Introduction to Deligne-Lusztig Theory

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Deligne (SGA $4\frac{1}{2}$, 1976). *"Les exposés I à VI de SGA 4 donnent la théorie générale des topologies de Grothendieck. Très détaillés, ils peuvent être précieux lors de l'étude de topologies exotiques, telle celle qui donne naissance à la topologie cristalline. Pour la topologie étale,* **si proche de l'intuition classique**, *un garde-fou si imposant n'est pas nécessaire : il suffit de connaître (par exemple), le livre de Godement, et d'avoir un peu de foi. (...)"*

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"Chapters I-VI of SGA 4 develop the general theory of Grothendieck topologies. Very detailed, they may be a valuable tool for studying exotic topologies, such as the one yielding the crystalline topology. For étale topology, **so close to classical intuition**, *such imposing safetynet is not necessary : it is sufficient to know (for instance), Godement's book, and to have some faith. (...)"*

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We denote by $H^*_c(\mathbf{V})$ the element of the Grothendieck group of the category of finite dimensional $\overline{\mathbb{Q}}_\ell\Gamma$-modules equal to

$$H^*_c(\mathbf{V}) = \sum_i (-1)^i \, [H^i_c(\mathbf{V})]$$

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$\mathbf{H}_c^i(\mathbf{P}^1)$?

$\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$: rule (3) $\Rightarrow$

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$$H^i_c(\mathbf{P}^1) = \begin{cases} \overline{\mathbb{Q}}_\ell & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$ as G-modules! (rule (7))

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Let $\mathbf{Z}$ a subvariety of $\mathbf{G}$ (locally closed): then $\mathbf{G}^F$ acts (on the left) on $\mathscr{L}^{-1}(\mathbf{Z})$.

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This is Deligne-Lusztig induction.

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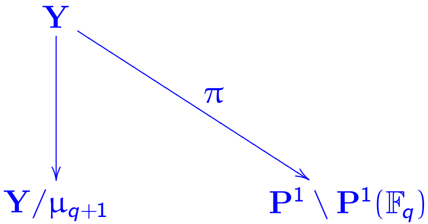

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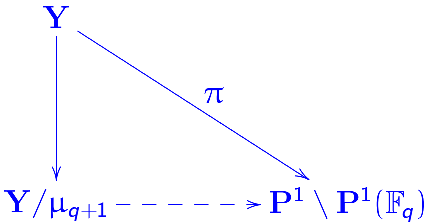

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$$\begin{array}{ccc} \mathbf{Y} & & \\ \downarrow & \searrow^{\pi} & \\ \mathbf{Y}/\mu_{q+1} & \overset{\sim}{\dashrightarrow} & \mathbf{P}^1 \setminus \mathbf{P}^1(\mathbb{F}_q) \end{array}$$

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$\Rightarrow$ $H_c^1(\mathbf{Y})_1 = \mathrm{St}_G$.

- $F(H_c^1(\mathbf{Y}))_\theta = H_c^1(\mathbf{Y})_{\theta^{-1}}$ (as G-modules)

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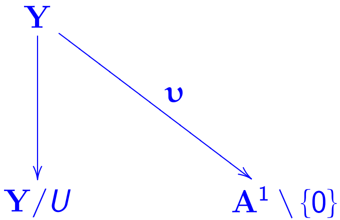

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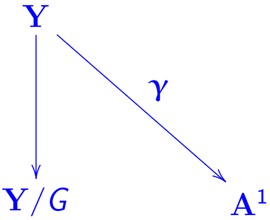

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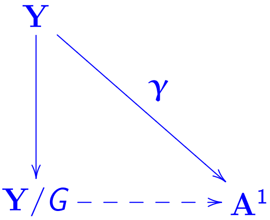

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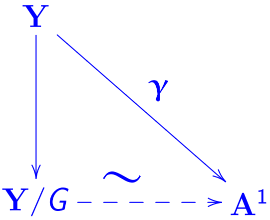

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Some references

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