ext^2(R(F_i), R(F_i))$
by restricting $\tilde{x}_i$ to $Rep \simeq Rep^0 \times Rep \subset Rep \times Rep$.

Relations of $\mathcal{B}$ can be proven for $\tilde{x}_i$ directly.

For $i \neq j$: $\mathcal{G}^{\alpha_i} \times_{Rep} \mathcal{G}^{\alpha_j} = \{(L \subset L' \subset L'') : L''/L' \in Rep^{\alpha_j}, L'/L \in Rep^{\alpha_i}\}$

$$\begin{tikzcd}
 & \mathcal{G}^{\alpha_i} \times_{Rep} \mathcal{G}^{\alpha_j} \arrow[dr, "\text{obvious}"] & \\
\left\{ L \subset L'' \,\middle|\, L''/L \simeq \mathbb{C}_{\alpha_i} \oplus \mathbb{C}_{\alpha_j} \right\} = Z \arrow[rr, hook] & & \{L \subset L'' \mid L''/L \in Rep^{\alpha_i+\alpha_j}\} \\
 & \mathcal{G}^{\alpha_j} \times_{Rep} \mathcal{G}^{\alpha_i} \arrow[ur, "\text{obvious}"'] &
\end{tikzcd}$$

$[\, Z = \{(L \subset L') \mid L'/L \simeq \mathbb{C}_{\alpha_i} \oplus \mathbb{C}_{\alpha_j}\} \,]$

Now cs $p^{(\alpha_i, \alpha_j)} : \mathcal{G}^{\alpha_i} \times_{Rep} \mathcal{G}^{\alpha_j} \longrightarrow Rep \times Rep$ & $F^{\alpha_i, \alpha_j} := p_!^{(\alpha_i, \alpha_j)} \mathbb{C} [shift]$

The previous diagram induces morphisms: $F^{\alpha_i, \alpha_j} \longrightarrow F^{\alpha_j, \alpha_i} [shift]$.

Geordie WILLIAMSON (w/ P. McNamara)

Really [say Ethan John]

Hard question in number theory: are there infinitely many prime numbers of the form:

* $n \in \mathbb{Z}_{\geq 0}$ — yes — Euclid.
* $an+b$, $a \wedge b = 1$ — yes — Dirichlet] additive group
* $2^n - 1$ (Mersenne) } — don't know.] $\mathbb{G}_m$ cases "the enemy"
* F_n (Fibonacci)

↳ big prime factors when not known prime

Def: Say n is r-almost prime if it contains at most r prime factors counted w/ multiplicity.

Ex {
* $\{\frac{1}{2} xy \mid \exists x, y, z \in \mathbb{Z}_{>0},$  $\}$
* entries of matrices in $\langle \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \rangle \subset GL_2(\mathbb{Z})$] $SL_2(\mathbb{C})$ examples

(Cayley graph expands a lot)

Affine sieve (Bourgain, Gamburd, Sarnak 2010)

Let $X \subset GL_n(\mathbb{Z})$ finite set, $\langle X \rangle$ generated gp, G = Zariski closure of $\langle X \rangle$ in $GL_n(\mathbb{C})$

Let $f : GL_n(\mathbb{Z}) \longrightarrow \mathbb{Z}$ polynomial.

Assume "no local destruction": $\forall q \geq 2$, $\gcd(q, f(x)) = 1$

"tori are the enemy": $R(G)$, the radical, contains no tori

Then $\exists r > 0$ s.th. $\{f(x), x \in \langle X \rangle\}$ has infinitely many r-~~almost~~-almost primes

Cs $\langle \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \rangle \subset GL_2(\mathbb{Z})$.

Rk: * tori are "thin" — sieving techniques won't work.

* the radical is a pb because can't conjugate / "expand".

<u>Hard question in rep'n theory:</u>

| p prime, $d \geq 1$; cs $\mathrm{Rep}(\mathbb{F}_p \mathfrak{S}_d)$

| Q: $\dim L(\lambda)$ (= simple module associated to a p-regular partition of d)?

— looks like FLT (useless in itself, though fruitful)

<u>Relevance of KLR algebras?</u>

Q = cyclic quiver w/ p vertices; vertex set I

Given $\alpha = \sum_{i \in I} \lambda_i\, i$ s.t. ~~…~~ let $\alpha = \sum_{i \in I} d_i$ (α = a dim'n vector),

For $d \geq 0$, set $H_d = \prod_{\substack{\alpha \\ \mathrm{ht}\,\alpha = d}} H_\alpha$ ← KLR algebra associated to α over $\mathbb{F}_p$.

<u>Brundan–Kleshchev isomorphism</u>

$$\begin{array}{ccc} \text{(explicit) (cyclotomic) quotient of } H_d & \xrightarrow{\sim} & \mathbb{F}_p \mathfrak{S}_d \\ \cup & & \cup \\ H_\alpha & \longrightarrow & \left(\begin{array}{c}\text{either a block} \\ \text{or zero}\end{array}\right) \end{array}$$

<u>Cor:</u> $\mathbb{F}_p \mathfrak{S}_d$ is graded!

Understanding projective modules for $\mathbb{F}_p \mathfrak{S}_d$ is a special case of understanding projectives for H_d.

~~The~~ <u>James conjecture</u> (1990) (prediction for the characters & $\dim L(\lambda)$ if $p > \sqrt{d}$) implies (he's lying badly) that ~~…~~ simple modules for $H_{d,\mathbb{Z}}$ should stay irreducible mod p if $p > \sqrt{d}$.

<u>Kleshchev–Ram conjecture</u> (~2010): If Q is a Dynkin quiver, representations of H_α are independent of characteristic.

("Baby James")

<u>Th</u> (W): Both conjectures are false.

In fact, BGS + McNamara ⟹ bad primes for A_n-quiver grow exponentially in n.

<u>Topology</u> (of the singularities we saw in the last four talks) (ah?)

. Q quiver w/ vertices I, ⇝ $\mathrm{Rep}\, Q$ (formal stack of rep'ns) $= \bigsqcup_\alpha \mathrm{Rep}_\alpha Q = \bigsqcup_{\alpha \in \sum_{i\in I} \lambda_i i} \prod_{i \to j} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) / \prod_i GL(\mathbb{C}^{d_i})$

. For $\underline{i} = (i_1, \ldots, i_m) \in I^m$,

cs $E_{\underline{i}}$ (smooth) $\xrightarrow{\pi_{\underline{i}}} \mathrm{Rep}_\alpha Q$ w/ $\alpha = \sum_{j \in I} i_j$

Fix a field of coefficients k. Co "Lusztig sheaves" $(\pi_{\underline{i}})_* \underline{k}_{E_{\underline{i}}}[?] =: \mathcal{L}_{\underline{i}}$

<u>Th</u> (Varagnolo–Vasserot, Rouquier if char $k = 0$; Maksimau $\forall k$)
Fix a dimension vector α, let $\mathrm{Seq}(\alpha) = \{(i_1, \ldots, i_m) : \sum_j i_j = \alpha\}$
Then: $\mathrm{Ext}^\bullet\left(\bigoplus_{\underline{i} \in \mathrm{Seq}(\alpha)} \mathcal{L}_{\underline{i}}\right) = H_\alpha$ and idempotents match.

~~KR conj …~~

<u>Cor:</u> James & ~~Kleshchev~~-Ram conjectures ⟺ (mean that) $\mathcal{L}_{\underline{i}}$'s should decompose the same way in char. p and char. 0.

This boils down to a question of torsion in cohomology of subvarieties in quiver rep'ns.

<u>Ex</u>: $\bullet \longleftarrow \bullet$ A_2, dim. vector $(2,2)$

Let $Z :=$ closure of the orbit of $(\mathbb{C} \xleftarrow{\sim} \mathbb{C}) \oplus (0 \leftarrow \mathbb{C}) \oplus (\mathbb{C} \leftarrow 0)$.

In a chart: ($\mathrm{rk}\, M \leq 0$) ⇕

cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ ⟶ $Z = \{M \in \mathcal{M}_2(\mathbb{C}) : \det M = 0\} = \{zy - xw\} \subset \mathbb{C}^4$.

Exercise: $H^*(Z \setminus \{0\}, \mathbb{Z}) =$

0	1	2	3	4	5
$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$

<u>Fact</u>: The fact that $H^*(Z \setminus \{0\}, \mathbb{Z})$ has no torsion confirms the KR-conjecture in this case.

<u>Ex 2</u>: $Q = \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l] \\ \bullet \arrow[r] & \bullet \arrow[u] \end{tikzcd}$ $\qquad \alpha = \begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix}$

Closure of orbit of $\begin{tikzcd} \mathbb{C}\oplus\mathbb{C} \arrow[d, "\sim"'] & \mathbb{C}\oplus\mathbb{C} \arrow[l, "\sim"'] \\ \mathbb{C}\oplus\mathbb{C} \arrow[r, "\sim"] & \mathbb{C}\oplus\mathbb{C} \arrow[u, "\beta"'] \end{tikzcd}$

Chart:

$$Z := \left\{ (M_i)_{i\in\mathbb{Z}/4\mathbb{Z}} \in \mathrm{Mat}_2(\mathbb{C})^4 \;\middle|\; \begin{array}{l} \forall i,\ \mathrm{rk}\, M_i = 1 \Leftrightarrow \det M_i = 0 \\ \cancel{M_i}\ M_i M_{i+1} = 0 \end{array} \right\}$$

$\uparrow$ obvious

$$\tilde{Z} := \left\{ \left((M_i)_{i\in\mathbb{Z}/4\mathbb{Z}}, (l_i)_{i\in\mathbb{Z}/4}\right) \in \mathcal{M}_2(\mathbb{C})^4 \times (\mathbb{P}^1)^4 \;\middle|\; \begin{array}{l} \text{same conditions} \\ + M_i l_i = 0 \end{array} \right\}$$

(very similar to Lusztig's resolution)

Much harder exercise:

$H^*(\tilde{Z} \setminus (\mathbb{P}^1)^4) =$

0	1	2	3	4	5	6	7	$\boxed{8}$	9 … 15
$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	$\mathbb{Z}^6$	0	$\mathbb{Z}^4$	0	$\boxed{\mathbb{Z}/2}$	…

This gives an ex. where a Lusztig sheaf decomposes differently in char. 2.

<u>Kashiwara - Saito</u>: Singularity Z ~~o~~ occurs in an A_5-quiver

$\Rightarrow$ counter-example to KR-conjecture.

<u>W.</u> Given an entry F in $\left\langle \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right\rangle_{\text{semi-group}}$,

one can produce(*) $\tilde{Z} \xrightarrow{\pi} Z$
$\qquad\qquad\qquad\qquad\quad\ \ 0 \in Z$ $\qquad$ (*) complicated way

sth $H^*(\tilde{Z} \setminus \pi^{-1}(0)) = \left[\quad \cdots \quad \mathbb{Z}/F\mathbb{Z} \right]$

$\boxed{\text{Q0}}$ Does not really understand geometrically.

— explains why he cares about F having large prime factors.

Finish up some more questions

$\boxed{\text{Q1}}$ Do the fibres of the maps $F_{\underline{i}} : E_{\underline{i}} \longrightarrow \mathrm{Rep}_\alpha Q$ have vanishing odd cohomology over $\mathbb{Z}$? Is H^{even} free?

[Maksimau: yes in type A or $\tilde{A}$. Not known for D_4?]

$\boxed{\text{Q2}}$ How different are the singularities in A_n and $\tilde{A}_m$ quivers?

[Idea: sth that happens in $\tilde{A}_m$ (w/ m small) might very well occur for A_n (w/ ~~large~~ large n)]