Cellular structures on Hecke algebras of type B

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- Cellular structure**s**
- Canonical basis of Fock space: Ariki's Theorem
- Kazhdan-Lusztig theory and cellular structures: Geck's Theorem

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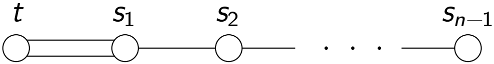

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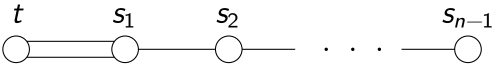

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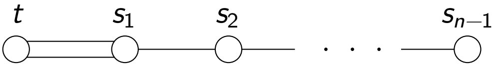

- $\ell : W_n \to \mathbb{N} = \{0, 1, 2, \dots\}$ length function
- $R = \mathbb{Z}[Q, Q^{-1}, q, q^{-1}]$, Q, q indeterminates
- $\mathcal{H}_n = \mathcal{H}_R(W_n, S_n, Q, q)$: Hecke algebra of type B_n with parameters Q and q.

- $K = \mathrm{Frac}(R)$, $K\mathcal{H}_n = K \otimes_R \mathcal{H}_n$ split semisimple

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Hypothesis and notation

- $Q_0^2 = -q_0^{2d}$, $d \in \mathbb{Z}$
- $e =$ order of q_0^2, $e > 2$.

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Uglov has constructed an involution $\bar{\ } : \mathcal{F}_r \to \mathcal{F}_r$ and there exists a *unique* $G(\lambda, r) \in \mathcal{F}_r$ such that

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$(|\lambda, r\rangle)_{\lambda \in Bip}$ is called the standard basis
$(G(\lambda, r))_{\lambda \in Bip}$ is called the Kashiwara-Lusztig canonical basis

Ariki's Theorem (Ariki, Uglov, Geck-Jacon). *Assume that* $r \equiv d \mod e$*. There exists a subset* $\mathrm{Bip}_{e,r}(n)$ *of* $\mathrm{Bip}(n)$ *and a bijection*

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REMARK - $d_{\lambda\mu}^r(v)$ is "computable"

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- $\mathrm{Bip}_{d_0,e}(n) = \{$FLOTW bipartitions$\}$ (Jacon). Here, $d_0 \equiv d \mod e$ and $d_0 \in \{0, 1, 2, \ldots, e-1\}$.

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- $R_{<_\theta 0} = \bigoplus_{\gamma \in \mathbb{Z}^2_{<_\theta 0}} \mathbb{Z} e^{\gamma}$.

Theorem (Kazhdan-Lusztig, 1979). *For each $w \in W_n$, there exists a* **unique** $C_w^\theta \in \mathcal{H}_n$ *such that*

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Theorem (Geck, 2007). *If Lusztig's conjectures (P1), (P2),…, (P14), (P*15^-*) hold, then* $(\pm C_w^\theta)_{w \in W_n}$ *is a* **cellular** *basis of* $\mathcal{H}_n$.

Case $n = 2$ (write $s = s_1$)

$$
\begin{aligned}
C_1^\theta &= 1 \\
C_t^\theta &= T_t + Q^{-1} \\
C_s^\theta &= T_s + q^{-1} \\
C_{st}^\theta &= T_{st} + Q^{-1} T_s + q^{-1} T_t + 1 \\
C_{ts}^\theta &= T_{ts} + Q^{-1} T_s + q^{-1} T_t + 1 \\
C_{sts}^\theta &= T_{sts} + q^{-1} (T_{st} + T_{ts}) + \\
&\quad \begin{cases} Q^{-1} q^{-1} (1 + q^2) T_s + q^{-2} T_t + Q^{-1} q^{-2} & \text{if } \theta > 1 \\ Q^{-1} q^{-1} T_s + Q^{-1} q^{-1} (1 - Q^2) T_t + Q^{-2} q^{-1} (1 - Q^2) & \text{if } 0 < \theta < 1 \end{cases} \\
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\end{aligned}$$

$$Q^{-1} q^{-1} (1 - q^2) = e^{(-1,-1)} - e^{(-1,1)} \in R_{<_\theta 0} \qquad \text{if } \theta > 1$$

Contents

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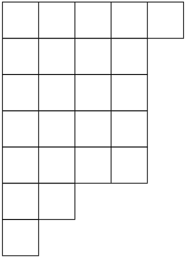

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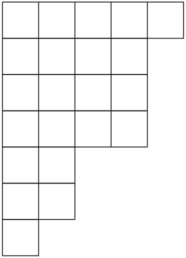

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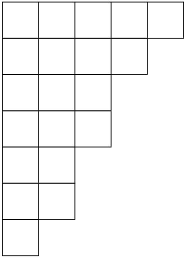

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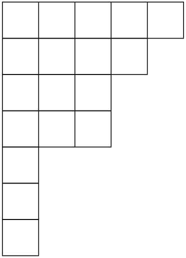

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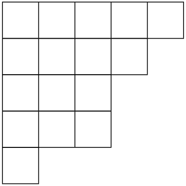

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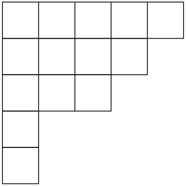

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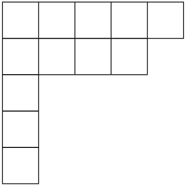

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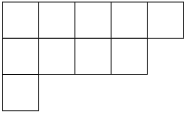

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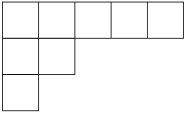

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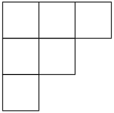

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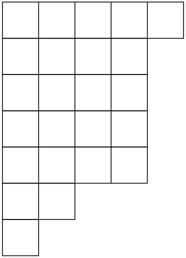

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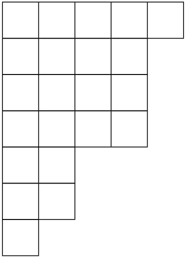

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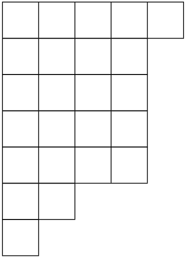

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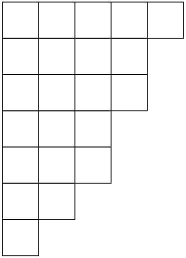

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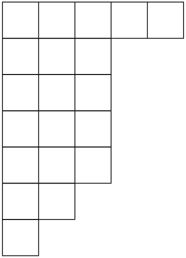

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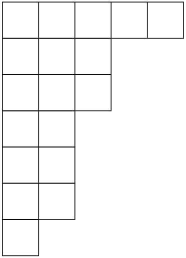

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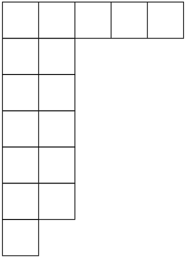

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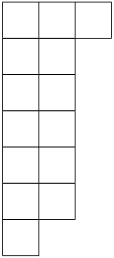

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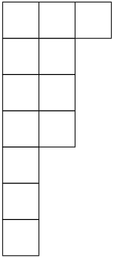

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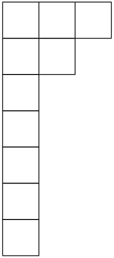

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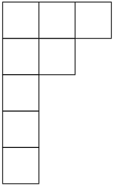

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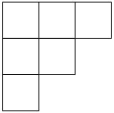

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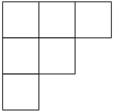

2-core of $\alpha = \delta_3 = (3, 2, 1)$

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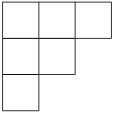

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2-weight of $\alpha = 9$

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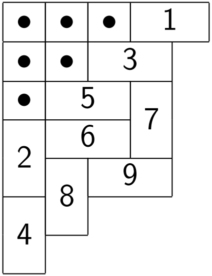
- If $r \geqslant n-1$, then $\trianglelefteq_r$ is the usual dominance order on bipartitions.

- **Domino tableaux**



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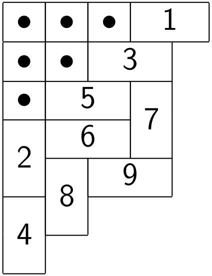

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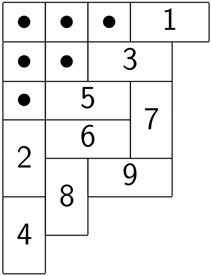

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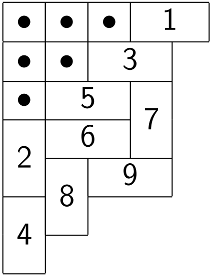

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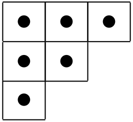
Example: $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & -8 & -9 & 1 & 6 & -4 & 5 & 3 & -2 \end{pmatrix}$



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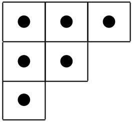

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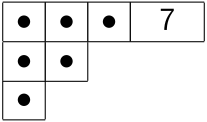

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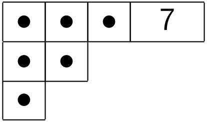

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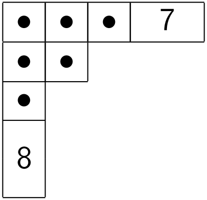

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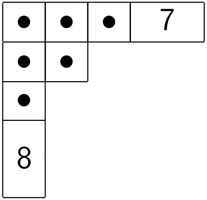

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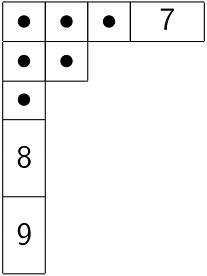

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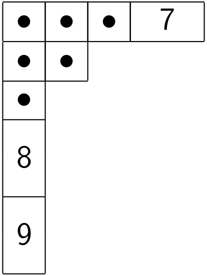

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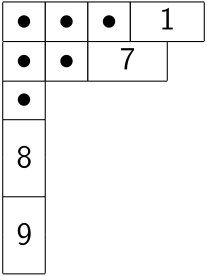

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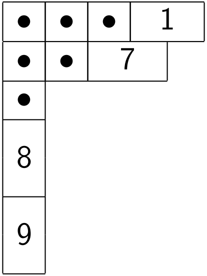

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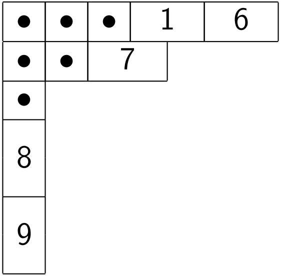

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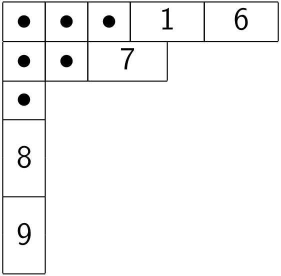

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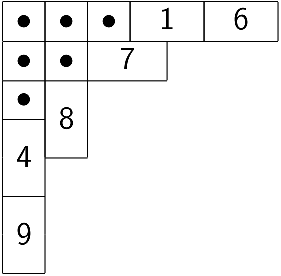

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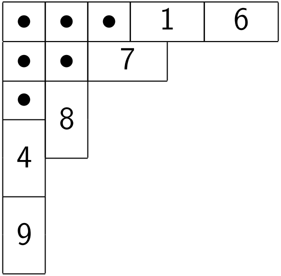

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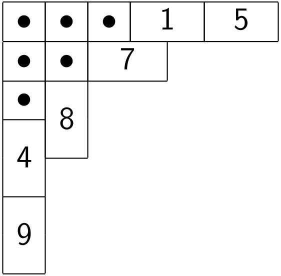

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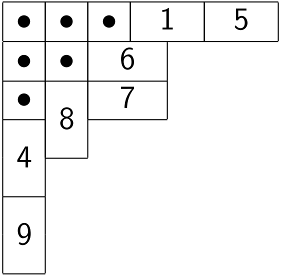

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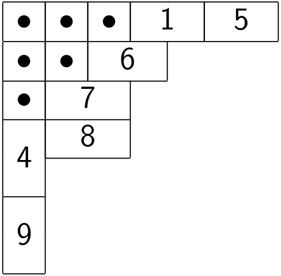

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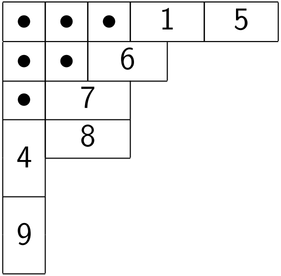

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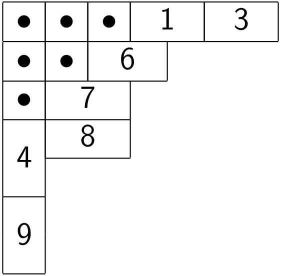

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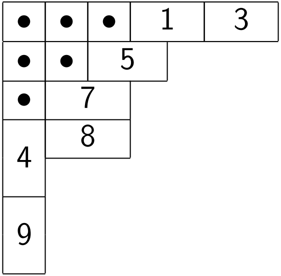

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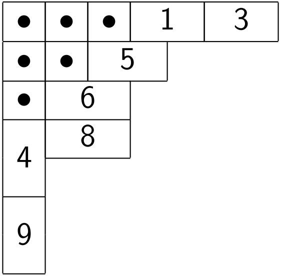

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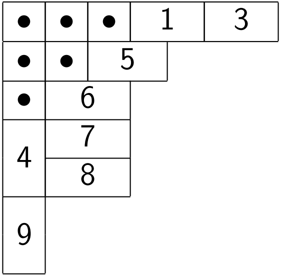

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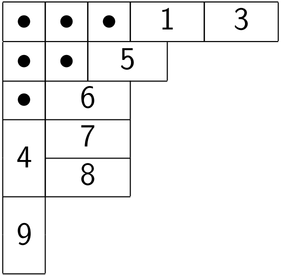

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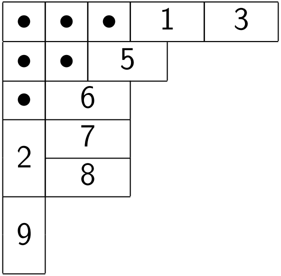

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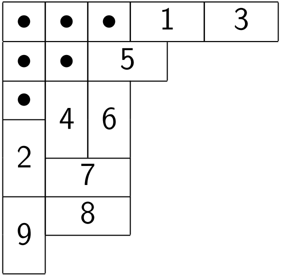

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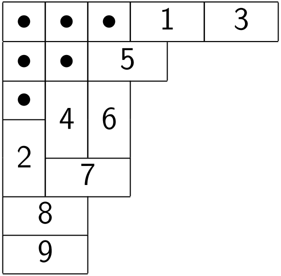
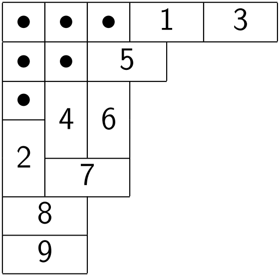


Example: $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & -8 & -9 & 1 & 6 & -4 & 5 & 3 & -2 \end{pmatrix}$

$D_3(w) =$



If $(A, B) \in ST_2(n) \times ST_2(n)$ is a pair of standard bitableaux of the same shape, and if $r < \theta < r + 1$, we set

$$C^{\theta}_{AB} = C^{\theta}_{w},$$

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Conjecture 1 (Geck-Iancu-Lam-B.). *If $r < \theta < r+1$, $\mathcal{C}_r = ((Bip(n), \trianglelefteq_r), ST_2, C^{\theta}, *)$ is a cell datum for $\mathcal{H}_n$ (in the sense of Graham and Lehrer). Here, $T^*_w = T_{w^{-1}}$.*

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- It is true if "$\theta = \frac{1}{2}$ or $\frac{3}{2}$" (Lusztig 2003)

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is bijective and coincides with the map in Ariki's Theorem. Moreover, the decomposition map is given by

$$\mathrm{d}_n[KS_\lambda^\theta] = \sum_{\mu \in \mathrm{Bip}_{e,r}(n)} d_{\lambda\mu}^r(1)[D_\mu^\theta].$$

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