Conforming finite element sequences for strain and curvature.

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Outline

- Elasticity complexes : strain and curvature
- New finite element discretization
- Finite element systems : sheaves
- Vector bundles : RM cochains
- de Rham theorem and Bianchi identity

Elasticity Strain Complex

- Continuous metrics:

$$\mathrm{H}^2(U,\mathbb{V}) \xrightarrow{\mathsf{def}} \mathrm{H}^1_{\mathsf{sven}}(U,\mathbb{S}) \xrightarrow{\mathsf{sven}} \mathrm{H}^0(U,\mathbb{R}). \tag{1}$$

with:

$$\mathrm{H}^1_{\mathsf{sven}}(U,\mathbb{S}) = \{u \in \mathrm{H}^1(U,\mathbb{S}) \ : \ \mathsf{sven}\, u \in \mathrm{H}^0(U,\mathbb{R})\}. \tag{2}$$

- Exactness and rigid motions.
- Saint Venant compatibility and linearized curvature.
- Lower regularity and partitions of unity.

New finite element

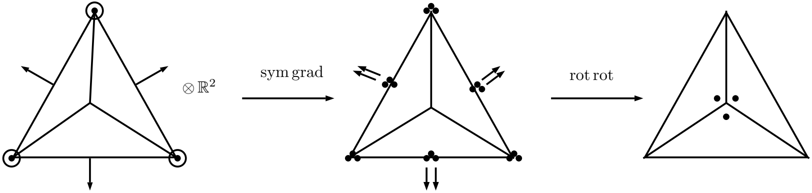


Figure: Strain complex with continuous metrics.

Spaces

- Vector valued Clough Tocher:

$$A^0(T) = \mathrm{C}^1\mathrm{P}^3(\mathcal{R}(T), \mathbb{V}), \tag{3}$$

- Continuous P^2 metrics with integrable sven (cont. $\partial_\nu u\tau \cdot \tau$):

$$A^1(T) = \mathrm{C}^0_{\mathsf{sven}}\mathrm{P}^2(\mathcal{R}(T), \mathbb{S}), \tag{4}$$

DoFs: – values at vertices (3×3),
– pairings with $M(E) \approx RM$ for each edge E (3×3),
– integral against normal vector on edges (3×2).

- Piecewise constants:

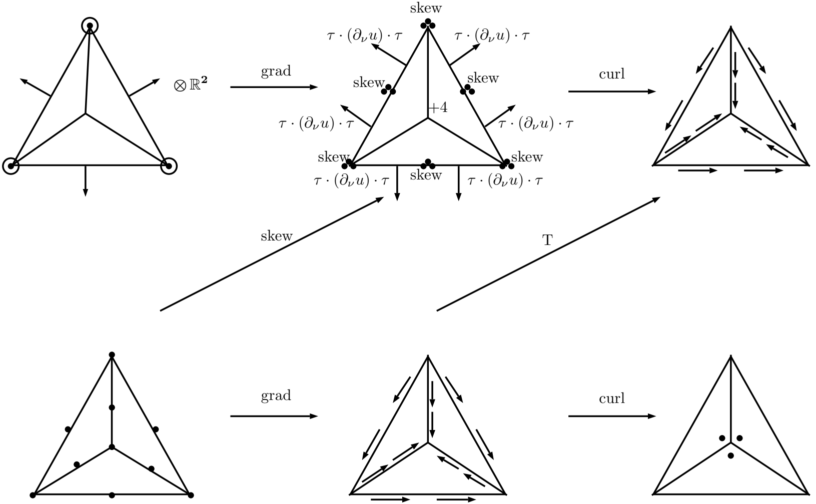
$$A^2(T) = \mathrm{P}^0(\mathcal{R}(T), \mathbb{R}), \tag{5}$$

DoFs: integration against affine functions $(\cdot \approx RM)$.

BGG

$$
\begin{array}{ccccc}
\mathrm{H}^2(U,\mathbb{V}) & \xrightarrow{\mathsf{grad}} & \mathrm{H}^1_{\mathsf{sven}}(U,\mathbb{M}) & \xrightarrow{\mathsf{curl}} & \mathrm{H}^0_{\mathsf{curl}\,\mathrm{T}}(U,\mathbb{V}), \\
 & \nearrow_{\mathsf{skew}} & & \nearrow_{\mathrm{T}} & \\
\mathrm{H}^1(U,\mathbb{R}) & \xrightarrow{\mathsf{grad}} & \mathrm{H}^0_{\mathsf{curl}}(U,\mathbb{V}^{\mathrm{T}}) & \xrightarrow{\mathsf{curl}} & \mathrm{H}^0(U,\mathbb{R}).
\end{array}
\tag{6}
$$

Discrete BGG



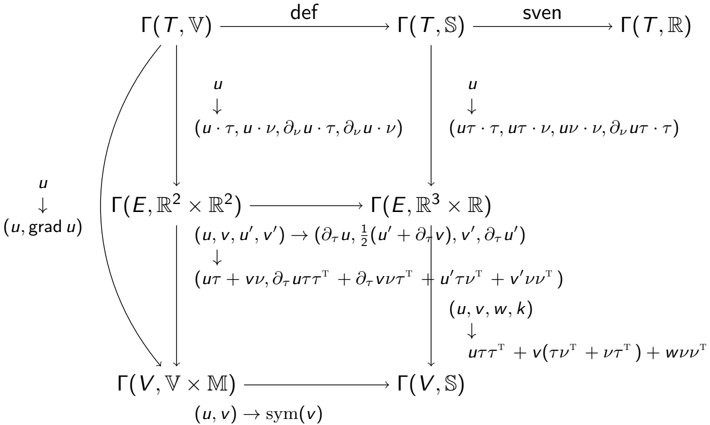
Finite element systems [C. 08, C.-Hu 18]

- Fix a cellular complex $\mathcal{T}$.
- A finite element system A is
  $A^k(T)$ for $k \in \mathbb{N}$ and $T \in \mathcal{T}$ of all dimensions.
  – differentials: $\mathrm{d} : A^k(T) \to A^{k+1}(T)$.
  – restrictions: $T' \subseteq T$ gives $\mathrm{r} : A^k(T) \to A^k(T')$.
  – de Rham map.
  commutation relations.
- Associated global space:

$$A^k(\mathcal{T}) = \{u \in \bigoplus_{T \in \mathcal{T}} A^k(T) \ : \ T' \subseteq T \Rightarrow u_T|_{T'} = u_{T'}\}.$$

  Encodes continuity.
- – FES is a contravariant functor
  from a cellular complex to differential complexes.
  – Global space is the inverse limit.

Induced operators (on faces)



Cochains with coefficients

- For each $T \in \mathcal{T}$, a vectorspace $L(T)$. A discrete vectorbundle.
- When T' is a codim 1 face of T, an isomorphism $\mathrm{t}_{TT'} : L(T') \to L(T)$. A discrete connection.
- Flatness:

$$\mathrm{t}_{TT'_0}\mathrm{t}_{T'_0T''} = \mathrm{t}_{TT'_1}\mathrm{t}_{T'_1T''}. \tag{7}$$

- Cochains $\mathcal{C}^k(\mathcal{T}, L)$: $(u(T))_{T \in \mathcal{T}^k}$ such that $u(T) \in L(T)$.
- Differential $\delta^k_{\mathrm{t}} : \mathcal{C}^k(\mathcal{T}, L) \to \mathcal{C}^{k+1}(\mathcal{T}, L)$ defined by:

$$(\delta^k_{\mathrm{t}} u)(T) = \sum_{T' \trianglelefteq T} \mathrm{o}(T, T')\mathrm{t}_{TT'} u(T'). \tag{8}$$

- Flatness gives $\delta^{k+1}_{\mathrm{t}} \circ \delta^k_{\mathrm{t}} = 0$.

FES and cochains

- $\mathsf{e} : A^k(T) \to L(T)$. Generalized Stokes: For $u \in A^{k-1}(T)$:

$$\mathsf{e}_T \mathsf{d}_T u = \sum_{T' \in \partial T} \mathrm{o}(T, T') \mathsf{t}_{TT'} \mathsf{e}_{T'} \mathsf{r}_{T'T} u. \tag{9}$$

- Commutes with differentials:

$$\mathsf{e} : A^\bullet(\mathcal{T}') \to \mathcal{C}^\bullet(\mathcal{T}', L). \tag{10}$$

Example

- Spaces:
  $M(T)$: affine functions, on T.
  $M(E)$: (u, v) with u affine, v constant, on E.
  $M(V)$: $\mathbb{R}^2 \times \mathbb{R}$.
- Restrictions:
  $M(T) \to M(E)$: $u \mapsto (u, \partial_\nu u)$ on E.
  $M(E) \to M(V)$: $(u, v) \mapsto (v\tau - \partial_\tau u \nu, u)$ on V.
- Vectorbundle with discrete connection by duality.
  Check flatness.

de Rham theorem

- The evaluation map $\mathrm{e} : A^\bullet(\mathcal{T}') \to \mathcal{C}^\bullet(\mathcal{T}', L)$ induces isomorphisms on cohomology groups.
- Proof: Induction on dimension: add top dimensional cells.
  Write Mayer Vietoris short exact sequences for A and $\mathcal{C}$, and connect them by e.
  Deduce long exact sequences that are connected by e.
  Use five lemma.

Bianchi identity

- Drop requirement of flatness, and introduce curvature:

$$\mathrm{c}_{\mathrm{t}}(T, T'') = \pm(\mathrm{t}_{TT'_0}\mathrm{t}_{T'_0T''} - \mathrm{t}_{TT'_1}\mathrm{t}_{T'_1T''}). \qquad (11)$$

- Then $\delta_{\mathrm{t}}^{k+1} \circ \delta_{\mathrm{t}}^{k} u(T) = \sum_{T''} \mathrm{c}_{\mathrm{t}}(T, T'') u(T'')$.
- Introduce cubical complex,
  and discrete connection on endomorphisms.
- Bianchi:
  The covariant exterior derivative of the curvature is 0.
  Combinatorial identity attached to cubes.