Characters of finite reductive groups: a survey

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Oberwolfach, April 2007

Contents

Contents

Contents

Contents

- $\mathbf{GL}_{2,3,4}(\mathbb{F}_q)$: Steinberg 1952
- $\mathbf{GL}_n(\mathbb{F}_q)$: Green 1955
- $\mathbf{Sp}_4(\mathbb{F}_q)$: Srinivasan 1968
- Deligne-Lusztig 1976
- Lusztig's conjecture 1990

- $\mathbf{G} = \mathbf{GL}_2(\overline{\mathbb{F}}_q)$, $F : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^q & b^q \\ c^q & d^q \end{pmatrix}$.
- $\mathbf{G}^F = \mathbf{GL}_2(\mathbb{F}_q)$
- $\mathbf{B}_0 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, $\mathbf{T}_0 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$, $\mathbf{T}_0^F \simeq \mathbb{F}_q^\times \times \mathbb{F}_q^\times$.

For α, $\beta \in \mathbb{F}_q^\times$, let $\theta_{\alpha,\beta}$:

$$\begin{array}{ccc} \mathbf{B}_0^F & \longrightarrow & \overline{\mathbb{Q}}_\ell^\times \\ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} & \longmapsto & \alpha(a)\beta(b) \end{array}$$

By the Mackey formula (and the Bruhat decomposition $\mathbf{G}^F = \mathbf{B}_0^F \coprod \mathbf{B}_0^F \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathbf{B}_0^F$), we have

$$\langle \mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} \theta_{\alpha,\beta}, \mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} \theta_{\alpha,\beta} \rangle = \begin{cases} 1 & \text{if } \alpha \neq \beta \\ 2 & \text{if } \alpha = \beta \end{cases}$$

- $\mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} \theta_{1,1} = \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F] = \overline{\mathbb{Q}}_\ell[\mathbf{P}^1(\mathbb{F}_q)] = 1_{\mathbf{G}} + \mathrm{St}_{\mathbf{G}}$, where $\mathrm{St}_{\mathbf{G}}$ is the Steinberg character.
- $\mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} \theta_{\alpha,\alpha} = (\alpha \circ \det) + \mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$

What about the others?

Several constructions: Steinberg ($\mathbf{GL}_{2,3,4}(\mathbb{F}_q)$, 1952), Green ($\mathbf{GL}_n(\mathbb{F}_q)$, 1955), Drinfeld (geometric, 1974)

$\longrightarrow$ parametrized by orbits (under $? \mapsto ?^q$) of $\omega \in (\mathbb{F}_{q^2}^\times)^\wedge$ such that $\omega \neq \omega^q$.

Contents

Harish-Chandra Theory

- $\mathbf{P} = \mathbf{LV}$ $\quad F$-stable Levi decomposition of an F-stable parabolic subgroup
- Harish-Chandra induction

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : \begin{array}{ccc} \overline{\mathbb{Q}}_\ell \mathbf{L}^F - \mathrm{mod} & \longrightarrow & \overline{\mathbb{Q}}_\ell \mathbf{G}^F - \mathrm{mod} \\ M & \longmapsto & \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \tilde{M} \end{array}$$

  where $\tilde{M}$ is the lift of M to $\mathbf{P}^F$

- Harish-Chandra restriction

$${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : \begin{array}{ccc} \overline{\mathbb{Q}}_\ell \mathbf{G}^F - \mathrm{mod} & \longrightarrow & \overline{\mathbb{Q}}_\ell \mathbf{L}^F - \mathrm{mod} \\ M & \longmapsto & M^{\mathbf{V}^F} \end{array}$$

- They are adjoint, transitive...

- An irreducible module M (resp. character γ) is called cuspidal if, for all **proper** F-stable parabolic subgroup $\mathbf{P} = \mathbf{LV}$, we have ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} M = 0$ (resp. ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \gamma = 0$).
- $\mathrm{CuspPairs}(\mathbf{G}, F) = \{(\mathbf{L}, \lambda) \mid \mathbf{L}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup of $\mathbf{G}$ and λ is an irreducible cuspidal character of $\mathbf{L}^F\}$.
- $\mathrm{Irr}(\mathbf{G}^F, \mathbf{L}, \lambda) = \{\chi \in \mathrm{Irr}\,\mathbf{G}^F \mid \langle \chi, R^{\mathbf{G}}_{\mathbf{L}} \lambda \rangle \neq 0\}$

Theorem (Harish-Chandra Theory)

$$\mathrm{Irr}\,\mathbf{G}^F = \coprod_{(\mathbf{L},\lambda)\in \mathrm{CuspPairs}(\mathbf{G},F)/\sim_{\mathbf{G}^F}} \mathrm{Irr}(\mathbf{G}^F, \mathbf{L}, \lambda)$$

Contents

An example: $\mathrm{Irr}(\mathbf{G}^F, \mathbf{T}_0, 1)$

We have $\mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} 1 = \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F]$.

$$\mathrm{End}_{\mathbf{G}^F} \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F] \simeq \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F]^* \otimes \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F]$$

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&\simeq \overline{\mathbb{Q}}_\ell[(\mathbf{G}^F/\mathbf{B}_0^F \times \mathbf{G}^F/\mathbf{B}_0^F)/\mathbf{G}^F]
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&\simeq \overline{\mathbb{Q}}_\ell[W^F]
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as a vector space!

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&\simeq \overline{\mathbb{Q}}_\ell[W^F]
\end{aligned}$$

as a vector space!

but also as an algebra!!! (Tits deformation theorem)

We get a bijection

$$\begin{array}{ccc} \operatorname{Irr} W^F & \longrightarrow & \operatorname{Irr}(\mathbf{G}^F, \mathbf{T}_0, 1) \\ \chi & \longmapsto & \gamma_\chi \end{array}$$

such that

$$\operatorname{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} 1 = \sum_{\chi \in \operatorname{Irr} W^F} \chi(1)\gamma_\chi.$$

Sub-example: $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$, $W = W^F = \mathfrak{S}_n$

$$\{\text{partitions of } n\} \longleftrightarrow \operatorname{Irr} \mathfrak{S}_n \longleftrightarrow \operatorname{Irr}(\mathbf{G}^F, \mathbf{T}_0, 1)$$

$$\lambda \vdash n \qquad \longmapsto \qquad \gamma_\lambda$$

$$\gamma_\lambda(u_\mu) = 0 \qquad \text{unless } \mu \trianglelefteq \lambda$$

- u_μ: unipotent element associated to μ
- $\trianglelefteq$: dominance order

Contents

- $(\mathbf{L}, \lambda) \in \mathrm{CuspPairs}(\mathbf{G}, F)$
- Let $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) = N_{\mathbf{G}^F}(\mathbf{L}, \lambda)/\mathbf{L}^F$

Theorem (Howlett-Lehrer, 1979).

$$\mathrm{End}_{\mathbf{G}^F} R_{\mathbf{L}}^{\mathbf{G}} \text{"}\lambda\text{"} \simeq \overline{\mathbb{Q}}_\ell^{[\omega]} W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$$

where ω is a 2-cocycle.

- **Lusztig (1984):** if the centre of $\mathbf{G}$ is connected, then ω is trivial
- **Geck (1992):** ω is always trivial

$$\mathrm{Irr}(\mathbf{G}^F, \mathbf{L}, \lambda) \longleftrightarrow \mathrm{Irr}\, W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$$

Contents

Drinfeld's example (1974):

- $\mathcal{C} = \{(x,y) \in \mathbf{A}^2(\overline{\mathbb{F}}_q) \mid (xy^q - yx^q)^{q-1} = 1\}$.
- $\mathbf{GL}_2(\mathbb{F}_q)$ acts linearly on $\mathbf{A}^2(\overline{\mathbb{F}}_q)$ and stabilizes $\mathcal{C}$.
- $\mathbb{F}_{q^2}^\times$ acts on $\mathbf{A}^2(\overline{\mathbb{F}}_q)$ by multiplication and stabilizes $\mathcal{C}$.
- These two actions commute.
- Let $\theta \in (\mathbb{F}_{q^2}^\times)^\wedge$ be such that $\theta \neq \theta^q$.
- Drinfeld proved that the θ-isotypic component of $H_c^1(\mathcal{C}, \overline{\mathbb{Q}}_\ell)$ is an irreducible cuspidal $\mathbf{G}^F$-module (he was 19 years-old!).

Contents

Deligne-Lusztig varieties:

- $\mathbf{P} = \mathbf{L}\mathbf{V}$ Levi decomposition with $F(\mathbf{L}) = \mathbf{L}$ but $\mathbf{P}$ is not necessarily F-stable!

$$\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V}\cdot F(\mathbf{V})\}.$$

- $\mathbf{G}^F$ acts on the left, $\mathbf{L}^F$ acts on the right; these actions commute.
- Deligne-Lusztig induction

$$\begin{array}{rccc} R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : & \mathbb{Z}\,\mathrm{Irr}\,\mathbf{L}^F & \longrightarrow & \mathbb{Z}\,\mathrm{Irr}\,\mathbf{G}^F \\ & [M] & \longmapsto & \sum(-1)^i[H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}},\overline{\mathbb{Q}}_\ell)\otimes_{\overline{\mathbb{Q}}_\ell\mathbf{L}^F} M] \end{array}$$

- Deligne-Lusztig restriction

$$\begin{array}{rccc} {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : & \mathbb{Z}\,\mathrm{Irr}\,\mathbf{G}^F & \longrightarrow & \mathbb{Z}\,\mathrm{Irr}\,\mathbf{L}^F \\ & [M] & \longmapsto & \sum(-1)^i[H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}},\overline{\mathbb{Q}}_\ell)^*\otimes_{\overline{\mathbb{Q}}_\ell\mathbf{G}^F} M] \end{array}$$

Examples:

(1) If $F(\mathbf{P}) = \mathbf{P}$, then $F(\mathbf{V}) = \mathbf{V}$ and
$\mathbf{Y}_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}} = (\mathbf{G}/\mathbf{V})^F = \mathbf{G}^F/\mathbf{V}^F$ is a finite set.
Then Deligne-Lusztig induction = Harish-Chandra induction

(2) $\mathbf{G}^F = \mathbf{GL}_2(\mathbb{F}_q)$: there is an F-stable maximal torus $\mathbf{T}'$ of a non-F-stable Borel subgroup $\mathbf{B}'$ of $\mathbf{G}$ such that $\mathbf{T}'^F \simeq \mathbb{F}_{q^2}^\times$ and $\mathbf{Y}_{\mathbf{T}' \subset \mathbf{B}'}^{\mathbf{G}} \simeq \boldsymbol{\mathcal{C}}$ (Drinfeld's variety).
Then, if $\theta \in (\mathbb{F}_{q^2}^\times)^\wedge = (\mathbf{T}'^F)^\wedge$ is such that $\theta \neq \theta^q$, then
$H_c^2(\boldsymbol{\mathcal{C}}, \overline{\mathbb{Q}}_\ell)_\theta = 0$ and

$$[H_c^1(\boldsymbol{\mathcal{C}}, \overline{\mathbb{Q}}_\ell)_\theta] = -R_{\mathbf{T}'}^{\mathbf{G}}(\theta)$$

Contents

Lusztig series:

- $(\mathbf{G}^*, F^*)$ dual to $(\mathbf{G}, F)$ (dual root system)
- $\nabla(\mathbf{G}, F) = \{(\mathbf{T}, \theta) \mid \mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in (\mathbf{T}^F)^\wedge\}$.
- $\nabla^*(\mathbf{G}, F) = \{(\mathbf{T}^*, s) \mid \mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ and $s \in \mathbf{T}^{*F^*}\}$.
- **Deligne-Lusztig:** $\nabla(\mathbf{G}, F)/\sim_{\mathbf{G}^F} \longleftrightarrow \nabla^*(\mathbf{G}, F)/\sim_{\mathbf{G}^{*F^*}}$.
  $\longrightarrow$ we set $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ if $(\mathbf{T}, \theta)$ is associated to $(\mathbf{T}^*, s)$.
- Let $\mathcal{E}(\mathbf{G}^F, s) = \{$irreducible components of virtual characters of the form $R_{\mathbf{T}^*}^{\mathbf{G}}(t)$ where $(\mathbf{T}^*, t) \in \nabla^*(\mathbf{G}, F)$ and s and t are $\mathbf{G}^{*F^*}$-conjugate$\}$: Lusztig series

Theorem (Deligne-Lusztig, 1976).

$$\operatorname{Irr}\mathbf{G}^F = \coprod_{s \in \mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim} \mathcal{E}(\mathbf{G}^F, s).$$

Theorem (Lusztig, 1976-77-78-79-80-81-82-83-84).

(a) If the centre of $\mathbf{G}$ is connected, then there is a bijection $\mathcal{E}(\mathbf{G}^F, s) \longleftrightarrow \mathcal{E}(\mathbf{G}(s)^F, 1)$ where $\mathbf{G}(s)$ is dual to $C_{\mathbf{G}^*}(s)$: **Jordan decomposition**

(b) Parametrization of $\mathcal{E}(\mathbf{G}^F, 1)$ by a set depending only on (W, F), not on q.

Remark - The Jordan decomposition "commutes" with Harish-Chandra theory

Contents

Unipotent characters

$$\mathcal{E}(\mathbf{G}^F, 1) = \{\text{unipotent characters}\}$$

If $w \in W$, let

$$\mathbf{X}(w) = \{g\mathbf{B}_0 \in \mathbf{G}/\mathbf{B}_0 \mid g^{-1}F(g) \in \mathbf{B}_0 w \mathbf{B}_0\}.$$

Then $\mathbf{G}^F$ acts on the left on $\mathbf{X}(w)$. Let

$$\mathcal{R}_w = \sum_{i \geqslant 0} (-1)^i [H_c^i(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell)] \in \mathbb{Z} \operatorname{Irr} \mathbf{G}^F.$$

Fact: $\gamma \in \operatorname{Irr} \mathbf{G}^F$ is unipotent if and only if it occurs in some $\mathcal{R}_w$.

Remarks - (1) $\mathbf{X}(1) = (\mathbf{G}/\mathbf{B}_0)^F = \mathbf{G}^F/\mathbf{B}_0^F$ so $\mathcal{R}_1 = \operatorname{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} 1$.

So all the $(\boldsymbol{\gamma}_\chi)_{\chi \in \operatorname{Irr} W^F}$ are unipotent characters.

(2) $\sum_{w \in W} \mathcal{R}_w = |W| 1_{\mathbf{G}^F}$.

Almost characters

From now on, we assume that (G, F) is split, so that F acts trivially on W.

$$\Rightarrow \quad \langle \mathcal{R}_w, \mathcal{R}_{w'} \rangle = \begin{cases} 0 & \text{if } w \not\sim w' \\ |C_W(w)| & \text{if } w \sim w' \end{cases} \qquad \text{(Deligne-Lusztig)}$$

If χ is a class function on W, let

$$R_\chi = \frac{1}{|W|} \sum_{w \in W} \chi(w) \mathcal{R}_w.$$

Then $\langle R_\chi, R_\psi \rangle = \langle \chi, \psi \rangle$. So, if $\chi \in \mathrm{Irr}\, W$, then $\langle R_\chi, R_\chi \rangle = 1$. But, in general, $R_\chi \neq \boldsymbol{\gamma}_\chi$!!!

If $\chi \in \mathrm{Irr}\, W$, R_χ is called an almost character.
Lusztig has constructed other (unipotent) almost characters: he has determined the transition matrix between almost characters and irreducible characters (diagonal by blocks with "small" blocks).

What happens in $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$?

$$\gamma_\chi = R_\chi$$

and

$$\mathcal{E}(\mathbf{G}^F, 1) = \{R_\chi \mid \chi \in \operatorname{Irr} W\}.$$

In general, character values of almost characters are much nicer than those of characters. Explanation (?): theory of character sheaves.

Contents

- Let $\mathbf{X}$ be a variety and $F : \mathbf{X} \longrightarrow \mathbf{X}$ be a Frobenius endomorphism
- Let $D^b(\mathbf{X})$ denote the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves.
- Let $\mathcal{D}_{\mathbf{X}} : D^b(\mathbf{X}) \longrightarrow D^b(\mathbf{X})$ be the Verdier duality.
- A bounded complex of $\overline{\mathbb{Q}}_\ell$-sheaves K is called perverse if $\dim \operatorname{supp}(\mathcal{H}^i K) \leqslant -i$ for all i and similarly for $\mathcal{D}_{\mathbf{X}}(K)$.
- Let $\mathcal{M}(\mathbf{X})$ denote the category of perverse sheaves: it is an abelian category.
- If C is a bounded complex of $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbf{X}$ and if $\tau : F^* C \longrightarrow C$ is an isomorphism, then, if $x \in \mathbf{X}$, then τ_x induces an isomorphism $(F^*C)_x = C_{F(x)} \simeq C_x$. If moreover $x \in \mathbf{X}^F$, we set

$$\mathcal{X}_{C,\tau}(x) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}(\mathcal{H}^i(\tau_x), \mathcal{H}^i(C_x)).$$

$\mathcal{X}_{C,\tau}$ is called the characteristic function of (C, t).

Contents

Grothendieck map:

- Let $\tilde{\mathbf{G}} = \{(g, x\mathbf{B}_0) \mid g \in x\mathbf{B}_0 x^{-1}\}$
- Let $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ be the second projection (Grothendieck map). It is projective.
- $K = R\pi_* \overline{\mathbb{Q}}_\ell[\dim \mathbf{G}]$ is a bounded complex of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbf{G}$, $\mathbf{G}$-equivariant.
- The canonical isomorphism $F^*\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell$ induces (by the proper base change theorem), an isomorphism $\varphi : F^*K \to K$.
- K is a perverse sheave (because π is a small map).
- By the Proper Base Change Theorem and Lefchetz Theorem,

$$\begin{aligned}\mathcal{X}_{K,\varphi}(g) &= \sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(F, H_c^i(\pi^{-1}(g), \overline{\mathbb{Q}}_\ell)) \\ &= |\pi^{-1}(g)^F| \\ &= (\operatorname{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} 1)(g).\end{aligned}$$

$$\mathcal{X}_{K,\varphi} = \operatorname{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} 1.$$

Decomposition of K:

$\mathbf{G}_{\mathrm{reg}} = \{$regular semisimple elements$\}$, $\tilde{\mathbf{G}}_{\mathrm{reg}} = \pi^{-1}(\mathbf{G}_{\mathrm{reg}})$.
The map

$$\begin{array}{ccc} \mathbf{G}/\mathbf{T}_0 \times (\mathbf{T}_0)_{\mathrm{reg}} & \longrightarrow & \tilde{\mathbf{G}}_{\mathrm{reg}} \\ (g\mathbf{T}_0, t) & \longmapsto & (gtg^{-1}, g\mathbf{B}_0) \end{array}$$

is an isomorphism. So W acts on $\tilde{\mathbf{G}}_{\mathrm{reg}}$ and $\pi_{\mathrm{reg}} : \tilde{\mathbf{G}}_{\mathrm{reg}} \to \mathbf{G}_{\mathrm{reg}}$ is a Galois étale covering with group W. Moreover,

$$K = IC(\mathbf{G}, (\pi_{\mathrm{reg}})_* \overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}].$$

But $(\pi_{\mathrm{reg}})_* \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\chi \in \mathrm{Irr}\, W} \mathcal{L}_\chi^{\oplus \chi(1)}$. So

$$K \simeq \bigoplus_{\chi \in \mathrm{Irr}\, W} K_\chi^{\oplus \chi(1)},$$

where $K_\chi = IC(\mathbf{G}, \mathcal{L}_\chi)[\dim \mathbf{G}]$. φ induces an isomorphism $\varphi_\chi : F^* K_\chi \to K_\chi$ and

$$\boxed{\mathcal{X}_{K_\chi, \varphi_\chi} = R_\chi}$$

Contents

There exists a family of "F-stable" $\mathbf{G}$-equivariant perverse sheaves on $\mathbf{G}$ which are indexed by the same set as the almost characters: the character sheaves. Their characteristic functions are class functions (by $\mathbf{G}$-equivariance) and form an orthonormal basis of the space of class functions on $\mathbf{G}^F$ (Lusztig).

Lusztig's conjecture: *The characteristic function of a character sheaf is, up to a root of unity, an almost character.*

Lusztig: Transition matrix between irreducible and almost characters

Lusztig (+Shoji): Algorithm for computing characteristic functions of character sheaves.

Proved:

- Shoji (1995): groups with connected centre in good characteristic
- Waldspurger (2003): symplectic, orthogonal groups in odd characteristic
- Shoji (2005): $\mathbf{SL}_n(\mathbb{F}_q)$
- Bonnafé (2006): $\mathbf{SL}_n(\mathbb{F}_q)$, $\mathbf{SU}_n(\mathbb{F}_q)$
- Shoji (today!): classical groups in characteristic 2