# MACKEY FORMULA IN TYPE $A$

CÉDRIC BONNAFÉ

Let $\mathbf{G}$ be a connected, reductive algebraic group defined over a finite field with $q$ elements and let $F : \mathbf{G} \to \mathbf{G}$ denote the corresponding Frobenius endomorphism. Let $\mathbf{P}$ and $\mathbf{Q}$ be two parabolic subgroups of $\mathbf{G}$ and assume that $\mathbf{P}$ and $\mathbf{Q}$ have $F$-stable Levi subgroups $\mathbf{L}$ and $\mathbf{M}$ respectively. By the Mackey formula (for the datum $(\mathbf{G}, \mathbf{L}, \mathbf{P}, \mathbf{M}, \mathbf{Q})$), we mean the following equality :

$$(*) \qquad {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} = \sum_{g \in \mathbf{L}^F \backslash \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{P}\cap{}^g\mathbf{M}} \circ (\mathrm{ad}\, g)_{\mathbf{M}}.$$

where ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and $R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}$ denote respectively Lusztig restriction functor and Luztig induction functor and $\mathcal{S}_{\mathbf{G}}(\mathbf{L}, \mathbf{M})$ denotes the set of $g \in \mathbf{G}$ such that $\mathbf{L}$ and ${}^g\mathbf{M}$ have a common maximal torus. This formula is of central interest in the knowledge of Lusztig induction functors and more generally, in the knowledge of the character table of $\mathbf{G}^F$. It is known to hold in many cases :

(a) Whenever $\mathbf{P}$ and $\mathbf{Q}$ are $F$-stable (Deligne [LS], Theorem 2.5),
(b) Whenever $\mathbf{L}$ or $\mathbf{M}$ is a maximal torus of $\mathbf{G}$ (Deligne and Lusztig [DL2], Theorem 7),
(c) Whenever $q$ is large enough (*cf.* [B1], Theorem 5.1.1).

It is conjectured that the Mackey formula holds in general. The aim of this article is to prove that the Mackey formula holds in groups of type $A$ without restriction on $q$ (we say that $\mathbf{G}$ *is of type* $A$ if all the irreducible components of its root system are of type $A$ ; this definition does not involve the rational structure of $\mathbf{G}$) : this is Theorem 5.2.1. The Mackey formula has many consequences : for instance, the independence of the Lusztig functor on the choice of the parabolic subgroup and the commutation (up to sign) of Lusztig functors with Alvis-Curtis duality. Moreover, in the case of groups of type $A$, we obtain, as another consequence, a positive answer to the following conjecture (*cf.* [L4], Section 1) :

**Conjecture :** *If $p$ is almost good for $\mathbf{G}$, then the family of characteristic functions of $F$-stable unipotent cuspidal pairs forms an orthogonal basis of the space of absolutely cuspidal functions with unipotent support on $\mathbf{G}^F$.*

See Subsection 3.1 and Subsection 3.2 for the definition of absolutely cuspidal functions, cuspidal pairs and of their characteristic function. This conjecture is known to hold in any group if $p$ is almost good and $q$ is large enough (G. Lusztig [L4], Theorem 1.14) or if the center of $\mathbf{G}$ is connected (Shoji [S1] and [S2], Theorem 4.2). A new example of a group for which both conjectures are solved in this paper is $\mathbf{SL}_n(\mathbb{F}_q)$.

In the first section, we introduce some notation and some preliminary results. The second section investigates the problem of the comparison of Lusztig functors and Green functions in groups having the same Dynkin diagram. These results are useful to reduce the proof of the

Received by the editors 11 janvier 1999.

Mackey formula to simpler cases. The third section is concerned with the question of the link between absolutely cuspidal functions with unipotent support and unipotent cuspidal pairs. We obtain some partial results for the case of unipotent cuspidal pairs supported by the class of regular unipotent elements : this case plays a central role for groups of type $A$ because all cuspidal unipotent pairs in groups of type $A$ are supported by the class of regular unipotent elements. In the fourth section, we give some consequences of the Mackey formula (*cf.* Theorem 4.1.1) and make these results more precise in the case of groups of type $A$. This is a crucial step in the proof by induction on the dimension of $\mathbf{G}$ of the Mackey formula in groups of type $A$. The fifth section is devoted to this proof : it starts by a series of results which could be useful to prove the Mackey formula in other groups. In the last section, we investigate the consequences of the Mackey formula, for example the solution to the preceding conjecture without any condition on $q$ (*cf.* Theorem 6.2.1).

## Notation

Let $p$ be a prime number and let $\mathbb{F}$ denote an algebraic closure of the finite field with $p$ elements $\mathbb{F}_p$. We fix a power $q$ of $p$ and we denote by $\mathbb{F}_q$ the subfield of $\mathbb{F}$ with $q$ elements. All algebraic varieties and all algebraic groups will be considered over $\mathbb{F}$. If $\mathbf{H}$ is an algebraic group, we denote by $\mathbf{H}^\circ$ its neutral component, by $\mathbf{H}_{\mathrm{uni}}$ the set of unipotent elements of $\mathbf{H}$ and by $\mathbf{H}_{\mathrm{sem}}$ the set of semisimple elements of $\mathbf{H}$. If $h \in \mathbf{H}$, we denote by $C_{\mathbf{H}}(h)$ the centralizer of $h$ in $\mathbf{H}$ and by $C^\circ_{\mathbf{H}}(h)$ the neutral component of $C_{\mathbf{H}}(h)$ (in other words, $C^\circ_{\mathbf{H}}(h) = C_{\mathbf{H}}(h)^\circ$). We also define $A_{\mathbf{H}}(h) = C_{\mathbf{H}}(h)/C^\circ_{\mathbf{H}}(h)$. We denote by $\mathbf{Z}(\mathbf{H})$ the center of $\mathbf{H}$. If $\mathbf{H}$ is defined over $\mathbb{F}_q$ and if $F : \mathbf{H} \to \mathbf{H}$ is the corresponding Frobenius endomorphism, $H^1(F, \mathbf{H})$ will denote the set of $F$-conjugacy classes of elements of $\mathbf{H}$ (if $\mathbf{H}$ is abelian, then $H^1(F, \mathbf{H})$ is a finite abelian group).

We fix another prime number $\ell$ different from $p$ and we denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the $\ell$-adic field $\mathbb{Q}_\ell$. If $H$ is a finite group, all representations and all characters will be considered over $\overline{\mathbb{Q}}_\ell$. By a $H$-module, we mean a $\overline{\mathbb{Q}}_\ell H$-module of finite type and we denote by $\mathcal{K}H$ the Grothendieck group of the category of $H$-modules. Let $\mathrm{Class}(H)$ denote the $\overline{\mathbb{Q}}_\ell$-vector space of class functions $H \to \overline{\mathbb{Q}}_\ell$ and let

$$\begin{array}{rccc} \langle,\rangle_H : & \mathrm{Class}(H) \times \mathrm{Class}(H) & \longrightarrow & \overline{\mathbb{Q}}_\ell \\ & (\eta, \eta') & \longmapsto & \displaystyle\frac{1}{|H|}\sum_{h \in H} \eta(h)\overline{\eta'(h)} \end{array}$$

where $\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$, $x \mapsto \overline{x}$ is an automorphism of the field $\overline{\mathbb{Q}}_\ell$ such that $\overline{\zeta} = \zeta^{-1}$ for any root of unity $\zeta$ : $\langle,\rangle_H$ is a scalar product on $\mathrm{Class}(H)$. The set $\mathrm{Irr}\, H$ of irreducible characters of $H$ is an orthonormal basis of $\mathrm{Class}(H)$ and we have a natural isomorphism between $\mathcal{K}H$ and $\mathbb{Z}\,\mathrm{Irr}\, H$. We denote by $H^\wedge$ the group of linear characters of $H$ (if $H$ is abelian, then $\mathrm{Irr}\, H = H^\wedge$).

If $\mathbf{X}$ is an algebraic variety and if $i$ is a natural number, we will denote by $H^i_c(\mathbf{X})$ the cohomology group $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$. Finally, by a local system on the variety $\mathbf{X}$, we mean a $\overline{\mathbb{Q}}_\ell$-local system.

## 1. Preliminaries

We fix once and for all a connected reductive group $\mathbf{G}$ defined over $\mathbb{F}_q$ : let $F : \mathbf{G} \to \mathbf{G}$ denote the Frobenius endomorphism corresponding to this rational structure. We will denote by $\mathbf{Z}$ the center of $\mathbf{G}$. If $g \in \mathbf{G}^F$, we denote by $\gamma_g^{\mathbf{G}}$ (or $\gamma_g$ is no confusion is possible) the characteristic function of the $\mathbf{G}^F$-conjugacy class of $g$.

**1.1. Lusztig functors.** Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and assume that $\mathbf{P}$ has a rational Levi subgroup $\mathbf{L}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$ and, following Lusztig [L1], define

$$\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}.$$

Then the group $\mathbf{G}^F$ (respectively $\mathbf{L}^F$) acts on $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ by left (respectively right) translation. We denote by $H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$ the (virtual) $\mathbf{G}^F$-module-$\mathbf{L}^F$

$$H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}) = \sum_{k \in \mathbb{N}} (-1)^k \ H_c^k(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}).$$

Similarly, we define

$$H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})^{\vee} = \sum_{k \in \mathbb{N}} (-1)^k \ H_c^k(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})^{\vee}$$

where $H_c^i(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})^{\vee}$ is the dual of $H_c^i(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$ (this is a $\mathbf{L}^F$-module-$\mathbf{G}^F$).

Using these bimodules, Lusztig [L1] defined two functors :

$$\begin{array}{rccc} R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : & \mathcal{K}\mathbf{L}^F & \longrightarrow & \mathcal{K}\mathbf{G}^F \\ & \Lambda & \longmapsto & H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}^F} \Lambda \end{array}$$

and

$$\begin{array}{rccc} {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : & \mathcal{K}\mathbf{G}^F & \longrightarrow & \mathcal{K}\mathbf{L}^F \\ & \Gamma & \longmapsto & H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})^{\vee} \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^F} \Gamma \end{array}$$

respectively called **Lusztig induction functor** and **Lusztig restriction functor**. These functors extend by linearity to functions (also denoted by $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$) between the $\overline{\mathbb{Q}}_\ell$-vector spaces $\mathrm{Class}(\mathbf{L}^F)$ and $\mathrm{Class}(\mathbf{G}^F)$ : they are adjoint with respect to the scalar products $\langle , \rangle_{\mathbf{L}^F}$ and $\langle , \rangle_{\mathbf{G}^F}$.

**1.2. Green functions.** We denote by $Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ the map

$$\begin{array}{rccc} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : & \mathbf{G}_{\mathrm{uni}}^F \times \mathbf{L}_{\mathrm{uni}}^F & \longrightarrow & \overline{\mathbb{Q}}_\ell \\ & (u,v) & \longmapsto & \mathrm{Tr}((u,v), H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})). \end{array}$$

It is called the **Green function** associated to $\mathbf{L}$, $\mathbf{P}$ and $\mathbf{G}$. If $u \in \mathbf{G}_{\mathrm{uni}}^F$, let $Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(u,.)$ denote the function

$$\begin{array}{rccc} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(u,.) : & \mathbf{L}^F & \longrightarrow & \overline{\mathbb{Q}}_\ell \\ & v & \longmapsto & \begin{cases} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(u,v) & \text{if } v \text{ is unipotent,} \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

If $v \in \mathbf{L}_{\mathrm{uni}}^F$, we define in a similar way a class function $Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(.,v)$ on $\mathbf{G}^F$.

**1.3. Unipotent support functions.** We say that a class function $\gamma : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ **has a unipotent support** if $\gamma(g) = 0$ for any non-unipotent element $g \in \mathbf{G}^F$. We denote by $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$ the subspace of $\mathrm{Class}(\mathbf{G}^F)$ consisting of functions with unipotent support. If $s \in \mathbf{G}^F_{\mathrm{sem}}$ and if $\gamma \in \mathrm{Class}(\mathbf{G}^F)$, we define

$$\begin{array}{rccc} d_s^{\mathbf{G}}\gamma : & C^\circ_{\mathbf{G}}(s)^F & \longrightarrow & \overline{\mathbb{Q}}_\ell \\ & u & \longmapsto & \begin{cases} \gamma(su) & \text{if } u \text{ is unipotent,} \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

This defines a linear map $d_s^{\mathbf{G}} : \mathrm{Class}(\mathbf{G}^F) \to \mathrm{Class}_{\mathrm{uni}}(C^\circ_{\mathbf{G}}(s)^F)$. Using this notation, the "character formulas" (*cf.* [DM], Proposition 12.2) can be written in the following way : if $s \in \mathbf{G}^F_{\mathrm{sem}}$ and $t \in \mathbf{L}^F_{\mathrm{sem}}$, we have

**(1.3.1)**
$$d_s^{\mathbf{G}} \circ R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = \frac{1}{|\mathbf{L}^F|.|C^\circ_{\mathbf{G}}(s)^F|} \sum_{\substack{g \in \mathbf{G}^F \\ s \in {}^g\mathbf{L}}} |C^\circ_{{}^g\mathbf{L}}(s)^F| \; R^{C^\circ_{\mathbf{G}}(s)}_{C^\circ_{{}^g\mathbf{L}}(s) \subset C^\circ_{{}^g\mathbf{P}}(s)} \circ d_s^{{}^g\mathbf{L}} \circ (\mathrm{ad}\, g)_{\mathbf{L}}$$

**(1.3.2)**
$$d_t^{\mathbf{L}} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = {}^*R^{C^\circ_{\mathbf{G}}(t)}_{C^\circ_{\mathbf{L}}(t) \subset C^\circ_{\mathbf{P}}(t)} \circ d_t^{\mathbf{G}}$$

where $(\mathrm{ad}\, g)_{\mathbf{L}} : \mathrm{Class}(\mathbf{L}^F) \to \mathrm{Class}({}^g\mathbf{L}^F)$ is the natural isometry induced by the isomorphism $\mathrm{ad}\, g : \mathbf{L}^F \to {}^g\mathbf{L}^F$, $l \mapsto {}^g l$.

**1.4. The Mackey formula.** Let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{G}$ and assume that $\mathbf{Q}$ has a rational Levi subgroup $\mathbf{M}$. We denote by $\mathcal{S}_{\mathbf{G}}(\mathbf{L}, \mathbf{M})$ the set of elements $g \in \mathbf{G}$ such that $\mathbf{L}$ and ${}^g\mathbf{M}$ have a common maximal torus. We call **the Mackey formula** the following identity :

$(*)$
$$ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} = \sum_{g \in \mathbf{L}^F \backslash \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F / \mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M} \subset \mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M} \subset \mathbf{P}\cap{}^g\mathbf{M}} \circ (\mathrm{ad}\, g)_{\mathbf{M}}.$$

It is known that the Mackey formula holds in many cases. More precisely, we have

**Proposition 1.4.1.** *The Mackey formula* $(*)$ *holds in the following cases :*
(i) *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are* $F$*-stable* (*Deligne, cf. [LS], Theorem 2.5*),
(ii) *If* $\mathbf{L}$ *or* $\mathbf{M}$ *is a maximal torus of* $\mathbf{G}$ (*cf. [DL2], Theorem 7*),
(iii) *If* $q > q_0(\mathbf{G})$ *where* $q_0(\mathbf{G})$ *is a constant depending only on the root datum associated to* $\mathbf{G}$ (*cf. [B1], Theorem 5.1.1*).

**Conjecture A :** *The Mackey formula holds.*

The aim of this article is to prove that the Mackey formula holds if all irreducible components of the root system associated to $\mathbf{G}$ are of type $A$.

We define
$$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} - \sum_{g \in \mathbf{L}^F \backslash \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F / \mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M} \subset \mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M} \subset \mathbf{P}\cap{}^g\mathbf{M}} \circ (\mathrm{ad}\, g)_{\mathbf{M}}.$$

Then $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} : \mathrm{Class}(\mathbf{M}^F) \longrightarrow \mathrm{Class}(\mathbf{L}^F)$ is a linear map. The Mackey formula $(*)$ is equivalent to the vanishing of $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}$.

**Definition 1.4.2.** *We say that* **the Mackey formula holds in G** *if, for any connected reductive* $F$*-stable subgroup* $\mathbf{G}'$ *of* $\mathbf{G}$ *which has* **the same rank** *as* $\mathbf{G}$ *and for any parabolic subgroups* $\mathbf{P}'$ *and* $\mathbf{Q}'$ *of* $\mathbf{G}'$ *having* $F$*-stable Levi subgroups* $\mathbf{L}'$ *and* $\mathbf{M}'$ *respectively, we have* $\Delta^{\mathbf{G}'}_{\mathbf{L}'\subset\mathbf{P}',\mathbf{M}'\subset\mathbf{Q}'} = 0$.

**1.5. The Mackey formula for Green functions.** Similarly to the preceding Subsection 1.4, we put, for all $v \in \mathbf{L}^F_{\text{uni}}$ and $w \in \mathbf{M}^F_{\text{uni}}$

$$\begin{aligned}\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(v,w) \quad &= \quad \langle Q^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(.,v^{-1}), Q^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}(.,w^{-1})\rangle_{\mathbf{G}^F} \\ &- \sum_{g\in\mathbf{L}^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} \langle Q^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q}}(v,.), Q^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{P}\cap{}^g\mathbf{M}}({}^g w,.)\rangle_{\mathbf{L}^F\cap{}^g\mathbf{M}^F}\end{aligned}$$

This defines a function $\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} : \mathbf{L}^F_{\text{uni}} \times \mathbf{M}^F_{\text{uni}} \longrightarrow \overline{\mathbb{Q}}_\ell$. We call **the Mackey formula for Green functions** the equality $\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = 0$.

**Conjecture B :** *The Mackey formula for Green functions holds.*

The Mackey formula for Green functions is known to be equivalent to the Mackey formula (F. Digne and J. Michel, *cf.* [B1], Proposition 2.3.6) or, in other words, conjecture A is equivalent to conjecture B.

**1.6. Lusztig series.** Let $\mathbf{T}_0$ be an $F$-stable maximal torus of $\mathbf{G}$. We denote by $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ a dual triple of $(\mathbf{G}, \mathbf{T}_0, F)$ in the sense of Deligne-Lusztig. If $\sigma$ is a semisimple element of $\mathbf{G}^{*F^*}$, we denote by $(\sigma)_{\mathbf{G}^{*F^*}}$ (or $(\sigma)$ if no confusion is possible) the $\mathbf{G}^{*F^*}$-conjugacy class of $\sigma$. To this conjugacy class is associated a subset $\mathcal{E}(\mathbf{G}^F, (\sigma))$ of $\operatorname{Irr}\mathbf{G}^F$, called the **(rational) Lusztig series** associated to $(\sigma)$. Then there is a partition (*cf.* [DM], Theorem 14.51) :

$$\operatorname{Irr}\mathbf{G}^F = \bigcup_{(\sigma)} \mathcal{E}(\mathbf{G}^F, (\sigma))$$

where $(\sigma)$ runs over the set of $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements of $\mathbf{G}^{*F^*}$. This implies that we have

$$\operatorname{Class}(\mathbf{G}^F) = \bigoplus_{(\sigma)} \overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{G}^F, (\sigma))$$

and this direct sum is orthogonal. If $\gamma \in \operatorname{Class}(\mathbf{G}^F)$, we denote by $\gamma_{(\sigma)}$ the orthogonal projection of $\gamma$ on $\overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{G}^F, (\sigma))$. We have :

$$\gamma = \sum_{(\sigma)} \gamma_{(\sigma)}.$$

Let $\mathbf{L}^*$ be an $F^*$-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ which is dual to $\mathbf{L}$. We have the following result (*cf.* [B1], Corollary 4.4.1) :

**Proposition 1.6.1.** *Let $\sigma$ be a semisimple element of $\mathbf{L}^{*F^*}$ and let $\lambda \in \overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{L}^F, (\sigma)_{\mathbf{L}^{*F^*}})$. Then $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\lambda) \in \overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{G}^F, (\sigma)_{\mathbf{G}^{*F^*}})$.*

**Corollary 1.6.2.** *Let $\sigma$ be a semisimple element of $\mathbf{G}^{*F^*}$ and let $\gamma \in \overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{G}^F, (\sigma)_{\mathbf{G}^{*F^*}})$. Then*

$${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\gamma) \in \bigoplus_{(\tau)_{\mathbf{L}^{*F^*}} \subset (\sigma)_{\mathbf{G}^{*F^*}}} \overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{L}^F, (\tau)_{\mathbf{L}^{*F^*}}).$$

**1.7. Action of $\mathbf{Z}^F$.** If $z \in \mathbf{Z}^F$ and if $\gamma \in \mathrm{Class}(\mathbf{G}^F)$, we define

$$\begin{array}{rccc} t_z^{\mathbf{G}}\gamma : & \mathbf{G}^F & \longrightarrow & \overline{\mathbb{Q}}_\ell \\ & g & \longmapsto & \gamma(zg). \end{array}$$

Then $t_z^{\mathbf{G}}\gamma$ is a class function on $\mathbf{G}^F$ and $t_z^{\mathbf{G}} : \mathrm{Class}(\mathbf{G}^F) \to \mathrm{Class}(\mathbf{G}^F)$ is an isometry. Moreover, we have

**(1.7.1)** $$t_z^{\mathbf{G}} \circ t_{z'}^{\mathbf{G}} = t_{zz'}^{\mathbf{G}}$$

for all $z$ and $z'$ in $\mathbf{Z}^F$. So this defines an action of $\mathbf{Z}^F$ on the $\overline{\mathbb{Q}}_\ell$-vector space $\mathrm{Class}(\mathbf{G}^F)$. If $\phi : \mathbf{Z}^F \to \overline{\mathbb{Q}}_\ell^{\times}$ is a linear character, we denote by $\mathrm{Class}(\mathbf{G}^F)^\phi$ the subspace of $\mathrm{Class}(\mathbf{G}^F)$ consisting of class functions $\gamma$ such that

$$t_z^{\mathbf{G}}\gamma = \phi(z)\gamma$$

for any $z \in \mathbf{Z}^F$. Then we have

**(1.7.2)** $$\mathrm{Class}(\mathbf{G}^F) = \bigoplus_{\phi \in (\mathbf{Z}^F)^\wedge} \mathrm{Class}(\mathbf{G}^F)^\phi$$

and this direct sum is orthogonal. More generally, if $W$ is a subspace of $\mathrm{Class}(\mathbf{G}^F)$ stable under the action of $\mathbf{Z}^F$ and if $\phi$ is a linear character of $\mathbf{Z}^F$, then we put $W^\phi = W \cap \mathrm{Class}(\mathbf{G}^F)^\phi$.

The following identities can be deduced immediately from the fact that the action of $\mathbf{Z}^F$ on $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ by left (or right) translation commutes with the actions of $\mathbf{G}^F$ and $\mathbf{L}^F$ respectively :

**(1.7.3)** $$t_z^{\mathbf{G}} \circ R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ t_z^{\mathbf{L}},$$

**(1.7.4)** $$t_z^{\mathbf{L}} \circ {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ t_z^{\mathbf{G}}$$

for all $z \in \mathbf{Z}^F$.

**Lemma 1.7.5.** *If $\sigma$ is a semisimple element of $\mathbf{G}^{*F^*}$, then there exists a unique linear character $\phi : \mathbf{Z}^F \to \overline{\mathbb{Q}}_\ell^{\times}$, depending only on the $\mathbf{G}^{*F^*}$-conjugacy class of $\sigma$, such that*

$$\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F, (\sigma)) \subset \mathrm{Class}(\mathbf{G}^F)^\phi.$$

Notation - The linear character $\phi$ of the preceding Lemma 1.7.5 will be denoted by $\hat{\sigma}$ or $\hat{\sigma}^{\mathbf{G}}$ if we need to specify the group.

**1.8. Action of $H^1(F, \mathbf{Z})$.** For each $a \in H^1(F, \mathbf{Z})$, we choose an element $l_a \in \mathbf{L}$ such that $l_a^{-1}F(l_a)$ is an element of $\mathbf{Z}$ representing $a$. Because $\mathbf{Z}$ is central in $\mathbf{G}$, conjugation by $l_a$ induces an automorphism of $\mathbf{L}$ commuting with $F$. In particular, $l_a$ normalizes $\mathbf{L}^F$.

If $m_a$ is another element of $\mathbf{L}$ such that $m_a^{-1}F(m_a)$ is an element of $\mathbf{Z}$ representing $a$, then there exist $z \in \mathbf{Z}$ and $l \in \mathbf{L}^F$ such that $m_a = zll_a$. This proves that the automorphism of $\mathbf{L}^F$ induced by conjugation by $l_a$ is well-defined up to an inner automorphism of $\mathbf{L}^F$. Hence the map

$$\begin{array}{ccc} \mathrm{Irr}(\mathbf{L}^F) & \longrightarrow & \mathrm{Irr}(\mathbf{L}^F) \\ \lambda & \longmapsto & {}^{l_a}\lambda = \lambda \circ \mathrm{ad}\, l_a \end{array}$$

depends only on $a$. We will denote it by $\hat{a} : \mathrm{Irr}(\mathbf{L}^F) \to \mathrm{Irr}(\mathbf{L}^F)$, $\lambda \mapsto {}^{\hat{a}}\lambda$. It is easily checked that

$$\hat{a}\hat{b} = \widehat{ab}$$

for all $a$ and $b$ in $H^1(F,\mathbf{Z})$. So this defines an action of $H^1(F,\mathbf{Z})$ on $\mathrm{Irr}(\mathbf{L}^F)$. This action extends naturally by linearity to an action by isometries on $\mathrm{Class}(\mathbf{L}^F)$ denoted in the same way. Similarly, we also have an action of $H^1(F,\mathbf{Z})$ on $\mathrm{Class}(\mathbf{G}^F)$, also denoted by

$$\begin{array}{ccc} H^1(F,\mathbf{Z}) \times \mathrm{Class}(\mathbf{G}^F) & \longrightarrow & \mathrm{Class}(\mathbf{G}^F) \\ (a,\gamma) & \longmapsto & {}^{\hat{a}}\gamma. \end{array}$$

The following equalities hold :

**(1.8.1)** $$\hat{a} \circ R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ \hat{a},$$

**(1.8.2)** $$\hat{a} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ \hat{a}$$

for any $a \in H^1(F,\mathbf{Z})$. Indeed, conjugation by $l_a$ induces an automorphism of the variety $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}$.

If $\zeta$ is a linear character of $H^1(F,\mathbf{Z})$, we denote by $\mathrm{Class}(\mathbf{G}^F)_\zeta$ the subspace of $\mathrm{Class}(\mathbf{G}^F)$ consisting of class functions $\gamma$ on $\mathbf{G}^F$ such that ${}^{\hat{a}}\gamma = \zeta(a)\gamma$ for all $a \in H^1(F,\mathbf{Z})$. We have

**(1.8.3)** $$\mathrm{Class}(\mathbf{G}^F) = \bigoplus_{\zeta \in H^1(F,\mathbf{Z})} \mathrm{Class}(\mathbf{G}^F)_\zeta$$

and this direct sum is orthogonal. As in the preceding subsection, if $W$ is a subspace of $\mathrm{Class}(\mathbf{G}^F)$ stable under the action of $H^1(F,\mathbf{Z})$ and if $\zeta$ is a linear character of $H^1(F,\mathbf{Z})$, we put $W_\zeta = W \cap \mathrm{Class}(\mathbf{G}^F)_\zeta$.

Example - *The subspace* $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$ *is stable under the action of* $H^1(F,\mathbf{Z})$. *In particular, we have*

$$\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F) = \bigoplus_{\zeta \in H^1(F,\mathbf{Z})} \mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)_\zeta$$

By the identities 1.8.1 and 1.8.2, we have the following

**Lemma 1.8.4.** *Let* $\zeta \in H^1(F,\mathbf{Z})^\wedge$.
(a) *If* $\lambda \in \mathrm{Class}(\mathbf{L}^F)_\zeta$, *then* $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\lambda \in \mathrm{Class}(\mathbf{G}^F)_\zeta$.
(b) *If* $\gamma \in \mathrm{Class}(\mathbf{G}^F)_\zeta$ *then* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\gamma \in \mathrm{Class}(\mathbf{L}^F)_\zeta$.

The natural map $\mathbf{Z} \to \mathbf{Z}(\mathbf{L})$ induces a surjective morphism $h^{\mathbf{G}}_{\mathbf{L}} : \mathbf{Z}/\mathbf{Z}^\circ \to \mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$ (*cf.* [DLM], Lemma 1.4) so it induces a surjective morphism $\mathbf{h}^{\mathbf{G}}_{\mathbf{L}} : H^1(F,\mathbf{Z}) \to H^1(F,\mathbf{Z}(\mathbf{L}))$.

**Lemma 1.8.5.** *The group* $\mathrm{Ker}\,\mathbf{h}^{\mathbf{G}}_{\mathbf{L}}$ *acts trivially on* $\mathrm{Class}(\mathbf{L}^F)$. *In particular, if* $\zeta$ *is a linear character of* $H^1(F,\mathbf{Z})$ *such that* $\mathrm{Ker}\,\mathbf{h}^{\mathbf{G}}_{\mathbf{L}} \not\subseteq \mathrm{Ker}\,\zeta$, *then* $\mathrm{Class}(\mathbf{L}^F)_\zeta = \{0\}$.

Proof - If $a \in \mathrm{Ker}\,\mathbf{h}^{\mathbf{G}}_{\mathbf{L}}$, then we can choose the element $l_a$ in $\mathbf{Z}(\mathbf{L})$. This proves the first assertion. The second follows immediately. ■

**Corollary 1.8.6.** *Let* $\zeta \in H^1(F,\mathbf{Z})^\wedge$ *be such that* $\mathrm{Ker}\,\mathbf{h}^{\mathbf{G}}_{\mathbf{L}} \not\subseteq \mathrm{Ker}\,\zeta$ *and let* $\gamma \in \mathrm{Class}(\mathbf{G}^F)_\zeta$. *Then* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\gamma = 0$.

Let $\mathbf{T}$ be an $F$-stable maximal torus of $\mathbf{G}$ and let $\theta : \mathbf{T}^F \to \overline{\mathbb{Q}}_\ell^{\times}$ be a linear character of $\mathbf{T}^F$. We have $\mathrm{Ker}\,\mathbf{h}^{\mathbf{G}}_{\mathbf{T}} = H^1(F,\mathbf{Z})$ so, by 1.8.1 and by Lemma 1.8.5, we have

$$ {}^{\hat{a}}R^{\mathbf{G}}_{\mathbf{T}}(\theta) = R^{\mathbf{G}}_{\mathbf{T}}(\theta).$$

Thus the action of $H^1(F,\mathbf{Z})$ on $\mathrm{Irr}\,\mathbf{G}^F$ stabilizes the Lusztig series. Moreover, if $\sigma$ is a semisimple element of $\mathbf{G}^{*F^*}$ then there is a surjective morphism

$$\psi^{\mathbf{G}}_\sigma : H^1(F,\mathbf{Z}) \to (A_{\mathbf{G}^*}(\sigma)^{F^*})^\wedge$$

(*cf.* , e.g, [DLM], 3.12) and the kernel of this morphism acts trivially on $\mathcal{E}(\mathbf{G}^F,(\sigma))$ by the same argument as in [DLM], proof of 3.12, p.171. In particular, we have

**(1.8.7)** *If* $\zeta \in H^1(F,\mathbf{Z})^\wedge$ *is such that* $\operatorname{Ker}\psi_\sigma^{\mathbf{G}} \not\subseteq \operatorname{Ker}\zeta$ *then* $\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F,(\sigma))_\zeta = \{0\}$.

Notation - The actions of $\mathbf{Z}^F$ and $H^1(F,\mathbf{Z})$ on $\operatorname{Class}(\mathbf{G}^F)$ commute. If $W$ is a subspace of $\operatorname{Class}(\mathbf{G}^F)$ stable under both actions and if $\phi$ and $\zeta$ are linear characters of $\mathbf{Z}^F$ and $H^1(F,\mathbf{Z})$ respectively, we put

$$W_\zeta^\phi = W^\phi \cap W_\zeta.$$

In particular, we have

$$\operatorname{Class}(\mathbf{G}^F) = \bigoplus_{\substack{\phi\in(\mathbf{Z}^F)^\wedge \\ \zeta\in H^1(F,\mathbf{Z})^\wedge}} \operatorname{Class}(\mathbf{G}^F)_\zeta^\phi.$$

## 2. Comparison of Lusztig functors in groups of the same type

Let $\tilde{\mathbf{G}}$ be another connected reductive group defined over $\mathbb{F}_q$, with Frobenius endomorphism also denoted by $F : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$. We assume that there is a morphism of algebraic groups $i : \mathbf{G} \to \tilde{\mathbf{G}}$ defined over $\mathbb{F}_q$ and satisfying the following conditions :

(a) $\operatorname{Ker} i$ *is central in* $\mathbf{G}$,

(b) $i(\mathbf{G})$ *contains the derived group of* $\tilde{\mathbf{G}}$.

We denote by $i_{\mathbf{G}} : \operatorname{Class}(\tilde{\mathbf{G}}^F) \to \operatorname{Class}(\mathbf{G}^F)$, $\gamma \mapsto \gamma \circ i$.

Let $\tilde{\mathbf{L}}$ be an $F$-stable Levi subgroup of a parabolic subgroup $\tilde{\mathbf{P}}$ of $\tilde{\mathbf{G}}$ and let

$$\mathbf{L} = i^{-1}(\tilde{\mathbf{L}}) \qquad \text{and} \qquad \mathbf{P} = i^{-1}(\tilde{\mathbf{P}}).$$

Then $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}$ is an $F$-stable Levi subgroup of $\mathbf{P}$. We denote by $\tilde{\mathbf{U}}$ (respectively $\mathbf{U}$) the unipotent radical of $\tilde{\mathbf{P}}$ (respectively $\mathbf{P}$). The goal of this section is to study the link between the Lusztig functors $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and $R_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}$ (or, equivalently, between ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and ${}^*R_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}$) and the link between Green functions $Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and $Q_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}$ (*cf.* [DM], Proposition 13.22 for another proof of Corollary 2.1.3).

### 2.1. Lusztig functors.

In this subsection, we assume that $\operatorname{Ker} i$ is connected (in particular, we have $i(\mathbf{G}^F) = i(\mathbf{G})^F$ and $i(\mathbf{L}^F) = i(\mathbf{L})^F$ by Lang's theorem). The algebra $\overline{\mathbb{Q}}_\ell\tilde{\mathbf{G}}^F$ (respectively $\overline{\mathbb{Q}}_\ell\tilde{\mathbf{L}}^F$) has a natural structure of $\mathbf{G}^F$-module-$\mathbf{G}^F$ (respectively $\mathbf{L}^F$-module-$\mathbf{L}^F$). We have the following lemma :

**Proposition 2.1.1.** *If the kernel of* $i$ *is connected, then* $i$ *induces an isomorphism of* $\tilde{\mathbf{G}}^F$*-module-*$\mathbf{L}^F$

$$H_c^k(\mathbf{Y}_{\tilde{\mathbf{U}}}^{\tilde{\mathbf{G}}}) \simeq \overline{\mathbb{Q}}_\ell\tilde{\mathbf{G}}^F \otimes_{\overline{\mathbb{Q}}_\ell\mathbf{G}^F} H_c^k(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$$

*and an isomorphism of* $\mathbf{G}^F$*-module-*$\tilde{\mathbf{L}}^F$

$$H_c^k(\mathbf{Y}_{\tilde{\mathbf{U}}}^{\tilde{\mathbf{G}}}) \simeq H_c^k(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}) \otimes_{\overline{\mathbb{Q}}_\ell\mathbf{L}^F} \overline{\mathbb{Q}}_\ell\tilde{\mathbf{L}}^F.$$

Proof - We just give a proof for the first isomorphism, the second one being proved in a similar way. Let $\mathbf{G}_1$ be the image of $i$. Then we denote by $i_1 : \mathbf{G} \to \mathbf{G}_1$ the morphism induced by $i$ and by $i_2 : \mathbf{G}_1 \to \tilde{\mathbf{G}}$ the natural injection. To prove Proposition 2.1.1, it is sufficient to prove it when $i = i_1$ and when $i = i_2$ by transitivity of the tensor product. In other words, it is sufficient to prove the following lemma :

**Lemma 2.1.2.** (a) *If $i$ is surjective and has a connected kernel, then it induces a bijective morphism of varieties $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}/(\operatorname{Ker} i)^F \longrightarrow \mathbf{Y}_{\tilde{\mathbf{U}}}^{\tilde{\mathbf{G}}}$. Hence we have an isomorphism of $\mathbf{G}^F$-module-$\mathbf{L}^F$*

$$H_c^k(\mathbf{Y}_{\tilde{\mathbf{U}}}^{\tilde{\mathbf{G}}}) \simeq H_c^k(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})^{(\operatorname{Ker} i)^F}$$

*for all $k \in \mathbb{N}$.*

(b) *If $\mathbf{G}$ is a closed subgroup of $\tilde{\mathbf{G}}$ and if $i : \mathbf{G} \to \tilde{\mathbf{G}}$ is the canonical injection, then*

$$\mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}} = \coprod_{g \in \tilde{\mathbf{G}}^F/\mathbf{G}^F} g.\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}} = \coprod_{l \in \tilde{\mathbf{L}}^F/\mathbf{L}^F} \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}.l$$

*(note that $\mathbf{U} = \tilde{\mathbf{U}}$).*

Proof of Lemma 2.1.2 - Let first prove (a). So assume that $i$ is surjective and has a connected kernel. Then the map $i$ induces a morphism of varieties $i' : \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}} \to \mathbf{Y}_{\tilde{\mathbf{U}}}^{\tilde{\mathbf{G}}}$. We first prove that $i'$ is surjective. Let $\tilde{g} \in \mathbf{Y}_{\tilde{\mathbf{U}}}^{\tilde{\mathbf{G}}}$ and let $g \in \mathbf{G}$ be such that $i(g) = \tilde{g}$. Then there exists $z \in \operatorname{Ker} i$ such that $g^{-1}F(g) \in z\mathbf{U}$. But, since $\operatorname{Ker} i$ is connected, there exists an element $s \in \operatorname{Ker} i$ such that $s^{-1}F(s) = z^{-1}$. Then $(gs)^{-1}F(gs) = s^{-1}F(s)g^{-1}F(g) \in \mathbf{U}$ because $\operatorname{Ker} i$ is central in $\mathbf{G}$. So $gs \in \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ and $i'(gs) = \tilde{g}$ which proves that $i'$ is surjective.

We now prove that fibers of $i'$ are $(\operatorname{Ker} i)^F$-orbits. Let $g$ and $h$ be two elements of $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ such that $i'(g) = i'(h)$ or equivalently, $i(g) = i(h)$. Then there exists $z \in \operatorname{Ker} i$ such that $g = hz$. So $g^{-1}F(g) = z^{-1}F(z)h^{-1}F(h) \in z^{-1}F(z)\mathbf{U} \cap \mathbf{U}$. So $F(z) = z$. This proves the first assertion of (a). The second follows from the first one and from [DM], Proposition 10.10, (i).

Let now prove (b). So assume that $\mathbf{G}$ is a closed subgroup of $\tilde{\mathbf{G}}$ and that $i : \mathbf{G} \to \tilde{\mathbf{G}}$ is the canonical injection. It is clear that $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ is contained in $\mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}}$. Moreover, $\mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}}$ is stable under left translations by an element of $\tilde{\mathbf{G}}^F$. So we have

$$\coprod_{g \in \tilde{\mathbf{G}}^F/\mathbf{G}^F} g.\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}} \subset \mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}}.$$

Conversely, let $\tilde{x} \in \mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}}$. Then $\tilde{x}^{-1}F(\tilde{x}) \in \mathbf{U} \subset \mathbf{G}$. So, by Lang's theorem, there exists $x \in \mathbf{G}$ such that $x^{-1}F(x) = \tilde{x}^{-1}F(\tilde{x})$. Let $g = \tilde{x}x^{-1}$. Then $g \in \tilde{\mathbf{G}}^F$, $x \in \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ and $\tilde{x} = gx$. This proves that

$$\mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}} \subset \coprod_{g \in \tilde{\mathbf{G}}^F/\mathbf{G}^F} g.\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$$

and the first equality of (b). The second one is proved in a similar way. ■

**Corollary 2.1.3.** *If the kernel of $i$ is connected, then*

(a) $i_{\mathbf{G}} \circ R_{\tilde{\mathbf{L}} \subset \tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}} = R_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}} \circ i_{\mathbf{L}}$.

(b) $i_{\mathbf{L}} \circ {}^*R_{\tilde{\mathbf{L}} \subset \tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}} = {}^*R_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}} \circ i_{\mathbf{G}}$.

Proof - This follows immediately from Proposition 2.1.1. ■

Remarks - (1) In [DM], Proposition 13.22, the preceding Corollary 2.1.3 is proved in an entirely different way.

(2) If $i$ is injective, then Corollary 2.1.3 can be written in the following way :

**(2.1.4)** $$\mathrm{Res}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} \circ R_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}} = R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ \mathrm{Res}_{\mathbf{L}^F}^{\tilde{\mathbf{L}}^F},$$

**(2.1.5)** $$\mathrm{Res}_{\mathbf{L}^F}^{\tilde{\mathbf{L}}^F} \circ {}^*R_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}} = {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ \mathrm{Res}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F}$$

where $\mathbf{G}^F$ is viewed as a subgroup of $\tilde{\mathbf{G}}^F$. By taking adjoints, we get also the following formulas :

**(2.1.6)** $$\mathrm{Ind}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} \circ R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = R_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}} \circ \mathrm{Ind}_{\mathbf{L}^F}^{\tilde{\mathbf{L}}^F},$$

**(2.1.7)** $$\mathrm{Ind}_{\mathbf{L}^F}^{\tilde{\mathbf{L}}^F} \circ {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = {}^*R_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}} \circ \mathrm{Ind}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F}.$$

**2.2. Green functions.** As a particular case of Lemma 2.1.2, we get the following

**Proposition 2.2.1.** *Let $u$ and $v$ be unipotent elements of $\mathbf{G}^F$ and $\mathbf{L}^F$ respectively. Then :*

(a) *If $i$ is surjective and has a connected kernel, then*

$$Q_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}(i(u),i(v)) = \frac{1}{|(\mathrm{Ker}\, i)^F|} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(u,v).$$

(b) *If $i$ is injective, then*

$$\begin{aligned} Q_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}(u,v) &= \sum_{g\in\tilde{\mathbf{G}}^F/\mathbf{G}^F} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}({}^g u,v) \\ &= \sum_{l\in\tilde{\mathbf{L}}^F/\mathbf{L}^F} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(u,{}^l v) \end{aligned}$$

*where $\mathbf{G}_{\mathrm{uni}}^F$ and $\tilde{\mathbf{G}}_{\mathrm{uni}}^F$ are identified via $i$.*

Proof - Let $u$ and $v$ be two unipotent elements of $\mathbf{G}^F$ and $\mathbf{L}^F$ respectively. Then, by [DM], Proposition 10.10, (ii), and by Lemma 2.1.2, (a), we have

$$\mathrm{Tr}(i(u),i(v)),H_c^*(\mathbf{Y}_{\tilde{\mathbf{U}}}^{\tilde{\mathbf{G}}})) = \frac{1}{|(\mathrm{Ker}\, i)^F|} \sum_{z\in(\mathrm{Ker}\, i)^F} \mathrm{Tr}((u,zv),H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})).$$

But, if $z\in(\mathrm{Ker}\, i)^F$ is different from 1, then $\mathrm{Tr}((u,zv),H_c^*(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}))=0$ (*cf.* [DM], Lemma 12.3) and (a) follows.

(b) follows immediately from Lemma 2.1.2, (a). ∎

We return to the general case. We have a morphism of groups $H^1(F,\mathrm{Ker}\, i)\to H^1(F,\mathbf{Z})$. So, if $z\in H^1(F,\mathrm{Ker}\, i)$, we also denote by $\hat{z}$ the action of the image of $z$ in $H^1(F,\mathbf{Z})$ on the conjugacy classes of $\mathbf{G}^F$ or $\mathbf{L}^F$.

**Proposition 2.2.2.** *Assume that $i$ is surjective. Let $u$ and $v$ be two unipotent elements of $\mathbf{G}^F$ and $\mathbf{L}^F$ respectively. Then*

$$\begin{aligned} Q_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}(i(u),i(v)) &= \frac{1}{|(\mathrm{Ker}\, i)^F|} \sum_{z\in H^1(F,\mathrm{Ker}\, i)} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}({}^{\hat{z}}u,v) \\ &= \frac{1}{|(\mathrm{Ker}\, i)^F|} \sum_{z\in H^1(F,\mathrm{Ker}\, i)} Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(u,{}^{\hat{z}}v). \end{aligned}$$

Proof - Let $\mathbf{G}_0 = \mathbf{G}/(\operatorname{Ker} i)^\circ$ and let $\pi : \mathbf{G} \to \mathbf{G}_0$ be the canonical surjection. We denote by $i_1 : \mathbf{G}_0 \to \tilde{\mathbf{G}}$ the morphism of algebraic groups induced by $i$. By Proposition 2.2.1, (a), we have

$$Q^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}(\pi(u),\pi(v)) = \frac{1}{|(\operatorname{Ker} i)^{\circ F}|} Q^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(u,v).$$

So it is sufficient to prove the Proposition 2.2.2 for the morphism $i_1$. In other words, we can assume that $\operatorname{Ker} i$ is finite. In this case, $|(\operatorname{Ker} i)^F| = |H^1(F,\operatorname{Ker} i)|$. Hence it is sufficient to prove that

$$\begin{aligned} Q^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}(i(u),i(v)) &= \frac{1}{|\operatorname{Ker} i|}\sum_{z\in\operatorname{Ker} i} Q^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}({}^{l_z}u,v) \\ &= \frac{1}{|\operatorname{Ker} i|}\sum_{z\in\operatorname{Ker} i} Q^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(u,{}^{l_z}v) \end{aligned}$$

where, for each $z \in \operatorname{Ker} i$, $l_z$ is an element of $\mathbf{L}$ such that $l_z^{-1}F(l_z) = z$.

The map $i$ induces a bijective morphism of varieties

$$\Big(\bigcup_{z\in\operatorname{Ker} i} l_z \mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}\Big)/\operatorname{Ker} i \longrightarrow \mathbf{Y}^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{U}}}.$$

Hence, we have (for example by [DM], Proposition 10.10, (ii)),

$$\operatorname{Tr}((u,v),H^*_c(\mathbf{Y}^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{U}}})) = \frac{1}{|\operatorname{Ker} i|}\sum_{z\in\operatorname{Ker} i}\operatorname{Tr}((u,zv),\bigoplus_{z'\in\operatorname{Ker} i} H^*_c(l_{z'}\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})).$$

But, if $z$ and $z'$ are in $\operatorname{Ker} i$, then $ul_{z'}\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}zv = l_{z'z^{-1}F(z)}\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}$. Hence, if $F(z) \neq z$, then

$$\operatorname{Tr}((u,zv),\bigoplus_{z'\in\operatorname{Ker} i} H^*_c(l_{z'}\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})) = 0.$$

So, we have

$$\begin{aligned} \operatorname{Tr}((u,v),H^*_c(\mathbf{Y}^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{U}}})) &= \frac{1}{|\operatorname{Ker} i|}\sum_{\substack{z\in(\operatorname{Ker} i)^F\\ z'\in\operatorname{Ker} i}}\operatorname{Tr}((u,zv),H^*_c(l_{z'}\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})) \\ &= \frac{1}{|\operatorname{Ker} i|}\sum_{\substack{z\in(\operatorname{Ker} i)^F\\ z'\in\operatorname{Ker} i}}\operatorname{Tr}(({}^{l_{z'}^{-1}}u,zv),H^*_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})) \\ &= \frac{1}{|\operatorname{Ker} i|}\sum_{z'\in\operatorname{Ker} i}\operatorname{Tr}(({}^{l_{z'}^{-1}}u,v),H^*_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})) \end{aligned}$$

because $\operatorname{Tr}(({}^{l_{z'}}u,zv),H^*_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})) = 0$ for all $z \in (\operatorname{Ker} i)^F$ different from 1 and $z' \in \operatorname{Ker} i$ (*cf.* [DM], Proposition 12.3). This proves the first equality of the Proposition and the second one follows by a similar argument. ■

**Corollary 2.2.3.** *If $i$ is surjective and* $\operatorname{Ker} i \subset \{z^{-1}F(z) \mid z \in \mathbf{Z}(\mathbf{L})\}$ *then*

$$Q^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}}}(i(u),i(v)) = \frac{1}{|(\operatorname{Ker} i)^{\circ F}|} Q^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(u,v)$$

*for all unipotent elements $u$ and $v$ in $\mathbf{G}^F$ and $\mathbf{L}^F$ respectively. In particular, this equality holds whenever* $\operatorname{Ker} i \subset \mathbf{Z}(\mathbf{L})^\circ$.

## 3. Absolutely cuspidal functions

In this section, we investigate the link between absolutely cuspidal functions and cuspidal local systems (*cf.* definitions below). The first two subsections recall general facts while the third deals with the particular case of cuspidal local systems supported by the conjugacy class of regular unipotent elements. We will use these results in the next sections to get further informations about groups of type $A$ : in these groups, it turns out that all cuspidal local systems are supported by the conjugacy class of regular unipotent elements.

**3.1. Definition.** A class function $\gamma$ on $\mathbf{G}^F$ is said **absolutely cuspidal** if ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\gamma) = 0$ for any $F$-stable Levi subgroup $\mathbf{L}$ of a proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. We denote by $\mathrm{Cus}(\mathbf{G}^F)$ the subspace of $\mathrm{Class}(\mathbf{G}^F)$ consisting of absolutely cuspidal functions. Recall that a semisimple element $s$ of $\mathbf{G}$ is said **isolated** if $C_{\mathbf{G}}^\circ(s)$ is not contained in any Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$.

**Proposition 3.1.1.** *Let $\gamma$ be an absolutely cuspidal function on $\mathbf{G}^F$. Then :*

(a) *Let $g \in \mathbf{G}^F$ be such that $\gamma(g) \neq 0$. Then the semisimple part of $g$ is isolated.*

(b) *If $s$ is a semisimple element of $\mathbf{G}^F$, then $d_s^{\mathbf{G}}\gamma$ is an absolutely cuspidal function on $C_{\mathbf{G}}^\circ(s)^F$.*

Proof - Let first prove (a). Let $g$ be an element of $\mathbf{G}^F$ having a semisimple part $s$ which is not isolated in $\mathbf{G}$. Let $\mathbf{L}$ be the intersection of all Levi subgroups of parabolic subgroups of $\mathbf{G}$ containing $C_{\mathbf{G}}^\circ(s)$. Then $\mathbf{L}$ is an $F$-stable Levi subgroup of a proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Moreover, $g \in C_{\mathbf{G}}^\circ(s)\subset\mathbf{L}$. Let $u$ be the unipotent part of $g$. We have, by Formula 1.3.2 :

$$\begin{aligned} 0 &= d_s^{\mathbf{L}}({}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\gamma)(u) \\ &= {}^*R_{C_{\mathbf{L}}^\circ(s)\subset C_{\mathbf{P}}^\circ(s)}^{C_{\mathbf{G}}^\circ(s)}(d_s^{\mathbf{G}}\gamma)(u) \\ &= d_s^{\mathbf{G}}\gamma(u) \\ &= \gamma(su) = \gamma(g), \end{aligned}$$

the third equality following from the fact that $C_{\mathbf{G}}^\circ(s)\subset\mathbf{L}\subset\mathbf{P}$. So (a) follows.

Let now prove (b). Let $\mathbf{L}$ be an $F$-stable Levi subgroup of a proper parabolic subgroup $\mathbf{P}$ of $C_{\mathbf{G}}^\circ(s)$. We define $\mathbf{M} = C_{\mathbf{G}}(\mathbf{Z}(\mathbf{L})^\circ)$. Then $\mathbf{M}$ is an $F$-stable Levi subgroup of a proper parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ and $\mathbf{M} \cap C_{\mathbf{G}}^\circ(s) = C_{\mathbf{M}}^\circ(s) = \mathbf{L}$ by classical properties of Levi subgroups. Moreover, we can choose $\mathbf{Q}$ such that $\mathbf{P}\subset\mathbf{Q}$ : hence, we have $C_{\mathbf{Q}}^\circ(s) = \mathbf{P}$. By Formula 1.3.2, we have :

$$\begin{aligned} {}^*R_{\mathbf{L}\subset\mathbf{P}}^{C_{\mathbf{G}}^\circ(s)}(d_s^{\mathbf{G}}\gamma) &= d_s^{\mathbf{M}}({}^*R_{\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}}\gamma) \\ &= 0 \end{aligned}$$

which proves that $d_s^{\mathbf{G}}\gamma$ is absolutely cuspidal. ■

Let $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$ denote the subspace of $\mathrm{Class}(\mathbf{G}^F)$ consisting of absolutely cuspidal class functions with unipotent support. In other words,

$$\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) = \mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F) \cap \mathrm{Cus}(\mathbf{G}^F).$$

If $s \in \mathbf{G}^F$ is semisimple, then Proposition 3.1.1 shows that the map $d_s^{\mathbf{G}}$ induces a map $d_s^{\mathbf{G}} : \mathrm{Cus}(\mathbf{G}^F) \to \mathrm{Cus}_{\mathrm{uni}}(C_{\mathbf{G}}^\circ(s)^F)$ which is zero is $s$ is not isolated.

The subspace $\mathrm{Cus}(\mathbf{G}^F)$ is stable under the actions of $\mathbf{Z}^F$ and $H^1(F,\mathbf{Z})$ by 1.7.4 and 1.8.2 respectively. However, $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$ is stable only under the action of $H^1(F,\mathbf{Z})$ and not under the action of $\mathbf{Z}^F$ if $\mathbf{Z}^F \neq \{1\}$. In particular, we have the following decompositions

$$\mathrm{Cus}(\mathbf{G}^F) = \bigoplus_{\substack{\phi\in(\mathbf{Z}^F)^\wedge \\ \zeta\in H^1(F,\mathbf{Z})^\wedge}} \mathrm{Cus}(\mathbf{G}^F)^\phi_\zeta$$

and.
$$\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) = \bigoplus_{\zeta\in H^1(F,\mathbf{Z})^\wedge} \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)_\zeta$$

These direct sums are orthogonal.

The following lemma follows immediately from Corollary 1.6.2 :

**Lemma 3.1.2.** *Let $\gamma \in \mathrm{Cus}(\mathbf{G}^F)$ and let $\sigma$ be a semisimple element of $\mathbf{G}^{*F^*}$. Then $\gamma_{(\sigma)}$ is also absolutely cuspidal.*

**3.2. Cuspidal local systems.** We denote by $\mathcal{U}(\mathbf{G})$ the set of (isomorphism classes of) pairs $(C,\mathcal{L})$ where $C$ is a unipotent class of $\mathbf{G}$ and $\mathcal{L}$ is a $\mathbf{G}$-equivariant irreducible local system on $C$. The pair $(C,\mathcal{L})$ is said **$F$-stable** if $F(C)=C$ and if there is an isomorphism $F^*\mathcal{L}\simeq\mathcal{L}$. We denote by $\mathcal{U}(\mathbf{G})^F$ the set of $F$-stable elements of $\mathcal{U}(\mathbf{G})$. If $(C,\mathcal{L})\in\mathcal{U}(\mathbf{G})^F$ and if we choose an isomorphism $\varphi : F^*\mathcal{L}\simeq\mathcal{L}$, we can define a class function $\mathfrak{Y}_{(C,\mathcal{L})}$ called the **characteristic function** of $(C,\mathcal{L})$ which is well-defined up to a scalar (depending on the choice of the isomorphism $F^*\mathcal{L}\simeq\mathcal{L}$). This function is given by the following formula :

$$\mathfrak{Y}_{(C,\mathcal{L})}(g) = \begin{cases} \mathrm{Tr}(\varphi_g,\mathcal{L}_g) & \text{if } g\in C^F, \\ 0 & \text{otherwise,} \end{cases}$$

for any $g\in\mathbf{G}^F$. The following fact is proved in [L3], 24.2.7 :

**(3.2.1)** *The family $(\mathfrak{Y}_\iota)_{\iota\in\mathcal{U}(\mathbf{G})^F}$ is a basis of* $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$.

We denote by $\mathcal{U}(\mathbf{G})_{\mathrm{cus}}$ the set of cuspidal pairs of $\mathcal{U}(\mathbf{G})$ in the sense of Lusztig (*cf.* [L2], Definition 2.4). In [L4], Section 1, Lusztig made the following conjecture :

**Conjecture C :** *If $p$ is* **almost good** *for $\mathbf{G}$, then $(\mathfrak{Y}_\iota)_{\iota\in\mathcal{U}(\mathbf{G})^F_{\mathrm{cus}}}$ is an orthogonal basis of $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$.*

**Theorem 3.2.2.** *Conjecture C holds in the following cases :*

(i) *If $q>q_1(\mathbf{G})$ where $q_1(\mathbf{G})$ is a constant depending only on the root datum associated to $\mathbf{G}$ (Lusztig [L4], Theorem 1.14).*

(ii) *If the center of $\mathbf{G}$ is connected (Shoji [S1] and [S2], Theorem 4.2).*

Remark - In Lusztig's paper [L4], the space $\mathrm{Cus}(\mathbf{G}^F)$ was denoted by $\bar{\mathbf{F}}^{\mathbf{G}}_{\mathbf{G}}$ and the span of $(\mathfrak{Y}_\iota)_{\iota\in\mathcal{U}(\mathbf{G})^F_{\mathrm{cus}}}$ was denoted by $\mathbf{F}^{\mathbf{G}}_{\mathbf{G}}$.

As a consequence of the Mackey formula for groups of type $A$, we will obtain that conjecture C holds for groups of type $A$ without hypothesis on $q$ (*cf.* Theorem 6.2.1).

**3.3. Cuspidal local system on the regular unipotent class.** In this subsection, we assume that $p$ is **good** for $\mathbf{G}$. We fix a regular unipotent element $u_1 \in \mathbf{G}^F$ and we denote by $C_{\mathrm{reg}}$ the conjugacy class of $u_1$ in $\mathbf{G}$. The group $A_{\mathbf{G}}(u_1)$ is naturally isomorphic to $\mathbf{Z}/\mathbf{Z}^\circ$ because $p$ is good for $\mathbf{G}$. Consequently, there is a natural bijection between isomorphism classes of irreducible local system on $C_{\mathrm{reg}}$ and the set of linear characters of $\mathbf{Z}/\mathbf{Z}^\circ$. If $\zeta$ is a linear character of $\mathbf{Z}/\mathbf{Z}^\circ$, then we denote by $\mathcal{L}_\zeta$ a corresponding local system on $C_{\mathrm{reg}}$.

If $\mathbf{L}$ is a Levi subgroup of a parabolic subgroup of $\mathbf{G}$, then we recall that the natural map $\mathbf{Z} \hookrightarrow \mathbf{Z}(\mathbf{L})$ induces a surjective map $h_{\mathbf{L}}^{\mathbf{G}} : \mathbf{Z}/\mathbf{Z}^\circ \to \mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$. The following result is well-known :

**Proposition 3.3.1.** *Let $\zeta$ be a linear character of $\mathbf{Z}/\mathbf{Z}^\circ$. Then the pair $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is cuspidal if and only if $\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}} \not\subseteq \operatorname{Ker}\zeta$ for any Levi subgroup $\mathbf{L}$ of a proper parabolic subgroup of $\mathbf{G}$.*

We denote by $(\mathbf{Z}/\mathbf{Z}^\circ)^\wedge_{\mathrm{cus}}$ the set of linear characters $\zeta$ of $\mathbf{Z}/\mathbf{Z}^\circ$ such that $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is cuspidal.

**Proposition 3.3.2.** *Let $\mathbf{L}$ and $\mathbf{M}$ be two Levi subgroups of parabolic subgroups of $\mathbf{G}$ and let $\zeta$ and $\zeta'$ be two linear characters of $\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$ and $\mathbf{Z}(\mathbf{M})/\mathbf{Z}(\mathbf{M})^\circ$ respectively. Assume that $\zeta \circ h_{\mathbf{L}}^{\mathbf{G}} = \zeta' \circ h_{\mathbf{M}}^{\mathbf{G}}$, that $\zeta \in (\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ)^\wedge_{\mathrm{cus}}$ and that $\zeta' \in (\mathbf{Z}(\mathbf{M})/\mathbf{Z}(\mathbf{M})^\circ)^\wedge_{\mathrm{cus}}$. Then there exists $g \in \mathbf{G}$ such that ${}^g\mathbf{L} = \mathbf{M}$. In particular, $\mathbf{Z} \cap \mathbf{Z}(\mathbf{L})^\circ = \mathbf{Z} \cap \mathbf{Z}(\mathbf{M})^\circ$.*

Proof - Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ and let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. We can assume that $\mathbf{L}$ and $\mathbf{M}$ are standard with respect to the pair $(\mathbf{T}, \mathbf{B})$. Let $\boldsymbol{\zeta} = \zeta \circ h_{\mathbf{L}}^{\mathbf{G}} = \zeta' \circ h_{\mathbf{M}}^{\mathbf{G}}$. Then $(\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}}).(\operatorname{Ker} h_{\mathbf{M}}^{\mathbf{G}}) \subset \operatorname{Ker}\boldsymbol{\zeta}$. But, by [DLM], Lemma 1.5, we have

$$\operatorname{Ker} h_{\mathbf{L}\cap\mathbf{M}}^{\mathbf{G}} = (\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}}).(\operatorname{Ker} h_{\mathbf{M}}^{\mathbf{G}}).$$

Hence $\operatorname{Ker} h_{\mathbf{L}\cap\mathbf{M}}^{\mathbf{G}} \subset \operatorname{Ker}\boldsymbol{\zeta}$ which implies that $\operatorname{Ker} h_{\mathbf{L}\cap\mathbf{M}}^{\mathbf{L}} \subset \operatorname{Ker}\zeta$. But $\zeta \in (\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ)^\wedge_{\mathrm{cus}}$ so $\mathbf{L} \cap \mathbf{M} = \mathbf{L}$ by Proposition 3.3.1 which implies that $\mathbf{L}$ is contained in $\mathbf{M}$. Similarly, $\mathbf{M}$ is contained in $\mathbf{L}$ : this proves the first assertion of the proposition. The second one follows easily. ■

If $\zeta$ is a linear character of $\mathbf{Z}/\mathbf{Z}^\circ$, then the pair $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is $F$-stable if and only if the linear character $\zeta$ is $F$-stable that is, if and only if $\zeta$ is a linear character of $H^1(F, \mathbf{Z})$. We make Proposition 3.3.1 more precise in the following :

**Proposition 3.3.3.** *Let $\zeta$ be an $F$-stable linear character of $\mathbf{Z}/\mathbf{Z}^\circ$. Then the following are equivalent :*

(a) *The pair $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is cuspidal ;*

(b) *For any Levi subgroup $\mathbf{L}$ of a proper parabolic subgroup of $\mathbf{G}$, we have $\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}} \not\subseteq \operatorname{Ker}\zeta$.*

(c) *For any rational Levi subgroup $\mathbf{L}$ of a proper parabolic subgroup of $\mathbf{G}$, we have $\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}} \not\subseteq \operatorname{Ker}\zeta$.*

(d) *For any rational Levi subgroup $\mathbf{L}$ of a proper rational parabolic subgroup of $\mathbf{G}$, we have $\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}} \not\subseteq \operatorname{Ker}\zeta$.*

Proof - (a) is equivalent to (b) by Proposition 3.3.1. It is clear that (b) implies (c) and that (c) implies (d). It remains to prove that (d) implies (b). Let assume that (d) holds and let $\mathbf{L}$ be a Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}} \subset \operatorname{Ker}\zeta$. We have to prove that $\mathbf{L} = \mathbf{G}$.

Let $\mathbf{B}$ be an $F$-stable Borel subgroup of $\mathbf{G}$ and let $\mathbf{T}$ be an $F$-stable maximal torus of $\mathbf{B}$. By replacing the pair $(\mathbf{L}, \mathbf{P})$ by a conjugate, we can assume that $\mathbf{B} \subset \mathbf{P}$ and that $\mathbf{T} \subset \mathbf{L}$. Let $n$ be a non-zero natural number such that $F^n(\mathbf{P}) = \mathbf{P}$ (or, equivalently, $F^n(\mathbf{L}) = \mathbf{L}$) and let

$$\mathbf{M} = \mathbf{L} \cap F(\mathbf{L}) \cap \cdots \cap F^{n-1}(\mathbf{L}) \qquad \text{and} \qquad \mathbf{Q} = \mathbf{P} \cap F(\mathbf{P}) \cap \cdots \cap F^{n-1}(\mathbf{P}).$$

Then $\mathbf{M}$ is a rational Levi subgroup of the rational parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ and, by [DLM], Lemma 1.5, we have

$$\operatorname{Ker} h^{\mathbf{G}}_{\mathbf{M}} = (\operatorname{Ker} h^{\mathbf{G}}_{\mathbf{L}}).(\operatorname{Ker} h^{\mathbf{G}}_{F(\mathbf{L})}) \dots (\ker h^{\mathbf{G}}_{F^{n-1}(\mathbf{L})}).$$

But, if $0 \leqslant i \leqslant n-1$, then $\operatorname{Ker} h^{\mathbf{G}}_{F^i(\mathbf{L})} = F^i(\operatorname{Ker} h^{\mathbf{G}}_{\mathbf{L}}) \subset \operatorname{Ker}\zeta$ because $\zeta$ is $F$-stable. Hence $\operatorname{Ker} h^{\mathbf{G}}_{\mathbf{M}} \subset \operatorname{Ker}\zeta$ which implies that $\mathbf{M} = \mathbf{G}$ because we are assuming that (d) holds. So $\mathbf{L} = \mathbf{G}$ as desired. ■

We denote by $H^1(F, \mathbf{Z})^{\wedge}_{\mathrm{cus}}$ the set of linear characters of the group $H^1(F, \mathbf{Z})$ such that $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is cuspidal.

The set of $\mathbf{G}^F$-conjugacy classes of regular unipotent elements of $\mathbf{G}^F$ is parametrized by $H^1(F, \mathbf{Z})$. If $a \in H^1(F, \mathbf{Z})$, we denote by $u_a$ a representative of the $\mathbf{G}^F$-conjugacy class of regular unipotent elements associated to $a$ : in fact, $\gamma^{\mathbf{G}}_{u_a} = {}^{\hat{a}^{-1}}\gamma^{\mathbf{G}}_{u_1}$ (recall that $\gamma^{\mathbf{G}}_g$ is the characteristic function of the $\mathbf{G}^F$-conjugacy class of $g \in \mathbf{G}^F$). If $\zeta$ is a linear character of $H^1(F, \mathbf{Z})$, we denote by $\Gamma^{\mathbf{G}}_\zeta$ the characteristic function of $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$. If we normalize $\Gamma^{\mathbf{G}}_\zeta$ by the condition $\Gamma^{\mathbf{G}}_\zeta(u_1) = 1$, we have

**(3.3.4)**
$$\Gamma^{\mathbf{G}}_\zeta = \sum_{a \in H^1(F, \mathbf{Z})} \zeta(a)\ \gamma^{\mathbf{G}}_{u_a}.$$

It is easy to check that

**(3.3.5)**
$$\Gamma^{\mathbf{G}}_\zeta \in \operatorname{Class}_{\mathrm{uni}}(\mathbf{G}^F)_\zeta.$$

**Proposition 3.3.6.** *Let $\zeta$ be a linear character of $H^1(F, \mathbf{Z})$. Then the pair $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is cuspidal if and only if the characteristic function $\Gamma^{\mathbf{G}}_\zeta$ is absolutely cuspidal.*

Proof - If the pair $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is cuspidal then, by Proposition 3.3.3, (c), and by Corollary 1.8.6, the function $\Gamma^{\mathbf{G}}_\zeta$ is absolutely cuspidal.

Conversely, assume that $\Gamma^{\mathbf{G}}_\zeta$ is absolutely cuspidal and let $\mathbf{L}$ be an $F$-stable Levi subgroup of a proper $F$-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Then ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\Gamma^{\mathbf{G}}_\zeta = 0$ or, in other words,

$$\sum_{a \in H^1(F, \mathbf{Z})} \zeta(a)\ {}^{\hat{a}^{-1}}{}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\gamma^{\mathbf{G}}_{u_1} = 0.$$

But, by [DLM], Proposition 5.3, ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\gamma^{\mathbf{G}}_{u_1} = \gamma^{\mathbf{L}}_v$ where $v$ is a regular unipotent element of $\mathbf{L}^F$. Moreover, ${}^{\hat{a}^{-1}}\gamma^{\mathbf{L}}_v = \gamma^{\mathbf{L}}_v$ if and only if $a \in \operatorname{Ker} \mathbf{h}^{\mathbf{G}}_{\mathbf{L}}$. Hence we have

$$\sum_{a \in \operatorname{Ker} \mathbf{h}^{\mathbf{G}}_{\mathbf{L}}} \zeta(a) = 0$$

that is $\operatorname{Ker} \mathbf{h}^{\mathbf{G}}_{\mathbf{L}} \not\subseteq \operatorname{Ker}\zeta$. So $(C_{\mathrm{reg}}, \mathcal{L}_\zeta)$ is cuspidal by Proposition 3.3.3 (equivalence between (a) and (d)). ■

Example : The case of type $A$ - *If $\mathbf{G}$ is of type $A$ that is, if all the irreducible components of the root system of $\mathbf{G}$ are of type $A$, and if $(C, \mathcal{L}) \in \mathcal{U}(\mathbf{G})_{\mathrm{cus}}$, then $C = C_{\mathrm{reg}}$ by [L2], Proposition 2.8.*

## 4. Some consequences of the Mackey formula

We will prove in the Section 5 the Mackey formula for groups of type $A$ by induction on the dimension of the group. That means that we will assume that the Mackey formula holds for smaller groups. This is the reason why we need to obtain *a priori* some consequences of the Mackey formula. In this section, we are particularly interested in its connection with conjecture C. We will obtain that, for groups of type $A$, the Mackey formula implies conjecture C (*cf.* Theorem 4.2.1).

**4.1. The Mackey formula and absolutely cuspidal functions.** Whenever the Mackey formula holds in $\mathbf{G}$ it is possible to prove a weaker result than conjecture C. More precisely :

**Theorem 4.1.1.** *If the Mackey formula holds in* $\mathbf{G}$ *and if* $\omega : \mathbf{G} \to \mathbf{G}$ *is an automorphism of* $\mathbf{G}$ *commuting with* $F$ *then* $\dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)^\omega$ *is equal to the number of orbits of* $\omega$ *in* $\mathcal{U}(\mathbf{G})^F_{\mathrm{cus}}$.

**Corollary 4.1.2.** *If the Mackey formula holds in* $\mathbf{G}$, *then*

(a) $\dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) = |\mathcal{U}(\mathbf{G})^F_{\mathrm{cus}}|$.

(b) *If* $\mathbf{G}$ *is a rational Levi subgroup of a parabolic subgroup of a connected reductive group* $\mathbf{H}$ *(endowed with a Frobenius endomorphism also denoted by* $F$*) then all cuspidal functions on* $\mathbf{G}^F$ *with unipotent support are invariant under the action of* $N_{\mathbf{H}^F}(\mathbf{G})$.

Proof of Theorem 4.1.1 and Corollary 4.1.2 - We will prove these results by induction on $\dim \mathbf{G}$. They are obvious if $\dim \mathbf{G}$ is small. Therefore, assume that Theorem 4.1.1 and Corollary 4.1.2 are true with $\mathbf{G}$ replaced by any rational Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$ (indeed, the Mackey formula holds in $\mathbf{L}$ by hypothesis). Note that assertion (a) of Corollary 4.1.2 follows immediately from Theorem 4.1.1 and that assertion (b) follows from Theorem 4.1.1 and from the following fact (*cf.* [L2], Theorem 9.2, (a)) :

**(4.1.3)** *If* $\mathbf{L}$ *is a Levi subgroup of a parabolic subgroup of* $\mathbf{G}$, *then* $N_{\mathbf{G}}(\mathbf{L})$ *acts trivially on* $\mathcal{U}(\mathbf{L})_{\mathrm{cus}}$.

Hence it is sufficient to prove the assertion of Theorem 4.1.1.

It is well-known that the Mackey formula implies that the Lusztig functors are independent of the choice of a parabolic subgroup. Consequently, if $\mathbf{L}$ is an $F$-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, we will denote by $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ the Lusztig functors $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$.

The Mackey formula in $\mathbf{G}$ implies that

$$\textbf{(4.1.4)} \qquad \mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F) = \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) \oplus \Big( \bigoplus_{\mathbf{L}\in\mathcal{A}} R_{\mathbf{L}}^{\mathbf{G}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F)) \Big)$$

where $\mathcal{A}$ is a set of representative of $\mathbf{G}^F$-conjugacy classes of $F$-stable Levi subgroups of proper parabolic subgroups of $\mathbf{G}$ such that $\mathcal{U}(\mathbf{L})^F_{\mathrm{cus}} \neq \varnothing$ (indeed, by the induction hypothesis, if $\mathcal{U}(\mathbf{L})^F_{\mathrm{cus}} = \varnothing$, then $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F) = \{0\}$). Moreover, this direct sum is orthogonal.

If $\mathbf{L} \in \mathcal{A}$, then $N_{\mathbf{G}^F}(\mathbf{L})$ acts trivially on $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F)$ by the induction hypothesis. Hence, by the Mackey formula, we have, for all $\lambda$ and $\lambda'$ in $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F)$,

$$\langle R_{\mathbf{L}}^{\mathbf{G}}\lambda, R_{\mathbf{L}}^{\mathbf{G}}\lambda' \rangle_{\mathbf{G}^F} = \frac{|N_{\mathbf{G}^F}(\mathbf{L})|}{|\mathbf{L}^F|} \langle \lambda, \lambda' \rangle_{\mathbf{L}^F}$$

so the map

$$\textbf{(4.1.5)} \qquad R_{\mathbf{L}}^{\mathbf{G}} : \mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F) \longrightarrow R_{\mathbf{L}}^{\mathbf{G}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F))$$

is an isomorphism.

We fix an $F$-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and an $F$-stable maximal torus $\mathbf{T}$ of $\mathbf{B}$. We denote by $W$ the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$ and by $S$ the set of simple reflections of $W$ with respect to the choice of $\mathbf{B}$. For each $I \subset S$, we denote by $W_I$ the subgroup of $W$ generated by $I$ and by $\mathbf{L}_I$ the Levi subgroup of the parabolic subgroup $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$ containing $\mathbf{T}$ ($W_I$ is the Weyl group of $\mathbf{L}_I$ relative to $\mathbf{T}$). We denote by $\mathcal{P}(S)$ the set of subsets of $S$ and by $\mathcal{P}(S)^F_{\mathrm{cus}}$ the set of $I \in \mathcal{P}(S)^F$ such that $\mathcal{U}(\mathbf{L}_I)^F_{\mathrm{cus}} \neq \varnothing$. Then by [L2], Theorem 9.2, we have

**(4.1.6)** *If $I,\ J \in \mathcal{P}(S)^F_{\mathrm{cus}}$ and if there exists $w \in W$ such that ${}^wI = J$, then $I = J$.*

**(4.1.7)** *Every $\mathbf{L} \in \mathcal{A}$ is geometrically conjugate to a unique $\mathbf{L}_I$ with $I \in \mathcal{P}(S)^F_{\mathrm{cus}}$.*

If $I \in \mathcal{P}(S)^F_{\mathrm{cus}}$, then the set of $\mathbf{G}^F$-conjugacy classes of $F$-stable Levi subgroups of parabolic subgroups of $\mathbf{G}$ geometrically conjugate to $\mathbf{L}_I$ is in one-to-one correspondence with the set $H^1(F, W_{\mathbf{G}}(\mathbf{L}_I))$ where $W_{\mathbf{G}}(\mathbf{L}_I) = N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I$. If $w \in H^1(F, W_{\mathbf{G}}(\mathbf{L}_I))$, then we denote by $\mathbf{L}_{I,w}$ a representative of the class associated to $w$ and by $\dot{w}$ a representative of $w$ in $N_{\mathbf{G}}(\mathbf{L}_I)$. With these notations, we have, by 4.1.3,

$$|\mathcal{U}(\mathbf{L}_{I,w})^F_{\mathrm{cus}}| = |\mathcal{U}(\mathbf{L}_I)^{\dot{w}F}_{\mathrm{cus}}| = |\mathcal{U}(\mathbf{L}_I)^F_{\mathrm{cus}}|.$$

We denote by $\mathcal{C}$ the set of pairs $(I, w)$ where $I \in \mathcal{P}(S)^F_{\mathrm{cus}}$, $I \neq S$ and $w \in H^1(F, W_{\mathbf{G}}(\mathbf{L}_I)$. Then, by 4.1.6 and 4.1.7, the decomposition 4.1.4 can be rewritten :

**(4.1.8)**
$$\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F) = \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) \oplus \Big( \bigoplus_{(I,w)\in\mathcal{C}} R^{\mathbf{G}}_{\mathbf{L}_{I,w}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{I,w}))\Big)$$

The automorphism $\omega$ is defined over $\mathbb{F}_q$ so $\omega(\mathbf{B})$ is an $F$-stable Borel subgroup of $\mathbf{G}$ and $\mathbf{T}$ is an $F$-stable maximal torus of $\mathbf{B}$. Thus there exists an element $g \in \mathbf{G}^F$ such that

$$\omega(\mathbf{B}) = {}^g\mathbf{B} \qquad \text{and} \qquad \omega(\mathbf{T}) = {}^g\mathbf{T}.$$

By replacing $\omega$ by $\mathrm{ad}\, g^{-1} \circ \omega$ if necessary, we can (and we will) assume that $\omega(\mathbf{B}) = \mathbf{B}$ and $\omega(\mathbf{T}) = \mathbf{T}$. If $X$ is a set on which $\omega$ acts, then we denote by $X/\omega$ the set of orbits of $\omega$ in $X$. If $x \in X/\omega$, then $\overline{x}$ will denote the orbit of $x$ under $\omega$.

If $\overline{(I,w)} \in \mathcal{C}/\omega$, we denote by

$$V_{\overline{(I,w)}} = \bigoplus_{n\in\mathbb{Z}} R^{\mathbf{G}}_{\mathbf{L}_{\omega^n(I),\omega^n(w)}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{\omega^n(I),\omega^n(w)})).$$

Then, by 4.1.8, we have

$$\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)^\omega = \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)^\omega \oplus \Big( \bigoplus_{\overline{(I,w)}\in\mathcal{C}/\omega} (V_{\overline{(I,w)}})^\omega\Big)$$

So

**(4.1.9)**
$$\dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)^\omega = \dim \mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)^\omega - \sum_{\overline{(I,w)}\in\mathcal{C}/\omega} \dim(V_{\overline{(I,w)}})^\omega.$$

Let $(I, w) \in \mathcal{C}$ and let $n = |\overline{(I,w)}|$. Then there exists $g \in \mathbf{G}^F$ such that $\omega^n(\mathbf{L}_{I,w}) = {}^g\mathbf{L}_{I,w}$. Let us denote by $\alpha = (\mathrm{ad}\, g)^{-1} \circ \omega^n$. Then $\omega^n$ and $\alpha$ stabilize the subspace $R^{\mathbf{G}}_{\mathbf{L}_{I,w}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{I,w}))$ and $R^{\mathbf{G}}_{\mathbf{L}_{I,w}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{I,w}))^\alpha = R^{\mathbf{G}}_{\mathbf{L}_{I,w}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{I,w}))^{\omega^n}$. But

$$\dim(V_{\overline{(I,w)}})^\omega = \dim R^{\mathbf{G}}_{\mathbf{L}_{I,w}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{I,w}))^{\omega^n}$$

and the map

$$R^{\mathbf{G}}_{\mathbf{L}_{I,w}} : \mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{I,w}) \longrightarrow R^{\mathbf{G}}_{\mathbf{L}_{I,w}}(\mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}^F_{I,w}))$$

is an isomorphism and commutes with the action of $\alpha$. So we have

$$\dim(V_{\overline{(I,w)}})^\omega = \dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}_{I,w}^F)^\alpha.$$

But, by the induction hypothesis, we have

$$\dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{L}_{I,w}^F)^\alpha = |\mathcal{U}(\mathbf{L}_{I,w})_{\mathrm{cus}}^F/\alpha| = |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\omega^n|.$$

On the other hand, by 3.2.1, we have

$$\dim \mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)^\omega = |\mathcal{U}(\mathbf{G})^F/\omega|.$$

So the equality 4.1.9 gives :

$$\textbf{(4.1.10)} \qquad \dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)^\omega = |\mathcal{U}(\mathbf{G})^F/\omega| - \sum_{\overline{(I,w)}\in\mathcal{C}/\omega} |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\omega^{|\overline{(I,w)}|}|.$$

Let $\mathcal{B}$ denote the set of pairs $(I,\chi)$ where $I \in \mathcal{P}(S)_{\mathrm{cus}}^F$, $I \neq S$ and $\chi \in (\mathrm{Irr}\, W_{\mathbf{G}}(\mathbf{L}_I))^F$. By generalized Springer correspondence, we have

$$|\mathcal{U}(\mathbf{G})^F/\omega| = |\mathcal{U}(\mathbf{G})_{\mathrm{cus}}^F/\omega| + \sum_{\overline{(I,\chi)}\in\mathcal{B}/\omega} |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\omega^{|\overline{(I,\chi)}|}|.$$

Hence

$$\textbf{(4.1.11)} \qquad |\mathcal{U}(\mathbf{G})_{\mathrm{cus}}^F/\omega| = |\mathcal{U}(\mathbf{G})^F/\omega| - \sum_{\overline{(I,\chi)}\in\mathcal{B}/\omega} |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\omega^{|\overline{(I,\chi)}|}|.$$

By comparison of 4.1.10 and 4.1.11, it is sufficient to prove that

$$(\triangle) \qquad \sum_{\substack{\overline{(I,\chi)}\in\mathcal{B}/\omega \\ I\neq S}} |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\omega^{|\overline{(I,\chi)}|}| = \sum_{\overline{(I,w)}\in\mathcal{C}/\omega} |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\omega^{|\overline{(I,w)}|}|.$$

Let $I \in \mathcal{P}(S)_{\mathrm{cus}}^F$ and let $\boldsymbol{\omega} = \omega^{|\overline{I}|}$. Then, to prove $(\triangle)$, it is sufficient to prove that

$$(\blacktriangle) \qquad \sum_{\overline{\chi}\in(\mathrm{Irr}\, W_{\mathbf{G}}(\mathbf{L}_I))^F/\boldsymbol{\omega}} |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\boldsymbol{\omega}^{|\overline{\chi}|}| = \sum_{\overline{w}\in H^1(F,W_{\mathbf{G}}(\mathbf{L}_I))/\boldsymbol{\omega}} |\mathcal{U}(\mathbf{L}_I)_{\mathrm{cus}}^F/\boldsymbol{\omega}^{|\overline{w}|}|.$$

But $(\blacktriangle)$ follows from the well-known fact that the $\boldsymbol{\omega}$-sets $(\mathrm{Irr}\, W_{\mathbf{G}}(\mathbf{L}_I))^F$ and $H^1(F, W_{\mathbf{G}}(\mathbf{L}_I))$ are isomorphic. ■

**4.2. The case of type** $A$**.** In this subsection, we assume that $\mathbf{G}$ is of type $A$. If the Mackey formula holds in $\mathbf{G}$ then we can make Theorem 4.1.1 more precise :

**Theorem 4.2.1.** *If* $\mathbf{G}$ *is of type* $A$ *and if the Mackey formula holds in* $\mathbf{G}$*, then* $(\mathfrak{Y}_\iota)_{\iota\in\mathcal{U}(\mathbf{G})_{\mathrm{cus}}^F}$ *is an orthogonal basis of* $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$ *or, in other words,* $(\Gamma_\zeta^{\mathbf{G}})_{\zeta\in H^1(F,\mathbf{Z})_{\mathrm{cus}}^\wedge}$ *is an orthogonal basis of* $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$*.*

Proof - Indeed, by Proposition 3.3.6, the functions $(\Gamma_\zeta)_{\zeta\in H^1(F,\mathbf{Z})_{\mathrm{cus}}^\wedge}$ are absolutely cuspidal. Moreover, by Theorem 4.1.1 and by the example following 3.3.6, we have $\dim \mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F) = |\mathcal{U}(\mathbf{G})_{\mathrm{cus}}^F| = |H^1(F,\mathbf{Z})_{\mathrm{cus}}^\wedge|$ and the theorem follows. ■

If $\phi$ is a linear character of $\mathbf{Z}^F$ and if $\zeta$ is a linear character of $H^1(F,\mathbf{Z})$, we put :

$$\textbf{(4.2.2)} \qquad \Gamma_{\phi,\zeta}^{\mathbf{G}} = \sum_{z\in\mathbf{Z}^F} \phi(z^{-1}) t_z^{\mathbf{G}} \Gamma_\zeta^{\mathbf{G}}.$$

It is clear that

$$\text{(4.2.3)} \qquad \Gamma^{\mathbf{G}}_{\phi,\zeta} \in \operatorname{Class}(\mathbf{G}^F)^{\phi}_{\zeta}.$$

**Corollary 4.2.4.** *If the group* $\mathbf{G}$ *is of type* $A$ *and if the Mackey formula holds in* $\mathbf{G}$ *then* $(\Gamma^{\mathbf{G}}_{\phi,\zeta})_{\substack{\phi\in(\mathbf{Z}^F)^\wedge \\ \zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}}}$ *is an orthogonal basis of* $\operatorname{Cus}(\mathbf{G}^F)$.

Proof - By Proposition 3.1.1, (a), an absolutely cuspidal function on $\mathbf{G}^F$ has its support in $\mathbf{Z}^F.\mathbf{G}^F_{\mathrm{uni}}$ because the isolated semisimple elements of $\mathbf{G}$ are central (because $\mathbf{G}$ is of type $A$). This implies that the family $(t^{\mathbf{G}}_z\Gamma^{\mathbf{G}}_{\zeta})_{\substack{z\in\mathbf{Z}^F \\ \zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}}}$ is an orthogonal basis of $\operatorname{Cus}(\mathbf{G}^F)$. The Corollary follows from the fact that

$$\text{(4.2.5)} \qquad t^{\mathbf{G}}_z\Gamma^{\mathbf{G}}_{\zeta} = \frac{1}{|\mathbf{Z}^F|}\sum_{\phi\in(\mathbf{Z}^F)^\wedge} \phi(z)\Gamma^{\mathbf{G}}_{\phi,\zeta}$$

for all $z \in \mathbf{Z}^F$ and $\zeta \in H^1(F,\mathbf{Z})^\wedge$. ■

**Corollary 4.2.6.** *If the group* $\mathbf{G}$ *is of type* $A$ *and if the Mackey formula holds in* $\mathbf{G}$*, then*

(a) *We have the following decompositions*

$$\operatorname{Cus}(\mathbf{G}^F) = \bigoplus_{\zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}} \operatorname{Class}(\mathbf{G}^F)_{\zeta}$$

*and*

$$\operatorname{Cus}(\mathbf{G}^F) = \bigoplus_{\substack{\phi\in(\mathbf{Z}^F)^\wedge \\ \zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}}} \operatorname{Class}(\mathbf{G}^F)^{\phi}_{\zeta}.$$

*Moreover* $\dim \operatorname{Class}(\mathbf{G}^F)^{\phi}_{\zeta} = 1$ *for all* $\phi \in (\mathbf{Z}^F)^\wedge$ *and* $\zeta \in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}$ *and* $\operatorname{Class}(\mathbf{G}^F)^{\phi}_{\zeta}$ *is generated by* $\Gamma^{\mathbf{G}}_{\phi,\zeta}$.

(b) *If* $\phi \in (\mathbf{Z}^F)^\wedge$ *and* $\zeta \in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}$ *then there exists a unique semisimple element* $\sigma$ *in* $\mathbf{G}^{*F^*}$ *(up to* $\mathbf{G}^{*F^*}$*-conjugacy) such that* $\Gamma^{\mathbf{G}}_{\phi,\zeta} \in \overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F,(\sigma))$.

Proof - (a) is clear from Corollary 4.2.4 and from Corollary 1.8.6. Let now prove (b). Let $\sigma$ and $\tau$ be two semisimple elements of $\mathbf{G}^{*F^*}$ such that the projections $\gamma_1$ and $\gamma_2$ of $\Gamma^{\mathbf{G}}_{\phi,\zeta}$ on $\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F,(\sigma))$ and $\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F,(\tau))$ respectively are non-zero. Then $\gamma_1$ and $\gamma_2$ belong to $\operatorname{Class}(\mathbf{G}^F)^{\phi}_{\zeta}$. By (a), this implies that they are proportional. If $\sigma$ is not $\mathbf{G}^{*F^*}$-conjugate to $\tau$, then there should be orthogonal, which is impossible. Hence $(\sigma) = (\tau)$ and (b) follows. ■

**Corollary 4.2.7.** *Assume that* $\mathbf{G}$ *is of type* $A$ *and that the Mackey formula holds in* $\mathbf{G}$. *Let* $\sigma$ *be a semisimple element of* $\mathbf{G}^{*F^*}$ *such that* $\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F,(\sigma))$ *contains a non-zero absolutely cuspidal functions. Then, for each Levi subgroup* $\mathbf{L}^*$ *of a proper parabolic subgroup of* $\mathbf{G}^*$ *such that* $\sigma \in \mathbf{L}^*$*, we have* $|A_{\mathbf{L}^*}(\sigma)| < |A_{\mathbf{G}^*}(\sigma)|$. *Moreover, if* $\mathbf{L}^*$ *is* $F^*$*-stable, then* $|A_{\mathbf{L}^*}(\sigma)^{F^*}| < |A_{\mathbf{G}^*}(\sigma)^{F^*}|$.

Proof - First, it is well-known that the natural morphism $C_{\mathbf{L}^*}(\sigma) \hookrightarrow C_{\mathbf{G}^*}(\sigma)$ induces an injective morphism $A_{\mathbf{L}^*}(\sigma) \hookrightarrow A_{\mathbf{G}^*}(\sigma)$. We identify $A_{\mathbf{L}^*}(\sigma)$ with the corresponding subgroup of $A_{\mathbf{G}^*}(\sigma)$. The surjective morphism $\psi^{\mathbf{G}}_{\sigma} : H^1(F,\mathbf{Z}) \to (A_{\mathbf{G}^*}(\sigma)^{F^*})^\wedge$ defined in Subsection 1.8 is induced by an injective morphism $\pi : A_{\mathbf{G}^*}(\sigma) \hookrightarrow (\mathbf{Z}/\mathbf{Z}^\circ)^\wedge$ (*cf.* [DM], Lemma 13.14). Moreover, the following diagram

$$\begin{tikzcd} A_{\mathbf{L}^*}(\sigma) \arrow[r] \arrow[d] & (\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ)^\wedge \arrow[d] \\ A_{\mathbf{G}^*}(\sigma) \arrow[r, "\pi"'] & (\mathbf{Z}/\mathbf{Z}^\circ)^\wedge \end{tikzcd}$$

is commutative and all maps are injective. Let $\gamma$ be an absolutely cuspidal function belonging to $\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F, (\sigma)_{\mathbf{G}^{*F^*}})$. By Corollary 4.2.6, there exists $\zeta \in H^1(F, \mathbf{Z})^\wedge_{\mathrm{cus}}$ such that the (orthogonal) projection of $\gamma$ on $\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F, (\sigma)_{\mathbf{G}^{*F^*}})_\zeta$ is non-zero. By 1.8.7, $\zeta$ is in the image of $\pi$. Let $\alpha \in A_{\mathbf{G}^*}(\sigma)$ be such that $\pi(\alpha) = \zeta$. If $\alpha \in A_{\mathbf{L}^*}(\sigma)$, it follows from the commutativity of the preceding diagram that $\operatorname{Ker} h_{\mathbf{L}}^{\mathbf{G}} \subset \operatorname{Ker} \zeta$ which is impossible. Hence $\alpha$ does not belong to $A_{\mathbf{L}^*}(\sigma)$ so $|A_{\mathbf{L}^*}(\sigma)| < |A_{\mathbf{G}^*}(\sigma)|$. This proves the first assertion of the corollary. The second can be proved by a similar argument by replacing the preceding diagram by the analogous diagram obtained by taking fixed points under Frobenius endomorphisms $F$ and $F^*$. ■

**Definition 4.2.8.** *If $\mathbf{G}$ is of type $A$, a semisimple element $\sigma \in \mathbf{G}^*$ is said* **cuspidal** *if, for all Levi subgroups $\mathbf{L}^*$ of a proper parabolic subgroup of $\mathbf{G}^*$ such that $\sigma \in \mathbf{L}^*$, we have $|A_{\mathbf{L}^*}(\sigma)| < |A_{\mathbf{G}^*}(\sigma)|$.*

**Lemma 4.2.9.** *If $\mathbf{G}$ is of type $A$ and if $\sigma$ is a cuspidal semisimple element of $\mathbf{G}^*$, then $\sigma$ is regular that is, $C^\circ_{\mathbf{G}^*}(\sigma)$ is a maximal torus of $\mathbf{G}^*$.*

Proof - Let $\mathbf{T}^*$ be a maximal torus of $C^\circ_{\mathbf{G}^*}(\sigma)$ and let $\mathbf{B}^*$ be a maximal torus of $C^\circ_{\mathbf{G}^*}(\sigma)$ containing $\mathbf{T}^*$. Let

$$A = \{g \in C_{\mathbf{G}^*}(\sigma) \mid {}^g(\mathbf{T}^*, \mathbf{B}^*) = (\mathbf{T}^*, \mathbf{B}^*)\}/\mathbf{T}^*.$$

Then $A$ is canonically isomorphic to $A_{\mathbf{G}^*}(\sigma)$. We put

$$\mathbf{L}^* = C_{\mathbf{G}^*}((\mathbf{T}^{*A})^\circ).$$

Then $\mathbf{L}^*$ is a Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ and, by [B2], Corollary 4.2.3, we have $C^\circ_{\mathbf{L}^*}(\sigma) = \mathbf{T}^*$. Moreover, $A_{\mathbf{L}^*}(\sigma) \simeq A_{\mathbf{G}^*}(\sigma)$ by construction. So $\mathbf{L}^* = \mathbf{G}^*$ because $\sigma$ is cuspidal. Hence $C^\circ_{\mathbf{G}^*}(\sigma) = \mathbf{T}^*$. ■

We end this subsection by stating a Lemma which will be crucial for the proof of the Mackey formula in type $A$ :

**Lemma 4.2.10.** *Assume that $\mathbf{G}$ is of type $A$. Let $\mathbf{L}$ be a Levi subgroup of a parabolic subgroup of $\mathbf{G}$ admitting a cuspidal pair and let $\sigma$ and $\tau$ be two cuspidal semisimple elements of $\mathbf{L}^*$ (where $\mathbf{L}^*$ is a Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ dual to $\mathbf{L}$). If $\sigma$ and $\tau$ are conjugate in $\mathbf{G}^*$, then they are conjugate in $N_{\mathbf{G}^*}(\mathbf{L}^*)$.*

Proof - Let $z$ be an element of the center of $\mathbf{G}^*$ such that $z\sigma \in \mathbf{L}^* \cap D(\mathbf{G}^*)$ where $D(\mathbf{G}^*)$ is the derived group of $\mathbf{G}^*$. Then $z\tau \in \mathbf{L}^* \cap D(\mathbf{G}^*)$. Moreover, $z\sigma$ and $z\tau$ are conjugate in $D(\mathbf{G}^*)$ and are cuspidal in $\mathbf{L}^* \cap D(\mathbf{G}^*)$. That shows that we can (and we will) assume that $\mathbf{G}^*$ is semisimple. Let $\pi : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ be a simply connected covering of $\mathbf{G}^*$. Let $g \in \mathbf{G}^*$ be such that $\tau = g\sigma g^{-1}$. Let $\tilde{g}$ and $\tilde{\sigma}$ be two elements of $\tilde{\mathbf{G}}^*$ such that $\pi(\tilde{g}) = g$ and $\pi(\tilde{\sigma}) = \sigma$. Let $\tilde{\tau} = \tilde{g}\tilde{\sigma}\tilde{g}^{-1}$. Then $\pi(\tilde{\tau}) = \tau$ and it is sufficient to prove that

$(\boldsymbol{\mathcal{P}})$ *$\tilde{\sigma}$ and $\tilde{\tau}$ are conjugate under $N_{\tilde{\mathbf{G}}^*}(\tilde{\mathbf{L}}^*)$ where $\tilde{\mathbf{L}}^* = \pi^{-1}(\mathbf{L}^*)$.*

• First, let us prove $(\mathcal{P})$ whenever $\mathbf{G}^*$ is adjoint. Then $\mathbf{G}^*$ is a product of projective general linear groups and it is sufficient to prove $(\mathcal{P})$ for each of the factors. Hence, we can assume that $\mathbf{G}^* = \mathbf{PGL}_n(\mathbb{F})$ and that $\tilde{\mathbf{G}}^* = \mathbf{SL}_n(\mathbb{F})$. Then, since $\mathbf{L}$ admits a cuspidal pair, we can assume that the group $\tilde{\mathbf{L}}^*$ is of the following form

$$\tilde{\mathbf{L}}^* = \Big(\underbrace{\mathbf{GL}_d(\mathbb{F}) \times \cdots \times \mathbf{GL}_d(\mathbb{F})}_{e \text{ times}}\Big) \cap \mathbf{SL}_n(\mathbb{F})$$

where $n = de$ and $p$ does not divide $d$ (*cf.* [L2], 10.3). We denote by $\tilde{\mathbf{T}}^*$ the maximal torus of $\tilde{\mathbf{L}}^*$ consisting of diagonal matrices. Up to conjugacy in $\tilde{\mathbf{L}}^*$, we can assume that $\tilde{\sigma} \in \tilde{\mathbf{T}}^*$. Write

$$\tilde{\sigma} = \mathrm{diag}(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ belong to $\mathbb{F}^\times$. For $1 \leqslant i \leqslant e$, we denote by $S_i = \{x_{(i-1)d+1}, x_{(i-1)d+2}, \ldots, x_{id}\}$. By Lemma 4.2.9, we have $|S_i| = d$. Moreover, by [DLM], 3.12, we have $A_{\mathbf{L}^*}(\sigma) \simeq \{z \in \operatorname{Ker} \pi \mid \tilde{\sigma}$ and $\tilde{\sigma} z$ are conjugate in $\tilde{\mathbf{L}}^*\}$. But $\operatorname{Ker} \pi \simeq \mathbb{F}^\times$ ; let $A$ denote the finite subgroup of $\mathbb{F}^\times$ image of $A_{\mathbf{L}^*}(\sigma)$ by the preceding isomorphism. Then $A$ acts on $S_i$ by multiplication. Let $\Omega$ be an orbit of $A$ in $S_i$ : we can assume that $\Omega = \{x_{(i-1)d+1}, \ldots, x_{(i-1)d+k}\}$ for some $1 \leqslant k \leqslant d$. If

$$\tilde{\mathbf{M}}^* = \Big(\big(\underbrace{\mathbf{GL}_d(\mathbb{F}) \times \cdots \times \mathbf{GL}_d(\mathbb{F})}_{(i-1)\ \text{times}}\big) \times \mathbf{GL}_k(\mathbb{F}) \times \mathbf{GL}_{d-k}(\mathbb{F}) \times \big(\underbrace{\mathbf{GL}_d(\mathbb{F}) \times \cdots \times \mathbf{GL}_d(\mathbb{F})}_{(e-i)\ \text{times}}\big)\Big) \cap \mathbf{SL}_n(\mathbb{F})$$

and $\mathbf{M}^* = \pi(\tilde{\mathbf{M}}^*)$, then $A_{\mathbf{M}^*}(\sigma) \simeq A_{\mathbf{L}^*}(\sigma)$ so $\mathbf{M}^* = \mathbf{L}^*$ because $\sigma$ is cuspidal. In other words, $k = d$ which means that $A$ acts transitively on $S_i$. In particular, $|A| = d$ and $S_i$ is of the following form :

$$S_i = \{a_i, a_i\zeta, \ldots, a_i\zeta^{d-1}\}$$

where $\zeta$ is a primitive $d$-th root of unity in $\mathbb{F}^\times$.

Let $J$ be the $d \times d$-matrix

$$J = \mathrm{diag}(1, \zeta, \ldots, \zeta^{d-1}).$$

By the preceding argument, $\tilde{\sigma}$ is, up to conjugacy in $\tilde{\mathbf{L}}^*$, of the following form :

$$\tilde{\sigma} = \begin{pmatrix} a_1 J & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & a_e J \end{pmatrix}$$

where $a_1, \ldots, a_e$ are in $\mathbb{F}^\times$ and $a_1 \ldots a_e (\det J)^e = 1$. Similarly, we can assume that

$$\tilde{\tau} = \begin{pmatrix} b_1 J & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & b_e J \end{pmatrix}$$

where $b_1, \ldots, b_e$ are in $\mathbb{F}^\times$ and $b_1 \ldots b_e (\det J)^e = 1$. The semisimple elements $\tilde{\sigma}$ and $\tilde{\tau}$ are conjugate under $\tilde{\mathbf{G}}^*$ so they have common eigenvalues. Hence, up to conjugacy by an element of $N_{\tilde{\mathbf{G}}^*}(\tilde{\mathbf{L}})$, we can assume that $a_1 = b_1$ and an easy induction argument proves that $\tilde{\sigma}$ and $\tilde{\tau}$ are conjugate under $N_{\tilde{\mathbf{G}}^*}(\tilde{\mathbf{L}})$.

• We now prove $(\mathcal{P})$ in the general case. Let $\overline{\mathbf{G}}^*$ be the adjoint group of $\mathbf{G}^*$ and let $\mathbf{G}^* \to \overline{\mathbf{G}}^*$, $g \mapsto \overline{g}$ be the canonical projection. Then, to reduce the general case to the case where $\mathbf{G}^*$ is adjoint, it is sufficient to prove that $\overline{\sigma}$ is cuspidal in $\overline{\mathbf{L}}^*$. Let $\overline{\mathbf{M}}^*$ be a Levi subgroup of a proper parabolic subgroup of $\overline{\mathbf{L}}^*$ and let $\mathbf{M}^*$ be its inverse image in $\mathbf{G}^*$. Assume that $\sigma \in \mathbf{M}^*$ (or, equivalently, $\overline{\sigma} \in \overline{\mathbf{M}}^*$). Then we have a commutative diagram of injective morphisms

$$\begin{tikzcd} A_{\mathbf{M}^*}(\sigma) \arrow[r] \arrow[d] & A_{\overline{\mathbf{M}}^*}(\overline{\sigma}) \arrow[d] \\ A_{\mathbf{L}^*}(\sigma) \arrow[r] & A_{\overline{\mathbf{L}}^*}(\overline{\sigma}). \end{tikzcd}$$

Moreover, if we identify all the groups involved in this diagram with subgroups of $A_{\overline{\mathbf{L}}^*}(\sigma)$, we have

$$A_{\mathbf{M}^*}(\sigma) = A_{\mathbf{L}^*}(\sigma) \cap A_{\overline{\mathbf{M}}^*}(\overline{\sigma}).$$

Indeed, by [DLM], 3.12, we have $A_{\mathbf{L}^*}(\sigma) = \{z \in \operatorname{Ker}\pi \mid \tilde{\sigma}$ and $\tilde{\sigma}z$ are conjugate in $\tilde{\mathbf{L}}^*\}$ and similar descriptions for other groups. That proves the result. ■

## 5. The Mackey formula in type $A$

This section is devoted to the proof of the main theorem of this article, that is the Mackey formula for groups of type $A$ (*cf.* Theorem 5.2.1). By using properties of the functions $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}$ and $\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}$ and by an induction argument, we prove that the class function $\lambda = \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu)$ on $\mathbf{L}^F$ (where $\mu$ is a class function on $\mathbf{M}^F$) is absolutely cuspidal and we reduce the general case to a particular one. Since the group $\mathbf{L}$ is of type $A$ whenever $\mathbf{G}$ is and since the dimension of $\mathbf{L}$ is smaller than the dimension of $\mathbf{G}$, all the absolutely cuspidal functions on $\mathbf{L}^F$ are described explicitly by induction and by Corollary 4.2.4. Then, using the decomposition of the function $\lambda$ according to rational Lusztig series, we prove that the inequality $\lambda \neq 0$ is in contradiction with Lemma 4.2.10.

**5.1. Some properties of $\Delta$ and $\Gamma$-functions.** Before proving the Mackey formula for groups of type $A$, we state many properties about $\Delta$ and $\Gamma$-functions which can be proved *a priori* and without hypothesis on the group $\mathbf{G}$. In particular, it could help to prove the Mackey formula for the other groups than the ones of type $A$. We fix two parabolic subgroups $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbf{G}$ and we assume that $\mathbf{P}$ and $\mathbf{Q}$ have $F$-stable Levi subgroups $\mathbf{L}$ and $\mathbf{M}$ respectively. We denote by $\mathbf{U}$ and $\mathbf{V}$ the unipotent radicals of $\mathbf{P}$ and $\mathbf{Q}$ respectively. Let $\mathbf{L}^*$ (respectively $\mathbf{M}^*$) be an $F^*$-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ which is dual to $\mathbf{L}$ (respectively $\mathbf{M}$).

**(5.1.1)** $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}$ *and* $\Delta^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q},\mathbf{L}\subset\mathbf{P}}$ *are adjoint with respect to* $\langle,\rangle_{\mathbf{L}^F}$ *and* $\langle,\rangle_{\mathbf{M}^F}$.

**(5.1.2)** *If $v$ and $w$ are unipotent elements of $\mathbf{L}^F$ and $\mathbf{M}^F$ respectively, then*

$$\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(v,w) = \Gamma^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q},\mathbf{L}\subset\mathbf{P}}(w,v).$$

**(5.1.3)** *If $z \in \mathbf{Z}^F$, then* $t^{\mathbf{G}}_z \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \circ t^{\mathbf{M}}_z$.

**(5.1.4)** *If $\zeta \in H^1(F,\mathbf{Z})^\wedge$ and if $\mu \in \operatorname{Class}(\mathbf{M}^F)_\zeta$ then* $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu) \in \operatorname{Class}(\mathbf{L}^F)_\zeta$.

**(5.1.5)** *If $s$ is a semisimple element of $\mathbf{L}^F$, then*

$$d^{\mathbf{L}}_s \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = \sum_{\substack{g\in\mathbf{G}^F\\ s\in{}^g\mathbf{M}}} \frac{|C^\circ_{{}^g\mathbf{M}}(s)^F|}{|\mathbf{M}^F|.|C^\circ_{\mathbf{G}}(s)^F|}\; \Delta^{C^\circ_{\mathbf{G}}(s)}_{C^\circ_{\mathbf{L}}(s)\subset C^\circ_{\mathbf{P}}(s),C^\circ_{{}^g\mathbf{M}}(s)\subset C^\circ_{{}^g\mathbf{Q}}(s)} \circ d^{{}^g\mathbf{M}}_s \circ (\operatorname{ad} g)_{\mathbf{M}}.$$

**(5.1.6)** *If $z \in \mathbf{Z}^F$, then* $d^{\mathbf{L}}_z \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \circ d^{\mathbf{M}}_z$.

**(5.1.7)** *Let $\mathbf{L}'$ be an $F-$stable Levi subgroup of a parabolic subgroup $\mathbf{P}'$ of $\mathbf{L}$. Then*

$$\begin{aligned} {}^*R^{\mathbf{L}}_{\mathbf{L}'\subset\mathbf{P}'} \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} &= \Delta^{\mathbf{G}}_{\mathbf{L}'\subset\mathbf{P}'\mathbf{U},\mathbf{M}\subset\mathbf{Q}} \\ &+ \sum_{g\in\mathbf{L}^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} \Delta^{\mathbf{L}}_{\mathbf{L}'\subset\mathbf{P}',\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{P}\cap{}^g\mathbf{M}} \circ (\mathrm{ad}\, g)_{\mathbf{M}}. \end{aligned}$$

**(5.1.8)** *Let $\mathbf{M}'$ be an $F-$stable Levi subgroup of a parabolic subgroup $\mathbf{Q}'$ of $\mathbf{M}$. Then*

$$\begin{aligned} \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \circ R^{\mathbf{M}}_{\mathbf{M}'\subset\mathbf{Q}'} &= \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}'\subset\mathbf{Q}'\mathbf{V}} \\ &+ \sum_{g\in\mathbf{L}^F\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q}} \circ \Delta^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q},{}^g\mathbf{M}'\subset{}^g\mathbf{Q}'} \circ (\mathrm{ad}\, g)_{\mathbf{M}'}. \end{aligned}$$

**(5.1.9)** *If $\sigma$ is a semisimple element of $\mathbf{M}^{*F^*}$ and if $\mu \in \overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{M}^F,(\sigma)_{\mathbf{M}^{*F^*}})$, then*

$$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu) \in \bigoplus_{(\tau)_{\mathbf{L}^{*F^*}}\subset(\sigma)_{\mathbf{G}^{*F^*}}} \overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{L}^F,(\tau)_{\mathbf{L}^{*F^*}}).$$

Proofs - • 5.1.1 follows from the fact that the Lusztig restriction functor and the Lusztig induction functor are adjoint and that the map $\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M}) \to \mathcal{S}_{\mathbf{G}}(\mathbf{M},\mathbf{L})$, $g \mapsto g^{-1}$ is bijective. The bijectivity of this map also implies 5.1.2.

• 5.1.3 follows from 1.7.3 and 1.7.4.

• 5.1.4 follows from Lemma 1.8.4, (a) and (b).

• We now prove 5.1.5. We denote by

$$\mathcal{A} = \{g \in \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M}) \mid s \in \mathbf{L}\cap{}^g\mathbf{M}\},$$

$$\mathcal{B} = \{(x,l) \mid x \in \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F \text{ and } l \in \mathbf{L}^F \text{ and } s \in {}^l(\mathbf{L}\cap{}^x\mathbf{M})\}$$

and $$\mathcal{C} = \{(g,y) \mid g \in \mathbf{G}^F \text{ and } s \in {}^g\mathbf{M} \text{ and } y \in \mathcal{S}_{C^\circ_{\mathbf{G}}(s)}(C^\circ_{\mathbf{L}}(s), C^\circ_{{}^g\mathbf{M}}(s))^F\}.$$

Note that the map $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}$ is also given by the following formula :

$$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} - \sum_{\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F} \frac{|\mathbf{L}^F \cap {}^g\mathbf{M}^F|}{|\mathbf{L}^F|.|\mathbf{M}^F|} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{P}\cap{}^g\mathbf{M}} \circ (\mathrm{ad}\, g)_{\mathbf{M}}.$$

Using the fact that the maps

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{A} \\ (x,l) & \longmapsto & lx \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{A} \\ (g,y) & \longmapsto & yg \end{array}$$

are surjective and have all their fibers of cardinality $|\mathbf{L}^F|$ and $|C^\circ_{\mathbf{G}}(s)^F|$ respectively, 5.1.5 follows by a straightforward application of identities 1.3.1 and 1.3.2 (*cf.* also [B1], proof of Lemma 2.3.4 for a similar argument).

• 5.1.6 is an immediate consequence of 5.1.5.

• The proof of 5.1.7 is analogous to [B1], proof of Lemma 3.2.1 and 5.1.8 is just the adjoint formula of 5.1.7.

• 5.1.9 follows from Proposition 1.6.1 and Corollary 1.6.2. ■

We now use the same notation as in Section 2 that is, we assume given a morphism of algebraic groups $i : \mathbf{G} \to \tilde{\mathbf{G}}$ defined over $\mathbb{F}_q$ and satisfying properties (a) and (b) described in Section 2. We denote by $\tilde{\mathbf{P}}$ (respectively $\tilde{\mathbf{Q}}$) the unique parabolic subgroup of $\tilde{\mathbf{G}}$ such that $i^{-1}(\tilde{\mathbf{P}}) = \mathbf{P}$ (respectively $i^{-1}(\tilde{\mathbf{Q}}) = \mathbf{Q}$) and by $\tilde{\mathbf{L}}$ (respectively $\tilde{\mathbf{M}}$) the unique Levi subgroup of $\tilde{\mathbf{P}}$ (respectively $\tilde{\mathbf{Q}}$) such that $i^{-1}(\tilde{\mathbf{L}}) = \mathbf{L}$ (respectively $i^{-1}(\tilde{\mathbf{M}}) = \mathbf{M}$).

**(5.1.10)** *If $i$ is injective, then* $\mathrm{Res}_{\mathbf{L}^F}^{\tilde{\mathbf{L}}^F} \circ \Delta^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}},\tilde{\mathbf{M}}\subset\tilde{\mathbf{Q}}} = \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \circ \mathrm{Res}_{\mathbf{M}^F}^{\tilde{\mathbf{M}}^F}$ *and*

**(5.1.11)**
$$\begin{aligned}\Gamma^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}},\tilde{\mathbf{M}}\subset\tilde{\mathbf{Q}}}(v,w) &= \frac{|\mathbf{G}^F|}{|\tilde{\mathbf{G}}^F|}\sum_{l\in\tilde{\mathbf{L}}^F/\mathbf{L}^F} \Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}({}^l v, w)\\ &= \frac{|\mathbf{G}^F|}{|\tilde{\mathbf{G}}^F|}\sum_{m\in\tilde{\mathbf{M}}^F/\mathbf{M}^F} \Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(v, {}^m w)\end{aligned}$$

*for all* $(v,w) \in \mathbf{L}^F_{\mathrm{uni}} \times \mathbf{M}^F_{\mathrm{uni}}$.

Proof - This follows from 2.1.4, 2.1.5 and 2.2.1, (b). ■

**(5.1.12)** *If $i$ is surjective and* $\mathrm{Ker}\, i \subset \mathbf{Z}(\mathbf{L})^\circ \cap \mathbf{Z}(\mathbf{M})^\circ$, *then*

$$\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(v,w) = \Gamma^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{L}}\subset\tilde{\mathbf{P}},\tilde{\mathbf{M}}\subset\tilde{\mathbf{Q}}}(i(v), i(w))$$

*for all* $(v,w) \in \mathbf{L}^F_{\mathrm{uni}} \times \mathbf{M}^F_{\mathrm{uni}}$.

Proof - This follows from Corollary 2.2.3 and from the fact that $|\tilde{\mathbf{G}}^F| = |\mathbf{G}^F|/|(\mathrm{Ker}\, i)^{\circ F}|$. ■

**5.2. The Mackey formula in type A.** We are ready to prove the following

**Theorem 5.2.1.** *If $\mathbf{G}$ is of type $A$, then conjectures $A$ and $B$ hold in $\mathbf{G}$. In other words, the Mackey formula and the Mackey formula for Green functions hold in $\mathbf{G}$.*

Proof - Assume that $\mathbf{G}$ is of type $A$. We fix two parabolic subgroups $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbf{G}$ and we assume that $\mathbf{P}$ and $\mathbf{Q}$ have $F$-stable Levi subgroups $\mathbf{L}$ and $\mathbf{M}$ respectively. Let $n = \dim \mathbf{G}$ and $m = \dim \mathbf{L} + \dim \mathbf{M}$. We endow $\mathbb{N}^* \times \mathbb{N}^*$ with the lexicographic order. Theorem 5.2.1 is equivalent to prove by induction on $(n,m)$ the following :

**(P)** $$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = 0 \qquad \text{and} \qquad \Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = 0.$$

If $(n,m)$ is small then it is clear that (P) holds. We fix $(n,m) \in \mathbb{N}^*$ and we assume that (P) holds for any pair in $\mathbb{N}^* \times \mathbb{N}^*$ strictly smaller than $(n,m)$. We can (and we will) assume that $\mathbf{L} \neq \mathbf{G}$ and $\mathbf{M} \neq \mathbf{G}$ (otherwise, (P) is obvious). Note that all $F$-stable connected reductive subgroup $\mathbf{G}'$ of the same rank of $\mathbf{G}$ are of type $A$. Hence (P) holds for such $\mathbf{G}'$ different from $\mathbf{G}$ by the induction hypothesis. In particular

**(5.2.2)** *The Mackey formula holds in $\mathbf{L}$ and $\mathbf{M}$.*

Let $\mu$ be a class function on $\mathbf{M}^F$ and let $v$ and $w$ be unipotent elements of $\mathbf{L}^F$ and $\mathbf{M}^F$ respectively. Then, by [B1], Corollary 2.3.5 and Proposition 2.3.6, and by the induction hypothesis, the equalities $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}=0$ and $\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}=0$ are equivalent. Hence it is sufficient to prove

$$\textbf{(Q)} \qquad \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu)=0 \qquad \text{or} \qquad \Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(v,w)=0.$$

Since (Q) holds if $\mathbf{P}$ and $\mathbf{Q}$ are $F$-stable, we can assume that $\mathbf{P}$ is not $F$-stable or that $\mathbf{Q}$ is not $F$-stable. By 5.1.1 or by 5.1.2 we can assume that $\mathbf{P}$ is not rational. This implies the following :

$$\textbf{(5.2.3)} \qquad \mathbf{Z}(\mathbf{L})^\circ \textit{ is not split so } |\mathbf{Z}(\mathbf{L})^{\circ F}|>1.$$

Let $\mathbf{L}^*$ (respectively $\mathbf{M}^*$) be an $F^*$-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ dual to $\mathbf{L}$ (respectively $\mathbf{M}$). Let $\sigma$ be a semisimple element of $\mathbf{M}^{*F^*}$ and $\zeta$ a linear character of $H^1(F,\mathbf{Z}(\mathbf{M}))^\wedge$. We can assume that $\mu\in\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{M}^F,(\sigma)_{\mathbf{M}^{*F^*}})_\zeta$. So, if we denote by $\boldsymbol{\zeta}=\zeta\circ\mathbf{h}^{\mathbf{G}}_{\mathbf{M}}$, we have, by 5.1.4,

$$\textbf{(5.2.4)} \qquad \lambda=\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu)\in\bigoplus_{(\tau)_{\mathbf{L}^{*F^*}}\subset(\sigma)_{\mathbf{G}^{*F^*}}}\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{L}^F,(\tau)_{\mathbf{L}^{*F^*}})_{\boldsymbol{\zeta}}.$$

If $\operatorname{Ker}\mathbf{h}^{\mathbf{G}}_{\mathbf{L}}\not\subseteq\operatorname{Ker}\boldsymbol{\zeta}$, then, by Lemma 1.8.5, we have $\lambda=0$. So we can assume that $\operatorname{Ker}\mathbf{h}^{\mathbf{G}}_{\mathbf{L}}\subset\operatorname{Ker}\zeta$. But, by 5.1.7 and by the induction hypothesis, $\lambda$ is absolutely cuspidal, so we can assume that $\boldsymbol{\zeta}\in H^1(F,\mathbf{Z}(\mathbf{L}))^\wedge_{\mathrm{cus}}$ by 5.2.2 and Corollary 4.2.6. Moreover, by 5.1.8, we can assume that $\mu$ is absolutely cuspidal. In particular, $\boldsymbol{\zeta}\in H^1(F,\mathbf{Z}(\mathbf{M}))^\wedge_{\mathrm{cus}}$. It follows from Proposition 3.3.2 that $\mathbf{Z}\cap\mathbf{Z}(\mathbf{L})^\circ=\mathbf{Z}\cap\mathbf{Z}(\mathbf{M})^\circ$. Let $Z=\mathbf{Z}\cap\mathbf{Z}(\mathbf{L})^\circ$ and let $i:\mathbf{G}\to\mathbf{G}/Z$. We have, by 5.1.12,

$$\Gamma^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(v,w)=\Gamma^{i(\mathbf{G})}_{i(\mathbf{L})\subset i(\mathbf{P}),i(\mathbf{M})\subset i(\mathbf{Q})}(i(v),i(w)).$$

Thus, by replacing $\mathbf{G}$ by $\mathbf{G}/Z$ if necessary, we can assume that

$$\textbf{(5.2.5)} \qquad \mathbf{Z}\cap\mathbf{Z}(\mathbf{L})^\circ=\{1\}.$$

Moreover, we have, by Proposition 1.6.1 and Corollary 1.6.2,

$$\textbf{(5.2.6)} \qquad \lambda\in\bigoplus_{\substack{(\tau)_{\mathbf{L}^{*F^*}}\subset(\sigma)_{\mathbf{G}^{*F^*}}\\ \tau \text{ cuspidal in } \mathbf{L}^*}}\overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{L}^F,(\tau)_{\mathbf{L}^{*F^*}})_{\boldsymbol{\zeta}}.$$

By 5.1.5 and by the induction hypothesis, the function $\lambda$ has support in $\mathbf{Z}^F.\mathbf{L}^F_{\mathrm{uni}}$ and, if we put $\theta=\hat{\sigma}^{\mathbf{G}}$, we have, by 5.1.3 and Lemma 1.7.5,

$$t^{\mathbf{L}}_z\lambda=\theta(z)\lambda$$

for all $z\in\mathbf{Z}^F$. Therefore, using Corollary 4.2.4, we deduce that there exists $c\in\overline{\mathbb{Q}}_\ell$ such that

$$\textbf{(5.2.7)} \qquad \lambda=c\sum_{\substack{\phi\in(\mathbf{Z}(\mathbf{L})^F)^\wedge\\ \operatorname{Res}^{\mathbf{Z}(\mathbf{L})^F}_{\mathbf{Z}^F}\phi=\theta}}\Gamma^{\mathbf{L}}_{\phi,\zeta}.$$

At this point, we only need to prove that

$$\textbf{(R)} \qquad c=0.$$

Assume that $c \neq 0$. By 5.2.5, we have $\mathbf{Z}(\mathbf{L})^F = \mathbf{Z}^F \times \mathbf{Z}(\mathbf{L})^{\circ F}$. Let $\varphi : \mathbf{Z}(\mathbf{L})^{\circ F} \to \overline{\mathbb{Q}}_\ell$ be a non-trivial linear character (there exists at least one by 5.2.3). By Corollary 4.2.6, (b), there exist semisimple elements $\tau$ and $\tau'$ such that

$$\Gamma^{\mathbf{L}}_{\theta\otimes 1,\zeta} \in \overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{L}^F, (\tau)_{\mathbf{L}^{*F^*}})_{\boldsymbol{\zeta}}$$

and
$$\Gamma^{\mathbf{L}}_{\theta\otimes \varphi,\zeta} \in \overline{\mathbb{Q}}_\ell \mathcal{E}(\mathbf{L}^F, (\tau')_{\mathbf{L}^{*F^*}})_{\boldsymbol{\zeta}}.$$

So, by 5.2.6 and 5.2.7, $\tau$ and $\tau'$ are cuspidal in $\mathbf{L}^*$ and conjugate under $\mathbf{G}^*$. Hence there exists $n^* \in N_{\mathbf{G}^*}(\mathbf{L}^*)$ such that $\tau' = {}^{n^*}\tau$ (*cf.* Lemma 4.2.7). Let $i \in \mathbb{N}^*$ be such that $F^i(n^*) = n^*$ and let $n \in N_{\mathbf{G}^{F^i}}(\mathbf{L})$ be an element corresponding to $n^*$. Then

$$\varphi \circ N_{F^i/F} = 1 \circ N_{F^i/F} \circ (\operatorname{ad} n^{-1}) = 1$$

where $N_{F^i/F} : \mathbf{Z}(\mathbf{L})^{\circ F^i} \to \mathbf{Z}(\mathbf{L})^{\circ F}$ is the norm map. But the norm map is surjective, so $\varphi = 1$ which is contrary to the hypothesis. That proves (R) and the theorem. ■

## 6. Many consequences of the Mackey formula

In this section, we assume that $\mathbf{G}$ is of type $A$. The fact that the Mackey formula holds in $\mathbf{G}$ has many consequences. First, it implies that the Lusztig functors are independent of the choice of a parabolic subgroup and that these functors commute with the Alvis-Curtis duality operator. Moreover, using the results of Section 4, we obtain that conjecture C holds in $\mathbf{G}$ (*cf.* Theorem 6.2.1).

**6.1. Classical consequences.** Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and assume that $\mathbf{P}$ has an $F$-stable Levi subgroup $\mathbf{L}$. Then the following propositions are well-known to be consequences of the Mackey formula.

**Proposition 6.1.1.** *Assume that* $\mathbf{G}$ *is of type* $A$. *Let* $\mathbf{P}'$ *be another parabolic subgroup of* $\mathbf{G}$ *having* $\mathbf{L}$ *as Levi subgroup. Then*

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'} = R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$$

*and*
$${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'} = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}.$$

Thanks to Proposition 6.1.1, we can denote $R^{\mathbf{G}}_{\mathbf{L}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}}$ the Lusztig functors $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ without ambiguity. We denote by $D_{\mathbf{G}} : \operatorname{Class}(\mathbf{G}^F) \to \operatorname{Class}(\mathbf{G}^F)$ the Alvis-Curtis duality operator and we put $\varepsilon_{\mathbf{G}} = (-1)^{\mathbb{F}_q\text{-rank of }\mathbf{G}}$.

**Proposition 6.1.2.** *Assume that* $\mathbf{G}$ *is of type* $A$. *Then*

$$\varepsilon_{\mathbf{G}} D_{\mathbf{G}} \circ R^{\mathbf{G}}_{\mathbf{L}} = \varepsilon_{\mathbf{L}} R^{\mathbf{G}}_{\mathbf{L}} \circ D_{\mathbf{L}}$$

*and*
$$\varepsilon_{\mathbf{G}} {}^*R^{\mathbf{G}}_{\mathbf{L}} \circ D_{\mathbf{G}} = \varepsilon_{\mathbf{L}} D_{\mathbf{L}} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}}.$$

**6.2. Absolutely cuspidal functions.** By Theorem 5.2.1, all the results stated in Subsection 4.2 hold without hypothesis. We summarize these results :

**Theorem 6.2.1.** *Assume that* $\mathbf{G}$ *is of type* $A$. *Then conjecture* $C$ *holds in* $\mathbf{G}$ *that is, the family* $(\mathfrak{Y}_\iota)_{\iota\in\mathcal{U}(\mathbf{G})^F_{\mathrm{cus}}}$ *is an orthogonal basis of* $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$. *In other words,* $(\Gamma^{\mathbf{G}}_\zeta)_{\zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}}$ *is an orthogonal basis of* $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{G}^F)$.

**Corollary 6.2.2.** *Assume that* $\mathbf{G}$ *is of type* $A$. *Then*

(a) *We have the following decompositions*

$$\mathrm{Cus}(\mathbf{G}^F) = \bigoplus_{\zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}} \mathrm{Class}(\mathbf{G}^F)_\zeta$$

*and*

$$\mathrm{Cus}(\mathbf{G}^F) = \bigoplus_{\substack{\phi\in(\mathbf{Z}^F)^\wedge \\ \zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}}} \mathrm{Class}(\mathbf{G}^F)^\phi_\zeta.$$

*Moreover* $\dim \mathrm{Class}(\mathbf{G}^F)^\phi_\zeta = 1$ *for all* $\phi \in (\mathbf{Z}^F)^\wedge$ *and* $\zeta \in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}$ *and* $\mathrm{Class}(\mathbf{G}^F)^\phi_\zeta$ *is generated by* $\Gamma^{\mathbf{G}}_{\phi,\zeta}$. *In particular,* $(\Gamma^{\mathbf{G}}_{\phi,\zeta})_{\substack{\phi\in(\mathbf{Z}^F)^\wedge \\ \zeta\in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}}}$ *is an orthogonal basis of* $\mathrm{Cus}(\mathbf{G}^F)$.

(b) *If* $\phi \in (\mathbf{Z}^F)^\wedge$ *and* $\zeta \in H^1(F,\mathbf{Z})^\wedge_{\mathrm{cus}}$ *then there exists a unique semisimple element* $\sigma$ *in* $\mathbf{G}^{*F^*}$ *(up to* $\mathbf{G}^{*F^*}$*-conjugacy) such that* $\Gamma^{\mathbf{G}}_{\phi,\zeta} \in \overline{\mathbb{Q}}_\ell\mathcal{E}(\mathbf{G}^F,(\sigma))$.

## References


[B1] C. Bonnafé, *Formule de Mackey pour q grand*, J. of Alg. **201** (1998), 207-232.

[B2] C. Bonnafé, *Produit en couronne de groupes linéaires*, Accepted for publication in J. of Alg.

[DL1] P. Deligne & G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), 103-161.

[DL2] P. Deligne & G. Lusztig, *Duality for representations of a reductive group over a finite field*, J. of Alg. **81** (1983), 540-545.

[DLM] F. Digne, G. Lehrer & J. Michel, *The characters of the group of rational points of a reductive group with non-connected centre*, J. Reine Angew Math. **425** (1992), 155-192.

[DM] F. Digne & J. Michel, *Representations of finite groups of Lie type*, in London Math. Soc. Students Texts, Vol. **21**, Cambridge University press, Cambridge, UK, 1991.

[L1] G. Lusztig, *On the finiteness of the number of unipotent classes*, Invent. Math. **34** (1976), 201-213.

[L2] G. Lusztig, *Intersection cohomology complexes on reductive groups*, Invent. Math. **75** (1984), 205-272.

[L3] G. Lusztig, *Character sheaves V*, Adv. in Math. **61** (1986), 103-155.

[L4] G. Lusztig, *Green functions and character sheaves*, Ann. of Math. **131** (1990), 355-408.

[LS] G. Lusztig & N. Spaltenstein, *Induced unipotent classes*, J. of the London. Math. Soc. (2) **19** (1979), 41-52.

[S1] T. Shoji, *Character sheaves and almost characters of reductive groups I, II*, Adv. in Math. **111** (1995), 244-313 and 314-354.

[S2] T. Shoji, *On the computation of unipotent characters of finite classical groups*, Applicable Algebra in Engineering, Communication and Computing **7** (1996), 165-174.