Representation theory of the Mantaci-Reutenauer algebra

Cédric Bonnafé

CNRS (UMR 6623) - Université de Franche-Comté (Besançon)

Groups in Galway, May 2006

Contents

Contents

Contents

Contents

Contents

Let (W, S) be a finite Coxeter group:

$$W =< S | \forall\ s, s' \in S,\ s^2 = (ss')^{m_{ss'}} = 1 >$$

Length function: $\ell : W \to \mathbb{N}$

Let (W, S) be a finite Coxeter group:

$$W =< S | \forall\ s, s' \in S,\ s^2 = (ss')^{m_{ss'}} = 1 >$$

Length function: $\ell : W \to \mathbb{N}$

Parabolic subgroups: $I \subset S$, $W_I =< I >$, (W_I, I) is a finite Coxeter group.

Let (W, S) be a finite Coxeter group:

$$W =< S | \forall\ s, s' \in S,\ s^2 = (ss')^{m_{ss'}} = 1 >$$

Length function: $\ell : W \to \mathbb{N}$

Parabolic subgroups: $I \subset S$, $W_I =< I >$, (W_I, I) is a finite Coxeter group.

$$X_I = \{x \in W \mid \forall\ w \in W_I,\ \ell(xw) \geqslant \ell(x)\}$$

$$\begin{array}{c} X_I \xrightarrow{\sim} W/W_I \\ x \mapsto xW_I \end{array}$$

Let $X_{IJ} = (X_I)^{-1} \cap X_J$

- $$\begin{aligned} X_{IJ} &\xrightarrow{\sim} W_I \backslash W / W_J \\ d &\mapsto W_I d W_J \end{aligned}$$
- If $d \in X_{IJ}$, then $W_I \cap {}^d W_J = W_{I \cap {}^d J}$
- $X_J = \coprod_{d \in X_{IJ}} X^I_{I \cap {}^d J} . d$

Let $X_{IJ} = (X_I)^{-1} \cap X_J$

- $$\begin{aligned} X_{IJ} &\xrightarrow{\sim} W_I \backslash W / W_J \\ d &\mapsto W_I d W_J \end{aligned}$$
- If $d \in X_{IJ}$, then $W_I \cap {}^d W_J = W_{I \cap {}^d J}$
- $X_J = \coprod_{d \in X_{IJ}} X^I_{I \cap {}^d J}.d$

Solomon (1976): $x_I = \sum_{w \in X_I} w \quad \in \mathbb{Z}W$

$$\Sigma(W) := \bigoplus_{I \subset S} \mathbb{Z}x_I \quad \subset \mathbb{Z}W$$

$$\begin{aligned} \theta : \quad \Sigma(W) &\longrightarrow \mathbb{Z}\,\mathrm{Irr}\, W \\ x_I &\longmapsto \mathrm{Ind}_{W_I}^W 1_I \end{aligned}$$

Let $X_{IJ} = (X_I)^{-1} \cap X_J$

- $$\begin{aligned} X_{IJ} &\xrightarrow{\sim} W_I \backslash W / W_J \\ d &\mapsto W_I d W_J \end{aligned}$$
- If $d \in X_{IJ}$, then $W_I \cap {}^d W_J = W_{I \cap {}^d J}$
- $X_J = \coprod_{d \in X_{IJ}} X^I_{I \cap {}^d J}.d$

Solomon (1976): $x_I = \sum_{w \in X_I} w \quad \in \mathbb{Z}W$

$$\Sigma(W) := \bigoplus_{I \subset S} \mathbb{Z} x_I \quad \subset \mathbb{Z}W$$

$$\begin{array}{rccc} \theta : & \Sigma(W) & \longrightarrow & \mathbb{Z}\,\mathrm{Irr}\,W \\ & x_I & \longmapsto & \mathrm{Ind}_{W_I}^W 1_I \end{array}$$

$$x_I x_J = \sum_{d \in X_{IJ}} x_I x^I_{I \cap {}^d J} d = \sum_{d \in X_{IJ}} x_{I \cap {}^d J} d = \sum_{d \in X_{IJ}} x_{{}^{d^{-1}} I \cap J}$$

Theorem (Solomon).

- $\Sigma(W)$ is a subalgebra of $\mathbb{Z}W$.
- θ is a morphism of algebras.
- $\operatorname{Ker}\theta = \sum_{I \equiv J} \mathbb{Z}(x_I - x_J)$.
- $\mathbb{Q}\operatorname{Ker}\theta$ is the radical of $\mathbb{Q}\Sigma(W)$.
- $\theta(w_0) = \varepsilon$.

$I \equiv J \Leftrightarrow W_I \sim W_J$.

w_0 is the longest element of W

Further works:

- Idempotents (Bergeron-Bergeron-Howlett-Taylor)
- Cartan matrix unitriangular (?)
- Modular representations (Atkinson-Pfeiffer-Van Willigenburg): simples, radical...
- Lie idempotents (Reutenauer, Erdmann-Schocker)
- Symmetry property (Blessenhohl-Hohlweg-Schocker): $\theta(x_I)(x_J) = \theta(x_J)(x_I)$
- Complex reflection groups (Mathas)
- Loewy length (B.-Pfeiffer): all cases except type D_{2n+1}
- ...

Further works:

- Idempotents (Bergeron-Bergeron-Howlett-Taylor)
- Cartan matrix unitriangular (?)
- Modular representations (Atkinson-Pfeiffer-Van Willigenburg): simples, radical...
- Lie idempotents (Reutenauer, Erdmann-Schocker)
- Symmetry property (Blessenhohl-Hohlweg-Schocker): $\theta(x_I)(x_J) = \theta(x_J)(x_I)$
- Complex reflection groups (Mathas)
- Loewy length (B.-Pfeiffer): all cases except type D_{2n+1}
- ...

(The **Loewy length** of an algebra A is the minimal natural number $k \geqslant 1$ such that $(\mathrm{Rad}\, A)^k = 0$)

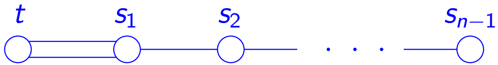
Theorem (B.-Pfeiffer, 2005). Let $\boldsymbol{\sigma}$ denote the automorphism of W induced by conjugacy by w_0. Then the Loewy length of $\mathbb{Q}\Sigma(W)^\sigma$ is $\lceil |S|/2 \rceil$.

Problem: θ is surjective if and only if W is a product of symmetric groups.

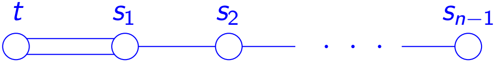
Contents

Let (W_n, S_n) be a Coxeter group of type B_n.

Let (W_n, S_n) be a Coxeter group of type B_n.

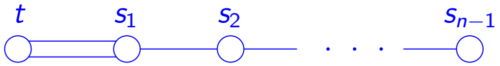


Let (W_n, S_n) be a Coxeter group of type B_n.



$$t_1 = t, \qquad t_{i+1} = s_i t_i s_i$$

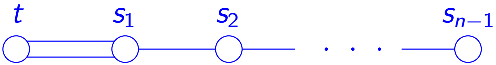
Let (W_n, S_n) be a Coxeter group of type B_n.



$$t_1 = t, \qquad t_{i+1} = s_i t_i s_i$$

$$I_n = \{1, 2, \dots, n\} \cup \{-1, -2, \dots, -n\}$$
$$W_n = \{\sigma : I_n \xrightarrow{\sim} I_n \mid \forall\ i \in I_n,\ \sigma(-i) = -\sigma(i)\}$$
$$s_i = (i, i+1)(-i, -i-1) \qquad t_i = (i, -i)$$

Let (W_n, S_n) be a Coxeter group of type B_n.



$$t_1 = t, \qquad t_{i+1} = s_i t_i s_i$$

$$I_n = \{1, 2, \ldots, n\} \cup \{-1, -2, \ldots, -n\}$$
$$W_n = \{\sigma : I_n \xrightarrow{\sim} I_n \mid \forall\ i \in I_n,\ \sigma(-i) = -\sigma(i)\}$$
$$s_i = (i, i+1)(-i, -i-1) \qquad t_i = (i, -i)$$

Let $\mathcal{T}_n : \mathbb{Z}W_n \to \mathbb{Z}$: canonical symmetrizing form

Contents

- $S_{-n} = \{s_1, \ldots, s_{n-1}\}$, $\qquad W_{-n} =< S_{-n} >\simeq \mathfrak{S}_n$

- $S_{-n} = \{s_1, \ldots, s_{n-1}\}$, $\qquad W_{-n} =< S_{-n} > \simeq \mathfrak{S}_n$
- $S'_n = S_n \cup \{t_1, \ldots, t_n\}$

$$W_n = \mathfrak{S}_n \ltimes < t_1, \ldots, t_n >$$

- $S_{-n} = \{s_1, \ldots, s_{n-1}\}$, $\qquad W_{-n} =< S_{-n} >\simeq \mathfrak{S}_n$
- $S'_n = S_n \cup \{t_1, \ldots, t_n\}$

$$W_n = \mathfrak{S}_n \ltimes < t_1, \ldots, t_n >$$

- $\mathrm{Comp}(n) = \{\text{signed compositions of } n\}$

$$|\mathrm{Comp}(n)| = 2.3^{n-1}$$

- $C = (c_1, \ldots, c_r)$, $\qquad |c_1| + \cdots + |c_r| = n$

$$W_C \simeq W_{c_1} \times \cdots \times W_{c_r} \subset W_n$$

- $S'_C = S'_n \cap W_C \quad \Rightarrow \quad W_C =< S'_C >$

Contents

- $X_C = \{x \in W_n \mid \forall\ w \in W_C,\ \ell(xw) \geqslant \ell(x)\}$
- $$\begin{aligned} X_C &\xrightarrow{\sim} W_n/W_C \\ x &\mapsto xW_C \end{aligned}$$

- $X_C = \{x \in W_n \mid \forall\, w \in W_C,\ \ell(xw) \geqslant \ell(x)\}$
- $$\begin{aligned} X_C &\xrightarrow{\sim} W_n/W_C \\ x &\mapsto xW_C \end{aligned}$$
- $X_{CD} = (X_C)^{-1} \cap X_D$
- $$\begin{aligned} X_{CD} &\xrightarrow{\sim} W_C\backslash W_n/W_D \\ d &\mapsto W_C d W_D \end{aligned}$$
- If $d \in X_{CD}$, then $W_C \cap {}^d W_D = W_{C \cap {}^d D}$

- $X_C = \{x \in W_n \mid \forall\ w \in W_C,\ \ell(xw) \geqslant \ell(x)\}$
- $$\begin{array}{c} X_C \xrightarrow{\sim} W_n/W_C \\ x \mapsto xW_C \end{array}$$
- $X_{CD} = (X_C)^{-1} \cap X_D$
- $$\begin{array}{c} X_{CD} \xrightarrow{\sim} W_C\backslash W_n/W_D \\ d \mapsto W_C d W_D \end{array}$$
- If $d \in X_{CD}$, then $W_C \cap {}^d W_D = W_{C \cap {}^d D}$

BUT (in general)

$$X_D \neq \coprod_{d \in X_{CD}} X^C_{C \cap {}^d D}.d$$

Let $x_C = \sum_{w \in X_C} w \quad \in \mathbb{Z}W_n$

Let $\Sigma'(W_n) := \bigoplus_{C \in \mathrm{Comp}(n)} \mathbb{Z}x_C \quad \subset \mathbb{Z}W_n$

Let

$$\begin{array}{rccc} \theta' : & \Sigma'(W_n) & \longrightarrow & \mathbb{Z}\,\mathrm{Irr}\,W_n \\ & x_C & \longmapsto & \mathrm{Ind}_{W_C}^{W_n} 1_C \end{array}$$

Let $x_C = \sum_{w \in X_C} w \quad \in \mathbb{Z}W_n$

Let $\Sigma'(W_n) := \bigoplus_{C \in \mathrm{Comp}(n)} \mathbb{Z}x_C \quad \subset \mathbb{Z}W_n$

Let

$$\begin{array}{rccc} \theta' : & \Sigma'(W_n) & \longrightarrow & \mathbb{Z}\,\mathrm{Irr}\,W_n \\ & x_C & \longmapsto & \mathrm{Ind}_{W_C}^{W_n} 1_C \end{array}$$

Theorem (B.-Hohlweg, 2004).

- $\Sigma'(W_n)$ is a subalgebra of $\mathbb{Z}W_n$.
- θ' is a surjective morphism of algebras.
- $\mathrm{Ker}\,\theta' = \sum_{C \equiv D} \mathbb{Z}(x_C - x_D)$.
- $\mathbb{Q}\,\mathrm{Ker}\,\theta'$ is the radical of $\mathbb{Q}\Sigma'(W_n)$.
- $\Sigma'(W_n) \simeq$ Mantaci-Reutenauer algebra.
- $\mathcal{T}_n(xy) = \langle \theta'(x), \theta'(y) \rangle$.

- $x_{-1,1}x_{-2} = 2x_{-1,-1} + 2(x_{-1,1} - x_{1,-1})$

- $x_{-1,1}x_{-2} = 2x_{-1,-1} + 2(x_{-1,1} - x_{1,-1})$
- If C is **parabolic** (i.e. $c_i < 0$ if $i \geqslant 2$) or if D is **almost positive** (i.e. $d_i \geqslant -1$), then

$$X_D = \coprod_{d \in X_{CD}} X^C_{C \cap {}^d D} . d$$

- $x_{-1,1}x_{-2} = 2x_{-1,-1} + 2(x_{-1,1} - x_{1,-1})$
- If C is **parabolic** (i.e. $c_i < 0$ if $i \geqslant 2$) or if D is **almost positive** (i.e. $d_i \geqslant -1$), then

$$X_D = \coprod_{d \in X_{CD}} X^C_{C \cap^d D} . d$$

- $\Sigma(W_n) \subset \Sigma'(W_n)$
- $\Sigma(\mathfrak{S}_n) \simeq \bigoplus_{C \in \mathrm{Comp}^+(n)} \mathbb{Z} x_C \subset \Sigma'(W_n)$

- $x_{-1,1}x_{-2} = 2x_{-1,-1} + 2(x_{-1,1} - x_{1,-1})$
- If C is **parabolic** (i.e. $c_i < 0$ if $i \geqslant 2$) or if D is **almost positive** (i.e. $d_i \geqslant -1$), then

$$X_D = \coprod_{d \in X_{CD}} X^C_{C \cap {}^d D}.d$$

- $\Sigma(W_n) \subset \Sigma'(W_n)$
- $\Sigma(\mathfrak{S}_n) \simeq \bigoplus_{C \in \mathrm{Comp}^+(n)} \mathbb{Z}x_C \subset \Sigma'(W_n)$

Notation: $C \subset D \quad \Leftrightarrow \quad W_C \subset W_D$

$$\Sigma'(W_D) = \bigoplus_{C \subset D} \mathbb{Z}x_C^D$$

- $x_{-1,1}x_{-2} = 2x_{-1,-1} + 2(x_{-1,1} - x_{1,-1})$
- If C is **parabolic** (i.e. $c_i < 0$ if $i \geqslant 2$) or if D is **almost positive** (i.e. $d_i \geqslant -1$), then

$$X_D = \coprod_{d \in X_{CD}} X^C_{C \cap {}^d D} . d$$

- $\Sigma(W_n) \subset \Sigma'(W_n)$
- $\Sigma(\mathfrak{S}_n) \simeq \bigoplus_{C \in \mathrm{Comp}^+(n)} \mathbb{Z} x_C \subset \Sigma'(W_n)$

Notation: $C \subset D \quad \Leftrightarrow \quad W_C \subset W_D$

$$\Sigma'(W_D) = \bigoplus_{C \subset D} \mathbb{Z} x^D_C$$

If $D = (d_1, \dots, d_r)$, then $\Sigma'(W_D) \simeq \Sigma'(W_{d_1}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \Sigma'(W_{d_r})$, where $\Sigma'(\mathfrak{S}_n) = \Sigma(\mathfrak{S}_n)$.

- $x_{-1,1}x_{-2} = 2x_{-1,-1} + 2(x_{-1,1} - x_{1,-1})$
- If C is **parabolic** (i.e. $c_i < 0$ if $i \geqslant 2$) or if D is **almost positive** (i.e. $d_i \geqslant -1$), then

$$X_D = \coprod_{d \in X_{CD}} X^C_{C \cap {}^d D} . d$$

- $\Sigma(W_n) \subset \Sigma'(W_n)$
- $\Sigma(\mathfrak{S}_n) \simeq \bigoplus_{C \in \mathrm{Comp}^+(n)} \mathbb{Z} x_C \subset \Sigma'(W_n)$

Notation: $C \subset D \quad \Leftrightarrow \quad W_C \subset W_D$

$$\Sigma'(W_D) = \bigoplus_{C \subset D} \mathbb{Z} x^D_C$$

If $D = (d_1, \ldots, d_r)$, then $\Sigma'(W_D) \simeq \Sigma'(W_{d_1}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \Sigma'(W_{d_r})$, where $\Sigma'(\mathfrak{S}_n) = \Sigma(\mathfrak{S}_n)$.

Remark: $x_D x^D_C = x_C$.

Contents

- cox_C: Coxeter element of W_C

- cox_C: Coxeter element of W_C
- $\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$

- cox_C: Coxeter element of W_C
- $\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$ (Example: $\boldsymbol{\lambda}(3,-1,-3,1,-2,-2,4) = (431;3221)$)

- cox_C: Coxeter element of W_C
- $\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$ (Example: $\boldsymbol{\lambda}(3,-1,-3,1,-2,-2,4) = (431;3221)$)
- $\mathrm{cox}_C \sim \mathrm{cox}_D \Leftrightarrow W_C \simeq W_D \Leftrightarrow C \equiv D \Leftrightarrow \boldsymbol{\lambda}(C) = \boldsymbol{\lambda}(D)$
- $\mathcal{C}_\lambda$: conjugacy class of cox_C if $\boldsymbol{\lambda}(C) = \lambda$; $\quad \mathrm{cox}_\lambda \in \mathcal{C}_\lambda$.

- cox_C: Coxeter element of W_C
- $\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$ (Example: $\boldsymbol{\lambda}(3,-1,-3,1,-2,-2,4) = (431;3221)$)
- $\mathrm{cox}_C \sim \mathrm{cox}_D \Leftrightarrow W_C \simeq W_D \Leftrightarrow C \equiv D \Leftrightarrow \boldsymbol{\lambda}(C) = \boldsymbol{\lambda}(D)$
- $\mathcal{C}_\lambda$: conjugacy class of cox_C if $\boldsymbol{\lambda}(C) = \lambda$; $\mathrm{cox}_\lambda \in \mathcal{C}_\lambda$.
- $$\begin{array}{ccc} \mathrm{Bip}(n) & \stackrel{\sim}{\longrightarrow} & W_n/\sim \\ \lambda & \longmapsto & \mathcal{C}_\lambda \end{array}$$

- cox_C: Coxeter element of W_C
- $\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$ (Example: $\boldsymbol{\lambda}(3,-1,-3,1,-2,-2,4) = (431;3221)$)
- $\mathrm{cox}_C \sim \mathrm{cox}_D \Leftrightarrow W_C \simeq W_D \Leftrightarrow C \equiv D \Leftrightarrow \boldsymbol{\lambda}(C) = \boldsymbol{\lambda}(D)$
- $\mathcal{C}_\lambda$: conjugacy class of cox_C if $\boldsymbol{\lambda}(C) = \lambda$; $\mathrm{cox}_\lambda \in \mathcal{C}_\lambda$.
- $\begin{array}{ccc} \mathrm{Bip}(n) & \xrightarrow{\sim} & W_n/\sim \\ \lambda & \longmapsto & \mathcal{C}_\lambda \end{array}$

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & -8 & 6 & -11 & 12 & -7 & -1 & 9 & 2 & -10 & -5 & -4 \end{pmatrix}$$

- cox_C: Coxeter element of W_C
- $\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$ (Example: $\boldsymbol{\lambda}(3,-1,-3,1,-2,-2,4) = (431;3221)$)
- $\mathrm{cox}_C \sim \mathrm{cox}_D \Leftrightarrow W_C \simeq W_D \Leftrightarrow C \equiv D \Leftrightarrow \boldsymbol{\lambda}(C) = \boldsymbol{\lambda}(D)$
- $\mathcal{C}_\lambda$: conjugacy class of cox_C if $\boldsymbol{\lambda}(C) = \lambda$; $\mathrm{cox}_\lambda \in \mathcal{C}_\lambda$.
- $$\begin{array}{ccc} \mathrm{Bip}(n) & \xrightarrow{\sim} & W_n/\sim \\ \lambda & \mapsto & \mathcal{C}_\lambda \end{array}$$

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & -8 & 6 & -11 & 12 & -7 & -1 & 9 & 2 & -10 & -5 & -4 \end{pmatrix}$$

- cox_C: Coxeter element of W_C
- $\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$ (Example: $\boldsymbol{\lambda}(3,-1,-3,1,-2,-2,4) = (431;3221)$)
- $\mathrm{cox}_C \sim \mathrm{cox}_D \Leftrightarrow W_C \simeq W_D \Leftrightarrow C \equiv D \Leftrightarrow \boldsymbol{\lambda}(C) = \boldsymbol{\lambda}(D)$
- $\mathcal{C}_\lambda$: conjugacy class of cox_C if $\boldsymbol{\lambda}(C) = \lambda$; $\quad \mathrm{cox}_\lambda \in \mathcal{C}_\lambda$.
- $$\begin{array}{ccc} \mathrm{Bip}(n) & \xrightarrow{\sim} & W_n/\sim \\ \lambda & \mapsto & \mathcal{C}_\lambda \end{array}$$

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & -8 & 6 & -11 & 12 & -7 & -1 & 9 & 2 & -10 & -5 & -4 \end{pmatrix}$$

$$w \in \mathcal{C}_{431;4}$$

- Let

$$\begin{array}{rccc} \pi_\lambda : & \Sigma'(W_n) & \longrightarrow & \mathbb{Z} \\ & x & \longmapsto & \theta'(x)(\mathrm{cox}_\lambda) \end{array}$$

This is a morphism of algebras.

- Let

$$\begin{array}{rccc} \pi_\lambda : & \Sigma'(W_n) & \longrightarrow & \mathbb{Z} \\ & x & \longmapsto & \theta'(x)(\mathrm{cox}_\lambda) \end{array}$$

  This is a morphism of algebras.
- If R is a commutative ring, $\pi_\lambda^R : R\Sigma'(W_n) \to R$ and $\mathcal{D}_\lambda^R$ is the $R\Sigma'(W_n)$-module which is R-free of rank 1 and on which $R\Sigma'(W_n)$ acts through π_λ^R.

- Let

$$\pi_\lambda : \begin{array}{ccc} \Sigma'(W_n) & \longrightarrow & \mathbb{Z} \\ x & \longmapsto & \theta'(x)(\mathrm{cox}_\lambda) \end{array}$$

This is a morphism of algebras.

- If R is a commutative ring, $\pi_\lambda^R : R\Sigma'(W_n) \to R$ and $\mathcal{D}_\lambda^R$ is the $R\Sigma'(W_n)$-module which is R-free of rank 1 and on which $R\Sigma'(W_n)$ acts through π_λ^R.

$$\mathrm{Irr}\,\mathbb{Q}\Sigma'(W_n) = \{\pi_\lambda^{\mathbb{Q}} \mid \lambda \in \mathrm{Bip}(n)\}$$

- Let
$$\begin{array}{rccc} \pi_\lambda : & \Sigma'(W_n) & \longrightarrow & \mathbb{Z} \\ & x & \longmapsto & \theta'(x)(\mathrm{cox}_\lambda) \end{array}$$
This is a morphism of algebras.
- If R is a commutative ring, $\pi_\lambda^R : R\Sigma'(W_n) \to R$ and $\mathcal{D}_\lambda^R$ is the $R\Sigma'(W_n)$-module which is R-free of rank 1 and on which $R\Sigma'(W_n)$ acts through π_λ^R.

$$\boxed{\mathrm{Irr}\,\mathbb{Q}\Sigma'(W_n) = \{\pi_\lambda^{\mathbb{Q}} \mid \lambda \in \mathrm{Bip}(n)\}}$$

Example: character table of $\mathbb{Q}\Sigma'(W_2)$

- Let
$$\begin{array}{rccc} \pi_\lambda : & \Sigma'(W_n) & \longrightarrow & \mathbb{Z} \\ & x & \longmapsto & \theta'(x)(\mathrm{cox}_\lambda) \end{array}$$
This is a morphism of algebras.
- If R is a commutative ring, $\pi_\lambda^R : R\Sigma'(W_n) \to R$ and $\mathcal{D}_\lambda^R$ is the $R\Sigma'(W_n)$-module which is R-free of rank 1 and on which $R\Sigma'(W_n)$ acts through π_λ^R.

$$\boxed{\mathrm{Irr}\,\mathbb{Q}\Sigma'(W_n) = \{\pi_\lambda^{\mathbb{Q}} \mid \lambda \in \mathrm{Bip}(n)\}}$$

Example: character table of $\mathbb{Q}\Sigma'(W_2)$

	x_2	$x_{\bar{2}}$	$x_{1,1}$	$x_{1,\bar{1}}$	$x_{\bar{1},\bar{1}}$
$\pi_{2;\varnothing}^{\mathbb{Q}}$	1	.	.	.	.
$\pi_{\varnothing;2}^{\mathbb{Q}}$	1	2	.	.	.
$\pi_{11;\varnothing}^{\mathbb{Q}}$	1	.	2	.	.
$\pi_{1;1}^{\mathbb{Q}}$	1	.	2	2	.
$\pi_{\varnothing;11}^{\mathbb{Q}}$	1	4	2	4	8

Contents

$$x_n = 1 = \sum_{\lambda \in \mathrm{Bip}(n)} E_\lambda$$

$$(E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda)$$

such that $\mathcal{P}_\lambda^{\mathbb{Q}} := \mathbb{Q}\Sigma'(W_n)E_\lambda$ is the projective cover of $\mathcal{D}_\lambda^{\mathbb{Q}}$

$$x_n = 1 = \sum_{\lambda \in \mathrm{Bip}(n)} E_\lambda$$

$$(E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda)$$

such that $\mathcal{P}_\lambda^{\mathbb{Q}} := \mathbb{Q}\Sigma'(W_n)E_\lambda$ is the projective cover of $\mathcal{D}_\lambda^{\mathbb{Q}}$
Note that

$\theta'(E_\lambda)$ is the characteristic function of $\mathcal{C}_\lambda$

$$x_n = 1 = \sum_{\lambda \in \mathrm{Bip}(n)} E_\lambda$$

$$(E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda)$$

such that $\mathcal{P}_\lambda^{\mathbb{Q}} := \mathbb{Q}\Sigma'(W_n)E_\lambda$ is the projective cover of $\mathcal{D}_\lambda^{\mathbb{Q}}$

Note that

$\theta'(E_\lambda)$ is the characteristic function of $\mathcal{C}_\lambda$

Moreover,

$$\dim_{\mathbb{Q}} \mathbb{Q}W_n E_\lambda = |\mathcal{C}_\lambda|$$

$$(\text{Indeed, } \dim_{\mathbb{Q}} \mathbb{Q}W_n E_\lambda = |W_n| \mathcal{T}_n(E_\lambda) = |W_n| \langle \theta'(E_\lambda), \theta'(1) \rangle)$$

$$x_n = 1 = \sum_{\lambda \in \mathrm{Bip}(n)} E_\lambda$$

$$(E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda)$$

such that $\mathcal{P}_\lambda^{\mathbb{Q}} := \mathbb{Q}\Sigma'(W_n)E_\lambda$ is the projective cover of $\mathcal{D}_\lambda^{\mathbb{Q}}$
Note that

$\theta'(E_\lambda)$ is the characteristic function of $\mathcal{C}_\lambda$

Moreover,

$$\dim_{\mathbb{Q}} \mathbb{Q}W_n E_\lambda = |\mathcal{C}_\lambda|$$

$$(\text{Indeed, } \dim_{\mathbb{Q}} \mathbb{Q}W_n E_\lambda = |W_n|\mathcal{T}_n(E_\lambda) = |W_n|\langle \theta'(E_\lambda), \theta'(1)\rangle)$$

Question: $\mathbb{Q}W_n E_\lambda \simeq \mathrm{Ind}_{C_{W_n}(\mathrm{cox}_\lambda)}^{W_n} \xi_\lambda$ for some linear character ξ_λ of $C_{W_n}(\mathrm{cox}_\lambda)$? (true for $n \leqslant 5$)

	$\mathcal{D}^{\mathbb{Q}}_{3;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{21;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;21}$	$\mathcal{D}^{\mathbb{Q}}_{1;1^2}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;3}$	$\mathcal{D}^{\mathbb{Q}}_{2;1}$	$\mathcal{D}^{\mathbb{Q}}_{1;2}$	$\mathcal{D}^{\mathbb{Q}}_{1^2;1}$	$\mathcal{D}^{\mathbb{Q}}_{1^3;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;1^3}$
$\mathcal{P}^{\mathbb{Q}}_{3;\varnothing}$	1	1	1	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{21;\varnothing}$	.	1	.	.	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;21}$	.	.	1	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{1;1^2}$	.	.	.	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;3}$	.	.	.	.	1	1	1	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{2;1}$	.	.	.	.	.	1	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{1;2}$	.	.	.	.	.	.	1	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{1^2;1}$	.	.	.	.	.	.	.	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{1^3;\varnothing}$	.	.	.	.	.	.	.	.	1	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;1^3}$	.	.	.	.	.	.	.	.	.	1

	4	31	$\varnothing$	$\varnothing$	21^2	2	1	1^2	$\varnothing$	3	1	2	21	1^2	$\varnothing$	1	1^3	2^2	1^4	$\varnothing$
	$\varnothing$	$\varnothing$	31	2^2	$\varnothing$	1^2	21	1^2	4	1	3	2	1	2	21^2	1^3	1	$\varnothing$	$\varnothing$	1^4
$4;\varnothing$	1	1	1	.	1	1	2	1	.	.	.	.	.	.	.	.	.	.	.	.
$31;\varnothing$	.	1	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\varnothing;31$	.	.	1	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\varnothing;2^2$	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$21^2;\varnothing$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$2;1^2$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$1;21$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$1^2;1^2$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\varnothing;4$	.	.	.	.	.	.	.	.	1	1	1	1	2	1	1	1	1	.	.	.
$3;1$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	1	1	.	.	.	.
$1;3$	.	.	.	.	.	.	.	.	.	.	1	.	1	1	.	.	1	.	.	.
$2;2$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
$21;1$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$1^2;2$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.
$\varnothing;21^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
$1;1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
$1^3;1$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
$2^2;\varnothing$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
$1^4;\varnothing$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
$\varnothing;1^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Theorem (B., 2005). Let λ, $\mu \in \mathrm{Bip}(n)$.

- $[\mathcal{P}_\lambda^{\mathbb{Q}} : \mathcal{D}_\lambda^{\mathbb{Q}}] = 1$
- If $\lambda \neq \mu$ and $[\mathcal{P}_\lambda^{\mathbb{Q}} : \mathcal{D}_\mu^{\mathbb{Q}}] \neq 0$, then
  - $\mathbf{lg}(\lambda) > \mathbf{lg}(\mu)$
  - $\mathbf{lg}(\lambda^-) \equiv \mathbf{lg}(\mu^-) \mod 2$

Theorem (B., 2005). Let $\lambda, \mu \in \mathrm{Bip}(n)$.

- $[\mathcal{P}_\lambda^{\mathbb{Q}} : \mathcal{D}_\lambda^{\mathbb{Q}}] = 1$
- If $\lambda \neq \mu$ and $[\mathcal{P}_\lambda^{\mathbb{Q}} : \mathcal{D}_\mu^{\mathbb{Q}}] \neq 0$, then
  - $\mathbf{lg}(\lambda) > \mathbf{lg}(\mu)$
  - $\mathbf{lg}(\lambda^-) \equiv \mathbf{lg}(\mu^-) \mod 2$

(Note that $\pi_\lambda(w_0) = (-1)^{n-\mathbf{lg}(\lambda^-)}$)

Contents

Let D be almost positive and let

$$\mathrm{Res}_D\, x_C := \sum_{d \in X_{CD}} x^D_{{}^{d^{-1}}C \cap D} \in \Sigma'(W_D) = \bigoplus_{C \subset D} \mathbb{Z} x^D_C$$

Let D be almost positive and let

$$\mathrm{Res}_D\, x_C := \sum_{d \in X_{CD}} x^D_{{}^{d^{-1}}C \cap D} \in \Sigma'(W_D) = \bigoplus_{C \subset D} \mathbb{Z} x^D_C$$

Then $x x_D = x_D \,\mathrm{Res}_D(x)$.
(Note that the map $\mathbb{Z}W_D \to \mathbb{Z}W_n$, $a \mapsto x_D a$ is injective)

Let D be almost positive and let

$$\mathrm{Res}_D\, x_C := \sum_{d \in X_{CD}} x^D_{{}^{d^{-1}}C \cap D} \in \Sigma'(W_D) = \bigoplus_{C \subset D} \mathbb{Z} x^D_C$$

Then $x x_D = x_D \,\mathrm{Res}_D(x)$.
(Note that the map $\mathbb{Z}W_D \to \mathbb{Z}W_n$, $a \mapsto x_D a$ is injective)

Theorem (B., 2005).

- Res_D is a morphism of algebras.
- $\theta'_D \circ \mathrm{Res}_D = \mathrm{Res}^{W_n}_{W_D} \circ \theta'$
- $\mathbb{Q}\Sigma'(W_n) = \mathrm{Ker}(\mathrm{Res}^{\mathbb{Q}}_D) \oplus \mathbb{Q}\Sigma'(W_n) x_D$

	4	31	$\varnothing$	$\varnothing$	21^2	2	1	1^2	$\varnothing$	3	1	2	21	1^2	$\varnothing$	1	1^3	2^2	1^4	$\varnothing$
	$\varnothing$	$\varnothing$	31	2^2	$\varnothing$	1^2	21	1^2	4	1	3	2	1	2	21^2	1^3	1	$\varnothing$	$\varnothing$	1^4
$4; \varnothing$	1	1	1	.	1	1	2	1	.	.	.	.	.	.	.	.	.	.	.	.
$31; \varnothing$	.	1	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\varnothing; 31$	.	.	1	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\varnothing; 2^2$	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$21^2; \varnothing$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$2; 1^2$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$1; 21$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$1^2; 1^2$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\varnothing; 4$	.	.	.	.	.	.	.	.	1	1	1	1	2	1	1	1	1	.	.	.
$3; 1$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	1	1	.	.	.	.
$1; 3$	.	.	.	.	.	.	.	.	.	.	1	.	1	1	.	.	1	.	.	.
$2; 2$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
$21; 1$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$1^2; 2$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.
$\varnothing; 21^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
$1; 1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
$1^3; 1$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
$2^2; \varnothing$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
$1^4; \varnothing$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
$\varnothing; 1^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

	4	31	∅	∅	21^2	**2**	**1**	**1^2**	∅	**3**	1	2	**21**	1^2	**∅**	**1**	**1^3**	2^2	1^4	**∅**
	∅	∅	**31**	2^2	∅	**1^2**	**21**	**1^2**	4	**1**	3	2	**1**	2	**21^2**	**1^3**	**1**	∅	∅	**1^4**
4; ∅	1	1	1	.	1	1	2	1	.	.	.	.	.	.	.	.	.	.	.	.
31; ∅	.	1	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
∅; 31	.	.	**1**	.	.	**1**	**1**	**1**	.	.	.	.	.	.	.	.	.	.	.	.
∅; 2^2	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
21^2; ∅	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2; 1^2	.	.	.	.	.	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1; 21	.	.	.	.	.	.	**1**	**1**	.	.	.	.	.	.	.	.	.	.	.	.
1^2; 1^2	.	.	.	.	.	.	.	**1**	.	.	.	.	.	.	.	.	.	.	.	.
∅; 4	.	.	.	.	.	.	.	.	1	1	1	1	2	1	1	1	1	.	.	.
3; 1	.	.	.	.	.	.	.	.	.	**1**	.	.	**1**	.	**1**	**1**	.	.	.	.
1; 3	.	.	.	.	.	.	.	.	.	.	1	.	1	1	.	.	1	.	.	.
2; 2	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
21; 1	.	.	.	.	.	.	.	.	.	.	.	.	**1**	.	.	.	.	.	.	.
1^2; 2	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.
∅; 21^2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**	**1**	.	.	.	.
1; 1^3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**	.	.	.	.
1^3; 1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**	.	.	.
2^2; ∅	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
1^4; ∅	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
∅; 1^4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**

	$\mathcal{D}^{\mathbb{Q}}_{3;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{21;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;21}$	$\mathcal{D}^{\mathbb{Q}}_{1;1^2}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;3}$	$\mathcal{D}^{\mathbb{Q}}_{2;1}$	$\mathcal{D}^{\mathbb{Q}}_{1;2}$	$\mathcal{D}^{\mathbb{Q}}_{1^2;1}$	$\mathcal{D}^{\mathbb{Q}}_{1^3;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;1^3}$
$\mathcal{P}^{\mathbb{Q}}_{3;\varnothing}$	1	1	1	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{21;\varnothing}$	.	1	.	.	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;21}$	.	.	1	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{1;1^2}$	.	.	.	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;3}$	.	.	.	.	1	1	1	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{2;1}$	.	.	.	.	.	1	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{1;2}$	.	.	.	.	.	.	1	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{1^2;1}$	.	.	.	.	.	.	.	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{1^3;\varnothing}$	.	.	.	.	.	.	.	.	1	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;1^3}$	.	.	.	.	.	.	.	.	.	1

The natural map $W_{n-1} \hookrightarrow W_n$ induces an **injective** map $\tau_n : \mathrm{Bip}(n-1) \to \mathrm{Bip}(n)$. In fact:

$$\tau_n((\lambda_1^+, \ldots, \lambda_r^+), (\lambda_1^-, \ldots, \lambda_s^-)) = ((\lambda_1^+, \ldots, \lambda_r^+), (\lambda_1^-, \ldots, \lambda_s^-, 1)).$$

The natural map $W_{n-1} \hookrightarrow W_n$ induces an **injective** map $\tau_n : \mathrm{Bip}(n-1) \to \mathrm{Bip}(n)$. In fact:

$$\tau_n((\lambda_1^+, \dots, \lambda_r^+), (\lambda_1^-, \dots, \lambda_s^-)) = ((\lambda_1^+, \dots, \lambda_r^+), (\lambda_1^-, \dots, \lambda_s^-, 1)).$$

Theorem (B., 2006).
$\mathrm{Res}_{n-1,-1} : \mathbb{Q}\Sigma'(W_n) \to \mathbb{Q}\Sigma'(W_{n-1})$ is surjective.

The natural map $W_{n-1} \hookrightarrow W_n$ induces an **injective** map $\tau_n : \mathrm{Bip}(n-1) \to \mathrm{Bip}(n)$. In fact:

$$\tau_n((\lambda_1^+, \dots, \lambda_r^+), (\lambda_1^-, \dots, \lambda_s^-)) = ((\lambda_1^+, \dots, \lambda_r^+), (\lambda_1^-, \dots, \lambda_s^-, 1)).$$

Theorem (B., 2006).
$\mathrm{Res}_{n-1,-1} : \mathbb{Q}\Sigma'(W_n) \to \mathbb{Q}\Sigma'(W_{n-1})$ is surjective.

It is probable that $\mathrm{Res}_{n-1,-1} : \Sigma'(W_n) \to \Sigma'(W_{n-1})$ is also surjective (this is true for $n \leqslant 5$). Note that $\mathrm{Res}_{W_{n-1}}^{W_n} : \mathbb{Z}\,\mathrm{Irr}\,W_n \to \mathbb{Z}\,\mathrm{Irr}\,W_{n-1}$ is surjective.

The natural map $W_{n-1} \hookrightarrow W_n$ induces an **injective** map $\tau_n : \mathrm{Bip}(n-1) \to \mathrm{Bip}(n)$. In fact:

$$\tau_n((\lambda_1^+, \ldots, \lambda_r^+), (\lambda_1^-, \ldots, \lambda_s^-)) = ((\lambda_1^+, \ldots, \lambda_r^+), (\lambda_1^-, \ldots, \lambda_s^-, 1)).$$

Theorem (B., 2006). $\mathrm{Res}_{n-1,-1} : \mathbb{Q}\Sigma'(W_n) \to \mathbb{Q}\Sigma'(W_{n-1})$ is surjective.

It is probable that $\mathrm{Res}_{n-1,-1} : \Sigma'(W_n) \to \Sigma'(W_{n-1})$ is also surjective (this is true for $n \leqslant 5$). Note that $\mathrm{Res}_{W_{n-1}}^{W_n} : \mathbb{Z}\,\mathrm{Irr}\, W_n \to \mathbb{Z}\,\mathrm{Irr}\, W_{n-1}$ is surjective.

"Corollary". If λ, $\mu \in \mathrm{Bip}(n-1)$, then $[\mathcal{P}_\lambda^{\mathbb{Q}}, \mathcal{D}_\mu^{\mathbb{Q}}] = [\mathcal{P}_{\tau_n(\lambda)}, \mathcal{D}_{\tau_n(\mu)}^{\mathbb{Q}}]$.

Contents

Let $\mathrm{Bip}_{p'}(n)$ be the set of bipartitions λ such that cox_λ is p-regular.

Let $\mathrm{Bip}_{p'}(n)$ be the set of bipartitions λ such that cox_λ is p-regular.

If $\lambda \in \mathrm{Bip}(n)$, let $\lambda_{p'}$ denote the unique bipartition in $\mathrm{Bip}_{p'}(n)$ such that $\mathrm{cox}_{\lambda_{p'}}$ is conjugate to the p'-part of cox_λ.

Let $\mathrm{Bip}_{p'}(n)$ be the set of bipartitions λ such that cox_λ is p-regular.

If $\lambda \in \mathrm{Bip}(n)$, let $\lambda_{p'}$ denote the unique bipartition in $\mathrm{Bip}_{p'}(n)$ such that $\mathrm{cox}_{\lambda_{p'}}$ is conjugate to the p'-part of cox_λ.

Let $\mathrm{Comp}_p(n) := \{C \in \mathrm{Comp}(n) \mid p \text{ divides } |N_{W_n}(W_C)/W_C|\}$.

Let $\mathrm{Bip}_{p'}(n)$ be the set of bipartitions λ such that cox_λ is p-regular.

If $\lambda \in \mathrm{Bip}(n)$, let $\lambda_{p'}$ denote the unique bipartition in $\mathrm{Bip}_{p'}(n)$ such that $\mathrm{cox}_{\lambda_{p'}}$ is conjugate to the p'-part of cox_λ.

Let $\mathrm{Comp}_p(n) := \{C \in \mathrm{Comp}(n) \mid p \text{ divides } |N_{W_n}(W_C)/W_C|\}$.

Theorem (B., 2005).

- $\pi_\lambda^{\mathbb{F}_p} = \pi_\mu^{\mathbb{F}_p} \Leftrightarrow \lambda_{p'} = \mu_{p'}$
- $\mathrm{Irr}\,\mathbb{F}_p\Sigma'(W_n) = \{\pi_\lambda^{\mathbb{F}_p} \mid \lambda \in \mathrm{Bip}_{p'}(n)\}$
- $\mathrm{Rad}\,\mathbb{F}_p\Sigma'(W_n) = \sum_{C \equiv D} \mathbb{F}_p(x_C - x_D) + \sum_{C \in \mathrm{Comp}_p(n)} \mathbb{F}_p x_C$

Contents

Let $\Delta_n(p)$ denote the matrix $(\delta_{\lambda_{p'},\mu})_{\lambda \in \mathrm{Bip}(n), \mu \in \mathrm{Bip}_{p'}(n)}$. This is the decomposition matrix from $\mathbb{Q}\Sigma'(W_n)$ to $\mathbb{F}_p\Sigma'(W_n)$.

Let $\Delta_n(p)$ denote the matrix $(\delta_{\lambda_{p'},\mu})_{\lambda\in\mathrm{Bip}(n),\mu\in\mathrm{Bip}_{p'}(n)}$. This is the decomposition matrix from $\mathbb{Q}\Sigma'(W_n)$ to $\mathbb{F}_p\Sigma'(W_n)$.

By a general result of Geck and Rouquier, we have

$$\mathrm{Cartan}(\mathbb{F}_p\Sigma'(W_n)) = {}^t\Delta_n(p) \times \mathrm{Cartan}(\mathbb{Q}\Sigma'(W_n)) \times \Delta_n(p)$$

Contents

Theorem (B., 2006). Let K be a field of characteristic $p \geqslant 0$. Then the Loewy length of $K \operatorname{Irr} W_n$ is equal to

$$\begin{cases} 1, & \text{if } p = 0; \\ n+1, & \text{if } p = 2; \\ [n/p]+1, & \text{if } p > 2. \end{cases}$$

Contents

Theorem (B., 2006). Let K be a field of characteristic $p \geqslant 0$. Then the Loewy length of $K\Sigma'(W_n)$ is equal to

$$\begin{cases} n, & \text{if } p \neq 2; \\ 2, & \text{if } p = 2 \text{ and } n = 1; \\ 2n-1 \quad ?, & \text{if } p = 2 \text{ and } n \geqslant 2. \end{cases}$$