CORRIGENDA : “MACKEY FORMULA IN TYPE A”

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The author recently noticed two errors in his paper [B3] (from which we keep all the notation). They concern Theorem 4.1.1 and Formulas 5.1.7 and 5.1.8 : however, they do not affect the validity of all other results in [B3] as is explained in this note.

1. About Formulas 5.1.7 and 5.1.8 in [B3].

The sign “+” in these formulas must be changed in “−”. This has no consequence concerning the results of [B3] since both formulas are used for the induction argument : in each case where they are used, all the terms involved are equal to 0. Because of these errors, we provide here a complete proof for both formulas.

Proposition 1. *Let* $\mathbf{P}$, $\mathbf{P}'$, $\mathbf{Q}$ *and* $\mathbf{Q}'$ *be four parabolic subgroups of* $\mathbf{G}$ *and let* $\mathbf{L}$, $\mathbf{L}'$, $\mathbf{M}$ *and* $\mathbf{M}'$ *be* F*-stable Levi subgroups of* $\mathbf{P}$, $\mathbf{P}'$, $\mathbf{Q}$ *and* $\mathbf{Q}'$ *respectively. We assume that* $\mathbf{P}\subset\mathbf{P}'$, $\mathbf{L}\subset\mathbf{L}'$, $\mathbf{Q}\subset\mathbf{Q}'$ *and* $\mathbf{M}\subset\mathbf{M}'$. *Then*

(a)
$$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}'\subset\mathbf{Q}'} \circ R^{\mathbf{M}'}_{\mathbf{M}\subset\mathbf{Q}\cap\mathbf{M}'}$$
$$+ \sum_{x\in\mathbf{L}^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M}')^F/\mathbf{M}'^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^x\mathbf{M}'\subset\mathbf{L}\cap{}^x\mathbf{Q}'} \circ \Delta^{{}^x\mathbf{M}'}_{\mathbf{L}\cap{}^x\mathbf{M}'\subset\mathbf{P}\cap{}^x\mathbf{M}',{}^x\mathbf{M}\subset{}^x(\mathbf{Q}\cap\mathbf{M}')} \circ (\mathrm{ad}\,x)_{\mathbf{M}}.$$

(b)
$$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = {}^*R^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}'} \circ \Delta^{\mathbf{G}}_{\mathbf{L}'\subset\mathbf{P}',\mathbf{M}\subset\mathbf{Q}}$$
$$+ \sum_{x\in\mathbf{L}'^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L}',\mathbf{M})^F/\mathbf{M}^F} \Delta^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}',\mathbf{L}'\cap{}^x\mathbf{M}\subset\mathbf{L}'\cap{}^x\mathbf{Q}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}'\cap{}^x\mathbf{M}\subset\mathbf{P}'\cap{}^x\mathbf{M}} \circ (\mathrm{ad}\,x)_{\mathbf{M}}.$$

(c)
$$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = {}^*R^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}'} \circ \Delta^{\mathbf{G}}_{\mathbf{L}'\subset\mathbf{P}',\mathbf{M}'\subset\mathbf{Q}'} \circ R^{\mathbf{M}'}_{\mathbf{M}\subset\mathbf{Q}\cap\mathbf{M}'}$$
$$+ \sum_{x\in\mathbf{L}'^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L}',\mathbf{M}')^F/\mathbf{M}'^F} {}^*R^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}'} \circ R^{\mathbf{L}'}_{\mathbf{L}'\cap{}^x\mathbf{M}'\subset\mathbf{L}'\cap{}^x\mathbf{Q}'} \circ \Delta^{{}^x\mathbf{M}'}_{\mathbf{L}'\cap{}^x\mathbf{M}'\subset\mathbf{P}'\cap{}^x\mathbf{M}',{}^x\mathbf{M}\subset{}^x(\mathbf{Q}\cap\mathbf{M}')} \circ (\mathrm{ad}\,x)_{\mathbf{M}}$$
$$+ \sum_{x\in\mathbf{L}'^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L}',\mathbf{M})^F/\mathbf{M}^F} \Delta^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}',\mathbf{L}'\cap{}^x\mathbf{M}\subset\mathbf{L}'\cap{}^x\mathbf{Q}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}'\cap{}^x\mathbf{M}\subset\mathbf{P}'\cap{}^x\mathbf{M}} \circ (\mathrm{ad}\,x)_{\mathbf{M}}.$$

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PROOF - Note that (b) follows from (a) by adjunction and that (c) follows by applying (a) and (b) successively. Now, let us prove (a). Let Δ_0 denote the right-hand side of the equality (a). By definition of the Δ-maps, we easily get

$$\Delta_0 = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}$$

$$+ \sum_{g\in\mathbf{L}^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M}')^F/\mathbf{M}'^F} \Big(-R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}'\subset\mathbf{L}\cap{}^g\mathbf{Q}'} \circ {}^*R^{{}^g\mathbf{M}'}_{\mathbf{L}\cap{}^g\mathbf{M}'\subset\mathbf{P}\cap{}^g\mathbf{M}'} \circ (\mathrm{ad}\, g)_{\mathbf{M}'} \circ R^{\mathbf{M}'}_{\mathbf{M}\subset\mathbf{Q}\cap\mathbf{M}'}$$

$$+R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}'\subset\mathbf{L}\cap{}^g\mathbf{Q}'} \circ {}^*R^{{}^g\mathbf{M}'}_{\mathbf{L}\cap{}^g\mathbf{M}'\subset\mathbf{P}\cap{}^g\mathbf{M}'} \circ (\mathrm{ad}\, g)_{\mathbf{M}'} \circ R^{\mathbf{M}'}_{\mathbf{M}\subset\mathbf{Q}\cap\mathbf{M}'}$$

$$- \sum_{y\in\mathbf{L}^F\cap{}^g\mathbf{M}'^F\backslash\mathcal{S}_{{}^g\mathbf{M}'}(\mathbf{L}\cap{}^g\mathbf{M}',{}^g\mathbf{M})^F/{}^g\mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^{yg}\mathbf{M}\subset\mathbf{L}\cap{}^{yg}\mathbf{Q}} \circ R^{{}^{yg}\mathbf{M}}_{\mathbf{L}\cap{}^{yg}\mathbf{M}\subset\mathbf{P}\cap{}^{yg}\mathbf{M}} \Big).$$

Therefore,

$$\Delta_0 = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}$$

$$- \sum_{g\in\mathbf{L}^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M}')^F/\mathbf{M}'^F} \Big(\sum_{y\in\mathbf{L}^F\cap{}^g\mathbf{M}'^F\backslash\mathcal{S}_{{}^g\mathbf{M}'}(\mathbf{L}\cap{}^g\mathbf{M}',{}^g\mathbf{M})^F/{}^g\mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^{yg}\mathbf{M}\subset\mathbf{L}\cap{}^{yg}\mathbf{Q}} \circ R^{{}^{yg}\mathbf{M}}_{\mathbf{L}\cap{}^{yg}\mathbf{M}\subset\mathbf{P}\cap{}^{yg}\mathbf{M}} \Big).$$

The argument at the end of the proof of [B1, Lemma 3.2.1] completes the proof of (a). ■

2. About Theorem 4.1.1 in [B3].

The second error is much more serious : Theorem 4.1.1 is false ! However, its corollary 4.1.2 is still correct ; it follows from Theorem 3 below. Fortunately, we use only Corollary 4.1.2 in the rest of [B3] (and not Theorem 4.1.1). This means that all the other results in [B3] are valid.

Our mistake in the proof of [B3, Theorem 4.1.1] is the following (here we keep the notation of this "theorem") : it may happen that ω stabilizes a cuspidal local system but that it acts on the characteristic function by multiplication by a scalar different from 1.

Let us first introduce some notation. If $\iota = (C,\mathcal{L}) \in \mathcal{U}(\mathbf{G})^F$, we fix once and for all an isomorphism $\varphi_\iota : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ and we denote by $\mathfrak{Y}_\iota$ (or $\mathfrak{Y}_\iota^{\mathbf{G}}$ if we need to make the ambient group precise) the characteristic function associated to this isomorphism. Let $\mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)$ denote the $\overline{\mathbb{Q}}_\ell$-vector subspace of $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$ generated by the functions $\mathfrak{Y}_\iota$ ($\iota \in \mathcal{U}(\mathbf{G})^F_{\mathrm{cus}}$). Let $\mathrm{Aut}(\mathbf{G},F)$ denote the group of automorphisms of $\mathbf{G}$ commuting with F. The group $\mathrm{Inn}(\mathbf{G}^F)$ of inner automorphisms of $\mathbf{G}^F$ is a normal subgroup of $\mathrm{Aut}(\mathbf{G},F)$. We set $\mathrm{Out}(\mathbf{G},F) = \mathrm{Aut}(\mathbf{G}^F)/\mathrm{Inn}(\mathbf{G}^F)$. It is clear that $\mathrm{Aut}(\mathbf{G},F)$ (or $\mathrm{Out}(\mathbf{G},F)$) acts on the vector spaces $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$, $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$ and $\mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)$.

We fix an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and an F-stable maximal torus $\mathbf{T}$ of $\mathbf{B}$. Let W denote the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$ and let S be the set of simple reflections in W corresponding to the choice of $\mathbf{B}$. If $I \subset S$, we denote by W_I the subgroup of W generated by I and we set $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$. We denote by $\mathbf{L}_I$ the Levi subgroup of $\mathbf{P}_I$ containing $\mathbf{T}$.

If I is a subset of S, we denote by A_I the stabilizer of $(\mathbf{B} \cap \mathbf{L}_I, \mathbf{T})$ in the group $\mathrm{Aut}(\mathbf{L}_I, F)$. We have $\mathrm{Aut}(\mathbf{G}, F) = \mathrm{Inn}(\mathbf{G}^F).A_S$. So studying the action of $\mathrm{Aut}(\mathbf{G}, F)$ on $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$, $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$ or $\mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)$ is equivalent to studying the action of A_S.

2.A. **Generalized Springer correspondence.** We denote by $\mathcal{P}(S)$ the set of subsets of S and by $\mathcal{P}(S)_{\mathrm{cus}}$ the set of subsets I of S such that $\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}} \neq \varnothing$. Note that F and A_S act on W, S, $\mathcal{P}(S)$ and $\mathcal{P}(S)_{\mathrm{cus}}$ and that these two actions commute.

Let $\mathcal{U}'(\mathbf{G})$ denote the set of triples (I, ι, ρ) where $I \subset S$, $\iota \in \mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}$ and $\rho \in \mathrm{Irr}\, W_{\mathbf{G}}(\mathbf{L}_I)$. The generalized Springer correspondence [L, Theorems 6.5 and 9.2] is a well-defined bijection $\psi : \mathcal{U}'(\mathbf{G}) \to \mathcal{U}(\mathbf{G})$. This bijection commutes with the actions of F and A_S.

2.B. **Action of automorphisms on characteristic functions of local systems.** The vector space $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$ admits $(\mathfrak{Y}_\iota)_{\iota \in \mathcal{U}(\mathbf{G}^F)}$ as a basis. With respect to this basis, the action of an element of $\mathrm{Aut}(\mathbf{G}, F)$ is monomial. We are interested here in the way to determine the non-zero coefficients of this monomial matrix. Since the characteristic function $\mathfrak{Y}_\iota$ (for $\iota = (C, \mathcal{L}) \in \mathcal{U}(\mathbf{G})^F$) depends on the choice of the isomorphism $\varphi_\iota : F^*\mathcal{L} \to \mathcal{L}$ that we have fixed once and for all, the interesting question is the following : if $\sigma \in \mathrm{Aut}(\mathbf{G}, F)$ and if $\iota \in \mathcal{U}(\mathbf{G})^F$ are such that $\sigma(\iota) = \iota$, then what is the root of unity $\xi_{\iota,\sigma}$ (or $\xi_{\iota,\sigma}^{\mathbf{G}}$ if we want to emphasize the ambient group) such that ${}^\sigma\mathfrak{Y}_\iota = \xi_{\iota,\sigma}\mathfrak{Y}_{\iota,\sigma}$?

2.B.1. *Permutation of unipotent classes in* $\mathbf{G}^F$. Let $\sigma \in \mathrm{Aut}(\mathbf{G}, F)$ and let $\iota = (C, \mathcal{L}) \in \mathcal{U}(\mathbf{G})^F$ be such that $\sigma(\iota) = \iota$. We fix $u \in C^F$ such that $\mathfrak{Y}_\iota(u) \neq 0$ and we denote by ζ the irreducible character of $A_{\mathbf{G}}(u)$ defined by $\mathcal{L}$. Let $\tilde{\zeta}$ denote the extension of ζ to the semi-direct product $A_{\mathbf{G}}(u) \rtimes < F >$ (here, $< F >$ is viewed as an infinite cyclic group) associated to the isomorphism φ_ι.

If $a \in H^1(F, A_{\mathbf{G}}(u))$, we denote by g_a an element of $\mathbf{G}$ such that $g_a^{-1}F(g_a) \in C_{\mathbf{G}}(u)$ and such that the image $\dot{a}$ of $g_a^{-1}F(g_a)$ in $A_{\mathbf{G}}(u)$ belongs to the class a. We set $u_a = g_a u g_a^{-1} \in C^F$. Then $\{u_a \mid a \in H^1(F, A_{\mathbf{G}}(u))\}$ is a set of representatives of $\mathbf{G}^F$-conjugacy classes in C^F and

(2) $$\mathfrak{Y}_\iota(u_a) = \tilde{\zeta}(\dot{a}F).$$

Therefore, if a_σ denotes the unique element of $H^1(F, A_{\mathbf{G}}(u))$ such that $\sigma^{-1}(u)$ is $\mathbf{G}^F$-conjugate to u_{a_σ}, we have

(3) $$\xi_{\iota,\sigma} = \frac{\tilde{\zeta}(\dot{a}_\sigma F)}{\tilde{\zeta}(F)}.$$

2.B.2. *Going down to cuspidal local systems.* Let $\iota \in \mathcal{U}(\mathbf{G})^F$. We denote by $A_{S,\iota}$ the stabilizer of ι in A_S and we set $\xi_\iota = \xi_\iota^{\mathbf{G}} : A_{S,\iota} \to \overline{\mathbb{Q}}_\ell^\times$, $\sigma \mapsto \xi_{\iota,\sigma}$. It is clear that ξ_ι is a linear character. Now, let $(I,\iota_0,\rho) = \psi^{-1}(\iota)$. Then $A_{S,\iota}$ stabilizes $\mathbf{L}_I$, $\mathbf{B} \cap \mathbf{L}_I$, $\mathbf{T}$, ι_0 and ρ. Therefore, we get a morphism $A_{S,\iota} \to A_{I,\iota_0}$.

Lemma 4. *With the above notation, we have* $\xi_\iota^{\mathbf{G}} = \operatorname{Res}_{A_{S,\iota}}^{A_{I,\iota_0}} \xi_{\iota_0}^{\mathbf{L}_I}$.

Proof - Let $\mathfrak{X}_{I,\iota_0}^{\mathbf{G}}$ denote the characteristic function of the restriction to the unipotent elements of the F-stable perverse sheaf defined by induction from the datum (I,ι_0). Then $\sigma \in A_{S,\iota}$ acts on $\mathfrak{X}_{I,\iota_0}^{\mathbf{G}}$ by multiplication $\xi_{\iota_0}^{\mathbf{L}_I}(\sigma)$. Moreover,

$$\mathfrak{X}_{I,\iota_0}^{\mathbf{G}} = \sum_{\rho \in (\operatorname{Irr} W_{\mathbf{G}}(\mathbf{L}_I))^F} n_\rho \mathfrak{X}_{I,\iota_0,\rho}^{\mathbf{G}}$$

where $\mathfrak{X}_{I,\iota_0,\rho}$ is the characteristic function of the F-stable perverse sheaf associated to (I,ι_0,ρ) via the generalized Springer correspondence and $n_\rho \in \overline{\mathbb{Q}}_\ell^\times$. Therefore, if ρ is σ-invariant, then σ acts on $\mathfrak{X}_{I,\iota_0,\rho}^{\mathbf{G}}$ by multiplication by $\xi_{\iota_0}^{\mathbf{L}_I}(\sigma)$ (indeed, the family $(\mathfrak{X}_{I,\iota_0,\rho})_{\rho \in (\operatorname{Irr} W_{\mathbf{G}}(\mathbf{L}_I))^F}$ is linearly independent).

But $\mathfrak{X}_{I,\iota_0,\rho}^{\mathbf{G}}$ and $\lambda \mathfrak{Y}_{\psi(I,\iota_0,\rho)}^{\mathbf{G}}$ coincide on C^F where $(C,\mathcal{L}) = \psi(I,\iota_0,\rho)$ for some $\lambda \in \overline{\mathbb{Q}}_\ell^\times$. So σ acts on $\mathfrak{Y}_{\psi(I,\iota_0,\rho)}^{\mathbf{G}}$ by multiplication by $\xi_{\iota_0}^{\mathbf{L}_I}(\sigma)$. □

2.B.3. *About cuspidal local systems.* Lemma 4 shows that, in order to determine the linear characters ξ_ι, we can restrict our attention to the case of cuspidal local systems. The first result in this direction is the following.

Lemma 5. *If* $\mathbf{L}$ *is a rational Levi subgroup of a parabolic subgroup of* $\mathbf{G}$*, then* $N_{\mathbf{G}^F}(\mathbf{L})$ *acts trivially on* $\mathcal{CUS}_{\mathrm{uni}}(\mathbf{L}^F)$.

Proof - Let $n \in N_{\mathbf{G}^F}(\mathbf{L})$, let $\iota = (C,\mathcal{L}) \in \mathcal{U}(\mathbf{L})_{\mathrm{cus}}^F$ and let $v \in C^F$. Then, by [B4, Proposition I.8.3], nvn^{-1} and v are $\mathbf{L}^F$-conjugate. This proves Lemma 5. ■

We close this section with a result concerning geometrically conjugate F-stable Levi subgroups. We need some further notation. Let $\mathcal{A}$ denote a set of representatives of $\mathbf{G}^F$-conjugacy classes of F-stable Levi subgroups $\mathbf{L}$ of proper parabolic subgroups of $\mathbf{G}$ such that $\mathcal{U}(\mathbf{L})_{\mathrm{cus}}^F \neq \varnothing$. By [L, Theorem 9.2], we have :

Lemma 6. (a) *If* I, $J \in \mathcal{P}(S)_{\mathrm{cus}}$ *and if there exists* $w \in W$ *such that* ${}^w I = J$*, then* $I = J$.

(b) *Every* $\mathbf{L} \in \mathcal{A}$ *is geometrically conjugate to a unique* $\mathbf{L}_I$ *with* $I \in \mathcal{P}(S)_{\mathrm{cus}}^F$.

If $I \in \mathcal{P}(S)^F_{\mathrm{cus}}$, then the set of $\mathbf{G}^F$-conjugacy classes of F-stable Levi subgroups (of parabolic subgroups of $\mathbf{G}$) geometrically conjugate to $\mathbf{L}_I$ are parametrized by $H^1(F, W_{\mathbf{G}}(\mathbf{L}_I))$ where $W_{\mathbf{G}}(\mathbf{L}_I) = N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I$. Let $\mathcal{C}$ be the set of pairs (I, w) such that $I \in \mathcal{P}(S)^F_{\mathrm{cus}}$, $I \neq S$ and $w \in H^1(F, W_{\mathbf{G}}(\mathbf{L}_I))$. We then have a bijection $\mathcal{C} \to \mathcal{A}$ denoted by $(I, w) \mapsto \mathbf{L}_{I,w}$.

We now fix in this subsection, and only in this subsection, an element $\sigma \in A$, a subset I of S and an element $w \in H^1(F, W_{\mathbf{G}}(\mathbf{L}_I))$ such that $\sigma(I, w) = (I, w)$. Let $g \in \mathbf{G}$ be such that $\mathbf{L}_{I,w} = {}^g\mathbf{L}_I$. We set $\dot{w} = g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{L}_I)$ ($\dot{w}$ is a representative in $N_{\mathbf{G}}(\mathbf{L}_I)$ of w). Then, conjugacy by g induces a bijection $\mathcal{U}(\mathbf{L}_I)^{\dot{w}F} \xrightarrow{\sim} \mathcal{U}(\mathbf{L}_{I,w})^F$, $\iota \mapsto {}^g\iota$.

Since the group $N_{\mathbf{G}}(\mathbf{L}_I)$ acts trivially on $\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}$ by [L, Theorem 9.2], we get a bijection $\mathcal{U}(\mathbf{L}_I)^F_{\mathrm{cus}} \xrightarrow{\sim} \mathcal{U}(\mathbf{L}_{I,w})^F_{\mathrm{cus}}$, $\iota \mapsto {}^g\iota$.

Since σ stabilizes w, there exists $x \in \mathbf{G}^F$ such that ${}^\sigma\mathbf{L}_{I,w} = {}^x\mathbf{L}_{I,w}$. We then set $\sigma' = \mathrm{Inn}(x^{-1}) \circ \sigma$ so that ${}^{\sigma'}\mathbf{L}_{I,w} = \mathbf{L}_{I,w}$.

Lemma 7. *Let $\iota \in \mathcal{U}(\mathbf{L}_I)^F_{\mathrm{cus}}$. Then :*

(a) *$\sigma(\iota) = \iota$ if and only if $\sigma'({}^g\iota) = {}^g\iota$.*

(b) *If $\sigma(\iota) = \iota$, then $\xi^{\mathbf{L}_I}_{\iota,\sigma} = \xi^{\mathbf{L}_{I,w}}_{{}^g\iota,\sigma'}$.*

Proof - Let $\tau = \mathrm{Inn}(g^{-1}) \circ \sigma' \circ \mathrm{Inn}(g)$. Then $\tau \in \mathrm{Aut}(\mathbf{L}_I, \mathrm{Inn}(\dot{w}) \circ F)$. Moreover, σ' stabilizes ${}^g\iota$ if and only if τ stabilizes ι. But $\tau = \mathrm{Inn}(g^{-1}x^{-1}\ {}^\sigma g) \circ \sigma$, so $g^{-1}x^{-1}\ {}^\sigma g \in N_{\mathbf{G}}(\mathbf{L}_I)$: this proves that $g^{-1}x^{-1}\ {}^\sigma g$ acts trivially on $\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}$ by [L, Theorem 9.2]. Therefore, τ stabilizes ι if and only if σ stabilizes ι. This proves (a).

Let us now prove (b). Let $\iota = (C, \mathcal{L}) \in \mathcal{U}(\mathbf{L}_I)^F_{\mathrm{cus}}$ be such that $\sigma(\iota) = \iota$. We fix an element $v \in C^F$ such that $\mathfrak{Y}^{\mathbf{L}_I}_\iota(v) \neq 0$.

We write $n = g^{-1}x^{-1}\ {}^\sigma g \in N_{\mathbf{G}}(\mathbf{L}_I)$. Then $\tau = \mathrm{Inn}(n) \circ \sigma$ commutes with $\mathrm{Inn}(\dot{w}) \circ F$. Since $N_{\mathbf{G}}(\mathbf{L}_I)$ stabilizes C and since $A_{\mathbf{L}_I}(v) = A_{\mathbf{G}}(v)$ (see [B2, Corollary to Proposition 1.1]), we may (and we will) assume that $\dot{w} \in N_{\mathbf{G}}(\mathbf{L}_I) \cap C^\circ_{\mathbf{G}}(v)$. Now, σ and n stabilize C. So there exists l and m in $\mathbf{L}_I$ such that $\sigma(v) = lvl^{-1}$ and $nvn^{-1} = mvm^{-1}$. So $m^{-1}n \in C_{\mathbf{G}}(v)$. Since $A_{\mathbf{L}_I}(v) = A_{\mathbf{G}}(v)$, we may (and we will) choose m in such a way that $m^{-1}n \in C^\circ_{\mathbf{G}}(v)$.

We have

$$l^{-1}F(l) \in C_{\mathbf{L}_I}(v), \qquad \tau(v) = \mathrm{Inn}(nln^{-1}m)(v)$$

and

$$(nln^{-1}m)^{-1}\dot{w}F(nln^{-1}m)\dot{w}^{-1} \in C_{\mathbf{L}_I}(v).$$

According to Formula 3, and since $\dot{w}$ acts trivially on $A_{\mathbf{L}_I}(v)$ (see [B4, Lemma I.3.12]), it is sufficient to prove that $l^{-1}F(l)$ and $(nln^{-1}m)^{-1}\dot{w}F(nln^{-1}m)\dot{w}^{-1}$ represent the same element of $A_{\mathbf{L}_I}(v)$. Since $A_{\mathbf{L}_I}(v) = A_{\mathbf{G}}(v)$, we need to determine the class in $A_{\mathbf{G}}(v)$ of $\mu = (nln^{-1}m)^{-1}\dot{w}F(nln^{-1}m)\dot{w}^{-1}$. But,

$$\mu = (m^{-1}n)l^{-1}n^{-1}\dot{w}F(nl)\dot{w}^{-1}(\dot{w}F(n^{-1}m)\dot{w}^{-1}),$$

$m^{-1}n \in C^\circ_{\mathbf{G}}(v)$ and $\dot{w}F(n^{-1}m)\dot{w}^{-1} \in C^\circ_{\mathbf{G}}(v)$ because $\mathrm{Inn}(\dot{w}) \circ F$ stabilizes v. Therefore, the class of μ in $A_{\mathbf{G}}(v)$ is equal to the class of $\mu' = l^{-1}n^{-1}\dot{w}F(nl)\dot{w}^{-1}$. It is also easily checked that $\dot{w}F(n) = n\ {}^\sigma\dot{w}$. Therefore,

$$\mu' = l^{-1}n^{-1}n\ {}^\sigma\dot{w}F(l)\dot{w}^{-1} = l^{-1}\ {}^\sigma\dot{w}ll^{-1}F(l)\dot{w}^{-1}.$$

But, $l^{-1}\ {}^\sigma\dot{w}l \in C^\circ_{\mathbf{G}}(v)$ because $l^{-1}\sigma(v)l = v$ and $\dot{w} \in C^\circ_{\mathbf{G}}(v)$. So the class of μ' in $A_{\mathbf{G}}(v)$ is equal to the class of $l^{-1}F(l)$, which is the desired result. ■

2.C. **The main result.** We recall (see for example [B3, Conjecture C]) that it is conjectured that $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) = \mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)$ whenever p is almost good for $\mathbf{G}$. The next theorem goes in this direction.

Theorem 8. *If the Mackey formula holds in* $\mathbf{G}$ *(in the sense of* [B3, Definition 1.4.2]*), then* $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$ *and* $\mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)$ *are isomorphic as* $\overline{\mathbb{Q}}_\ell\,\mathrm{Out}(\mathbf{G},F)$*-modules.*

Proof - We proceed as for the proof of [B3, Theorem 4.1.1]. But we avoid the mistake mentioned above ! So we assume that the Mackey formula holds in $\mathbf{G}$. Note that this implies that the Lusztig induction and restriction maps do not depend on the choice of the parabolic subgroup. Therefore, if $\mathbf{L}$ is an F-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, we will denote by $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ the maps $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$.

We argue by induction on $\dim \mathbf{G}$. The result is obvious if $\mathbf{G}$ is a torus. Therefore, we may assume that Theorem 8 holds for every F-stable Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$. Since $\mathrm{Out}(\mathbf{G},F)$ acts on $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$ and $\mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)$ through a finite quotient (namely its image in $\mathrm{Out}(\mathbf{G}^F)$), it is sufficient to prove the following : if $\sigma \in A_S$, then

$$(*) \qquad \mathrm{Tr}(\sigma, \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)) = \mathrm{Tr}(\sigma, \mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)).$$

First step. Let us first evaluate the right-hand side of $(*)$. Let $\mathcal{U}'(\mathbf{G})_*$ denote the set of $(I,\iota,\rho) \in \mathcal{U}'(\mathbf{G})$ such that $I \neq S$. Then, since $(\mathfrak{Y}_{\psi(I,\iota,\rho)})_{(I,\iota,\rho)\in\mathcal{U}'(\mathbf{G})^F}$ is a basis of $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$, we have

$$\mathrm{Tr}(\sigma, \mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)) = \mathrm{Tr}(\sigma, \mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)) + \sum_{\substack{(I,\iota,\rho)\in\mathcal{U}'(\mathbf{G})_*^F \\ \sigma(I,\iota,\rho)=(I,\iota,\rho)}} \xi^{\mathbf{G}}_{\psi(I,\iota,\rho)}(\sigma).$$

If we denote by $\mathcal{E}$ the set of pair (I,ι) such that $I \in \mathcal{P}(S)^F_{\mathrm{cus}}$, $I \neq S$ and $\iota \in \mathcal{U}(\mathbf{L}_I)^F_{\mathrm{cus}}$, and if we use Lemma 4, we get :

$$\text{(A)} \qquad \begin{aligned} \mathrm{Tr}(\sigma, \mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)) &= \mathrm{Tr}(\sigma, \mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)) \\ &\quad - \sum_{\substack{(I,\iota)\in\mathcal{E} \\ \sigma(I,\iota)=(I,\iota)}} \xi^{\mathbf{L}_I}_{\iota,\sigma}.|\{\rho \in (\mathrm{Irr}\, W_{\mathbf{G}}(\mathbf{L}_I))^F \mid \sigma(\rho)=\rho\}|. \end{aligned}$$

Second step. We now evaluate the left-hand side of $(*)$. If $\mathbf{L} \in \mathcal{A}$, then $N_{\mathbf{G}^F}(\mathbf{L})$ acts trivially on $\mathcal{CUS}_{\text{uni}}(\mathbf{L}^F)$ by Lemma 5, so it acts trivially on $\text{Cus}_{\text{uni}}(\mathbf{G}^F)$ by the induction hypothesis. So, since Mackey formula holds in $\mathbf{G}$, we have :

$$\text{Class}_{\text{uni}}(\mathbf{G}^F) = \text{Cus}_{\text{uni}}(\mathbf{G}^F) \oplus \big(\bigoplus_{(I,w)\in\mathcal{C}} R_{\mathbf{L}_{I,w}}^{\mathbf{G}}(\text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F))\big),$$

and the map $R_{\mathbf{L}_{I,w}}^{\mathbf{G}} : \text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F) \to R_{\mathbf{L}_{I,w}}^{\mathbf{G}}(\text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F))$ is an isomorphism. Note that this isomorphism commutes with every element of $\text{Aut}(\mathbf{G}, F)$ stabilizing $\mathbf{L}_{I,w}$. Therefore,

$$\text{(B)} \quad \text{Tr}(\sigma, \text{Cus}_{\text{uni}}(\mathbf{G}^F)) = \text{Tr}(\sigma, \text{Class}_{\text{uni}}(\mathbf{G}^F)) - \sum_{\substack{(I,w)\in\mathcal{C} \\ \sigma(I,w)=(I,w)}} \text{Tr}(\sigma, R_{\mathbf{L}_{I,w}}^{\mathbf{G}}(\text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F))).$$

Let $(I, w) \in \mathcal{C}$ be such that $\sigma(I, w) = (I, w)$. Then there exists $x \in \mathbf{G}^F$ such that ${}^{\sigma}\mathbf{L}_{I,w} = {}^{x}\mathbf{L}_{I,w}$. We set $\sigma' = \text{Inn}(x)^{-1} \circ \sigma$. Then σ' stabilizes $\mathbf{L}_{I,w}$ and

$$\text{Tr}(\sigma, R_{\mathbf{L}_{I,w}}^{\mathbf{G}}(\text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F))) = \text{Tr}(\sigma', R_{\mathbf{L}_{I,w}}^{\mathbf{G}}(\text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F))),$$

so $\text{Tr}(\sigma, R_{\mathbf{L}_{I,w}}^{\mathbf{G}}(\text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F))) = \text{Tr}(\sigma', \text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F))$. But, by the induction hypothesis, we get that $\text{Tr}(\sigma', \text{Cus}_{\text{uni}}(\mathbf{L}_{I,w}^F)) = \text{Tr}(\sigma', \mathcal{CUS}_{\text{uni}}(\mathbf{L}_{I,w}^F))$. Moreover, by Lemma 7, we have $\text{Tr}(\sigma', \mathcal{CUS}_{\text{uni}}(\mathbf{L}_{I,w}^F)) = \text{Tr}(\sigma, \mathcal{CUS}_{\text{uni}}(\mathbf{L}_I^F))$. So we deduce from (B) that

$$\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(\mathbf{G}^F)) = \text{Tr}(\sigma, \text{Class}_{\text{uni}}(\mathbf{G}^F)) - \sum_{\substack{(I,w)\in\mathcal{C} \\ \sigma(I,w)=(I,w)}} \text{Tr}(\sigma, \mathcal{CUS}_{\text{uni}}(\mathbf{L}_I^F)).$$

In other words,

$$\begin{aligned}\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(\mathbf{G}^F)) &= \text{Tr}(\sigma, \text{Class}_{\text{uni}}(\mathbf{G}^F)) \\ &- \sum_{\substack{I\in\mathcal{P}(S)^F_{\text{cus}},\ I\neq S \\ \sigma(I)=I}} \text{Tr}(\sigma, \mathcal{CUS}_{\text{uni}}(\mathbf{L}_I^F)).|\{w \in H^1(F, W_{\mathbf{G}}(\mathbf{L}_I)) \mid \sigma(w) = w\}|.\end{aligned}$$

Finally, we get

$$\text{(C)} \quad \begin{aligned}\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(\mathbf{G}^F)) &= \text{Tr}(\sigma, \text{Class}_{\text{uni}}(\mathbf{G}^F)) \\ &- \sum_{\substack{(I,\iota)\in\mathcal{E} \\ \sigma(I,\iota)=(I,\iota)}} \xi_{\iota,\sigma}^{\mathbf{L}_I}.|\{w \in H^1(F, W_{\mathbf{G}}(\mathbf{L}_I)) \mid \sigma(w) = w\}|.\end{aligned}$$

Third step. Let $I \in \mathcal{P}(S)^F$ be such that $\sigma(I) = I$. Then σ acts on $W_{\mathbf{G}}(\mathbf{L}_I)$ and this action commutes with the action of F. Therefore,

$$\text{(D)} \quad |\{w \in H^1(F, W_{\mathbf{G}}(\mathbf{L}_I)) \mid \sigma(w) = w\}| = |\{\rho \in (\text{Irr}\, W_{\mathbf{G}}(\mathbf{L}_I))^F \mid \sigma(\rho) = \rho\}|.$$

The proof of (D) is similar to the proof of the well-known theorem of Brauer [I, Theorem 6.32]. By applying (A), (C) and (D), we get $(*)$. ■

2.D. **Some consequences of Theorem 8.** In [B3, §1.8], we defined a morphism of groups $H^1(F, \mathbf{Z}) \to \mathrm{Out}(\mathbf{G}, F)$. So Theorem 8 immediately implies the following result :

Corollary 9. *Let $\zeta \in H^1(F, \mathbf{Z})^\wedge$. If the Mackey formula holds in $\mathbf{G}$, then*

$$\dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)_\zeta = \dim \mathcal{CUS}_{\mathrm{uni}}(\mathbf{G}^F)_\zeta.$$

The last result says that [B3, Corollary 4.1.2] is correct. It is just a straightforward consequence of Theorem 8 and Lemma 5. Note that in [B3, Corollary 4.1.2 (b)], the term "cuspidal function" must be replaced by "absolutely cuspidal function".

Corollary 10. *If the Mackey formula holds in $\mathbf{G}$, then*

(a) $\dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) = |\mathcal{U}(\mathbf{G})^F_{\mathrm{cus}}|$.

(b) *If $\mathbf{G}$ is a rational Levi subgroup of a parabolic subgroup of a connected reductive group $\mathbf{H}$ (endowed with a Frobenius endomorphism also denoted by F) then all absolutely cuspidal functions on $\mathbf{G}^F$ with unipotent support are invariant under the action of $N_{\mathbf{H}^F}(\mathbf{G})$.*

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