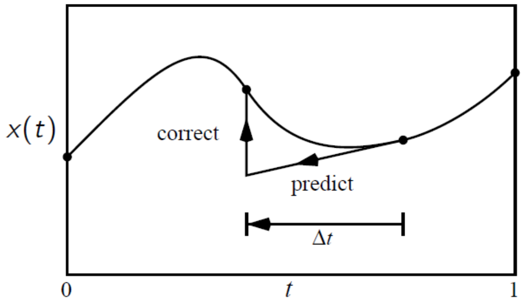
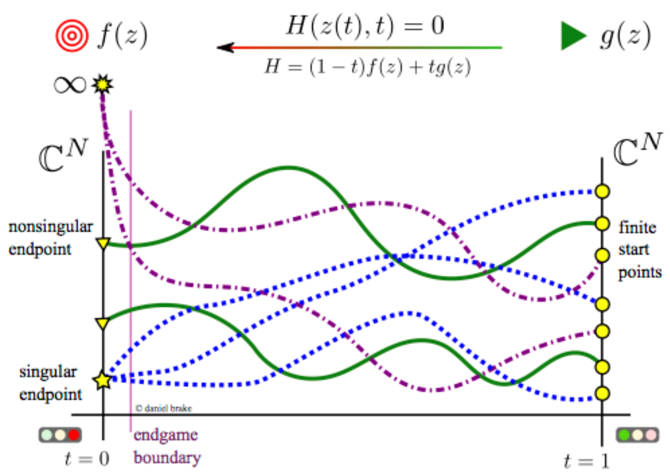
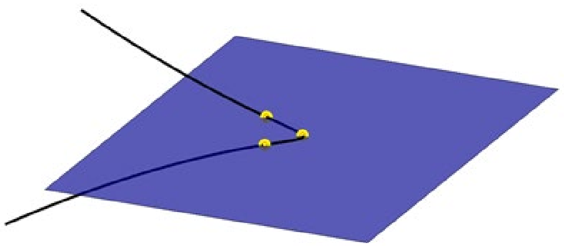
Introduction to Numerical Algebraic Geometry







Jonathan Hauenstein

March 2020

Setting the Table

Solve
$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_N) \\ \vdots \\ f_n(x_1, \ldots, x_N) \end{bmatrix} = 0$$

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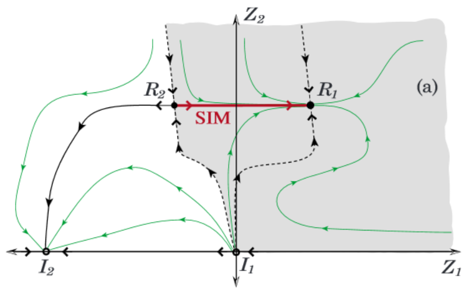
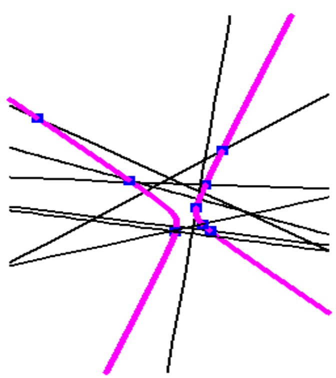
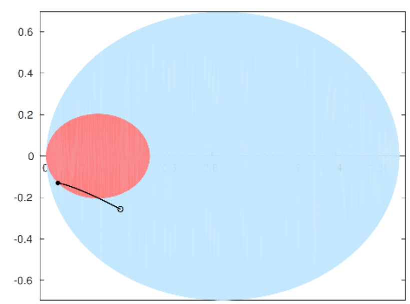


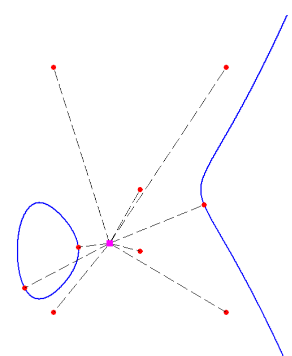
- Equilibrium and transition states

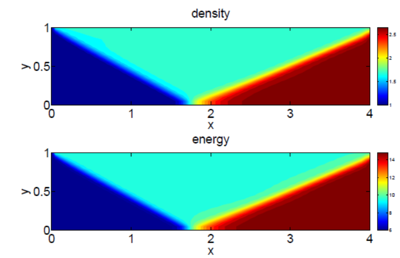


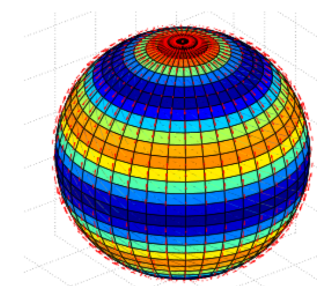
- Real enumerative geometry



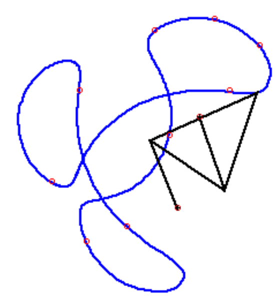
- Optimization

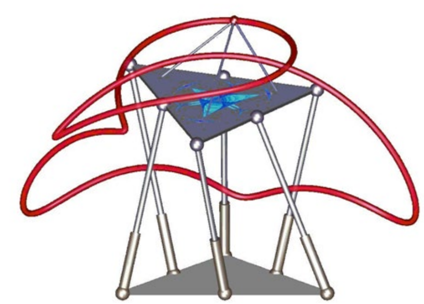






- Solving differential equations

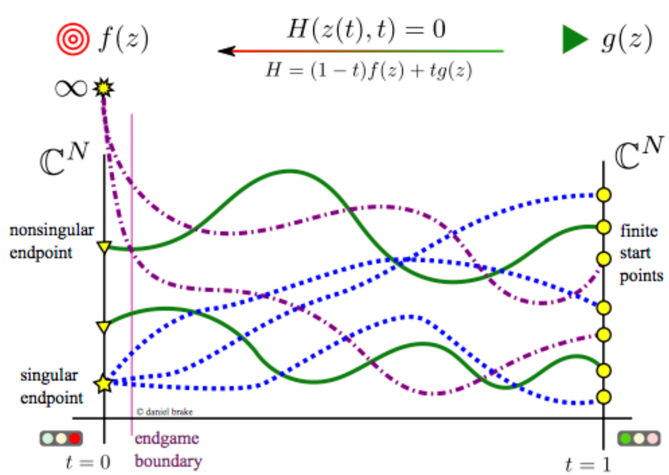




- Mechanism design

Setting the Table

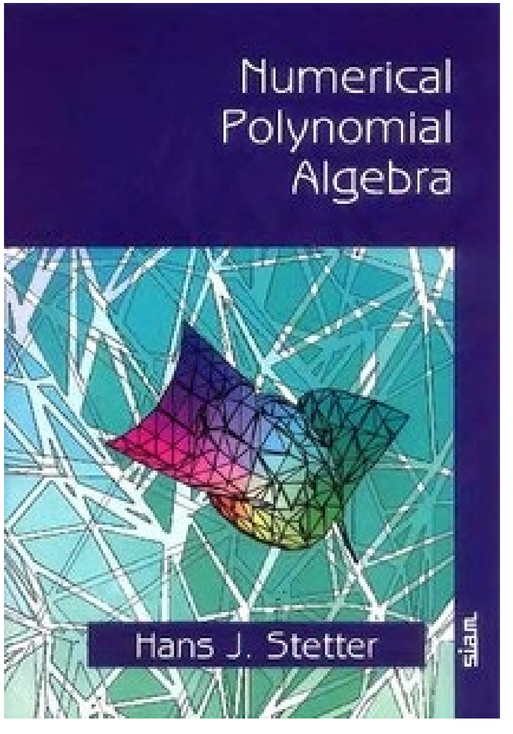
- **Overview** of homotopy continuation and num. alg. geom.
- Historical perspective
- Utilize `Bertini` but there are many other packages, e.g.:
  - `PHCpack`, `Hom4PS`, `NAG4M2`, `HomotopyContinuation.jl`



Algebra vs. Geometry

Algebra:

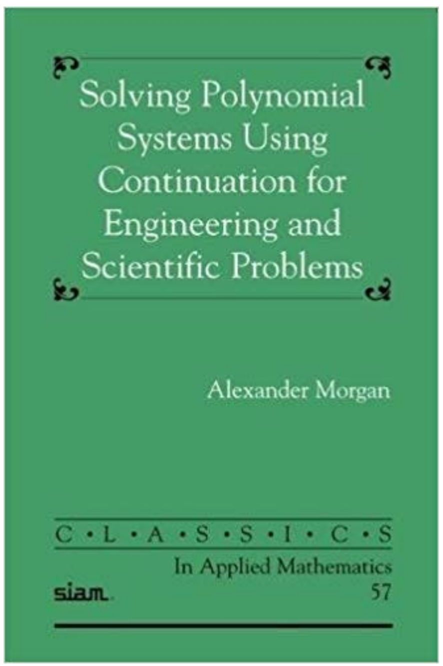
- "Numerical Polynomial Algebra" by Hans Stetter
- Normal forms, eigenvectors/eigenvalues, border basis, ...
- K. Batselier, B. De Moor, P. Dreesen, B. Mourrain, S. Telen, M. Van Barel, ...

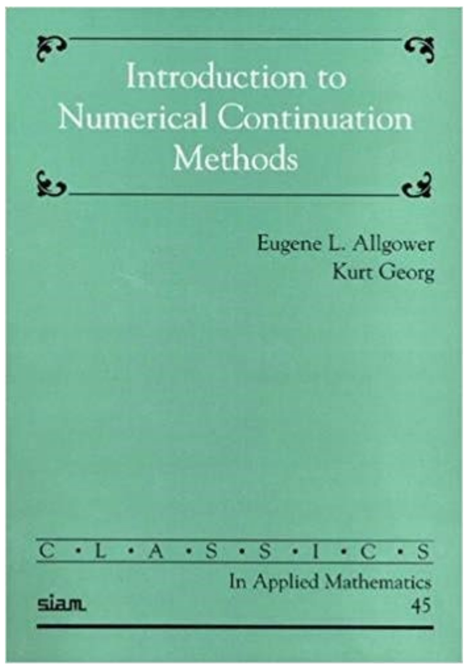


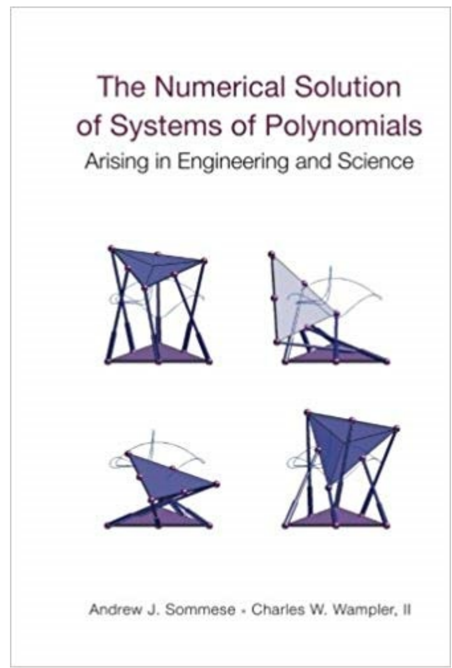
Algebra vs. Geometry

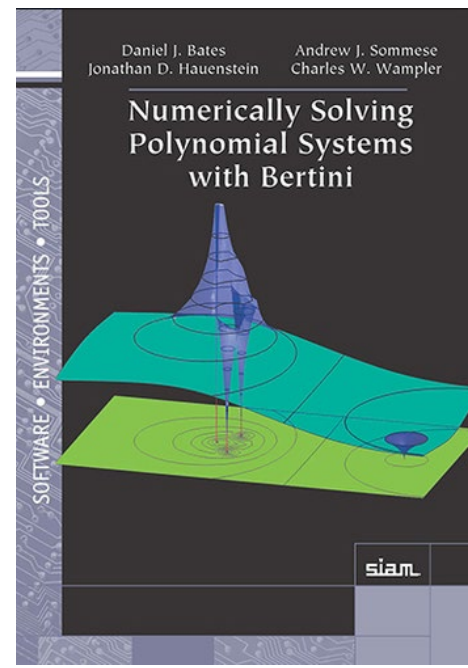
Geometry:

- Homotopy continuation and numerical algebraic geometry
- Morgan (1987), Allgower-Georg (1990), Sommese-Wampler (2005) Bates-H.-Sommese-Wampler (2013)









Algebra vs. Geometry

Generally speaking:

- Algebraic methods prefer vastly over-determined systems
  - fewer "new" polynomials to compute
  - Bardet-Faugere-Salvy (2004)

- Numerical algebraic geometry prefers well-constrained systems of low degrees with coefficients of roughly unit magnitude
  - codimension $=$ # equations
  - stable under perturbations

Early History of Solving

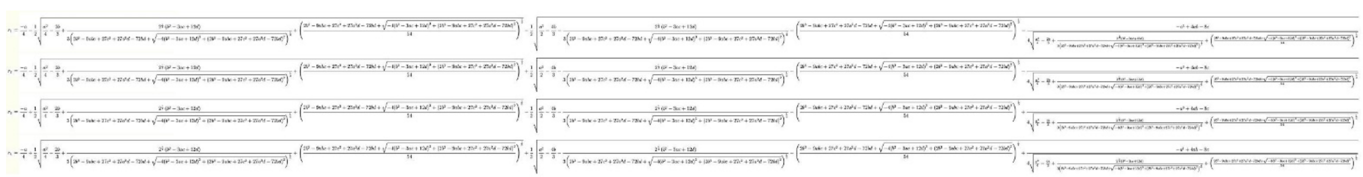
Early History

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d = 0$$

- $d = 1$: $x = \dfrac{-a_0}{a_1}$
- $d = 2$: $x = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$
- $d = 3$:

$$\begin{aligned}
x_1 &= -\frac{b}{3a} \\
&\quad - \frac{1}{3a}\sqrt[3]{\tfrac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
&\quad - \frac{1}{3a}\sqrt[3]{\tfrac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
x_2 &= -\frac{b}{3a} \\
&\quad + \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\tfrac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
&\quad + \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\tfrac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
x_3 &= -\frac{b}{3a} \\
&\quad + \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\tfrac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
&\quad + \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\tfrac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]}
\end{aligned}$$

- $d = 4$:



Early History

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d = 0$$

- Abel-Ruffini Theorem (1824)
  - No algebraic solution (using radicals) to general polynomial equations of degree 5 or higher with arbitrary coefficients
- What does it mean to "solve $x^5 - x + 1 = 0$"?

Early History

- What does it mean to "solve $x^5 - x + 1 = 0$"?

Maple

$> solve(x^5 - x + 1);$

$RootOf(\_Z^5 - \_Z + 1, index = 1), RootOf(\_Z^5 - \_Z + 1, index = 2), RootOf(\_Z^5 - \_Z + 1, index = 3),$
$RootOf(\_Z^5 - \_Z + 1, index = 4), RootOf(\_Z^5 - \_Z + 1, index = 5)$

Early History

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> solve(x^5 - x + 1);
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```
> fsolve(x^5 - x + 1);
```

$$-1.167303978$$

Early History

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> $fsolve(x^5 - x + 1);$

-1.167303978

> $evalf(solve(x^5 - x + 1));$

$0.764884433600585 + 0.352471546031726\,I, -0.181232444469875 + 1.08395410131771\,I,$
$-1.16730397826142, -0.181232444469875 - 1.08395410131771\,I, 0.764884433600585$
$- 0.352471546031726\,I$

Early History

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0.764884433600585 + 0.352471546031726 I, −0.181232444469875 + 1.08395410131771 I,
−1.16730397826142, −0.181232444469875 − 1.08395410131771 I, 0.764884433600585
− 0.352471546031726 I

Bertini

input

```
variable_group x;
function f;
f = x^5 - x + 1;
```

finite_solutions

```
5

7.648844336005847e-01 -3.524715460317264e-01

7.648844336005849e-01 3.524715460317262e-01

-1.812324444698754e-01 1.083954101317711e+00

-1.167303978261419e+00 -2.220446049250313e-16

-1.812324444698754e-01 -1.083954101317711e+00
```

Early History

Vast generalization of the meaning of "solve":

- Early history: find a solution and study local properties
- Late 20^{th} century: find all isolated solutions
- Early 21^{st} century: describe all solutions
  - isolated and positive-dimensional components

Early History

Lack of exact formula for solutions $\longrightarrow$ iterative refinement

- Newton (1643-1727), Raphson (1648-1715), Simpson (1710-1761)
- compute solution to arbitrary accuracy given approximation

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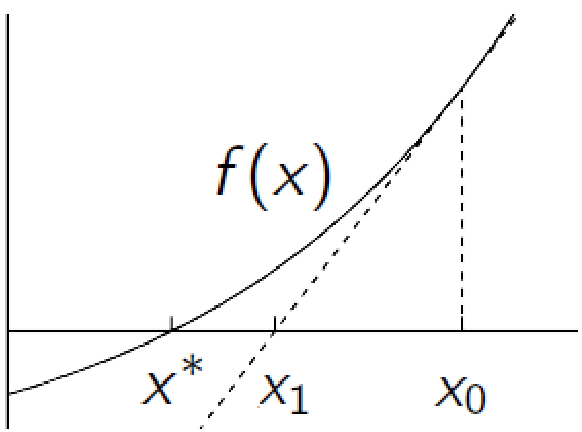
Newton's method:

- $f : \mathbb{C}^n \to \mathbb{C}^n$ with Jacobian $Jf : \mathbb{C}^n \to \mathbb{C}^{n\times n}$
- Given approximation x_0, compute $x_1, x_2, x_3, \ldots$ via

$$x_{k+1} = x_k - Jf(x_k)^{-1} f(x_k)$$



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If $f(x^*) = 0$, $Jf(x^*)^{-1}$ exists (nonsingular), and $\|x_0 - x^*\|$ small,

$$x_k \to x^* \text{ quadratically.}$$

Early History

Example

Approximate $x^* = \sqrt{2}$ by solving $f(x) = x^2 - 2 = 0$ with $x_0 = 1$:

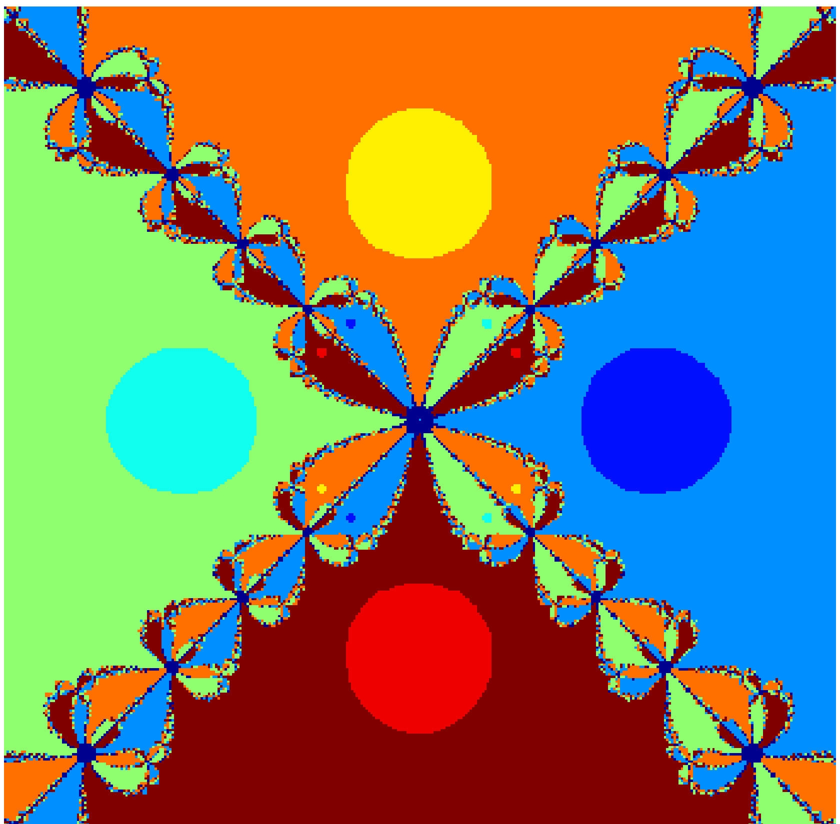
$$x_{k+1} = x_k - Jf(x_k)^{-1} f(x_k) = x_k - \frac{x_k^2 - 2}{2x_k}$$

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1.5 \\
x_2 &= 1.416666666666666666666666666666666666666667 \\
x_3 &= 1.414215686274509803921568627450980392156862745098 0 \\
x_4 &= 1.414213562374689910626295578890134910116559622115 7 \\
x_5 &= 1.414213562373095048801689623502530243614981925776 2 \\
x_6 &= 1.414213562373095048801688724209698078569671875377 2 \\
\sqrt{2} &= 1.414213562373095048801688724209698078569671875376 9
\end{aligned}$$

Early History

Double-edged sword of Newton's method:

- Qudaratic convergence near nonsingular solutions
- Slow convergence or divergence near singular solutions
- Difficulty away solutions (chaos, limit cycles, etc)



$$f(x) = x^4 - 1$$

Early History

Double-edged sword of Newton's method:

- Qudaratic convergence near nonsingular solutions
- Slow convergence or divergence near singular solutions
- Difficulty away solutions (chaos, limit cycles, etc)

Goal

- Use continuation methods to stay near solutions
- Use deflation to restore quadratic convergence for sing. solns.
  - Ojika-Watanabe-Mitsui (1983), Ojika (1987), Leykin-Verschelde-Zhao (2006,2008), Dayton-Zeng (2005), Mantzaflaris-Mourrain (2011), Guisti-Yakoubsohn (2013), H.-Wampler (2013), H.-Mourrain-Szanto (2017), ...

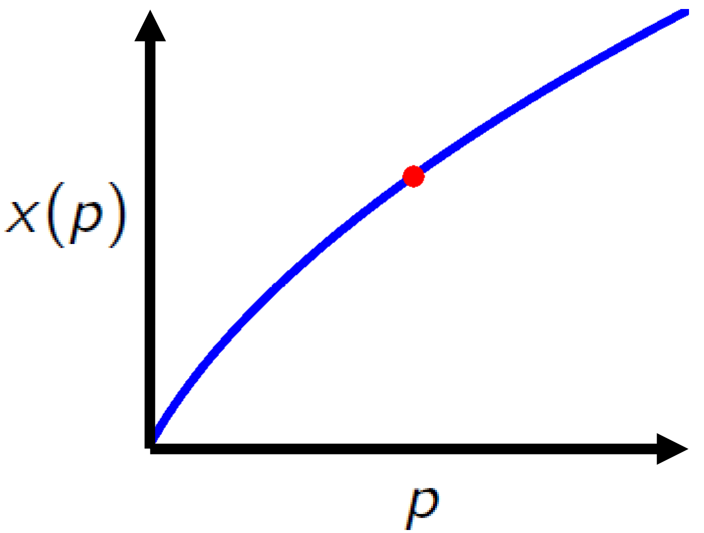
Early History

Continuation from complex analysis:

- Cauchy (1789-1857), Riemann (1826-1866), Mittag-Leffler (1846-1927)
- Implicit function theorem
- Analytic extension of functions (analytic continuation)

Big picture idea:

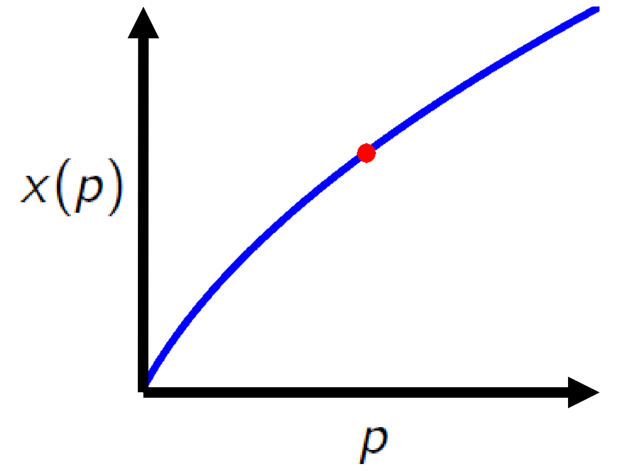
- solutions "continue" locally under small parameter changes



Early History

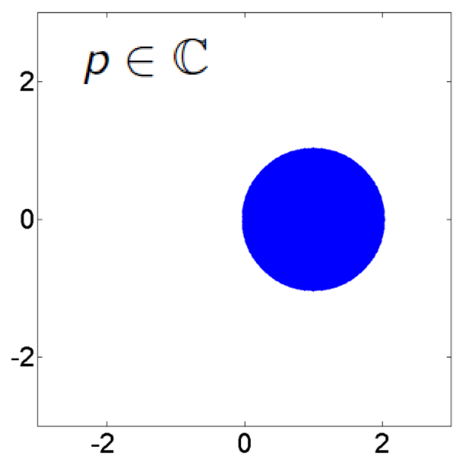
Example

$$f(x;p) = x^2 - p = 0$$



- Starting at $(x,p) = (1,1)$, IFT provides that there is an analytic function $x(p)$ with $x(1) = 1$ such that $f(x(p),p) = 0$.

$$x(p) = \sqrt{p} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (1-2n)(n!)^2} (p-1)^n$$



- converges for $|p-1| \leq 1$

Early History

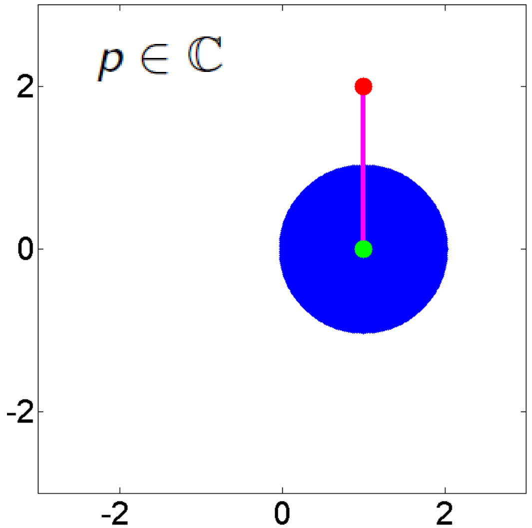
Example

$f(x;p) = x^2 - p = 0$

- Extend beyond original domain using continuation

Compute $x(1+2i) = \sqrt{1+2i}$ via the path $x(1+(1-t)\cdot 2i)$:

- $t = 1$: $x(1) = 1$ is known
- $t = 0$: $x(1+2i)$ is what we want to compute



Early History

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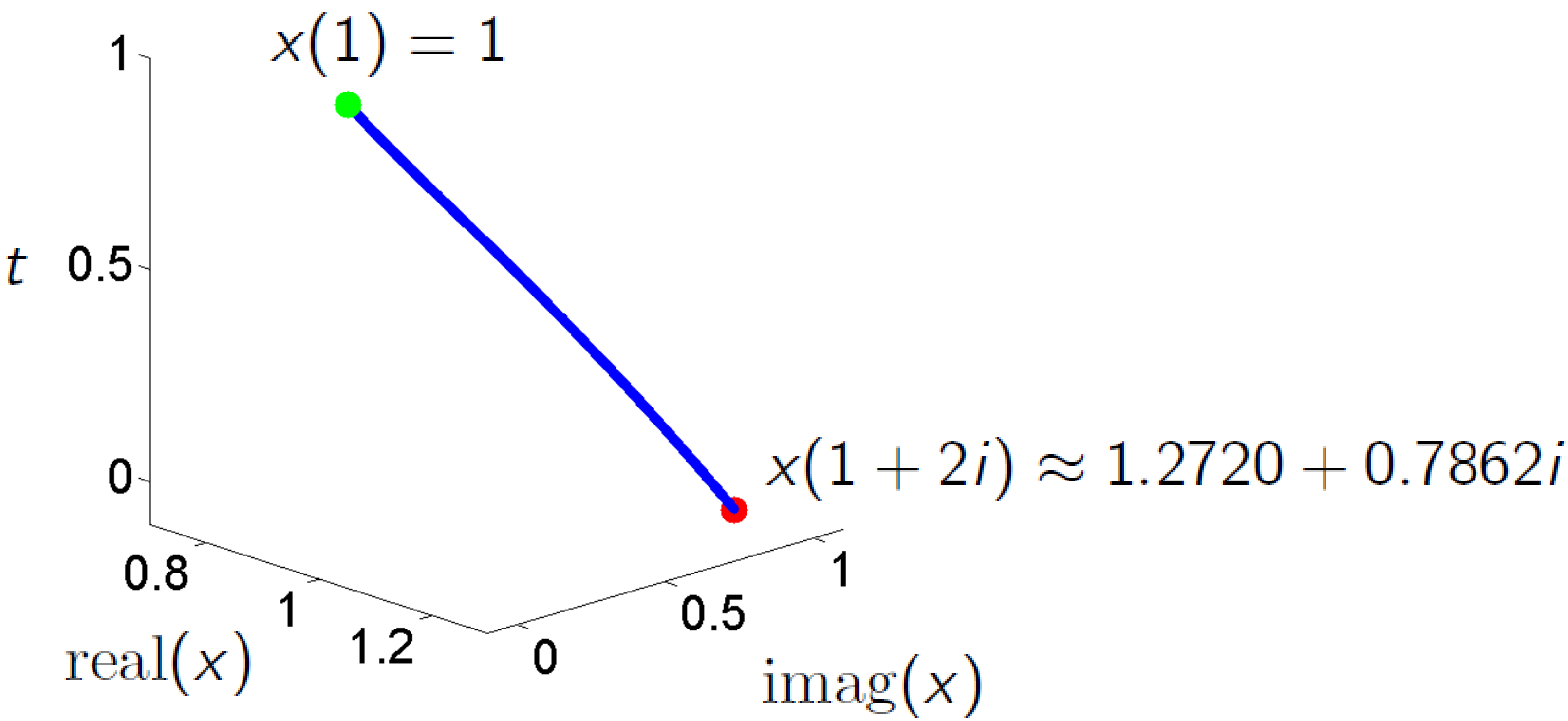


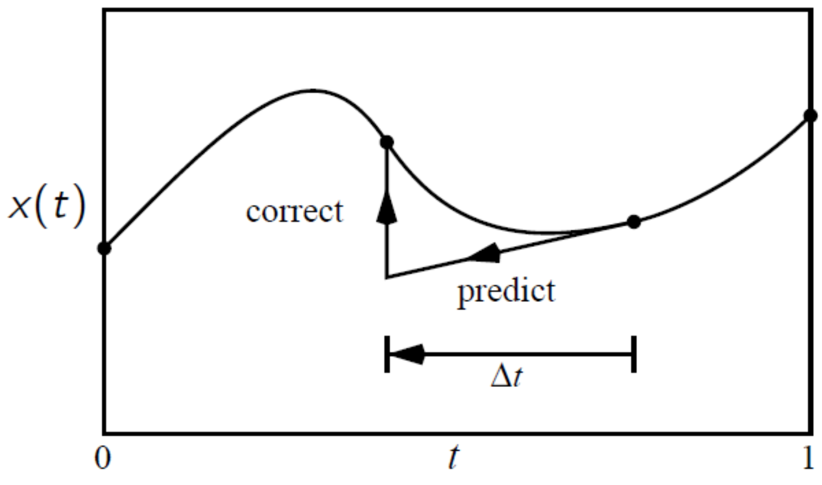
Early History

Numerically track along the path $x(t)$ satisfying $f(x(t), t) = 0$:

- (Predictor) Estimate $x(t + \Delta t)$ from $x(t)$ by discretizing using the Davidenko differential equation (1953):

$$f = 0 \quad \longrightarrow \quad \frac{d}{dt} f = 0 \quad \longrightarrow \quad \dot{x}(t) = -J_x f(x(t), t)^{-1} J_t f(x(t), t)$$

  - Constant, Euler, Heun, Runge-Kutta, Runge-Kutta-Fehlberg, ...



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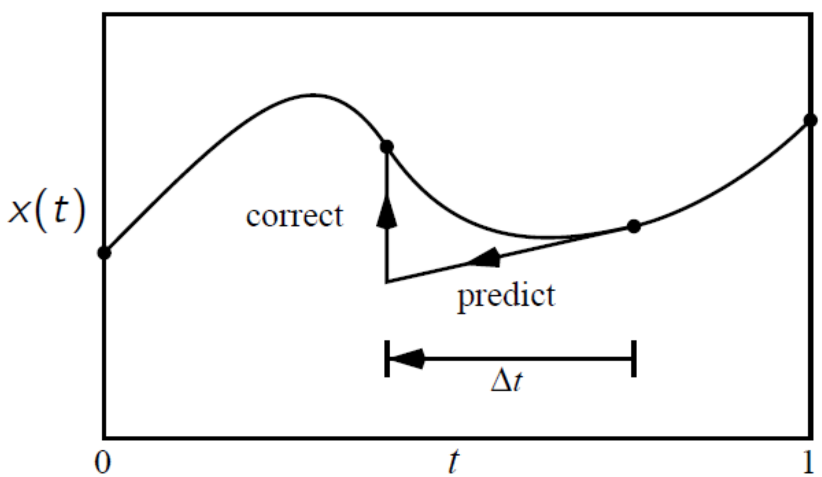
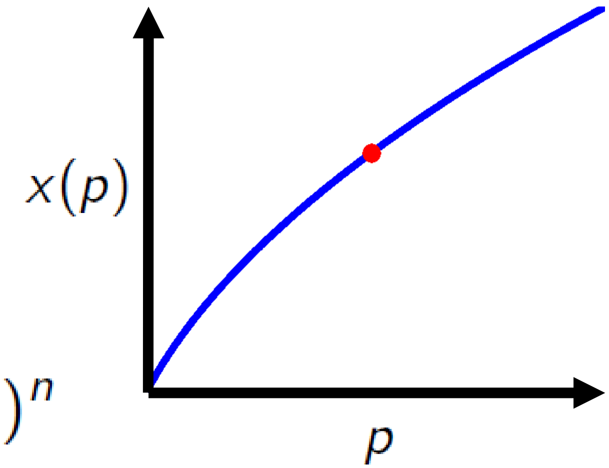
  - Constant, Euler, Heun, Runge-Kutta, Runge-Kutta-Fehlberg, ...

- (Corrector) for each t, apply Newton's method to $f(\bullet, t) = 0$



Early History

Example

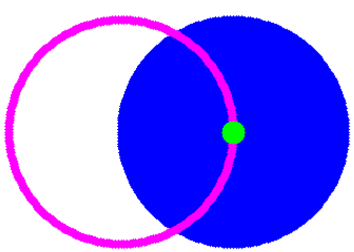


$$f(x;p) = x^2 - p = 0$$

$$x(p) = \sqrt{p} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (1-2n)(n!)^2} (p-1)^n$$

Track around a loop: $\quad x(e^{i\theta})$

$p \in \mathbb{C}$



Early History

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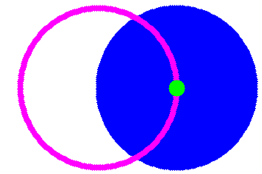
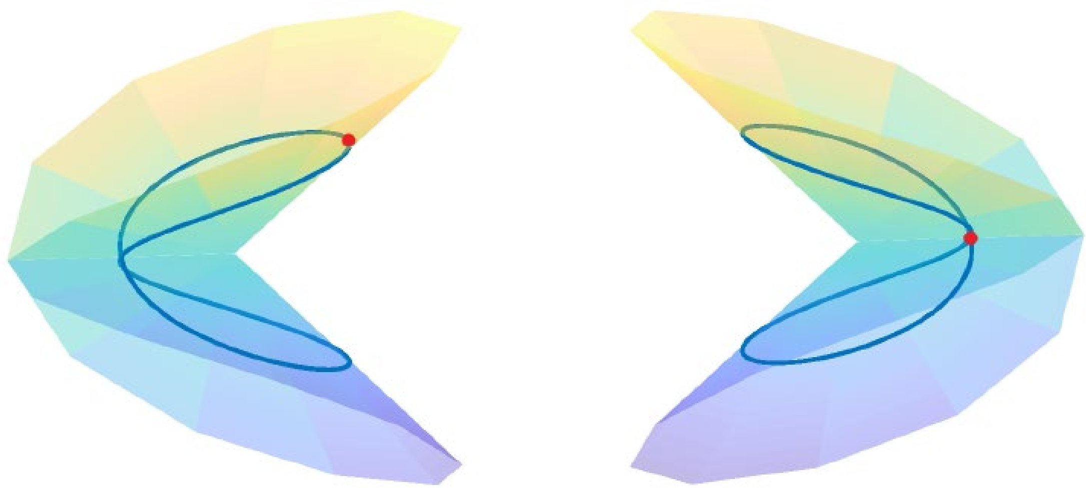


$$f(x; p) = x^2 - p = 0$$

Track around a loop: $x(e^{i\theta})$

- $\theta = 0$: $x = 1$
- $\theta = 2\pi$: $x = -1$
- $\theta = 4\pi$: $x = 1$



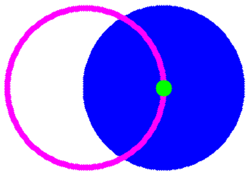
real(x) imag(x)

cycle number = winding number = 2

Early History

Example

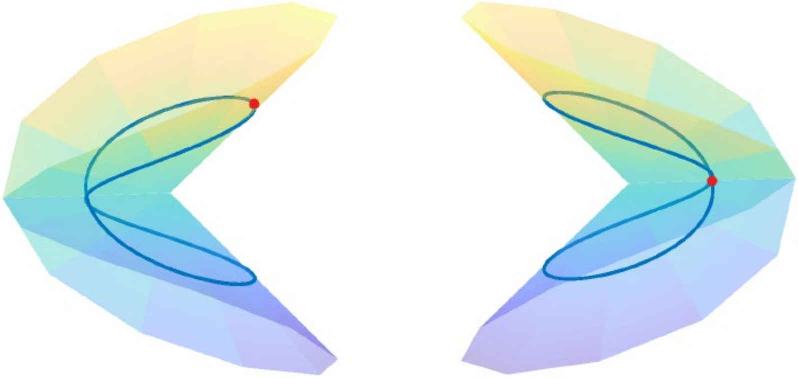
$p \in \mathbb{C}$



$$f(x;p) = x^2 - p = 0$$

Track around a loop: $x(e^{i\theta})$

- monodromy action: permutation of solutions along loop
  - compute other solutions
  - decompose solution sets



real(x) imag(x)

Early History

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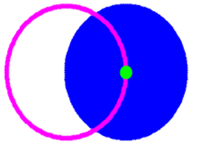
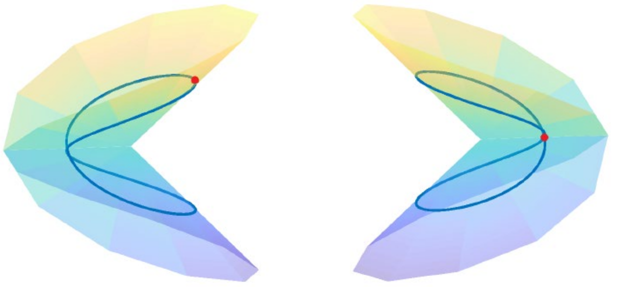


$$f(x;p) = x^2 - p = 0$$

Track around a loop: $\quad x(e^{i\theta})$



- Cauchy integral theorem: computing singular endpoints
  - cycle number c
  - sufficiently small radius $r > 0$

$$x(0) = \frac{1}{2\pi c} \int_0^{2\pi c} x(re^{i\theta})d\theta$$

- Cauchy endgame: Morgan-Sommese-Wampler (1991)

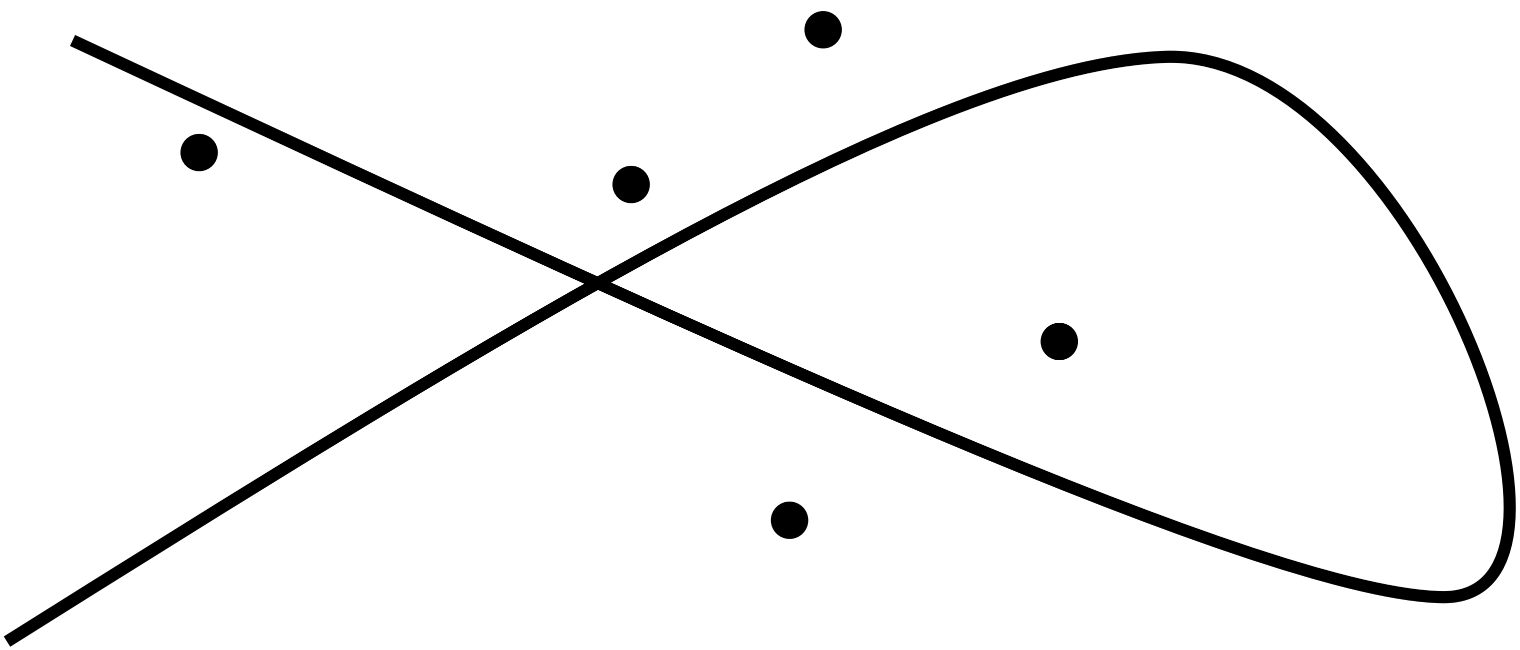
Late 20^{th} Century

1970s – 1990s

Isolated Solutions

Find all isolated solutions of

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0$$



Isolated Solutions

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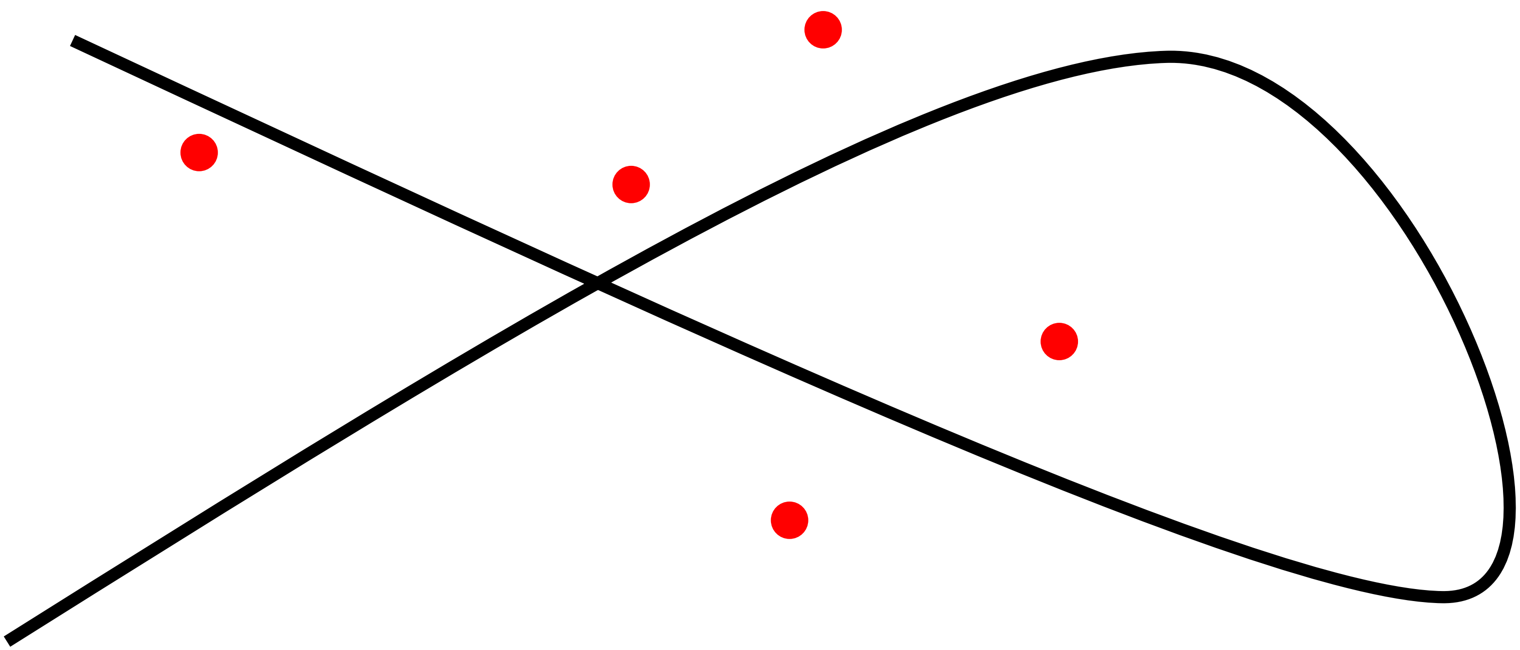

Isolated Solutions

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Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to "continue"
   - think of f as a member of a family $\mathcal{F}$

Isolated Solutions

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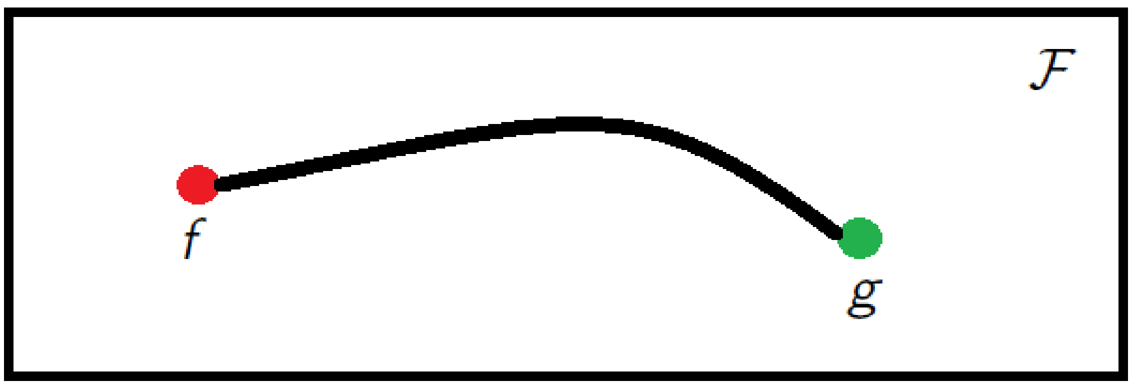
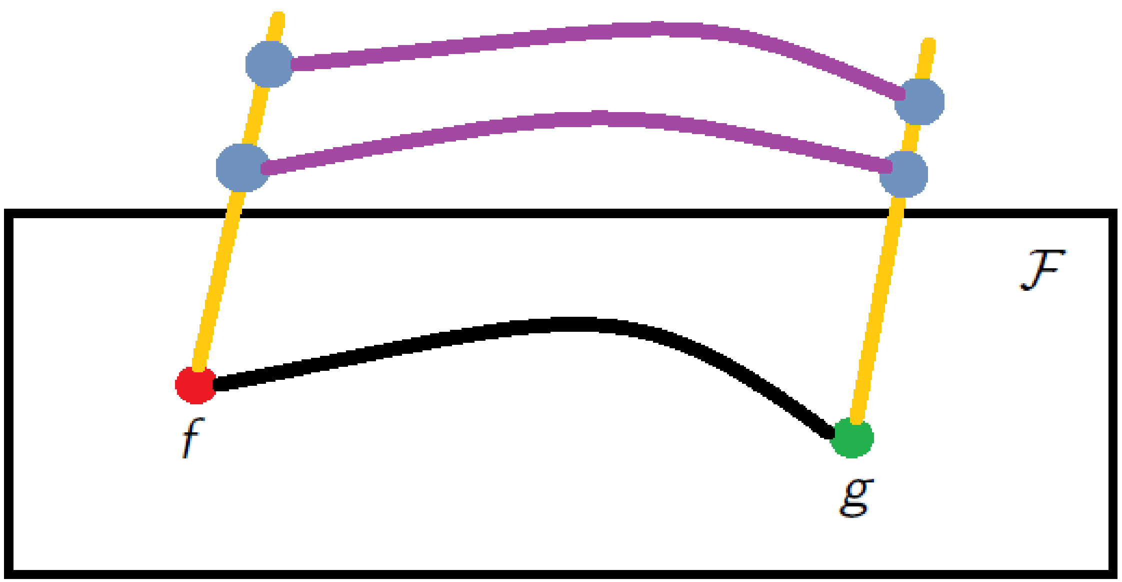
   - construct a deformation inside of $\mathcal{F}$ that ends at f



Isolated Solutions

Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to "continue"
   - think of f as a member of a family $\mathcal{F}$
2. homotopy that describes the deformation of the parameters
   - construct a deformation inside of $\mathcal{F}$ that ends at f
3. start points to track along paths as parameters deform
   - parallelize computation – track each path independently

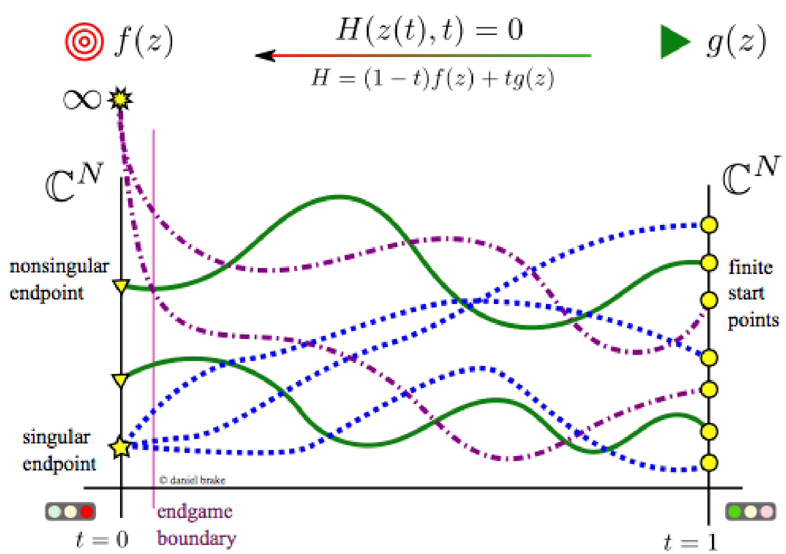


Isolated Solutions

Theorem

For properly constructed homotopies, with finite endpoints $S \subset \mathbb{C}^n$:

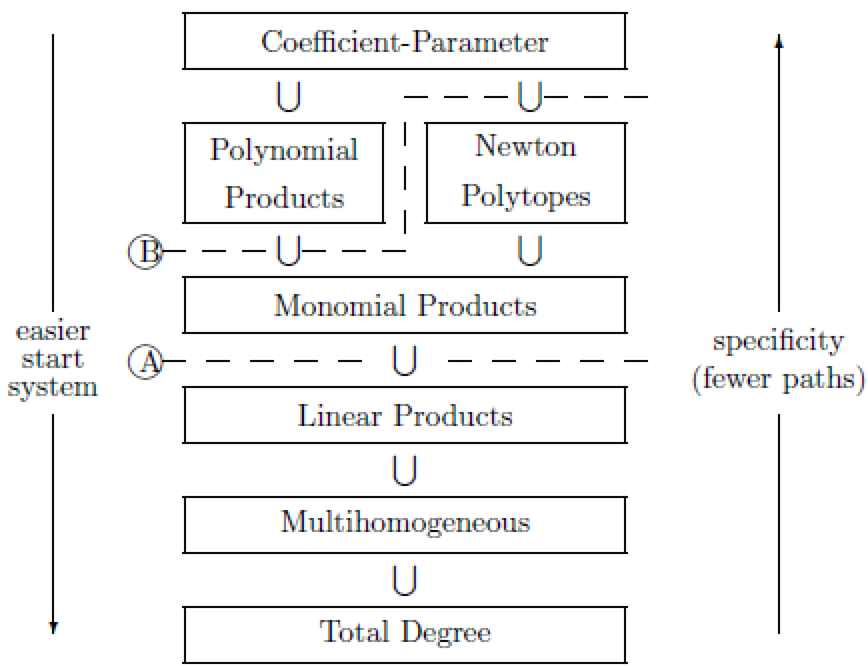
- *each isolated solution is contained in S*
  - *in fact, S contains a point on every connected component*
- *for square systems, multiplicity = number of paths if isolated.*
  - *Local dimension test to identify nonisolated solutions (Bates-H.-Peterson-Sommese (2009))*

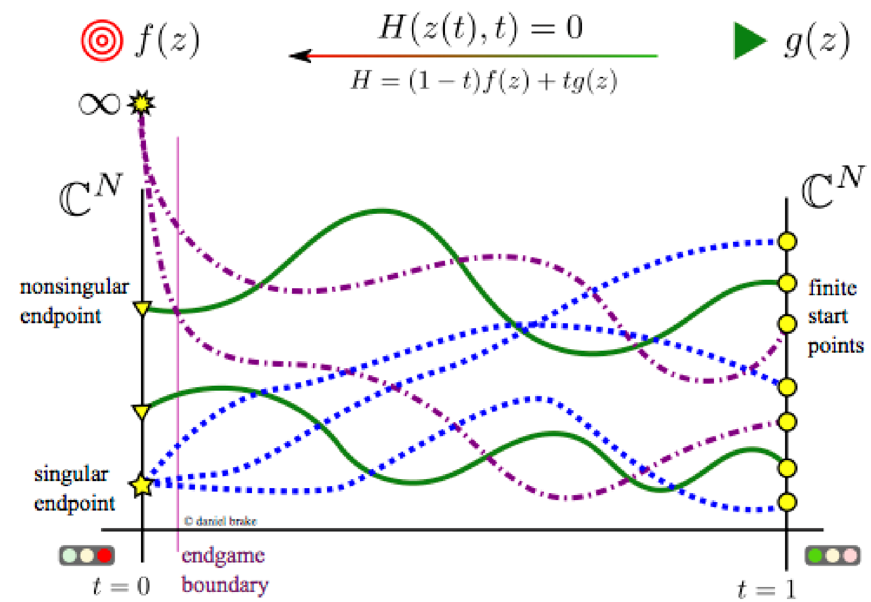


Isolated Solutions

Art in the construction of family $\mathcal{F}$:

- number of start points
- ease to compute start points





Each method is sharp for generic members of $\mathcal{F}$.

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

Isolated Solutions

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$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Bézout family (total degree):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix} : \deg g_i = 2 \right\} \qquad g = \begin{bmatrix} x^2 - 1 \\ y^2 - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g: 4

$$H = (1-t)\cdot f + \gamma t \cdot g$$

- $\gamma \in \mathbb{C}$ is used to create a general deformation
    - avoid singularities that arise from tracking over real numbers

Isolated Solutions

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Number of paths = number of isolated solutions for g: 4

Bertini

input

```
variable_group x,y;
function f1,f2;
f1 = x^2 + 2*x - 8;
f2 = x*y + 2*x + 4*y - 3;
```

finite_solutions

```
1

2.000000000000000e+00 0.000000000000000e+00
-1.666666666666667e-01 0.000000000000000e+00
```

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Multihomogeneous Bézout family (Morgan-Sommese (1987)):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x) \\ g_2(x,y) \end{bmatrix} : \begin{array}{l} \deg_x g_1 = 2, \\ \deg_x g_2 = \deg_y g_2 = 1 \end{array} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x-2)(y-1) \end{bmatrix} \qquad H = (1-t) \cdot f + \gamma t \cdot g$$

Number of paths = number of isolated solutions for g: 2

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

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```
Bertini
  input   variable_group x;
          variable_group y;
          function f1,f2;
          f1 = x^2 + 2*x - 8;
          f2 = x*y + 2*x + 4*y - 3;
```

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Polyhedral (BKK, Huber-Sturmfels (1995)):

$$\mathcal{F} = \left\{ \begin{bmatrix} a_1 x^2 + a_2 x + a_3 \\ a_4 xy + a_5 x + a_6 y + a_7 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ y - 1 \end{bmatrix} \qquad H = (1 - t) \cdot f + \gamma t \cdot g$$

Number of paths = number of isolated solutions for g: 2

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Extra structure in the coefficients of f.

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1 a_2 \\ (x - a_1)y + a_3 x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 1)y - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g: 1

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1 a_2 \\ (x - a_1)y + a_3 x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 1)y - 1 \end{bmatrix}$$

Since $\mathcal{F}$ is no longer linear, use a parameter homotopy:

$$H = p(x, y; a(t))$$

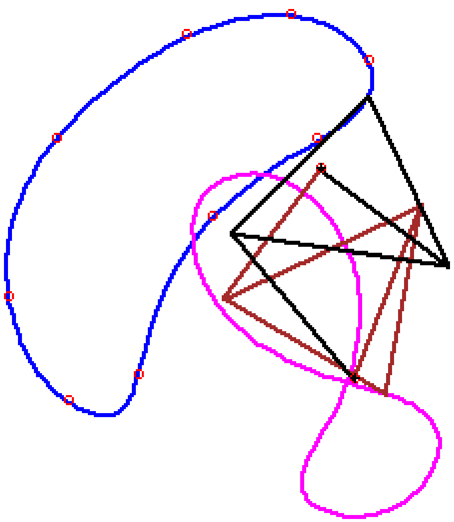
where $a(t) = (1 - \tau(t))(-4, 2, 2, -3) + \tau(t)(1, -1, 0, -1)$

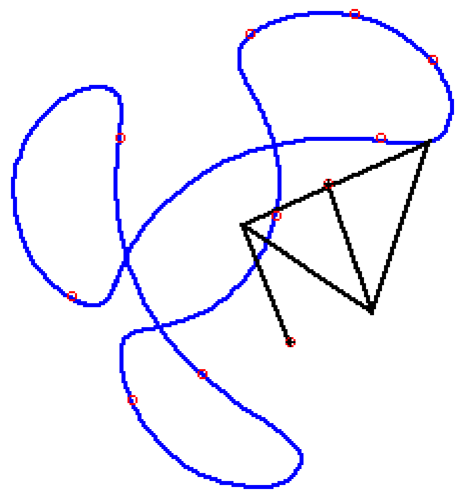
$$\tau(t) = \frac{\gamma t}{1 - t + \gamma t}$$

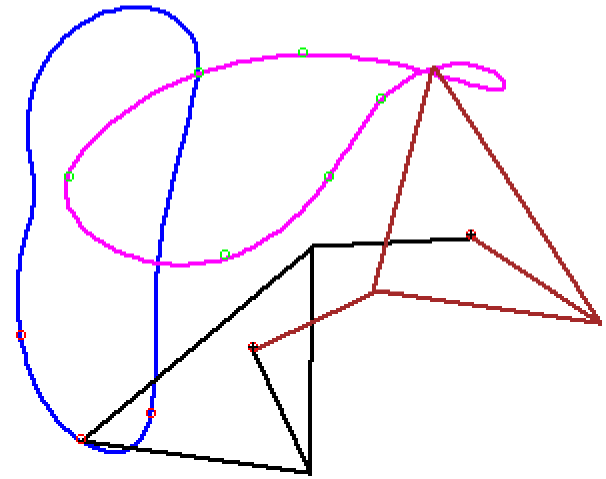
Isolated Solutions

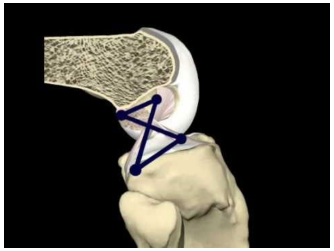
Example (Alt's problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.









Isolated Solutions

Example (Alt's problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.

- $8652 = 6 \cdot 1442$ (Wampler-Morgan-Sommese (1992))

Their polynomial system: 4 quadratics and 8 quartics

Bézout	1,048,576	$= 2^4 \cdot 4^8$
M-hom Bézout	286,720	$= 2^{12} \cdot \binom{8}{4}$
Polyhedral	79,135	
Product decomp.	18,700	
Actual	8,652	

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To date: only verification via numerical algebraic geometry

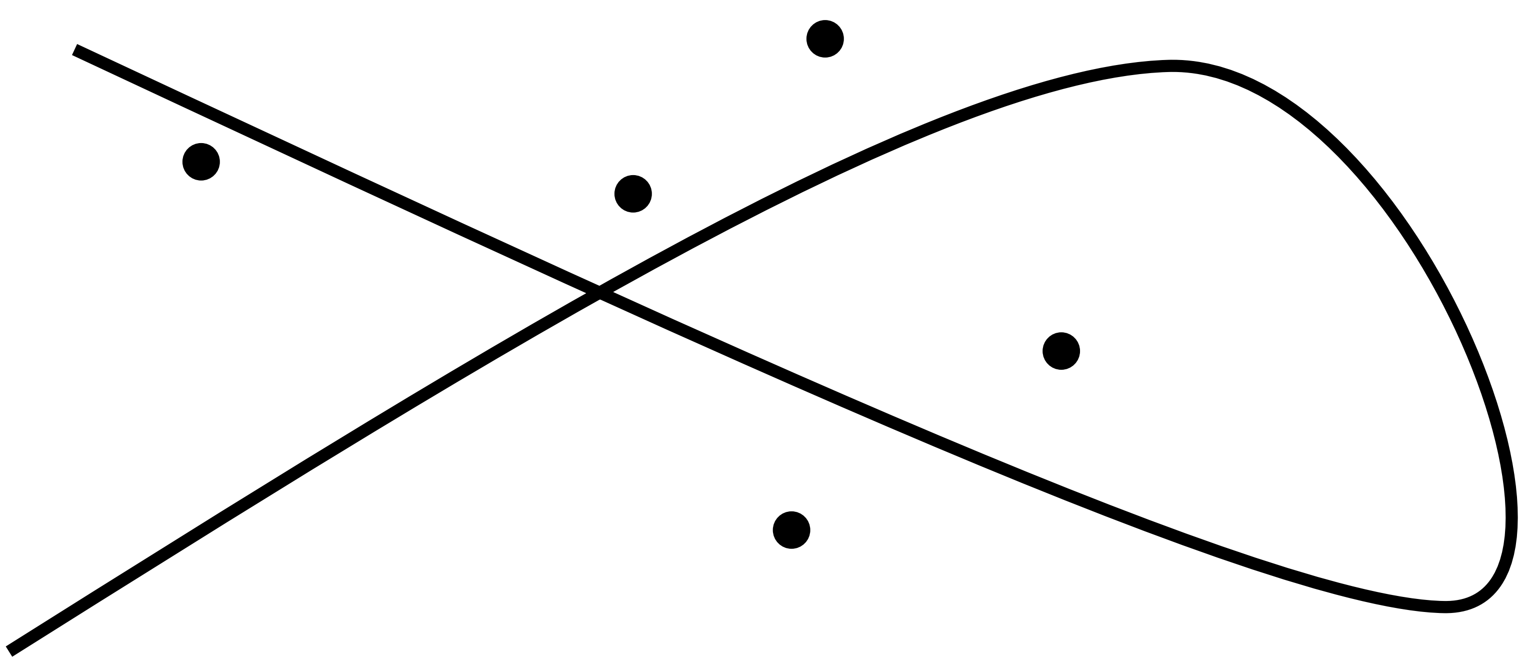
- What structure can be exploited to prove 8,652 is correct?

Early 21st Century

Witness Set

Describe all solutions of

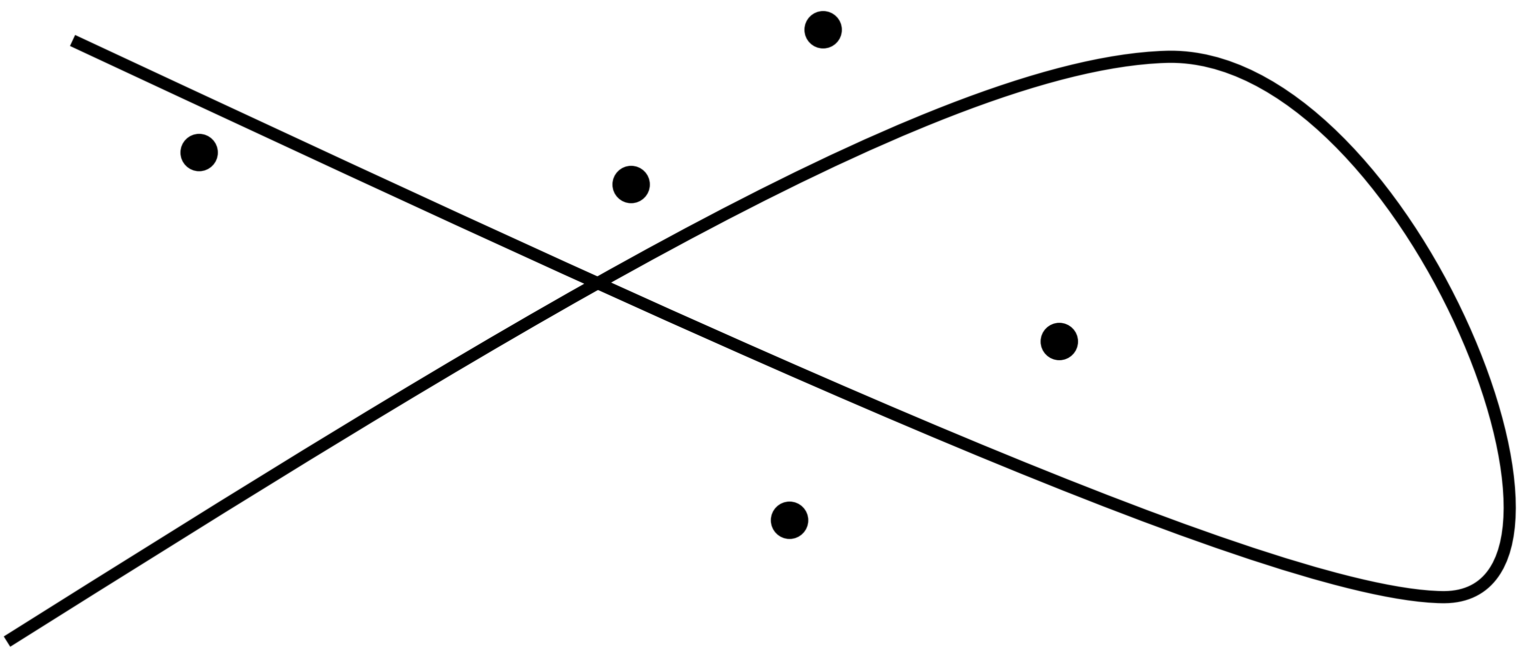
$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_k(x_1, \ldots, x_n) \end{bmatrix} = 0$$



Witness Set

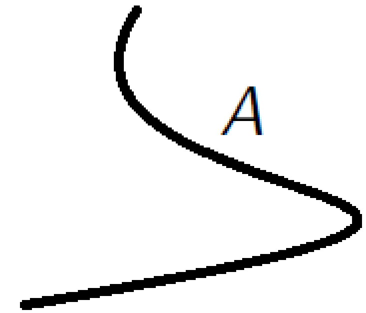
Numerical irreducible decomposition:

- decompose into irreducible components
- provide a numerical description of each irreducible component



Witness Set

How to represent an irreducible algebraic variety A on a computer?



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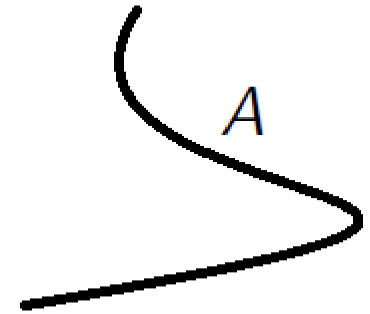


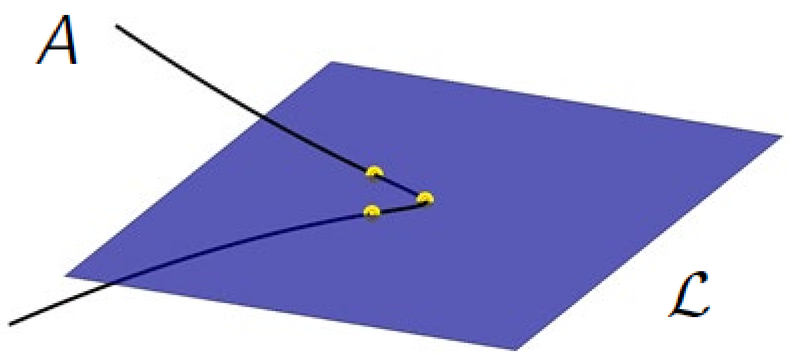
- algebraic: prime ideal $I(A) = \{g \mid g(a) = 0 \text{ for all } a \in A\}$
  - Hilbert Basis Theorem (1890): there exists $f_1, \ldots, f_k$ such that

$$I(A) = \langle f_1, \ldots, f_k \rangle$$

Witness Set

How to represent an irreducible algebraic variety A on a computer?

- geometric: witness set $\{f, \mathcal{L}, W\}$ where
  - f is polynomial system where A is an irred. component of $\mathcal{V}(f)$
  - $\mathcal{L}$ is a linear space with $\mathrm{codim}\ \mathcal{L} = \dim\ A$
  - $W = \mathcal{L} \cap A$ where $\#W = \deg\ A$

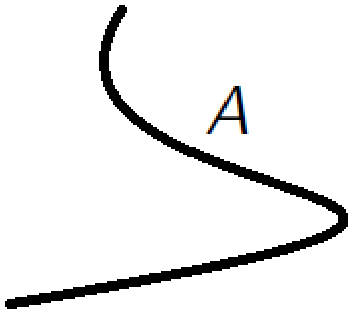


Witness Set

Example

$A = \{[s^3, s^2t, st^2, t^3] \mid [s,t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$ – twisted cubic curve

- $I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle$



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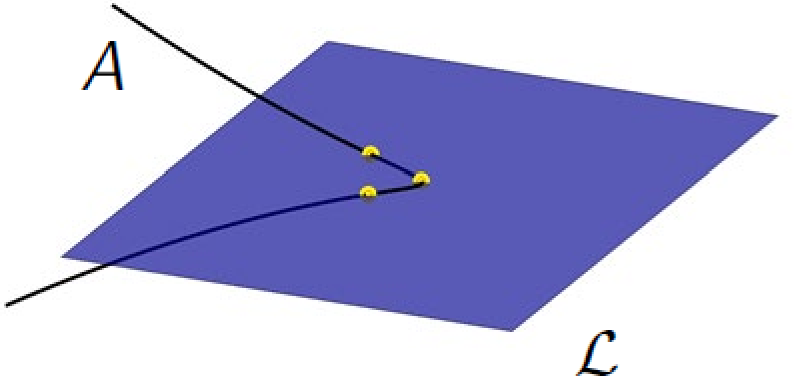


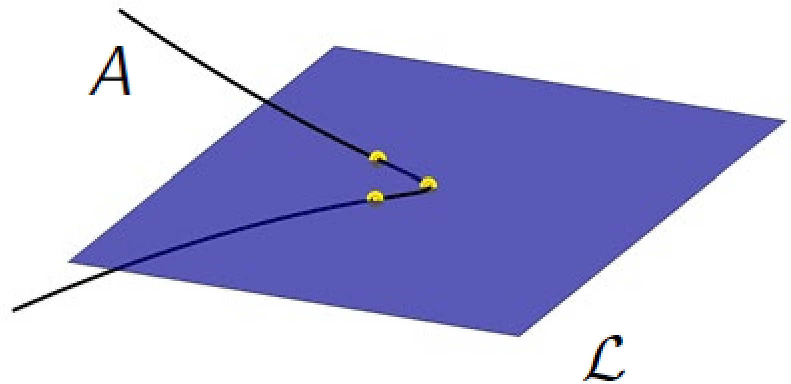
- $\{f, \mathcal{L}, W\}$ where
  - $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$
  - $\mathcal{L} = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid 6x_0 - 6x_1 - 2x_2 + x_3 = 0\} \subset \mathbb{P}^3$
    - $\operatorname{codim} \mathcal{L} = \dim A = 1$
  - $W = \left\{ \begin{array}{l} [1, 3.2731, 10.7130, 35.0644], \\ [1, 0.8596, 0.7389, 0.6351], \\ [1, -2.1326, 4.5481, -9.6995] \end{array} \right\}$
    - $\deg A = 3$

Witness Set

Example

$A = \{[s^3, s^2t, st^2, t^3] \mid [s,t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$ – twisted cubic curve

- $I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle$
- $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$



$\mathcal{V}(f) = A \cup \{x_0 = x_1 = 0\}$

- Witness sets "localize" computations to A effectively ignoring the other irreducible components.
- Sample points from A by moving the linear slice $\mathcal{L}$.

Witness Set

Example

- $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$

Bertini

input

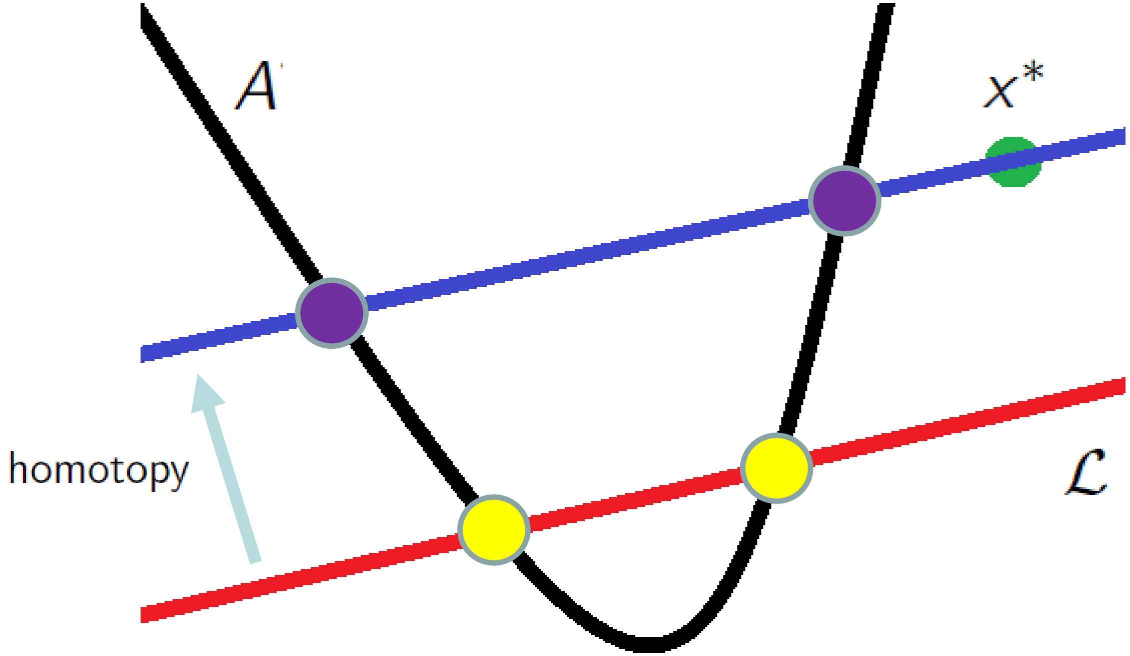
```
CONFIG
 TrackType: 1;
END;
INPUT
 hom_variable_group x0,x1,x2,x3;
 function f1,f2;
 f1 = x1^2 - x0*x2;
 f2 = x1*x2 - x0*x3;
END;
```

```
Dimension 1: 2 classified components
-------------------------------------
   degree 1: 1 component
   degree 3: 1 component
```

Witness Set

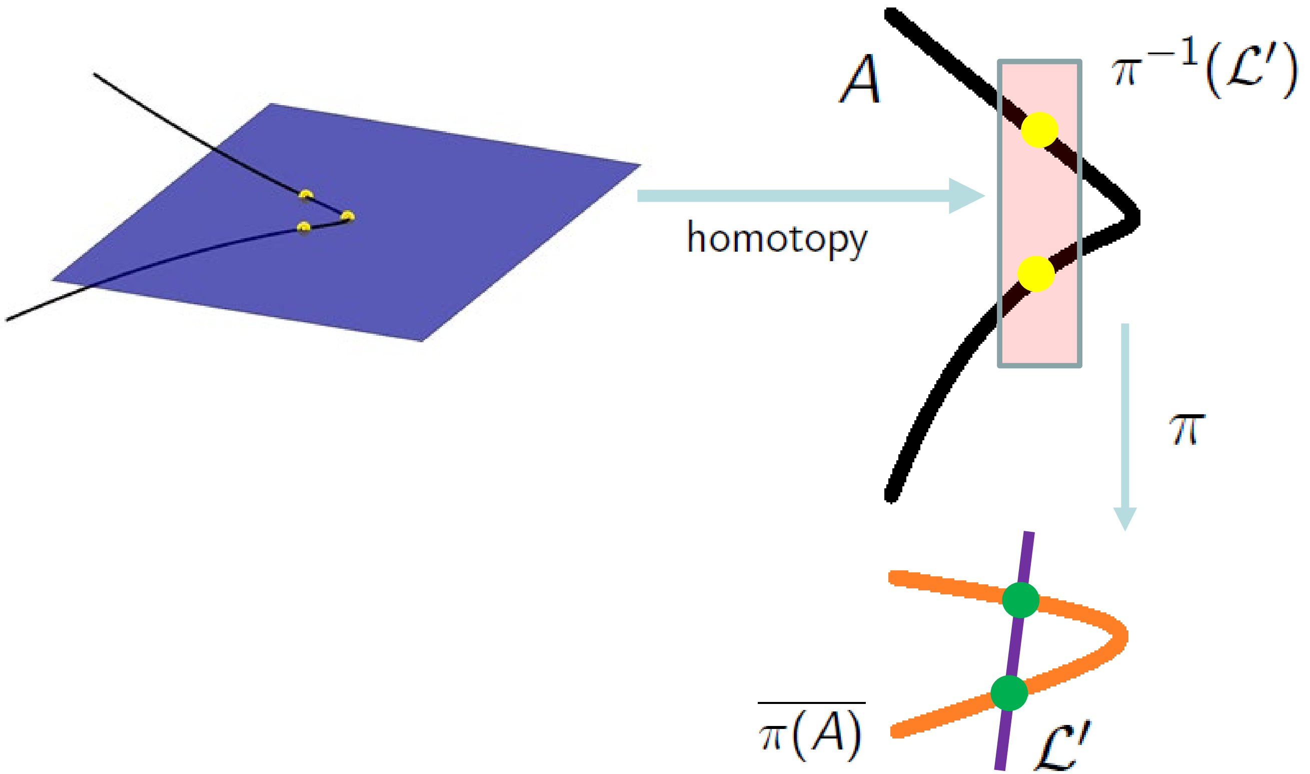
Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- membership testing: is $x^* \in A$?
  - decide if $g(x^*) = 0$ for every $g \in I(A)$ without knowing $I(A)$



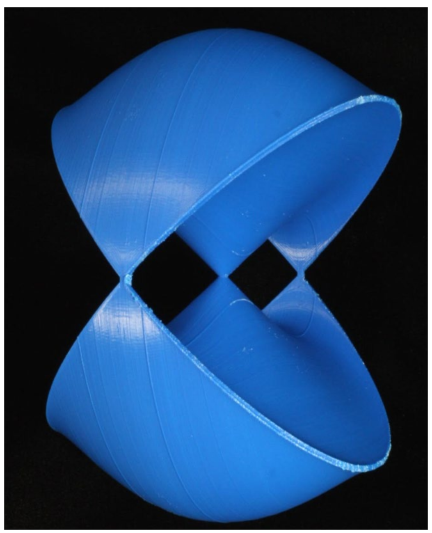
Witness Set

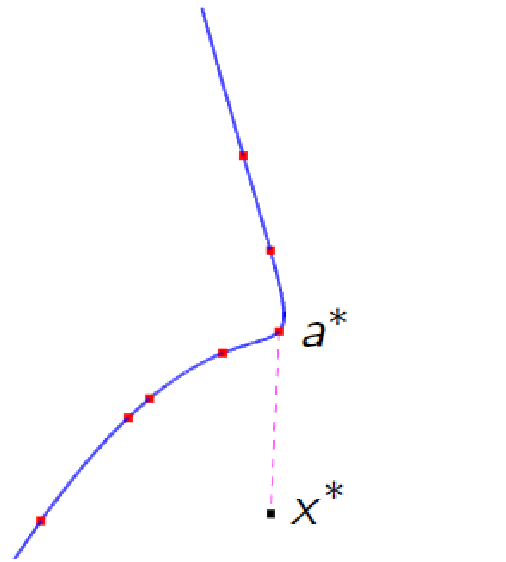
- projection: $\overline{\pi(A)}$
  - perform computations on $\overline{\pi(A)}$ without knowing *any* polynomials that vanish on $\overline{\pi(A)}$



Witness Set

- intersection: $A \cap B$
  - special case is *regeneration*
    - $\mathcal{V}(f_1, \ldots, f_k, f_{k+1}) = \mathcal{V}(f_1, \ldots, f_k) \cap \mathcal{V}(f_{k+1})$ via witness sets
  - compute A_{sing}
  - compute critical points of optimization problem



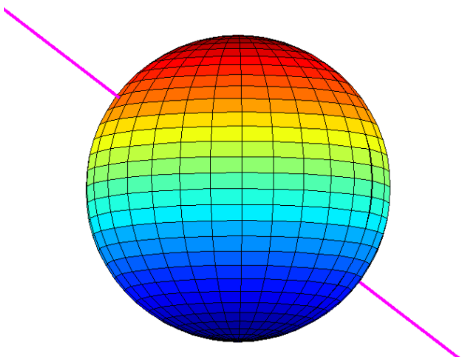


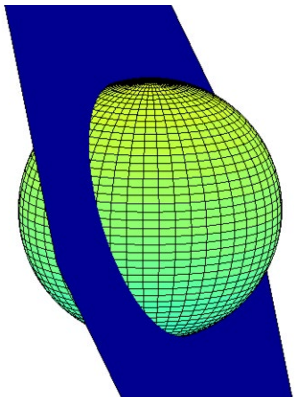
$\min \|x^* - a\|_2$ such that $a \in A \cap \mathbb{R}^n$

Witness Set

Test other algebraic properties of A

- is A arithmetically Cohen Macaulay?
- is A arithmetically Gorenstein?
- is A a complete intersection?





Witness Set

Example

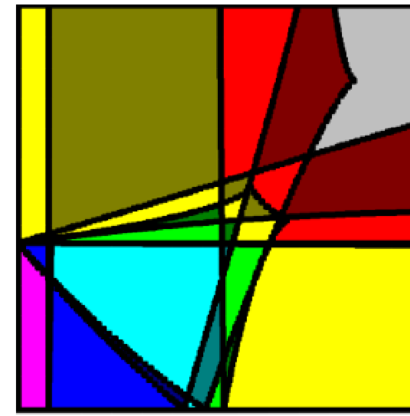
$$A = \sigma_4(\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^4) \subset \mathbb{P}^{35}$$

- $\dim A = 31$
- $\deg A = 345$

- $I(A)$ contains 10 poly. of degree 6 and 20 poly. of degree 9
  - Bates-Oeding (2011), Friedland-Gross (2012)
- used sampling to show that A was aCM and that these polynomials generate $I(A)$

N.S. Daleo and J.D. Hauenstein,
Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme.
J. Symb. Comput., 72, 128-146, 2016.

Future?



- Specialized/structured homotopies
- Real solutions especially over parameter spaces
- Certification for singular and positive-dimensional sets
- Many applications in math, stats, science, and engineering
  - Local methods (too many solutions to find all of them?)

Summary

Numerical algebraic geometry provides a toolbox for solving polynomial systems.

- "If a problem was easy, someone else would have solved it."
  - Gröbner basis computation probably did not terminate
- think carefully about what information you want/need
- art in building efficient homotopies that incorporate structure
- preconditioning is important
  - transform problem into form suitable for num. computations