Complexes from complexes

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University of Minnesota

NEMESIS Workshop, Montpellier
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Joint work with Kaibo Hu, Oxford

1. **A peek at FEEC**
2. **Hilbert complexes and the de Rham complex**
3. **New complexes from old**
4. **Applications**

Finite Element Exterior Calculus in a nutshell

FEEC is a prime example of a *structure-preserving discretization* for PDEs.

It applies to the many PDE problems that relate to a Hilbert complex, with the resulting cohomology and Hodge theory.

FEEC designs discretizations that faithfully capture these structures at the discrete level, and, in this way, achieve stable and convergent numerical methods.

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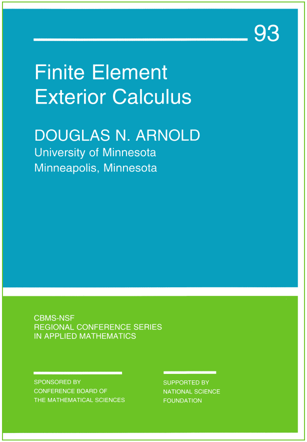
DNA plenary @ ICM 2002

DNA, Falk, Winther: Acta Numerica 2006; Bulletin of the AMS 2010

Numerical analysis antecedents:
Raviart-Thomas 1975; Nédélec 1980, 1986; Bossavit 1988; Hiptmair 2000

Topology antecedents: Whitney 1957; Sullivan 1978; Dodziuk, Patodi 1976



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Finite Element
Exterior Calculus

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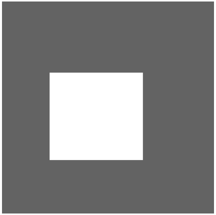
Eigenvalues of the Laplacian



$$\text{minimize} \int |\operatorname{grad} u|^2 \quad \text{subject to} \int |u|^2 = 1,\, u = 0 \text{ on boundary}$$

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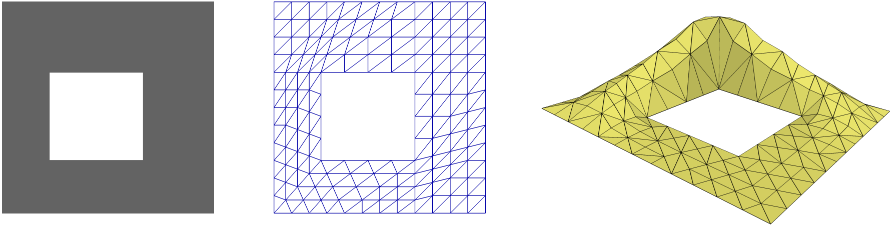

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Finite element discretization

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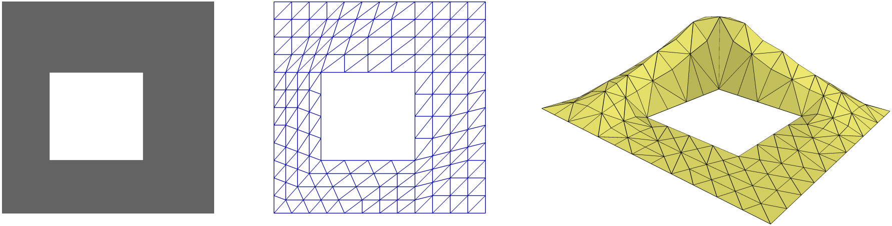

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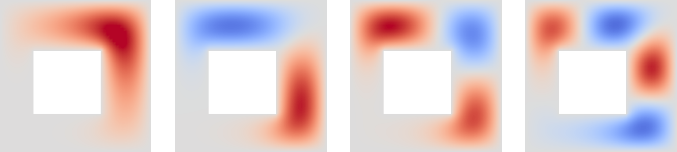
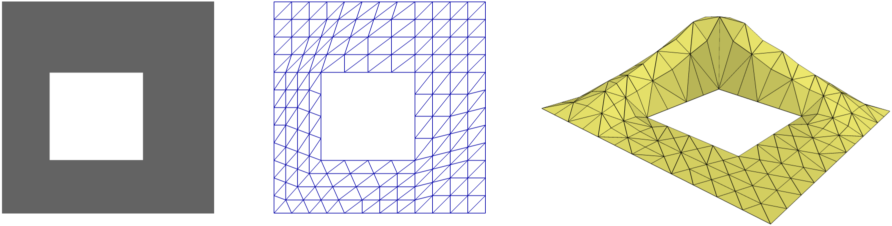


4K $\mathcal{P}_1$ elements: $\lambda_1 = 9.279$ $\lambda_2 = 11.245$ $\lambda_3 = 12.453$ $\lambda_4 = 15.835$

Eigenvalues of the Laplacian



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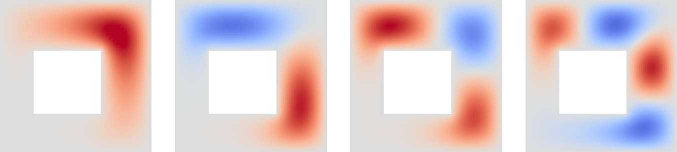
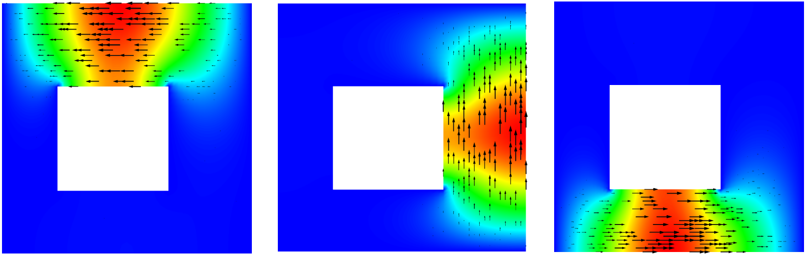
Finite element discretization



4K $\mathcal{P}_1$ elements:	$\lambda_1 = 9.279$	$\lambda_2 = 11.245$	$\lambda_3 = 12.453$	$\lambda_4 = 15.835$
16K $\mathcal{P}_4$ elements:	9.190	11.166	12.327	15.678

Eigenvalues of the 1-form Laplacian

minimize $\int (|\operatorname{curl} u|^2 + |\operatorname{div} u|^2)$ subject to $\int |u|^2 = 1$, $u \cdot n = \operatorname{curl} u = 0$ on bdry



$\mathcal{P}_1$ 4K elts $\lambda_1 = 1.94$ $\lambda_2 = 2.02$ $\lambda_3 = 2.26$

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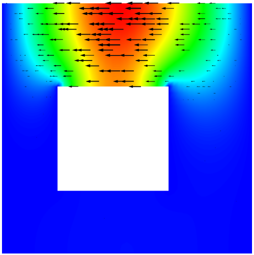
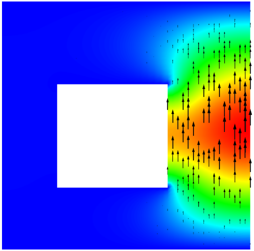


$\lambda_1 = 1.94$



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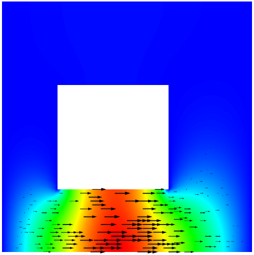
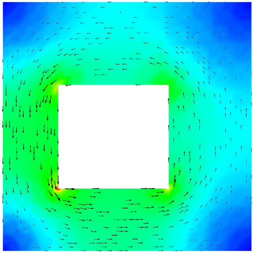
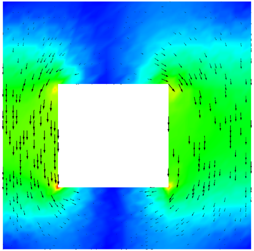


$\lambda_3 = 2.26$

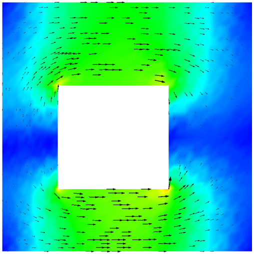
FEEC



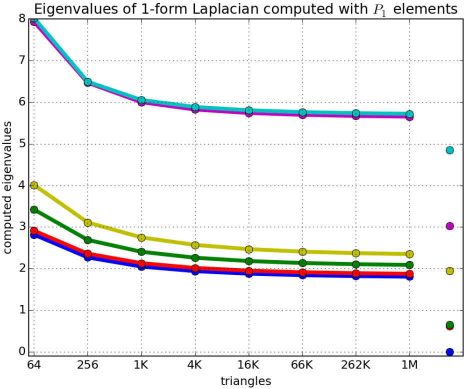
$\lambda_1 = 0$



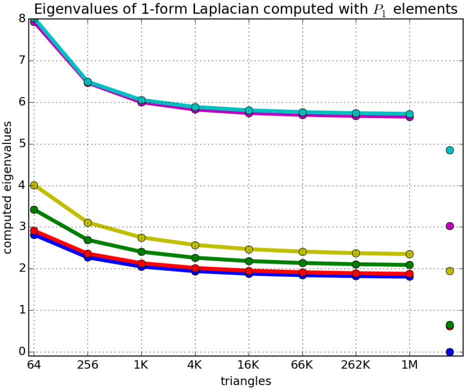
$\lambda_1 = 0.617$



$\lambda_2 = 0.658$



$\mathcal{P}_1$ FEM



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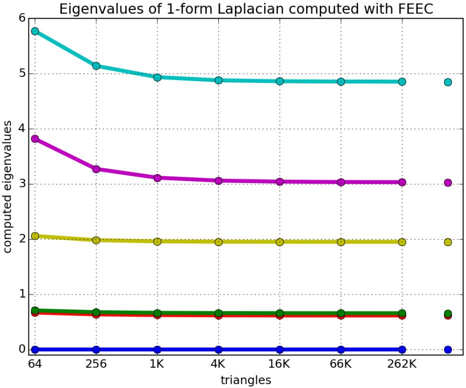


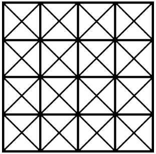
FEEC

The Maxwell eigenvalue problem with P_1 elements

minimize $\int |\operatorname{curl} u|^2$ subject to $\int |u|^2 = 1, \ldots$

For $\Omega = (0,\pi) \times (0,\pi), \quad \lambda = m^2 + n^2, \quad m, n > 0$

elts: 16	64	256	1024	4096
2.2606	2.0679	2.0171	2.0043	2.0011
4.8634	5.4030	5.1064	5.0267	5.0067
5.6530	5.4030	5.1064	5.0267	5.0067
5.6530	5.6798	5.9230	5.9807	5.9952
11.3480	9.0035	8.2715	8.0685	8.0171
11.3480	11.3921	10.4196	10.1067	10.0268
12.2376	11.4495	10.4197	10.1067	10.0268
12.2376	11.6980	13.7043	13.1804	13.0452
12.9691	11.6980	13.7043	13.1804	13.0452
13.9508	15.4308	13.9669	14.7166	14.9272



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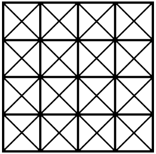


Discretization

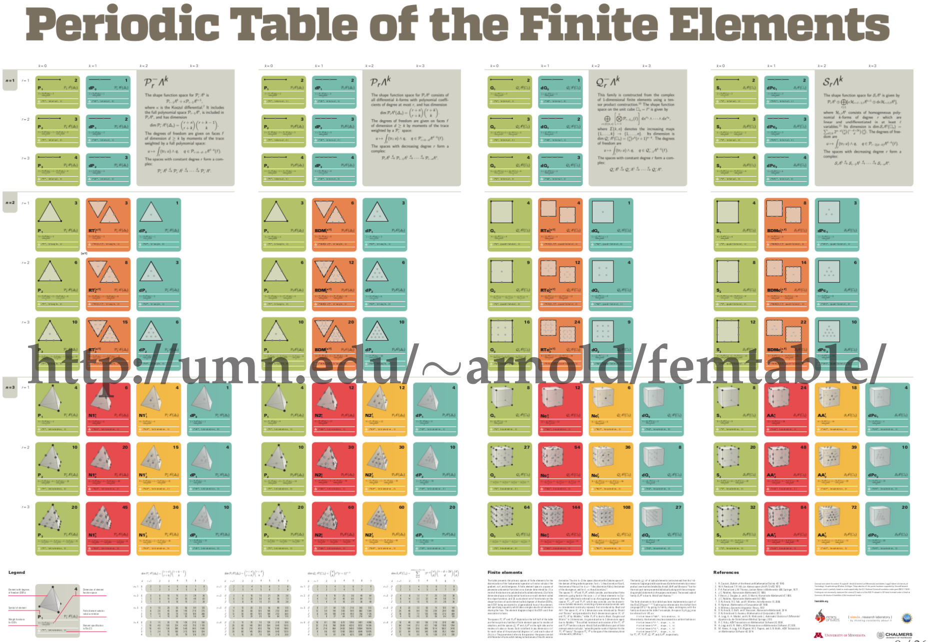
A key result of FEEC are conditions on the finite element spaces required for stable discretizations of problems arising from Hilbert complexes. The key requirements are that

- The subspaces form a Hilbert subcomplex.
- There exist bounded commuting projections from the Hilbert complex onto the subcomplex.

These conditions have a host of desirable consequences.

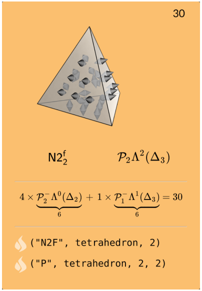
In the case of finite elements for the de Rham complexes, such spaces can be constructed systematically, giving rise to the Periodic Table of the Finite Elements.

Structure-preserving finite elements for the de Rham complex



http://umn.edu/∼arnold/femtable/

FEEC elements



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$\mathrm{N2}_2^{\mathrm{f}}$ $\mathcal{P}_2\Lambda^2(\Delta_3)$

$$4 \times \underbrace{\mathcal{P}_2^-\Lambda^0(\Delta_2)}_{6} + 1 \times \underbrace{\mathcal{P}_1^-\Lambda^1(\Delta_3)}_{6} = 30$$

```
("N2F", tetrahedron, 2)
("P", tetrahedron, 2, 2)
```

- element shape: tetrahedron
- shape functions: $\mathcal{P}_2\Lambda^2$ (quadratic polynomial 2-forms)
- DOFs: 6 per facet + 6 per tetrahedron
- total DOFs: 30 per tetrahedron
- FEEC name: $\mathcal{P}_2\Lambda^2(\Delta_3)$
- Traditional name: $\mathrm{N2}_2^{\mathrm{f}}$ (Nédélec face elements of 2nd kind of deg 2)

The construction of such structure-preserving elements has involved techniques from many branches of mathematics:

- numerical analysis
- functional analysis
- algebraic topology
- differential geometry
- homological algebra
- representation theory

Key tools:

- Hodge theory
- the Koszul complex
- chain homotopy
- Bernstein–Gelfand–Gelfand resolution

1. A peek at FEEC

2. **Hilbert complexes and the de Rham complex**

3. **New complexes from old**

4. **Applications**

The de Rham complex

The de Rham complex on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$:

$$0 \longrightarrow \Lambda^0(\Omega) \xrightarrow{d^0} \Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Lambda^n(\Omega) \longrightarrow 0$$

$\Lambda^k(\Omega) := C^\infty(\Omega, \mathrm{Alt}^k\mathbb{R}^n) = C^\infty(\Omega) \otimes \mathrm{Alt}^k\mathbb{R}^n$, $\quad d =$ exterior derivative

$$\boxed{d^k \circ d^{k-1} = 0}$$

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De Rham cohomology:

$\mathscr{H}^k = \mathcal{N}(d^k) / \mathcal{R}(d^{k-1}), \quad$ dim $= k$th Betti number

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De Rham cohomology:

$$\mathscr{H}^k = \mathcal{N}(d^k) \,/\, \mathcal{R}(d^{k-1}), \quad \text{dim} = k\text{th Betti number}$$

In 3D, via scalar and vector proxies ($\mathbb{V} = \mathbb{R}^3$):

$$0 \longrightarrow C^\infty \xrightarrow{\text{grad}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{div}} C^\infty \longrightarrow 0$$

Complexes and PDEs

The de Rham complex underlies many of important PDEs of mathematical physics:

- Laplacian, heat equation, wave equation, nonlinear variants, …
- Maxwell's equations, MHD, curl curl problems, …
- Darcy flow, div–curl problems…

Other PDEs require other complexes:

- Hessian complex (elastic plates, Einstein–Bianchi eqs of GR)

$$0 \longrightarrow C^\infty \xrightarrow{\text{hess}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{T} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

symm. matrices

trace-free matrices

- Elasticity complex (elasticity, plasticity, dislocations, GR)

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

symm.grad

curl T curl

matrix div

- div div complex, grad curl complex, …

The Sobolev–de Rham complex

There is a bounded Hilbert complex version of the de Rham complex for each Sobolev index $s \in \mathbb{R}$:

$$\begin{tikzcd} 0 \arrow[r] & H^s\Lambda^0 \arrow[r, "d"] & H^{s-1}\Lambda^1(\Omega) \arrow[r, "d"] & \cdots \arrow[r, "d"] & H^{s-n}\Lambda^n \arrow[r] & 0 \end{tikzcd}$$

These encode that d is 1st order. They all give the same cohomology.

THEOREM (UNIFORM REP'N OF COHOMOLOGY, COSTABEL–MCINTOSH '10)

Let Ω be a bounded Lipschitz domain. Then there exist finite dimensional spaces $\mathfrak{H}^k_\infty \subset \Lambda^k(\Omega)$, $k = 0, \dots, n$, independent of s, which represent the cohomology of every Sobolev–de Rham complex:

$$\mathcal{N}(d, H^s\Lambda^k) = \mathcal{R}(d, H^{s+1}\Lambda^{k-1}) \oplus \mathfrak{H}^k_\infty.$$

The L^2 de Rham complex

Another variant, the L^2 de Rham complex, is most relevant to FEEC:

$$0 \longrightarrow L^2\Lambda^0 \xrightarrow{d^0} L^2\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} L^2\Lambda^n \longrightarrow 0$$

This is an unbounded Hilbert complex:

$d^k : L^2\Lambda^k \to L^2\Lambda^{k+1}$ is a closed densely defined operator. It's domain is

$$H\Lambda^k = \{\, u \in L^2\Lambda^k \,|\, du \in L^2\Lambda^{k+1} \,\}.$$

The L^2 de Rham complex again gives the same de Rham cohomology.

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The L^2 de Rham complex again gives the same de Rham cohomology.

It also gives rise to a dual complex:

$$0 \longleftarrow L^2\Lambda^0 \xleftarrow{d_1^*} L^2\Lambda^1 \xleftarrow{d_2^*} \cdots \xleftarrow{d_n^*} L^2\Lambda^n \longleftarrow 0$$

Using it we obtain an elliptic PDE at each level, the Hodge Laplacian:

$$d^{k-1}d_k^* + d_{k+1}^*d^k : L^2\Lambda^k \to L^2\Lambda^k.$$

Closed Hilbert complexes

A crucial property for a Hilbert complex is that it be closed, i.e., that each d has closed range.

We shall see that the de Rham complex is closed. This has many consequences, crucial to FEEC:

- Harmonic forms: $\mathcal{N}(d, H\Lambda^k) = \mathcal{R}(d, H\Lambda^{k-1}) \oplus \mathfrak{H}$, $\mathfrak{H} = \mathcal{N} \cap \mathcal{R}^\perp$
- Duality: $\mathcal{N}(d, H\Lambda^k)^\perp = \mathcal{R}(d^*, H^*\Lambda^{k+1})$
- Hodge decomposition: $L^2\Lambda^k = \mathcal{R}(d, H\Lambda^{k-1}) \oplus \mathfrak{H} \oplus \mathcal{R}(d^*, H^*\Lambda^{k+1})$
- Poincaré inequality: $\|u\| \le c\|du\| \quad \forall u \perp \mathcal{N}(d)$
- Hodge Laplacian BVP: $dd^*u + d^*d\,u = f$
  is well-posed up to harmonic forms

Finite dimensional cohomology implies closed range

Directly proving that a Hilbert complex is closed can be difficult. But it is sufficient to prove that the cohomology is finite-dimensional.

THEOREM

If $S : X \to Y$ is a bounded linear operator on Banach spaces and $\dim(Y/\mathcal{R}(S)) < \infty$, then $\mathcal{R}(S)$ is closed in Y.

COROLLARY

If a complex $X \xrightarrow{S} Y \xrightarrow{T} Z$ has finite dimensional cohomology, then $\mathcal{R}(S)$ is closed in Y.

Consequences of the Costabel–McIntosh theorem

For the de Rham complex, all the needed properties are consequences of the Costabel–McIntosh theorem:

- Cohomology dimensions are finite & independent of the smoothness.
- The Sobolev–de Rham complexes are *closed* for all s.
- The L^2 de Rham complex is *closed.*
  $\Longrightarrow$ harmonic forms, Hodge decomp., Poincaré ineq., well-posed Hodge Laplacian
- Regular potentials: $dH\Lambda^k = dH^1\Lambda^k$.
- Regular decomposition: $H\Lambda^k = dH^1\Lambda^{k-1} + H^1\Lambda^k$
- Compactness property: $H\Lambda^k \cap H^*\Lambda^k \hookrightarrow L^2\Lambda^k$ is compact

1. A peek at FEEC

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4. **Applications**

Systematic derivation of complexes and their properties

What other complexes are there? Do they possess the properties needed for their analysis and discretization? E.g., are they closed? Is the cohomology independent of the Sobolev regularity?

- Inspired by the BGG resolution, we give a systematic derivation of numerous complexes in n-dimensions.
- Starting with two input Hilbert complexes satisfying certain assumptions, we construct an output complex which inherits their properties.
- The cohomology dimensions of the output complex $\leq$ the sum of those of the original complexes.
- With an additional assumption, we get equality. In fact, the output cohomology spaces are isomorphic to the direct sum of those of the input complexes.

Complexes from complexes in four easy steps

1. Input complexes: two bounded Hilbert complexes,

$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}$$

where

$$Z^k = V^k \otimes \mathbb{E}^k, \quad \tilde{Z}^k = \tilde{V}^k \otimes \tilde{\mathbb{E}}^k$$

for Hilbert spaces V^k, $\tilde{V}^k = V^{k+1}$ and *finite dimensional* Hilbert spaces $\mathbb{E}^k$, $\tilde{\mathbb{E}}^k$.

For example, if the Z complex is the Sobolev–de Rham complex, then $Z^k = H^{s-k}\Lambda^k$, so $V^k = H^{s-k}$ and $\mathbb{E}^k = \mathrm{Alt}^k$.

Linking maps

2. We also require linear operators $s^k : \tilde{\mathbb{E}}^k \to \mathbb{E}^{k+1}$ from which we obtain *linking maps* $\quad S^k = \mathrm{id} \otimes s^k : \tilde{Z}^k \to Z^{k+1}$.

$$\begin{tikzcd}
0 \arrow[r] & V^0 \otimes \mathbb{E}^0 \arrow[r, "D^0"] & V^1 \otimes \mathbb{E}^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{V}^1 \otimes \tilde{\mathbb{E}}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{V}^n \otimes \tilde{\mathbb{E}}^n \arrow[r] & 0
\end{tikzcd}$$

We further require

- *Anticommutivity:* $\quad S^{k+1}\tilde{D}^k = -D^{k+1}S^k$
- *Injectivity/surjectivity:* $\exists\, J < n$ s.t. s^k is $\begin{cases} \text{injective}, & 0 \le k \le J, \\ \text{surjective} & k \ge J \end{cases}$

Reduction

3. Reduce the first $J+1$ spaces of the first complex from $Z^k = V^k \otimes \mathbb{E}^k$ to $V^k \otimes \mathbb{F}^k$ where $\mathbb{F}^k = \mathcal{R}(s^{k-1})^{\perp} \subset \mathbb{E}^k$ is the cokernel.

$$0 \longrightarrow V^0 \otimes \mathbb{F}^0 \xrightarrow{P_{\mathbb{F}} D^0} V^1 \otimes \mathbb{F}^1 \xrightarrow{P_{\mathbb{F}} D^1} \cdots \xrightarrow{P_{\mathbb{F}} D^{J-1}} V^J \otimes \mathbb{F}^J$$

Similarly, reduce the final $n-J$ spaces of the second complex from $\tilde{Z}^k = V^{k+1} \otimes \tilde{\mathbb{E}}^k$ to $V^{k+1} \otimes \tilde{\mathbb{F}}^k$ where $\tilde{\mathbb{F}}^k = \mathcal{N}(s^k) \subset \tilde{\mathbb{E}}^k$:

$$\tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1} \xrightarrow{\tilde{D}^{J+1}} \tilde{V}^{J+2} \otimes \tilde{\mathbb{F}}^{J+2} \xrightarrow{\tilde{D}^{J+2}} \cdots \xrightarrow{\tilde{D}^{J}} \tilde{V}^n \otimes \tilde{\mathbb{F}}^n \longrightarrow 0$$

Splicing

4. Finally, splice together the two sequences

$$0 \longrightarrow V^0 \otimes \mathbb{F}^0 \xrightarrow{P_{\mathbb{F}} D^0} V^1 \otimes \mathbb{F}^1 \xrightarrow{P_{\mathbb{F}} D^1} \cdots \xrightarrow{P_{\mathbb{F}} D^{J-1}} V^J \otimes \mathbb{F}^J$$

$$\tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1} \xrightarrow{\tilde{D}^{J+1}} \tilde{V}^{J+2} \otimes \tilde{\mathbb{F}}^{J+2} \xrightarrow{\tilde{D}^{J+2}} \cdots \xrightarrow{\tilde{D}^{J}} \tilde{V}^{n} \otimes \tilde{\mathbb{F}}^{n} \longrightarrow 0$$

through this diagram:

$$\begin{tikzcd}
V^J \otimes \mathbb{F}^J \arrow[r, "D^J"] & V^{J+1} \otimes \mathbb{E}^{J+1} \\
\tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[ru, "S^J"] \arrow[r, "\tilde{D}^J"] & \tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1}
\end{tikzcd}$$

$$\Longrightarrow \quad V^J \otimes \mathbb{F}^J \xrightarrow{\tilde{D}^J (S^J)^{-1} D^J} \tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1}$$

Summary of the construction

$$0 \to V^0 \otimes \mathbb{E}^0 \longrightarrow \cdots \longrightarrow V^J \otimes \mathbb{E}^J \longrightarrow V^{J+1} \otimes \mathbb{E}^{J+1} \longrightarrow \cdots \longrightarrow V^n \otimes \mathbb{E}^n \to 0$$

$$0 \to \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \longrightarrow \cdots \longrightarrow \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \longrightarrow \tilde{V}^{J+1} \otimes \tilde{\mathbb{E}}^{J+1} \longrightarrow \cdots \longrightarrow \tilde{V}^n \otimes \tilde{\mathbb{E}}^n \to 0$$

1. Input complexes (of tensor product form)

Summary of the construction

$$\begin{tikzcd}
0 \arrow[r] & V^0 \otimes \mathbb{E}^0 \arrow[r] & \cdots \arrow[r] & V^J \otimes \mathbb{E}^J \arrow[r] & V^{J+1} \otimes \mathbb{E}^{J+1} \arrow[r] & \cdots \arrow[r] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[r] \arrow[ur, "S", "\cong"'] & \tilde{V}^{J+1} \otimes \tilde{\mathbb{E}}^{J+1} \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^n \otimes \tilde{\mathbb{E}}^n \arrow[r] & 0
\end{tikzcd}$$

1. Input complexes (of tensor product form)
2. Linking maps (anticommuting and *J*-surjective/injective)

Summary of the construction

$$\begin{tikzcd}
0 \arrow[r] & \boxed{V^0 \otimes \mathbb{F}^0} \arrow[r] & \cdots \arrow[r] & \boxed{V^J \otimes \mathbb{F}^J} \arrow[r] & V^{J+1} \otimes \mathbb{E}^{J+1} \arrow[r] & \cdots \arrow[r] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[r] \arrow[ur, "S", "\cong"'] & \boxed{\tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1}} \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \boxed{\tilde{V}^n \otimes \tilde{\mathbb{F}}^n} \arrow[r] & 0
\end{tikzcd}$$

1. Input complexes (of tensor product form)
2. Linking maps (anticommuting and *J*-surjective/injective)
3. Space reduction

Summary of the construction

$$\begin{tikzcd}
0 \arrow[r] & V^0 \otimes \mathbb{F}^0 \arrow[r] & \cdots \arrow[r] & V^J \otimes \mathbb{F}^J \arrow[r] & V^{J+1} \otimes \mathbb{E}^{J+1} \arrow[r] & \cdots \arrow[r] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[r] \arrow[ur, "S", "\cong"'] & \tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1} \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^n \otimes \tilde{\mathbb{F}}^n \arrow[r] & 0
\end{tikzcd}$$

1. Input complexes (of tensor product form)
2. Linking maps (anticommuting and J-surjective/injective)
3. Space reduction
4. Splicing

Cohomology of the output complex

THEOREM

The dimension of the kth cohomology space of the output complex is bounded by the sum of the dimensions of the kth cohomology spaces of the two input complexes. Equality holds iff the linking maps induce the zero map on cohomology, i.e., $S^k(\mathcal{N}(\tilde{D}^k)) \subset \mathcal{R}(D^k)$.

COROLLARY

If the input complexes have finite dimensional cohomology, so does the output complex. Consequently, it has closed ranges.

THEOREM

With an additional assumption ($\exists K^i : \tilde{Z}^i \to Z^i$ with $S = DK - K\tilde{D}$), there is a cochain map from the direct sum of the input complexes to the output complexes which induces an isomorphism on cohomology.

1. A peek at FEEC

2. Hilbert complexes and the de Rham complex

3. New complexes from old

4. **Applications**

The first family of applications

Fix $0 \leq J < n$. For the two input complexes take Sobolev–de Rham complexes tensored with Alt^J and Alt^{J+1}, respectively:

$$0 \to H^s\Lambda^0 \otimes \mathrm{Alt}^J \xrightarrow{d} H^{s-1}\Lambda^1 \otimes \mathrm{Alt}^J \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n}\Lambda^n \otimes \mathrm{Alt}^J \to 0$$

$$0 \to H^{s-1}\Lambda^0 \otimes \mathrm{Alt}^{J+1} \xrightarrow{d} H^{s-2}\Lambda^1 \otimes \mathrm{Alt}^{J+1} \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n-1}\Lambda^n \otimes \mathrm{Alt}^{J+1} \to 0$$

$$V^i = \tilde{V}^{i-1} = H^{s-i}, \quad \mathbb{E}^i = \mathrm{Alt}^i \otimes \mathrm{Alt}^J, \quad \tilde{\mathbb{E}}^i = \mathrm{Alt}^i \otimes \mathrm{Alt}^{J+1}$$

The linking maps $s^i : \mathrm{Alt}^i \otimes \mathrm{Alt}^{J+1} \to \mathrm{Alt}^{i+1} \otimes \mathrm{Alt}^J$ are naturally defined in terms of components:

$$M_{\underbrace{k_1 \cdots k_i}_{\text{antisym}} \underbrace{l_1 \cdots l_{J+1}}_{\text{antisym}}} \quad \overset{s^i}{\longmapsto} \quad M_{\underbrace{[k_1 \cdots k_i l_1]}_{\text{antisymmetrize}} l_2 \cdots l_{J+1}}$$

The linking maps

THEOREM

1. *The algebraic maps* $s^i : Alt^i \otimes Alt^{J+1} \to Alt^{i+1} \otimes Alt^J$ *are injective for* $i \leq J$ *and surjective for* $i \geq J$.
2. *The maps* $S^i = \mathrm{id} \otimes s^i$ *satisfy the anticommutativity and* J*-injectivity/surjectivity assumptions.*

The s^i in terms of vector proxies. In 3D, Alt^k, $k = 0, 1, 2, 3$, identifies with $\mathbb{R}$, $\mathbb{V}$, $\mathbb{V}$, $\mathbb{R}$.

$$\begin{tikzcd}
J \backslash i & 0 & 1 & 2 & 3 \\
0 & \mathbb{R} & \mathbb{V} & \mathbb{V} & \mathbb{R} \\
1 & \mathbb{V} \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[ur, "\mathrm{vskw}"] & \mathbb{M} \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \\
2 & \mathbb{V} \arrow[ur, "\mathrm{mskw}"] & \mathbb{M} \arrow[ur, "S^1"] & \mathbb{M} \arrow[ur, "\mathrm{vskw}"] & \mathbb{V} \\
3 & \mathbb{R} \arrow[ur, "\iota"] & \mathbb{V} \arrow[ur, "\mathrm{mskw}"] & \mathbb{V} \arrow[ur, "\mathrm{id}"] & \mathbb{R}
\end{tikzcd}$$

$\mathrm{vskw} : \mathbb{M} \to \mathbb{V}$, axial vector of the skew part

$\mathrm{tr} : \mathbb{M} \to \mathbb{R}$, trace

$\mathrm{mskw} : \mathbb{V} \to \mathbb{M}$, axial vector to matrix

$\iota : \mathbb{R} \to \mathbb{M}$, scalar to scalar matrix: $c \mapsto c\,I$

$S^1 : \mathbb{M} \to \mathbb{M}$, $S^1\tau = \tau^T - \mathrm{tr}(\tau)I$ (bijection)

The output complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The output complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The Hessian complex:

$$0 \to H^s \xrightarrow{\text{hess}} H^{s-2} \otimes \mathbb{S} \xrightarrow{\text{curl}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\text{div}} H^{s-4} \otimes \mathbb{V} \to 0$$

The output complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The elasticity complex:

$$0 \to H^{s-1} \otimes \mathbb{V} \xrightarrow{\text{def}} H^{s-2} \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{s-4} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{s-5} \otimes \mathbb{V} \to 0$$

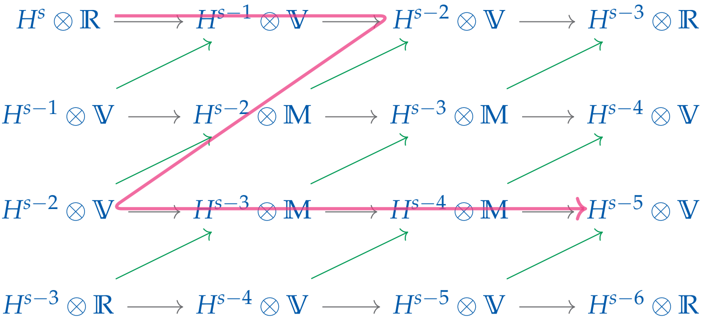
The output complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[dl, magenta] \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The div-div complex:

$$0 \to H^{s-2} \otimes \mathbb{V} \xrightarrow{\text{dev grad}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\text{sym curl}} H^{s-4} \otimes \mathbb{S} \xrightarrow{\text{div div}} H^{s-6} \to 0$$

The output complexes in 3D



$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The grad-curl complex:

$$0 \to H^s \xrightarrow{\text{grad}} H^{s-1} \otimes \mathbb{V} \xrightarrow{\text{grad curl}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\text{curl}} H^{s-4} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{s-5} \otimes \mathbb{V} \to 0$$

The output complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] \arrow[rrr, thick] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[ddll, thick] \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] \arrow[rr, thick] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The curl-div complex:

$$0 \to H^{s-1} \otimes \mathbb{V} \xrightarrow{\text{grad}} H^{s-2} \otimes \mathbb{M} \xrightarrow{\text{dev curl}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\text{curl div}} H^{s-5} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{s-6} \otimes \mathbb{V} \to 0$$

The output complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The grad-div complex:

$$0 \rightarrow H^s \xrightarrow{\text{grad}} H^{s-1} \otimes \mathbb{V} \xrightarrow{\text{curl}} H^{s-2} \otimes \mathbb{V} \xrightarrow{\text{grad div}} H^{s-4} \otimes \mathbb{V} \xrightarrow{\text{curl}} H^{s-5} \otimes \mathbb{V} \xrightarrow{\text{div}} H^{s-6} \rightarrow 0$$

Complexes from complexes from complexes

We can iterate this procedure, using the output complexes as inputs to derive more complexes. For example,

elasticity complex $+$ Hessian complex $\Longrightarrow$ conformal elasticity complex

which involves the deviatoric strain tensor $\operatorname{dev}\operatorname{def} u$, and a third order differential operator known as the Cotton tensor and arising in relativity.

$$\begin{tikzcd}
0 \arrow[r] & H^s \otimes \mathbb{V} \arrow[r, "\operatorname{def}"] & H^{s-1} \otimes \mathbb{S} \arrow[r, "\operatorname{inc}"] & H^{s-3} \otimes \mathbb{S} \arrow[r, "\operatorname{div}"] & H^{s-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{s-1} \arrow[r, "\operatorname{hess}"] \arrow[ur, "\iota"] & H^{s-3} \otimes \mathbb{S} \arrow[r, "\operatorname{curl}"] \arrow[ur, "S^1"] & H^{s-4} \otimes \mathbb{T} \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{vskw}"] & H^{s-5} \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}$$

$$\Downarrow$$

$$0 \to H^s \otimes \mathbb{V} \xrightarrow{\operatorname{dev}\operatorname{def}} H^{s-1} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\operatorname{Cott}} H^{s-4} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\operatorname{div}} H^{s-5} \otimes \mathbb{V} \to 0$$

The first Poincaré inequality for this complex is a strengthened Korn's inequality:

$$\|u\|_{H^1} \leq c(\|u\|_{L^2} + \|\operatorname{dev}\operatorname{def} u\|_{L^2})$$

A 2D example, continuous and discrete

The construction gives us tools to extend the periodic table of finite elements to more complexes. We end with a simple example.

Beginning with an H^2 de Rham complex and an H^1 vector-valued de Rham complex, we get a 2D elasticity complex:

$$\begin{tikzcd}
0 \arrow[r] & H^2 \arrow[r, "\mathrm{curl}"] & H^1 \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & L^2 \arrow[r] & 0 \\
0 \arrow[r] & H^1 \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{id}"] & L^2 \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{skw}"] & H^{-1} \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
0 \arrow[r] & H^2 \arrow[r, "\mathrm{curl}\,\mathrm{curl}"] & L^2 \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] & H^{-1} \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}$$

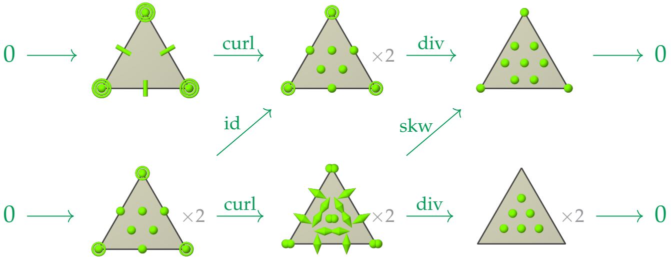
The corresponding L^2 Hilbert complex is what we need to discretize mixed elasticity finite elements.

$$\begin{tikzcd}
0 \arrow[r] & H^2 \arrow[r, "\mathrm{curl}\,\mathrm{curl}"] & H(\mathrm{div}, \mathbb{S}) \arrow[r, "\mathrm{div}"] & L^2 \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}$$

Finite element discretization

Discretize with finite elements on a triangulation:

- H^2: Argyris quintic 1968 ($\mathcal{P}_5$)
- $H^1 \otimes \mathbb{V}$: 2 copies of Hermite quartic elements ($\mathcal{P}_4 \otimes \mathbb{V}$)
- L^2: piecewise cubics with vertex continuity ($\mathcal{P}_3$)
- $L^2 \otimes \mathbb{M}$: 2 copies of “nonstandard” variant of BDM, Stenberg 2010 ($\mathcal{P}_3 \otimes \mathbb{M}$)
- $H^{-1} \otimes \mathbb{V}$: 2 copies of DG2 ($\mathcal{P}_2 \otimes \mathbb{V}$)

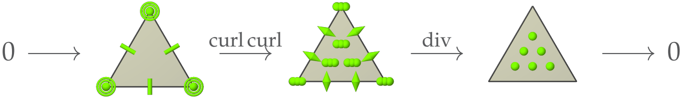


The top sequence was studied by Falk and Neilan 2013 as a discretization for the Stokes complex.

The resulting complex

The resulting output complex is a discretization of the 2D elasticity complex.

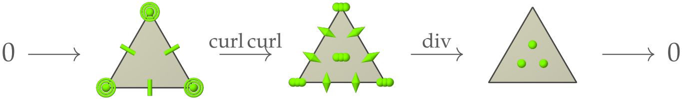
- The scalar space (Airy potential) is Argyris.
- Stress: symmetric matrices with rows in Stenberg $\mathcal{P}_3$ space
- Displacement: DG2 vectors



mixed elasticity elements proposed by J. Hu–S. Zhang 2015

Other elements/other complexes

A similar construction may be given for the original mixed elasticity elements of Arnold–Winther 2002 (cf. Arnold–Falk–Winter, IMA vol. 142).



Lots of recent progress for other complexes:

- elasticity complex in 3D: Chen–Huang 2021
- Hessian complex: Hu–Liang 2020, Chen–Huang 2020 (VEM)
- div div complex: Chen–Huang 2020, Hu–Liang–Ma (2021)

Will there be a periodic table of finite elements for these other complexes?

Conclusions

- A closed Hilbert complex has a rich structure which underlies the behavior of many fundamental PDE.
- Discretizing the complex in a structure-preserving fashion provides stable convergent numerical methods, which can otherwise be difficult to find.
- This program is quite complete for the de Rham complex, with applications to Darcy flow, electromagnetism, MHD, etc.
- Via a BGG-inspired construction, we start from well-understood complexes and systematically derive new ones, extending the applications of FEEC to elasticity, plate theory, GR, …
- Cohomology and other structural aspects of the output complexes are determined from that of the input complexes.
- There is still work to be done to produce the "periodic table of finite elements" for the newly derived complexes.

DNA and K. Hu, Found. Comput. Math. 2021