Equivariant cohomology and fixed points of smooth Calogero-Moser spaces

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CNRS (UMR 5149) - Université de Montpellier

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- $\mathbb{C}[h]/\langle h - \xi \rangle \otimes \mathrm{Rees}_{\mathscr{F}}(A) \simeq \begin{cases} A & \text{if } \xi \neq 0, \\ \mathrm{Grad}_{\mathscr{F}}(A) & \text{if } \xi = 0. \end{cases}$

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Theorem (Fong-Srinivasan, 1982)

Assume that $o(q \bmod \ell) = d$. Two unipotent characters ρ_λ and ρ_μ of $\mathbf{GL}_n(\mathbb{F}_q)$ lie in the same ℓ-block if and only if $\gamma_d(\lambda) = \gamma_d(\mu)$.

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Theorem (B.-Maksimau-Shan, 2017)

$$\varphi_\gamma(\mathscr{F}_i(\mathrm{Z}(\mathbb{C}\mathfrak{S}_n)) \subset \mathscr{F}_i(\mathrm{Z}(\mathbb{C}G(d,1,r)))$$

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$$(i.e. \quad (\mathrm{Id}_{\mathbb{C}[h]} \otimes \varphi_\gamma)(\mathrm{Rees}_{\mathscr{F}} \mathrm{Z}(\mathbb{C}\mathfrak{S}_n)) \subset \mathrm{Rees}_{\mathscr{F}} \mathrm{Z}(\mathbb{C}G(d,1,r)) \quad).$$

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$$\mathbf{H}_c = \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \qquad \text{(as a vector space)}$$

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Theorem (Gordon, 2003)

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$$\mathscr{Z}_c^{\mathbb{C}^\times} \stackrel{\sim}{\longleftrightarrow} \mathrm{Irr}(W)$$

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Remark - $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{X}^{\mathbb{C}^\times}) = \mathbb{C}[h] \otimes \mathrm{H}^{2\bullet}(\mathscr{X})$.

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Sketch of the proof. By folklore Theorem and Etingof-Ginzburg Theorem,

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An easy argument based on comparison of dimensions shows that it is sufficient to prove that

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$\Rightarrow$ equivariant Chern character $\mathrm{ch}(e\mathbf{H}_c \otimes_{\mathbb{C}W} E) \in \hat{\mathrm{H}}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_c)$.

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Theorem (Bellamy, 2009)

$$i^*_{\mathscr{Z}_c}(\mathrm{ch}(e\mathbf{H}_c \otimes_{\mathbb{C}W} E)) = \sum_{\chi \in \mathrm{Irr}(W)} q^{b_\chi} \frac{\langle \chi, \mathbb{C}[V]^{\mathrm{co}(W)} \otimes E \rangle^{\mathrm{gr}}_W}{\langle \chi, \mathbb{C}[V]^{\mathrm{co}(W)} \rangle^{\mathrm{gr}}_W} e^W_\chi$$

where $q = \exp(h)$ and $b_\chi = \mathrm{val}\langle \chi, \mathbb{C}[V]^{\mathrm{co}(W)} \rangle^{\mathrm{gr}}_W$.

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$\Rightarrow$ $(\bigstar)$ holds (!).

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Fix such a d-core γ. By functoriality of equivariant cohomology, we get two maps

$$\begin{array}{ccc}
\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_1(\mathfrak{S}_n)) & \xrightarrow{\varphi^a_\gamma} & \mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_{c_\gamma}(G(d,1,r_\gamma))) \\
\big\downarrow & & \big\downarrow \\
\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_1(\mathfrak{S}_n)^{\mathbb{C}^\times}) & \xrightarrow{\varphi^b_\gamma} & \mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_{c_\gamma}(G(d,1,r_\gamma))^{\mathbb{C}^\times})
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Using a theorem of Przezdziecki (2016), $\varphi_\gamma^b = \mathrm{Id}_{\mathbb{C}[h]} \otimes \varphi_\gamma$ (!).

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