Arbitrary-order polytopal schemes for the Yang–Mills equations

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Outline

Notations

- Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$
  - Vector space $\mathfrak{g}$
  - Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, antisymmetric, bilinear etc.
- $(e_I)_I$ a basis of the Lie algebra

Example

- $\mathfrak{g} = \mathfrak{su}(2)$ matrix Lie algebra, $[A, B] := AB - BA$

$$e_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; e_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; e_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Decomposition of $\mathfrak{g}$-valued function $f = \sum_I f^I \otimes e_I$
- Inner product $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ s.t.

$$\langle [a, b], c\rangle = \langle a, [b, c]\rangle \qquad \forall a, b, c \in \mathfrak{g}.$$

Yang–Mills equations

- $\boldsymbol{A}(\boldsymbol{x}, t)$ - Gauge potential
- The 'electric field' $\boldsymbol{E}$ and 'magnetic field' $\boldsymbol{B}$ defined by

$$\boldsymbol{E} = -\partial_t \boldsymbol{A}$$
$$\boldsymbol{B} = \textbf{curl}\,\boldsymbol{A} + \frac{1}{2}\star[\boldsymbol{A}, \boldsymbol{A}]$$

Yang–Mills equations:

Evolution:

$$\partial_t \boldsymbol{E} = \textbf{curl}\,\boldsymbol{B} + \star[\boldsymbol{A}, \boldsymbol{B}]$$
$$\partial_t \boldsymbol{B} = -\textbf{curl}\,\boldsymbol{E} - \star[\boldsymbol{A}, \boldsymbol{E}]$$

Constraint:

$$\operatorname{div} \boldsymbol{E} + \star[\boldsymbol{A}, \star\boldsymbol{E}] = 0$$
$$\operatorname{div} \boldsymbol{B} + \star[\boldsymbol{A}, \star\boldsymbol{B}] = 0$$

Non-linear terms

The non-linear terms appearing in the Yang–Mills equations are the Lie algebra generalisation of some familiar operators...

Non-linear operators:

$$
\begin{aligned}
[q, \boldsymbol{v}] &:= \sum_{I,J} q^I \boldsymbol{v}^J \otimes [e_I, e_J] \\
\star[\boldsymbol{v}, \boldsymbol{w}] &:= \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \otimes [e_I, e_J] \\
\star[\boldsymbol{v}, \star\boldsymbol{w}] &:= \sum_{I,J} (\boldsymbol{v}^I \cdot \boldsymbol{w}^J) \otimes [e_I, e_J]
\end{aligned}
$$

L^2-inner products

Scalar and vector L^2-products:

$$\int_U \langle q, r \rangle := \sum_{I,J} \int_U q^I r^J \langle e_I, e_J \rangle, \qquad \int_U \langle \boldsymbol{v}, \boldsymbol{w} \rangle := \sum_{I,J} \int_U \boldsymbol{v}^I \cdot \boldsymbol{w}^J \langle e_I, e_J \rangle$$

Properties:

$$\int_U \langle \star[\boldsymbol{u}, \boldsymbol{v}], \boldsymbol{w} \rangle = \int_U \langle \boldsymbol{u}, \star[\boldsymbol{v}, \boldsymbol{w}] \rangle$$

$$\int_U \langle \boldsymbol{v}, [\boldsymbol{w}, q] \rangle = \int_U \langle \star[\boldsymbol{v}, \star\boldsymbol{w}], q \rangle$$

L^2-inner products

Scalar and vector L^2-products:

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Properties:

$$\int_U \langle \star[\boldsymbol{u}, \boldsymbol{v}], \boldsymbol{w} \rangle = \int_U \langle \boldsymbol{u}, \star[\boldsymbol{v}, \boldsymbol{w}] \rangle$$

$$\int_U \langle \boldsymbol{v}, [\boldsymbol{w}, q] \rangle = \int_U \langle \star[\boldsymbol{v}, \star\boldsymbol{w}], q \rangle$$

Consequence:

$$\int_U \langle \boldsymbol{v}, [\boldsymbol{v}, q] \rangle = \int_U \langle \star[\boldsymbol{v}, \star\boldsymbol{v}], q \rangle = 0$$

Weak formulation

Integration by parts:

- Find $(\boldsymbol{A}, \boldsymbol{E}) : [0, T] \to (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g})$ s.t.

$$\partial_t \boldsymbol{A} = -\boldsymbol{E},$$

$$\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle = \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g},$$

  where $\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A} + \frac{1}{2} \star [\boldsymbol{A}, \boldsymbol{A}]$
- Initial conditions: $\boldsymbol{A}(0) = \boldsymbol{A}_0$, $\boldsymbol{E}(0) = \boldsymbol{E}_0$ in U
- Constraint (was $\operatorname{div} \boldsymbol{E} + \star[\boldsymbol{A}, \star\boldsymbol{E}] = 0$):

$$\int_U \langle \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle = 0, \qquad \forall q \in H^1(U) \otimes \mathfrak{g},$$

Constraint preservation

$$
\begin{aligned}
\partial_t \boldsymbol{A} &= -\boldsymbol{E}, \\
\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle &= \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}, \\
\boldsymbol{B} &= \mathbf{curl}\, \boldsymbol{A} + \frac{1}{2} \star [\boldsymbol{A}, \boldsymbol{A}].
\end{aligned}
$$

$$
\partial_t \int_U \langle \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle = \int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle + \int_U \langle \boldsymbol{E}, [\partial_t \boldsymbol{A}, q] \rangle
$$

Constraint preservation

$$
\begin{aligned}
\partial_t \boldsymbol{A} &= -\boldsymbol{E}, \\
\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle &= \int_U \langle \boldsymbol{B}, \operatorname{\mathbf{curl}} \boldsymbol{v} + \star [\boldsymbol{A}, \boldsymbol{v}] \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{\mathbf{curl}}; U) \otimes \mathfrak{g}, \\
\boldsymbol{B} &= \operatorname{\mathbf{curl}} \boldsymbol{A} + \frac{1}{2} \star [\boldsymbol{A}, \boldsymbol{A}].
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \int_U \langle \boldsymbol{E}, \operatorname{\mathbf{grad}} q + [\boldsymbol{A}, q] \rangle &= \int_U \langle \partial_t \boldsymbol{E}, \operatorname{\mathbf{grad}} q + [\boldsymbol{A}, q] \rangle + \int_U \langle \boldsymbol{E}, [\partial_t \boldsymbol{A}, q] \rangle \\
&= \int_U \langle \partial_t \boldsymbol{E}, \operatorname{\mathbf{grad}} q + [\boldsymbol{A}, q] \rangle - \cancel{\int_U \langle \boldsymbol{E}, [\boldsymbol{E}, q] \rangle}
\end{aligned}
$$

Constraint preservation

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\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle &= \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}, \\
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\end{aligned}$$

$$\begin{aligned}
\partial_t \int_U \langle \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle &= \int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle + \int_U \langle \boldsymbol{E}, [\partial_t \boldsymbol{A}, q] \rangle \\
&= \int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle - \cancel{\int_U \langle \boldsymbol{E}, [\boldsymbol{E}, q] \rangle}
\end{aligned}$$

$$\mathbf{curl}(\mathbf{grad}\, q + [\boldsymbol{A}, q]) + \star[\boldsymbol{A}, \mathbf{grad}\, q + [\boldsymbol{A}, q]] = [\boldsymbol{B}, q]$$

Constraint preservation

$$
\begin{aligned}
\partial_t \boldsymbol{A} &= -\boldsymbol{E}, \\
\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle &= \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}, \\
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$$
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&= \int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle - \cancel{\int_U \langle \boldsymbol{E}, [\boldsymbol{E}, q] \rangle} \\
&= \int_U \langle \boldsymbol{B}, [\boldsymbol{B}, q] \rangle
\end{aligned}
$$

$$
\mathbf{curl}(\mathbf{grad}\, q + [\boldsymbol{A}, q]) + \star[\boldsymbol{A}, \mathbf{grad}\, q + [\boldsymbol{A}, q]] = [\boldsymbol{B}, q]
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Constraint preservation

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$$\mathbf{curl}(\mathbf{grad}\, q + [\boldsymbol{A}, q]) + \star[\boldsymbol{A}, \mathbf{grad}\, q + [\boldsymbol{A}, q]] = [\boldsymbol{B}, q]$$

Some more details...

$$
\begin{aligned}
&\Big\{ \mathbf{curl}(\,\mathbf{grad}\, q + [\boldsymbol{A}, q]) + \star[\boldsymbol{A}, \mathbf{grad}\, q + [\boldsymbol{A}, q]] \Big\}_{\alpha} \\
&\quad = \varepsilon_{\alpha\mu\nu} \partial^{\mu} [\boldsymbol{A}^{\nu}, q] + \varepsilon_{\alpha\mu\nu} [\boldsymbol{A}^{\mu}, \partial^{\nu} q + [\boldsymbol{A}^{\nu}, q]] \\
&\quad = \varepsilon_{\alpha\mu\nu} [\partial^{\mu} \boldsymbol{A}^{\nu}, q] + \varepsilon_{\alpha\mu\nu} [\boldsymbol{A}^{\nu}, \partial^{\mu} q] + \varepsilon_{\alpha\mu\nu} [\boldsymbol{A}^{\mu}, \partial^{\nu} q] + \varepsilon_{\alpha\mu\nu} [\boldsymbol{A}^{\mu}, [\boldsymbol{A}^{\nu}, q]] \\
&\quad = \left[\varepsilon_{\alpha\mu\nu} \partial_{\mu} \boldsymbol{A}^{\nu} + \frac{1}{2} \varepsilon_{\alpha\mu\nu} [\boldsymbol{A}^{\mu}, \boldsymbol{A}^{\nu}], q \right] \\
&\quad = \Big\{ [\boldsymbol{B}, q] \Big\}_{\alpha}.
\end{aligned}
$$

Constrained formulation

From [Christiansen and Winther, 2006], a *constrained formulation* for the Yang–Mills equations is:

Find
$(\boldsymbol{A}, \boldsymbol{E}, \lambda) : [0, T] \to (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (H^1(U) \otimes \mathfrak{g})$
s.t. $\forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}$, $\forall q \in H^1(U) \otimes \mathfrak{g}$

$$\begin{aligned} \partial_t \boldsymbol{A} &= -\boldsymbol{E} \\ \int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle + \int_U \langle \mathbf{grad}\, \lambda + [\boldsymbol{A}, \lambda], \boldsymbol{v} \rangle &= \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle \\ \int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle &= 0 \end{aligned}$$

where $\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A} + \frac{1}{2} \star [\boldsymbol{A}, \boldsymbol{A}]$

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where $\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A} + \frac{1}{2}\star[\boldsymbol{A}, \boldsymbol{A}]$

- Choosing $\boldsymbol{v} = \mathbf{grad}\, \lambda + [\boldsymbol{A}, \lambda]$:

$$\mathbf{grad}\, \lambda + [\boldsymbol{A}, \lambda] = 0$$

Outline

Outline

Design

- Continuous de Rham complex:

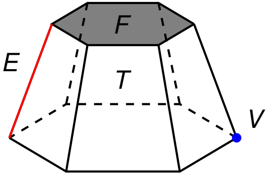
$$\mathbb{R} \xrightarrow{i_U} H^1(U) \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl}; U) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\text{div}; U) \xrightarrow{\text{div}} L^2(U) \xrightarrow{0} \{0\}.$$

- Discrete De Rham complex
  [Di Pietro and Droniou, 2021, Di Pietro et al., 2020]:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\textbf{grad},h}} \underline{X}^k_{\textbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\textbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\text{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Principle of DDR construction:

- A *mesh* is a collection of sets of elements $\mathcal{T}_h$, faces $\mathcal{F}_h$, edges $\mathcal{E}_h$ and vertices $\mathcal{V}_h$



Design

- Continuous de Rham complex:

$$\mathbb{R} \xrightarrow{i_U} H^1(U) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}; U) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}; U) \xrightarrow{\mathrm{div}} L^2(U) \xrightarrow{0} \{0\}.$$

- Discrete De Rham complex
  [Di Pietro and Droniou, 2021, Di Pietro et al., 2020]:

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Principle of DDR construction:

- Replace continuous spaces by fully discrete ones made of vectors of polynomials,
- Polynomials attached to geometric entities to emulate expected continuity properties of each space,
- Create discrete operators between them.

Discrete de Rham (DDR) complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Discrete spaces: Vector of values attached specific mesh entities:

Space	V	E	F	T
$\underline{X}^k_{\mathbf{grad},h}$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$	$\mathcal{RT}^k(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$			$\mathcal{P}^k(F)$	$\mathcal{N}^k(T)$
$\mathcal{P}^k(\mathcal{T}_h)$				$\mathcal{P}^k(T)$

Discrete de Rham (DDR) complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Discrete differential operators:

- Mimic integration-by-parts formula to define, from the DOFs, discrete operators in full polynomial spaces (*gradient: edge/face/element; curl: face/element; divergence: element*)
- Project these polynomials on DOFs of next space to create discrete differential operators, e.g. $\underline{\boldsymbol{C}}^k_h : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$.

Discrete de Rham (DDR) complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Reconstructions and L^2-products on each space, e.g.:

- Potential reconstructions $\boldsymbol{P}^k_{\mathbf{curl},T} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^k(T)$
- Tangential trace reconstruction $\gamma^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)$
- L^2-inner product $(\cdot,\cdot)_{\mathbf{curl},h} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$

Lie algebra DDR (LADDR) complex

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathbf{grad},h}^{k,\mathfrak{g}}} \underline{X}_{\mathbf{grad},h}^{k} \otimes \mathfrak{g} \xrightarrow{\underline{\boldsymbol{G}}_h^{k,\mathfrak{g}}} \underline{\boldsymbol{X}}_{\mathbf{curl},h}^{k} \otimes \mathfrak{g} \xrightarrow{\underline{\boldsymbol{C}}_h^{k,\mathfrak{g}}} \underline{\boldsymbol{X}}_{\mathrm{div},h}^{k} \otimes \mathfrak{g} \xrightarrow{D_h^{k,\mathfrak{g}}} \mathcal{P}^k(\mathcal{T}_h) \otimes \mathfrak{g} \xrightarrow{0} \{0\}$$

Lie algebra extension of spaces and operators

- Lie algebra values attached to mesh entities
- Tangential trace: $\gamma_{\mathrm{t},F}^{k,\mathfrak{g}} : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k} \otimes \mathfrak{g} \to \mathcal{P}^k(F) \otimes \mathfrak{g}$

$$\gamma_{\mathrm{t},F}^{k,\mathfrak{g}} \underline{\boldsymbol{v}}_F := \sum_I (\gamma_{\mathrm{t},F}^{k} \underline{\boldsymbol{v}}_F^I) \otimes e_I$$

- For all $\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^{k} \otimes \mathfrak{g}$

$$(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h)_{\mathbf{curl},\mathfrak{g},h} = \sum_{I,J} (\underline{\boldsymbol{v}}_h^I, \underline{\boldsymbol{w}}_h^J)_{\mathbf{curl},h} \langle e_I, e_J \rangle.$$

Discretising the constrained Yang–Mills equations

$(\boldsymbol{A}, \boldsymbol{E}, \lambda) : [0, T] \to (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (H^1(U) \otimes \mathfrak{g})$
s.t. $\forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}$, $\forall q \in H^1(U) \otimes \mathfrak{g}$:

$$\begin{aligned} \partial_t \boldsymbol{A} &= -\boldsymbol{E} \\ \int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle + \int_U \langle \mathbf{grad}\, \lambda + [\boldsymbol{A}, \lambda], \boldsymbol{v} \rangle &= \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle \\ \int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle &= 0 \end{aligned}$$

where $\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A} + \frac{1}{2} \star [\boldsymbol{A}, \boldsymbol{A}]$

Discretising the constrained Yang–Mills equations

$(\boldsymbol{A}, \boldsymbol{E}, \lambda) : [0,T] \to (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (H^1(U) \otimes \mathfrak{g})$
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where $\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A} + \frac{1}{2} \star [\boldsymbol{A}, \boldsymbol{A}]$

- Replace spaces/operators by LADDR spaces/operators $\rightsquigarrow$ deals with linear terms.

Discretising the constrained Yang–Mills equations

$(\boldsymbol{A}, \boldsymbol{E}, \lambda) : [0,T] \to (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (H^1(U) \otimes \mathfrak{g})$
s.t. $\forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}$, $\forall q \in H^1(U) \otimes \mathfrak{g}$:

$$\partial_t \boldsymbol{A} = -\boldsymbol{E}$$
$$\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle + \int_U \langle \mathbf{grad}\,\lambda + [\boldsymbol{A}, \lambda], \boldsymbol{v} \rangle = \int_U \langle \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle$$
$$\int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\,q + [\boldsymbol{A}, q] \rangle = 0$$

where $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A} + \frac{1}{2}\star[\boldsymbol{A}, \boldsymbol{A}]$

- Replace spaces/operators by LADDR spaces/operators $\rightsquigarrow$ deals with linear terms.
- What about the brackets?

Outline

Product bracket

$$\partial_t \boldsymbol{A} = -\boldsymbol{E}$$
$$\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle + \int_U \langle \mathbf{grad}\, \lambda + [\boldsymbol{A}, \lambda], \boldsymbol{v} \rangle = \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle$$
$$\int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q] \rangle = 0$$

- q scalar, $\boldsymbol{v}$ vector Lie algebra-valued functions, $(e_I)_I$ a basis of $\mathfrak{g}$
- Decompose $q = \sum_I q^I \otimes e_I$, where q^I is real-valued (resp. $\boldsymbol{v}$)

$$[q, \boldsymbol{v}] := \sum_{I,J} q^I \boldsymbol{v}^J \otimes [e_I, e_J]$$

Product bracket

$$\partial_t \boldsymbol{A} = -\boldsymbol{E}$$

$$\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle + \int_U \langle \mathbf{grad}\, \lambda + [\boldsymbol{A}, \lambda], \boldsymbol{v} \rangle = \int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle$$

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- q scalar, $\boldsymbol{v}$ vector Lie algebra-valued functions, $(e_I)_I$ a basis of $\mathfrak{g}$
- Decompose $q = \sum_I q^I \otimes e_I$, where q^I is real-valued (resp. $\boldsymbol{v}$)

$$[q, \boldsymbol{v}] := \sum_{I,J} q^I \boldsymbol{v}^J \otimes [e_I, e_J]$$

- Discretisation requires

$$(\underline{\boldsymbol{v}}_h, [\underline{\boldsymbol{v}}_h, \underline{q}_h])_{\mathbf{curl},\mathfrak{g},h} = 0$$

Product bracket

$$\partial_t \boldsymbol{A} = -\boldsymbol{E}$$

$$\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v}\rangle + \int_U \langle \mathbf{grad}\,\lambda + [\boldsymbol{A}, \lambda], \boldsymbol{v}\rangle = \int_U \langle \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}]\rangle$$

$$\int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\,q + [\boldsymbol{A}, q]\rangle = 0$$

- q scalar, $\boldsymbol{v}$ vector Lie algebra-valued functions, $(e_I)_I$ a basis of $\mathfrak{g}$
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- Discretisation requires

$$(\underline{\boldsymbol{v}}_h, [\underline{\boldsymbol{v}}_h, \underline{q}_h])_{\mathbf{curl},\mathfrak{g},h} = 0$$

- Choice:

$$\int_U \langle \boldsymbol{v}, [\boldsymbol{w}, q]\rangle \rightsquigarrow \int_U \langle \boldsymbol{P}^{k,\mathfrak{g}}_{\mathbf{curl},h} \underline{\boldsymbol{v}}_h, [\boldsymbol{P}^{k,\mathfrak{g}}_{\mathbf{curl},h} \underline{\boldsymbol{w}}_h, P^{k+1,\mathfrak{g}}_{\mathbf{grad},h} \underline{q}_h]\rangle$$

Cross product bracket

$$\partial_t \boldsymbol{A} = -\boldsymbol{E}$$
$$\int_U \langle \partial_t \boldsymbol{E}, \boldsymbol{v} \rangle + \int_U \langle \mathbf{grad}\,\lambda + [\boldsymbol{A}, \lambda], \boldsymbol{v} \rangle = \int_U \langle \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle$$
$$\int_U \langle \partial_t \boldsymbol{E}, \mathbf{grad}\,q + [\boldsymbol{A}, q] \rangle = 0$$

$\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A} + \frac{1}{2}\star[\boldsymbol{A}, \boldsymbol{A}].$

- $\boldsymbol{v}$, $\boldsymbol{w}$ vector Lie algebra-valued functions:

$$\star[\boldsymbol{v}, \boldsymbol{w}] := \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \otimes [e_I, e_J].$$

- $\boldsymbol{B} \in \boldsymbol{H}(\mathrm{div}; U) \otimes \mathfrak{g}$.

Option 1 (O1): discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

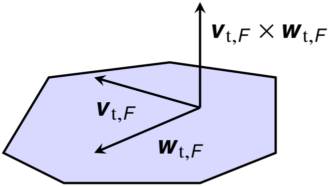
$$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}.$$

Option 1 (O1): discrete bracket in $\underline{\boldsymbol{X}}^k_{\text{div},h} \otimes \mathfrak{g}$

$$\star[\cdot,\cdot]^{\text{div},k,h} : (\underline{\boldsymbol{X}}^k_{\textbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\textbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\text{div},h} \otimes \mathfrak{g}.$$

- Face value in $\underline{\boldsymbol{X}}^k_{\text{div},h} \otimes \mathfrak{g}$ represents

$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \boldsymbol{n}_F = \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \cdot \boldsymbol{n}_F \otimes [e_I, e_J]$$



$$\star[\boldsymbol{v},\boldsymbol{w}]\cdot\boldsymbol{n}_F = \star[\boldsymbol{v}_{t,F},\boldsymbol{w}_{t,F}]\cdot\boldsymbol{n}_F$$

Option 1 (O1): discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

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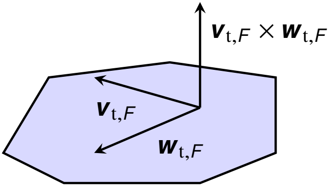

$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \boldsymbol{n}_F = \star[\boldsymbol{v}_{\mathrm{t},F}, \boldsymbol{w}_{\mathrm{t},F}] \cdot \boldsymbol{n}_F$$

- Leads to setting $\left(\star[\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h]^{\mathrm{div},k,h}\right)_F = \pi^k_{\mathcal{P},F}(\star[\gamma^{k,\mathfrak{g}}_{\mathrm{t},F}\underline{\boldsymbol{v}}_F, \gamma^{k,\mathfrak{g}}_{\mathrm{t},F}\underline{\boldsymbol{w}}_F] \cdot \boldsymbol{n}_F)$

Option 1 (O1): discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

$$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}.$$

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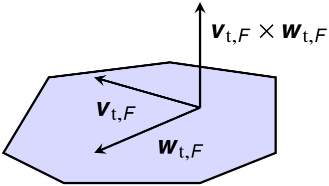

$$\star[\boldsymbol{v},\boldsymbol{w}]\cdot\boldsymbol{n}_F = \star[\boldsymbol{v}_{\mathrm{t},F},\boldsymbol{w}_{\mathrm{t},F}]\cdot\boldsymbol{n}_F$$

- Leads to setting $\left(\star[\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{w}}_h]^{\mathrm{div},k,h}\right)_F = \pi^k_{\mathcal{P},F}(\star[\gamma^{k,\mathfrak{g}}_{\mathrm{t},F}\underline{\boldsymbol{v}}_F, \gamma^{k,\mathfrak{g}}_{\mathrm{t},F}\underline{\boldsymbol{w}}_F] \cdot \boldsymbol{n}_F)$
- Element value built using $\boldsymbol{P}^k_{\mathbf{curl},T}$.

O1: discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

$$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}.$$

Then

$$\int_U \langle \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{v} + \star[\boldsymbol{A}, \boldsymbol{v}] \rangle \rightsquigarrow (\underline{\boldsymbol{B}}_h, \underline{\boldsymbol{C}}_h^{\mathfrak{g},k} \underline{\boldsymbol{v}}_h + \star[\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{v}}_h]^{\mathrm{div},k,h})_{\mathrm{div},\mathfrak{g},h}$$

with $\underline{\boldsymbol{B}}_h = \underline{\boldsymbol{C}}_h^{\mathfrak{g},k} \underline{\boldsymbol{A}}_h + \frac{1}{2}\star[\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{A}}_h]^{\mathrm{div},k,h} \in \underline{\boldsymbol{X}}^k_{\mathrm{div},k} \otimes \mathfrak{g}$.

Option 2 (O2): potentials in continuous bracket

With $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A} + \frac{1}{2}\star[\boldsymbol{A},\boldsymbol{A}]$:

$$\int_U \langle \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{v} + \star[\boldsymbol{A},\boldsymbol{v}]\rangle = \int_U \langle \mathbf{curl}\,\boldsymbol{A}, \mathbf{curl}\,\boldsymbol{v}\rangle + \int_U \langle \mathbf{curl}\,\boldsymbol{A}, \star[\boldsymbol{A},\boldsymbol{v}]\rangle + \int_U \langle \frac{1}{2}\star[\boldsymbol{A},\boldsymbol{A}], \mathbf{curl}\,\boldsymbol{v} + \star[\boldsymbol{A},\boldsymbol{v}]\rangle$$

discretised as

$$(\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h)_{\mathrm{div},\mathfrak{g},h} + \int_U \langle \boldsymbol{P}_{\mathrm{div},h}^{\mathfrak{g},k}\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h]\rangle$$
$$+ \int_U \langle \frac{1}{2}\star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},h}\underline{\boldsymbol{A}}_h], \boldsymbol{P}_{\mathrm{div},h}^{\mathfrak{g},k}\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h + \star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h]\rangle.$$

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With $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A} + \frac{1}{2}\star[\boldsymbol{A},\boldsymbol{A}]$:

$$\int_U \langle \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{v} + \star[\boldsymbol{A},\boldsymbol{v}]\rangle = \int_U \langle \mathbf{curl}\,\boldsymbol{A}, \mathbf{curl}\,\boldsymbol{v}\rangle + \int_U \langle \mathbf{curl}\,\boldsymbol{A}, \star[\boldsymbol{A},\boldsymbol{v}]\rangle + \int_U \langle \frac{1}{2}\star[\boldsymbol{A},\boldsymbol{A}], \mathbf{curl}\,\boldsymbol{v} + \star[\boldsymbol{A},\boldsymbol{v}]\rangle$$

discretised as

$$(\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h)_{\mathrm{div},\mathfrak{g},h} + \int_U \langle \boldsymbol{P}_{\mathrm{div},h}^{\mathfrak{g},k}\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h]\rangle + \int_U \langle \frac{1}{2}\star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},h}\underline{\boldsymbol{A}}_h], \boldsymbol{P}_{\mathrm{div},h}^{\mathfrak{g},k}\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h + \star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h]\rangle.$$

- Magnetic field not in the discrete $\boldsymbol{H}(\mathrm{div};U)\otimes\mathfrak{g}$, but piecewise polynomial:

$$\boldsymbol{B}_h = \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h + \frac{1}{2}\star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h].$$

Outline

Illustration with O1 (discrete bracket)

Time discretisation:

$$\delta_t^{n+1}\boldsymbol{Z} = \frac{1}{\delta t}(\boldsymbol{Z}^{n+1} - \boldsymbol{Z}^n)$$

Scheme: Find families $(\underline{\boldsymbol{A}}_h^n)_n$, $(\underline{\boldsymbol{E}}_h^n)_n$, $(\underline{\lambda}_h^n)_n$ such that for all n, $(\underline{\boldsymbol{A}}_h^n, \underline{\boldsymbol{E}}_h^n, \underline{\lambda}_h^n) \in (\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \otimes \mathfrak{g}) \times (\underline{X}_{\mathbf{grad},h}^k \otimes \mathfrak{g})$, and

$$\delta_t^{n+1}\underline{\boldsymbol{A}}_h = -\underline{\boldsymbol{E}}_h^{n+1},$$

$$(\delta_t^{n+1}\underline{\boldsymbol{E}}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},\mathfrak{g},h} + (\underline{\boldsymbol{G}}_h^{\mathfrak{g},k}\underline{\lambda}_h^{n+1}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},\mathfrak{g},h} + \int_U \langle [\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^{n+1}, P_{\mathbf{grad},h}^{\mathfrak{g},k}\underline{\lambda}_h^{n+1}], \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h\rangle$$

$$= (\underline{\boldsymbol{B}}_h^{n+1}, \underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{v}}_h + \star[\underline{\boldsymbol{A}}_h^{n+\frac{1}{2}}, \underline{\boldsymbol{v}}_h]^{\mathrm{div},k,h})_{\mathrm{div},\mathfrak{g},h}, \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \otimes \mathfrak{g},$$

$$(\delta_t^{n+1}\underline{\boldsymbol{E}}_h, \underline{\boldsymbol{G}}_h^{\mathfrak{g},k}\underline{q}_h)_{\mathbf{curl},\mathfrak{g},h} + \int_U \langle \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}(\delta_t^{n+1}\underline{\boldsymbol{E}}_h), [\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^{n}, P_{\mathbf{grad},h}^{\mathfrak{g},k}\underline{q}_h]\rangle$$

$$= 0, \qquad \forall \underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k \otimes \mathfrak{g}.$$

Properties

See [Droniou et al., 2023, Droniou and Qian, 2023].

- Preservation of discrete constraint:

$$\mathfrak{C}^n(\underline{q}_h) := (\underline{\boldsymbol{E}}_h^n, \underline{\boldsymbol{G}}_h^{\mathfrak{g},k}\underline{q}_h)_{\mathbf{curl},\mathfrak{g},h} + \int_U \langle \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{E}}_h^n, [\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^n, P_{\mathbf{grad},h}^{\mathfrak{g},k}\underline{q}_h]\rangle, \quad \forall \underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k \otimes \mathfrak{g}.$$

Recall the weak form of continuous constraint:

$$\int_U \langle \boldsymbol{E}, \mathbf{grad}\, q + [\boldsymbol{A}, q]\rangle.$$

Properties

See [Droniou et al., 2023, Droniou and Qian, 2023].

- Preservation of discrete constraint:

$$\mathfrak{C}^n(\underline{q}_h) := (\underline{\boldsymbol{E}}_h^n, \underline{\boldsymbol{G}}_h^{\mathfrak{g},k}\underline{q}_h)_{\mathbf{curl},\mathfrak{g},h} + \int_U \langle \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{E}}_h^n, [\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^n, P_{\mathbf{grad},h}^{\mathfrak{g},k}\underline{q}_h]\rangle, \quad \forall \underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k \otimes \mathfrak{g}.$$

- O1: Energy bound, for $(\underline{\boldsymbol{A}}_h^0, \underline{\boldsymbol{E}}_h^0)$ s.t. $\mathfrak{C}^0 \equiv 0$:

$$\frac{1}{2}\|\underline{\boldsymbol{E}}_h^n\|_{\mathbf{curl},\mathfrak{g},h}^2 + \frac{1}{2}\|\underline{\boldsymbol{B}}_h^n\|_{\mathrm{div},\mathfrak{g},h}^2 \leq \frac{1}{2}\|\underline{\boldsymbol{E}}_h^0\|_{\mathbf{curl},\mathfrak{g},h}^2 + \frac{1}{2}\|\underline{\boldsymbol{B}}_h^0\|_{\mathrm{div},\mathfrak{g},h}^2.$$

Properties

See [Droniou et al., 2023, Droniou and Qian, 2023].

- Preservation of discrete constraint:

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- O2: Energy bound, for $(\underline{\boldsymbol{A}}_h^0, \underline{\boldsymbol{E}}_h^0)$ s.t. $\mathfrak{C}^0 \equiv 0$:

$$\frac{1}{2}\|\underline{\boldsymbol{E}}_h^n\|_{\mathbf{curl},\mathfrak{g},h}^2 + \frac{1}{2}\|\boldsymbol{B}_h^n\|_{L^2(U)\otimes\mathfrak{g}}^2 + \mathrm{stab}(\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^n, \underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^n) \leq \frac{1}{2}\|\underline{\boldsymbol{E}}_h^0\|_{\mathbf{curl},\mathfrak{g},h}^2 + \frac{1}{2}\|\boldsymbol{B}_h^0\|_{L^2(U)\otimes\mathfrak{g}}^2 + \mathrm{stab}(\underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^0, \underline{\boldsymbol{C}}_h^{\mathfrak{g},k}\underline{\boldsymbol{A}}_h^0).$$

Outline

Numerical tests

- $k = 0$
- Domain $U = (0,1)^3$, $t \in [0,1]$
- Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$

$$e_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ e_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Mesh sequences
  - Voronoi polytopal meshes
  - Tetrahedral meshes
  - Cubic meshes
- Newton iterations
  - Stopping criterion $\epsilon = 10^{-6}$
  - Timestep $\delta t = \min\{0.1, 0.2h\}$
  - Direct solver

Newton iterations (O1)

	Voronoi mesh				
	1	2	3	4	5
h	0.83	0.45	0.31	0.22	0.18
δt	0.1	0.083	0.059	0.043	0.034
N_{avg} $(\epsilon = 10^{-6})$	2	2	2	2.6	1.4
N_{avg} $(\epsilon = 10^{-10})$	2.3	2.3	2.1	3.3	2

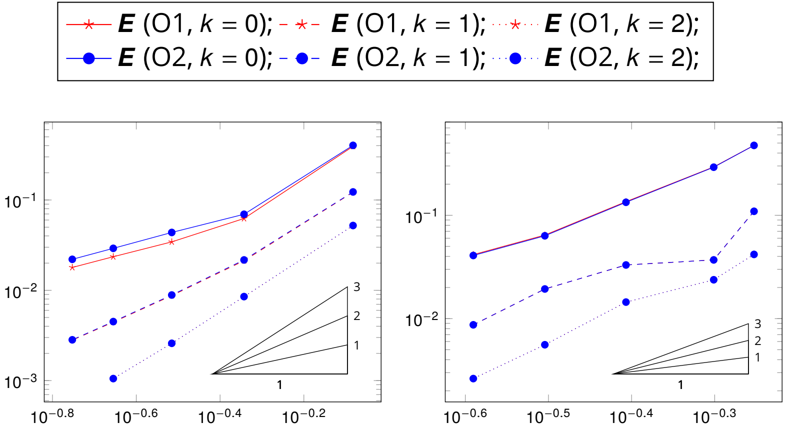
	Tetrahedral mesh				
	1	2	3	4	5
h	0.56	0.50	0.39	0.31	0.26
δt	0.1	0.091	0.077	0.063	0.05
N_{avg} $(\epsilon = 10^{-6})$	2	2	2	1.9	1.6
N_{avg} $(\epsilon = 10^{-10})$	2	2	2	2	2

Comparison of total runtimes between O1 and O2

	Voronoi mesh		Tetrahedral mesh		Cubic mesh	
O1	1	3	2	4	1	3
$k = 0$	5.00865	145.77	4.70017	21.1378	0.564665	35.2541
$k = 1$	35.9231	2836.36	50.997	360.943	3.58296	588.679
$k = 2$	198.435	43303.9	578.499	5732.1	14.6638	14337.4
O2	1	3	2	4	1	3
$k = 0$	4.57515	135.231	4.16998	19.7708	0.53204	32.879
$k = 1$	34.2036	2814.14	51.3249	340.817	3.62877	631.546
$k = 2$	190.447	42083.9	634.162	6421.87	13.8524	14414.9

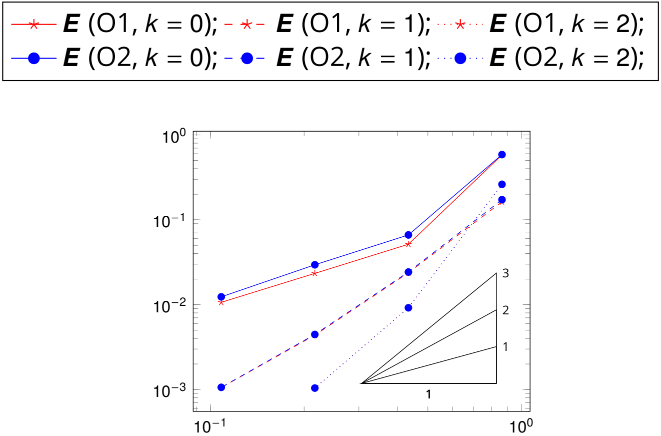
Table: Total runtime for each test in seconds

Errors on $\boldsymbol{E}$



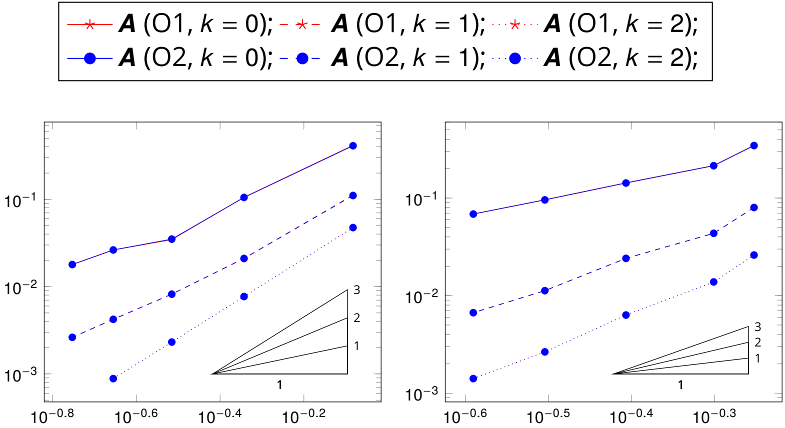
(a) “Voro-small-0” mesh (b) “Tetgen-Cube-0” mesh

Errors on $\boldsymbol{E}$



(a) “Cubic-Cells” mesh

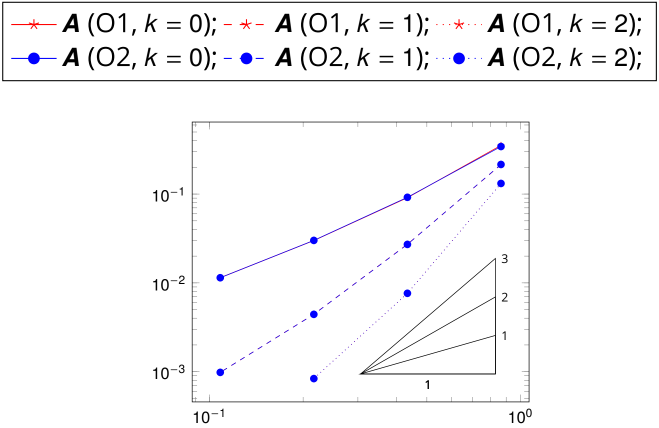
Errors on $\boldsymbol{A}$



(a) “Voro-small-0” mesh

(b) “Tetgen-Cube-0” mesh

Errors on $\boldsymbol{A}$



(a) “Cubic-Cells” mesh

Constraint preservation

	Voronoi mesh		Tetrahedral mesh	
O1	1	3	2	4
$k = 0$	8.47329e-15	3.05676e-14	2.06362e-14	4.75412e-14
$k = 1$	1.4144e-13	8.93075e-13	3.67781e-13	1.81426e-12
$k = 2$	3.69918e-12	1.18207e-10	3.82037e-12	2.77407e-11
O2	1	3	2	4
$k = 0$	8.16124e-15	3.13617e-14	2.01667e-14	4.77633e-14
$k = 1$	9.8531e-14	8.83056e-13	3.69107e-13	1.81787e-12
$k = 2$	4.16428e-12	7.99416e-11	3.81537e-12	2.77391e-11

Table: Maximum over n of the difference $\mathfrak{C}^n - \mathfrak{C}^0$ measured in the dual norm

Constraint preservation

	Cubic mesh	
O1	1	3
$k = 0$	3.52318e-15	2.16527e-14
$k = 1$	2.33678e-14	6.44019e-13
$k = 2$	4.45312e-14	6.48608e-12
O2	1	3
$k = 0$	4.22851e-15	2.11083e-14
$k = 1$	2.52977e-14	6.48934e-13
$k = 2$	4.51672e-14	6.48285e-12

Table: Maximum over n of the difference $\mathfrak{C}^n - \mathfrak{C}^0$ measured in the dual norm

Conclusion

Highlights

- Arbitrary order
- Polytopal meshes

Questions

- Scheme to preserve constraint on weak (not constrained) formulation?
- Convergence analysis (error estimates)?
- Other models?

Conclusion

Highlights

- Arbitrary order
- Polytopal meshes

Questions

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Thank you!

- Notes and series of introductory lectures to DDR:

  `https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html`

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

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Two arbitrary-order constraint-preserving schemes for the yang–mills equations on polyhedral meshes.
page 21p.