

An introduction to Discrete de Rham (DDR) methods

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- 1 Three model problems and their well-posedness
- 2 Discrete de Rham (DDR) complexes
- 3 Application to magnetostatics

Setting I

- Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral domain with **Betti numbers** b_i
- We have $b_0 = 1$ (number of connected components) and $b_3 = 0$
- b_1 accounts for the number of **tunnels** crossing Ω



$$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$$

- b_2 is the number of **voids** encapsulated by Ω



$$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$$

Setting II

- We consider PDE models that hinge on the **vector calculus operators**:

$$\mathbf{grad} q = \begin{pmatrix} \partial_1 q \\ \partial_2 q \\ \partial_3 q \end{pmatrix}, \quad \mathbf{curl} \mathbf{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}, \quad \operatorname{div} \mathbf{w} = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$$

for smooth enough functions

$$q : \Omega \rightarrow \mathbb{R}, \quad \mathbf{v} : \Omega \rightarrow \mathbb{R}^3, \quad \mathbf{w} : \Omega \rightarrow \mathbb{R}^3$$

- The corresponding L^2 -graph (domain) spaces are

$$\begin{aligned} H^1(\Omega) &:= \{q \in L^2(\Omega) : \mathbf{grad} q \in L^2(\Omega) := L^2(\Omega)^3\}, \\ \mathbf{H}(\mathbf{curl}; \Omega) &:= \{\mathbf{v} \in L^2(\Omega) : \mathbf{curl} \mathbf{v} \in L^2(\Omega)\}, \\ \mathbf{H}(\operatorname{div}; \Omega) &:= \{\mathbf{w} \in L^2(\Omega) : \operatorname{div} \mathbf{w} \in L^2(\Omega)\} \end{aligned}$$

- Assume for the moment that Ω has **trivial topology** (i.e., $b_1 = b_2 = 0$)



Three model problems

The Stokes problem in curl-curl formulation

- Given a real number $\nu > 0$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the Stokes problem reads:
Find the **velocity** $\mathbf{u} : \Omega \rightarrow \mathbb{R}^3$ and **pressure** $p : \Omega \rightarrow \mathbb{R}$ s.t.

$$\begin{aligned} \overbrace{\nu(\mathbf{curl} \mathbf{curl} \mathbf{u} - \mathbf{grad} \operatorname{div} \mathbf{u})}^{-\nu \Delta \mathbf{u}} + \mathbf{grad} p &= \mathbf{f} && \text{in } \Omega, && \text{(momentum conservation)} \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)} \\ \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 &&& \text{on } \partial\Omega, && \text{(boundary conditions)} \\ \int_{\Omega} p &= 0 \end{aligned}$$

- Weak formulation:** Find $(\mathbf{u}, p) \in \mathbf{H}(\mathbf{curl}; \Omega) \times H^1(\Omega)$ s.t. $\int_{\Omega} p = 0$ and

$$\begin{aligned} \int_{\Omega} \nu \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} + \int_{\Omega} \mathbf{grad} p \cdot \mathbf{v} &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} && \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega), \\ - \int_{\Omega} \mathbf{u} \cdot \mathbf{grad} q &= 0 && \forall q \in H^1(\Omega) \end{aligned}$$

Three model problems

The magnetostatics problem

- For $\mu > 0$ and $\mathbf{J} \in \mathbf{curl} \mathbf{H}(\mathbf{curl}; \Omega)$, the magnetostatics problem reads:
Find the **magnetic field** $\mathbf{H} : \Omega \rightarrow \mathbb{R}^3$ and **vector potential** $\mathbf{A} : \Omega \rightarrow \mathbb{R}^3$ s.t.

$$\begin{aligned}\mu \mathbf{H} - \mathbf{curl} \mathbf{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)} \\ \mathbf{curl} \mathbf{H} &= \mathbf{J} && \text{in } \Omega, && \text{(Ampère's law)} \\ \operatorname{div} \mathbf{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)} \\ \mathbf{A} \times \mathbf{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}\end{aligned}$$

- Weak formulation:** Find $(\mathbf{H}, \mathbf{A}) \in \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\operatorname{div}; \Omega)$ s.t.

$$\begin{aligned}\int_{\Omega} \mu \mathbf{H} \cdot \boldsymbol{\tau} - \int_{\Omega} \mathbf{A} \cdot \mathbf{curl} \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \mathbf{H}(\mathbf{curl}; \Omega), \\ \int_{\Omega} \mathbf{curl} \mathbf{H} \cdot \boldsymbol{\nu} + \int_{\Omega} \operatorname{div} \mathbf{A} \operatorname{div} \boldsymbol{\nu} &= \int_{\Omega} \mathbf{J} \cdot \boldsymbol{\nu} && \forall \boldsymbol{\nu} \in \mathbf{H}(\operatorname{div}; \Omega)\end{aligned}$$

Three model problems

The Darcy problem in velocity-pressure formulation

- Given $\kappa > 0$ and $f \in L^2(\Omega)$, the Darcy problem reads:

Find the **velocity** $\mathbf{u} : \Omega \rightarrow \mathbb{R}^3$ and **pressure** $p : \Omega \rightarrow \mathbb{R}$ s.t.

$$\begin{aligned}\kappa^{-1} \mathbf{u} - \mathbf{grad} p &= 0 && \text{in } \Omega, && \text{(Darcy's law)} \\ -\operatorname{div} \mathbf{u} &= f && \text{in } \Omega, && \text{(mass conservation)} \\ p &= 0 && \text{on } \partial\Omega && \text{(boundary condition)}\end{aligned}$$

- Weak formulation:** Find $(\mathbf{u}, p) \in \mathbf{H}(\operatorname{div}; \Omega) \times L^2(\Omega)$ s.t.

$$\begin{aligned}\int_{\Omega} \kappa^{-1} \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} p \operatorname{div} \mathbf{v} &= 0 && \forall \mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega), \\ -\int_{\Omega} \operatorname{div} \mathbf{u} q &= \int_{\Omega} f q && \forall q \in L^2(\Omega)\end{aligned}$$

A unified view

- The above problems are **mixed formulations** involving two fields
- They can be recast into the abstract setting: Find $(\sigma, u) \in \Sigma \times U$ s.t.

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= f(\tau) \quad \forall \tau \in \Sigma, \\ -b(\sigma, v) + c(u, v) &= g(v) \quad \forall v \in U, \end{aligned}$$

or, equivalently, in variational formulation,

$$\mathcal{A}((\sigma, u), (\tau, v)) = f(\tau) + g(v) \quad \forall (\tau, v) \in \Sigma \times U$$

with

$$\mathcal{A}((\sigma, u), (\tau, v)) := a(\sigma, \tau) + b(\tau, u) - b(\sigma, v) + c(u, v) = f(\tau) + g(v)$$

- Well-posedness holds under an **inf-sup condition on \mathcal{A}**

A unified tool for well-posedness: The de Rham complex

$$\{0\} \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} \mathbf{H}(\text{curl}; \Omega) \xrightarrow{\text{curl}} \mathbf{H}(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties, possibly depending on the topology of Ω :

$$\text{Im grad} \subset \text{Ker curl},$$

$$\text{Im curl} \subset \text{Ker div},$$

$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \text{Im div} = L^2(\Omega) \quad (\text{Darcy, magnetostatics})$$

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- Key properties, possibly depending on the topology of Ω :

no tunnels crossing Ω ($b_1 = 0$) \implies **Im grad = Ker curl** (Stokes)

no voids contained in Ω ($b_2 = 0$) \implies **Im curl = Ker div** (magnetostatics)

$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) \implies **Im div = $L^2(\Omega)$** (Darcy, magnetostatics)

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$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) \implies **Im div = $L^2(\Omega)$** (Darcy, magnetostatics)

- When $b_1 \neq 0$ or $b_2 \neq 0$, **de Rham's cohomology** characterizes

$$\text{Ker curl} / \text{Im grad} \quad \text{and} \quad \text{Ker div} / \text{Im curl}$$

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$$\text{Ker curl} / \text{Im grad} \quad \text{and} \quad \text{Ker div} / \text{Im curl}$$

- **Emulating these algebraic properties is key for stable discretizations**

Generalization through differential forms

- The de Rham complex generalizes to **domains of \mathbb{R}^n** or smooth manifolds
- Denoting by d the **exterior derivative** and by $H\Lambda(\Omega)$ its L^2 -domain,

$$H\Lambda^0(\Omega) \xrightarrow{d^0} \dots \xrightarrow{d^{k-1}} H\Lambda^k(\Omega) \xrightarrow{d^k} \dots \xrightarrow{d^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the vector calculus version is recovered through **vector proxies**

$$\begin{array}{ccccccc} H\Lambda^0(\Omega) & \xrightarrow{d^0} & H\Lambda^1(\Omega) & \xrightarrow{d^1} & H\Lambda^2(\Omega) & \xrightarrow{d^2} & H\Lambda^3(\Omega) \longrightarrow \{0\} \\ \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong \\ H^1(\Omega) & \xrightarrow{\text{grad}} & \mathbf{H}(\text{curl}; \Omega) & \xrightarrow{\text{curl}} & \mathbf{H}(\text{div}; \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \longrightarrow \{0\} \end{array}$$

The (trimmed) Finite Element way

Local spaces

- Let $T \subset \mathbb{R}^3$ be a polyhedron and set, for any $k \geq -1$,

$$\mathcal{P}^k(T) := \{\text{restrictions of 3-variate polynomials of degree } \leq k \text{ to } T\}$$

- Fix $k \geq 0$ and write, denoting by \mathbf{x}_T a point inside T ,

$$\begin{aligned}\mathcal{P}^k(T)^3 &= \mathbf{grad} \mathcal{P}^{k+1}(T) \oplus (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-1}(T)^3 =: \mathcal{G}^k(T) \oplus \mathcal{G}^{c,k}(T) \\ &= \mathbf{curl} \mathcal{P}^{k+1}(T)^3 \oplus (\mathbf{x} - \mathbf{x}_T) \mathcal{P}^{k-1}(T) =: \mathcal{R}^k(T) \oplus \mathcal{R}^{c,k}(T)\end{aligned}$$

- Define the **trimmed spaces** that sit between $\mathcal{P}^k(T)^3$ and $\mathcal{P}^{k+1}(T)^3$:

$$\begin{aligned}\mathcal{N}^{k+1}(T) &:= \mathcal{G}^k(T) \oplus \mathcal{G}^{c,k+1}(T) \\ \mathcal{RT}^{k+1}(T) &:= \mathcal{R}^k(T) \oplus \mathcal{R}^{c,k+1}(T)\end{aligned}$$

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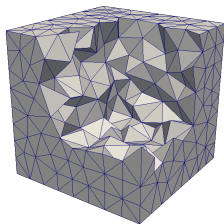
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- $\mathcal{P}^{-,k} \Lambda^r(f)$ generalizes to r -forms on d -faces f through **Koszul complements**

The (trimmed) Finite Element way

Global complex



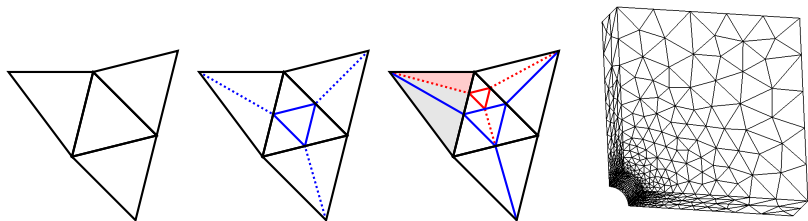
- Let \mathcal{T}_h be a **conforming tetrahedral mesh** of Ω and let $k \geq 0$
- Local spaces can be **glued together** to form a **global FE complex**:

$$\begin{array}{ccccccccc} \mathcal{P}_c^{k+1}(\mathcal{T}_h) & \xrightarrow{\text{grad}} & \mathcal{N}^{k+1}(\mathcal{T}_h) & \xrightarrow{\text{curl}} & \mathcal{RT}^{k+1}(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}^k(\mathcal{T}_h) & \longrightarrow & \{0\} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H^1(\Omega) & \xrightarrow{\text{grad}} & \mathbf{H}(\text{curl}; \Omega) & \xrightarrow{\text{curl}} & \mathbf{H}(\text{div}; \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \longrightarrow & \{0\} \end{array}$$

- **The gluing only works on conforming meshes (simplicial complexes)!**

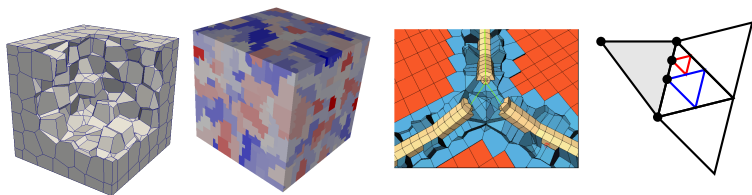
The Finite Element way

Shortcomings



- Approach limited to **conforming meshes** with **standard elements**
 - ⇒ local refinement requires to **trade mesh size for mesh quality**
 - ⇒ complex geometries may require a **large number of elements**
 - ⇒ the element shape cannot be **adapted to the solution**
- Need for (global) basis functions
 - ⇒ significant increase of DOFs on hexahedral elements

The discrete de Rham (DDR) approach I

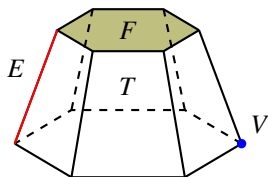


- **Key idea:** replace both spaces and operators by discrete counterparts:

$$\underline{X}_{\text{grad},h}^k \xrightarrow{\underline{G}_h^k} \underline{X}_{\text{curl},h}^k \xrightarrow{\underline{C}_h^k} \underline{X}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \longrightarrow \{0\}$$

- Support of **polyhedral meshes (CW complexes)** and **high-order**
- Several strategies to **reduce the number of unknowns** on general shapes
- Natural generalization to the **de Rham complex of differential forms**

The discrete de Rham (DDR) approach II



- DDR spaces are spanned by **vectors of polynomials**
- Polynomial components enable **consistent reconstructions** of
 - Vector calculus operators
 - The corresponding scalar or vector potentials
- These reconstructions emulate **integration by parts (Stokes) formulas**

References for this presentation

- Vector FE spaces [Raviart and Thomas, 1977, Nédélec, 1980]
- FEEC [Arnold, Falk, Winther, 2006–pres.]
- Introduction of DDR [DP, Droniou, Rapetti, 2020]
- DDR with Koszul complements [DP and Droniou, 2023a]
- Algebraic properties (general topology) [DP, Droniou, Pitassi, 2023]
- Bridges with VEM [Beirão da Veiga, Dassi, DP, Droniou, 2022]
- Polytopal Exterior Calculus (PEC) [Bonaldi, DP, Droniou, Hu, 2023]
- C++ open-source implementation available in **HArDCore3D**

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The two-dimensional case

Continuous exact complex

- With F mesh face let, for $q : F \rightarrow \mathbb{R}$ and $\mathbf{v} : F \rightarrow \mathbb{R}^2$ smooth enough,

$$\mathbf{rot}_F q := (\mathbf{grad}_F q)^\perp \quad \mathbf{rot}_F \mathbf{v} := \mathbf{div}_F(\mathbf{v}^\perp)$$

- We start by deriving a discrete counterpart of the 2D de Rham complex:

$$H^1(F) \xrightarrow{\mathbf{grad}_F} \mathbf{H}(\mathbf{rot}; F) \xrightarrow{\mathbf{rot}_F} L^2(F) \longrightarrow \{0\}$$

- We will need the following decomposition of $\mathcal{P}^k(F)^2$:

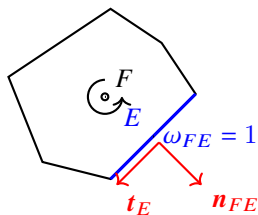
$$\mathcal{P}^k(F)^2 = \mathbf{rot}_F \mathcal{P}^{k+1}(F) \oplus (\mathbf{x} - \mathbf{x}_F) \mathcal{P}^{k-1}(F) =: \mathcal{R}^k(F) \oplus \mathcal{R}^{c,k}(F),$$

and recall the 2D Raviart–Thomas space

$$\mathcal{RT}^{k+1}(F) := \mathcal{R}^k(F) \oplus \mathcal{R}^{c,k+1}(F)$$

The two-dimensional case

A key remark

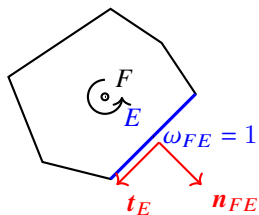


- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\mathbf{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F q \cdot \mathbf{v} = - \int_F q \operatorname{div}_F \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q|_{\partial F} (\mathbf{v} \cdot \mathbf{n}_{FE})$$

The two-dimensional case

A key remark

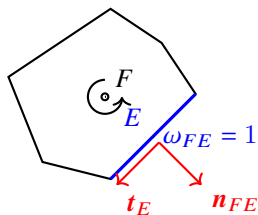


- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\mathbf{v} \in \mathcal{P}^k(F)^2$, we have

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The two-dimensional case

A key remark



- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\mathbf{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F q \cdot \mathbf{v} = - \int_F \underbrace{\pi_{\mathcal{P},F}^{k-1} q}_{\in \mathcal{P}^{k-1}(F)} \operatorname{div}_F \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q|_{\partial F} (\mathbf{v} \cdot \mathbf{n}_{FE})$$

- Hence, $\mathbf{grad}_F q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q|_{\partial F}$

The two-dimensional case

Discrete $H^1(F)$ space

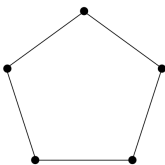
- Based on this remark, we take as discrete counterpart of $H^1(F)$

$$\underline{X}_{\text{grad},F}^k := \left\{ \underline{q}_F = (q_F, q_{\partial F}) : q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}_c^{k+1}(\mathcal{E}_F) \right\}$$

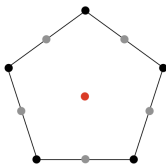
- Let $\underline{I}_{\text{grad},F}^k : C^0(\bar{F}) \rightarrow \underline{X}_{\text{grad},F}^k$ be s.t., $\forall q \in C^0(\bar{F})$,

$$\underline{I}_{\text{grad},F}^k q := (\pi_{\mathcal{P},F}^{k-1} q, q_{\partial F}) \text{ with}$$

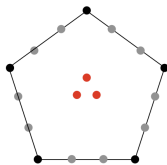
$$\pi_{\mathcal{P},E}^{k-1}(q_{\partial F})|_E = \pi_{\mathcal{P},E}^{k-1} q|_E \quad \forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\mathbf{x}_V) = q(\mathbf{x}_V) \quad \forall V \in \mathcal{V}_F$$



$k = 0$



$k = 1$



$k = 2$

The two-dimensional case

Reconstructions in $\underline{X}_{\text{grad},F}^k$

- For all $E \in \mathcal{E}_F$, the **edge gradient** $G_E^k : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^k(E)$ is s.t.

$$G_E^k \underline{q}_F := (q_{\partial F})'|_E$$

- The **face gradient** $G_F^k : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^k(F)^2$ is s.t., $\forall \mathbf{v} \in \mathcal{P}^k(F)^2$,

$$\int_F G_F^k \underline{q}_F \cdot \mathbf{v} = - \int_F q_F \operatorname{div}_F \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\mathbf{v} \cdot \mathbf{n}_{FE})$$

- The **scalar trace** $\gamma_F^{k+1} : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^{k+1}(F)$ is s.t., for all $\mathbf{v} \in \mathcal{R}^{c,k+2}(F)$,

$$\int_F \gamma_F^{k+1} \underline{q}_F \operatorname{div}_F \mathbf{v} = - \int_F G_F^k \underline{q}_F \cdot \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F q_{\mathcal{E}_F} (\mathbf{v} \cdot \mathbf{n}_{FE})$$

- By construction, we have **polynomial consistency**: For all $q \in \mathcal{P}^{k+1}(F)$,

$$G_F^k (I_{\underline{\text{grad},F}^k}^k q) = \operatorname{grad}_F q \quad \text{and} \quad \gamma_F^{k+1} (I_{\underline{\text{grad},F}^k}^k q) = q$$

The two-dimensional case

Discrete $\mathbf{H}(\text{rot}; F)$ space

- We start from: $\forall \mathbf{v} \in \mathcal{N}^{k+1}(F) := \mathcal{RT}^{k+1}(F)^\perp, \forall q \in \mathcal{P}^k(F),$

$$\int_F \text{rot}_F \mathbf{v} \, q = \int_F \mathbf{v} \cdot \underbrace{\text{rot}_F q}_{\in \mathcal{RT}^k(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\mathbf{v} \cdot \mathbf{t}_E) q|_E$$

The two-dimensional case

Discrete $\mathbf{H}(\text{rot}; F)$ space

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$$\int_F \text{rot}_F \mathbf{v} \cdot q = \int_F \underbrace{\boldsymbol{\pi}_{\mathcal{RT}, F}^k \mathbf{v}}_{\in \mathcal{RT}^k(F)} \cdot \underbrace{\text{rot}_F q}_{\in \mathcal{RT}^k(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\mathbf{v} \cdot \mathbf{t}_E)}_{\in \mathcal{P}^k(E)} q|_E$$

The two-dimensional case

Discrete $\mathbf{H}(\text{rot}; F)$ space

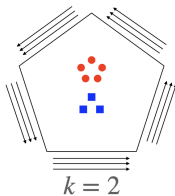
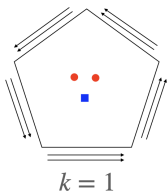
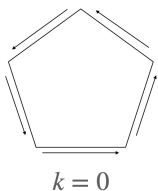
- We start from: $\forall \mathbf{v} \in \mathcal{N}^{k+1}(F) := \mathcal{RT}^{k+1}(F)^\perp, \forall q \in \mathcal{P}^k(F),$

$$\int_F \text{rot}_F \mathbf{v} \, q = \int_F \underbrace{\pi_{\mathcal{RT}, F}^k \mathbf{v}}_{\in \mathcal{RT}^k(F)} \cdot \underbrace{\text{rot}_F q}_{\in \mathcal{RT}^k(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\mathbf{v} \cdot \mathbf{t}_E)}_{\in \mathcal{P}^k(E)} q|_E$$

- This leads to the following discrete counterpart of $\mathbf{H}(\text{rot}; F)$:

$$\underline{\mathbf{X}}_{\text{curl}, F}^k := \left\{ \underline{\mathbf{v}}_F = (\mathbf{v}_F, (\mathbf{v}_E)_{E \in \mathcal{E}_F}) : \mathbf{v}_F \in \mathcal{RT}^k(F) \text{ and } \mathbf{v}_E \in \mathcal{P}^k(E) \forall E \in \mathcal{E}_F \right\}$$

- $\underline{\mathbf{I}}_{\text{rot}, F}^k : H^1(F)^2 \rightarrow \underline{\mathbf{X}}_{\text{curl}, F}^k$ is obtained collecting L^2 -orthogonal projections



The two-dimensional case

Reconstructions in $\underline{\mathbf{X}}_{\text{curl},F}^k$

- The **face curl operator** $C_F^k : \underline{\mathbf{X}}_{\text{curl},F}^k \rightarrow \mathcal{P}^k(F)$ is s.t.,

$$\int_F C_F^k \underline{\mathbf{v}}_F q = \int_F \mathbf{v}_F \cdot \text{rot}_F q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{v}_E q \quad \forall q \in \mathcal{P}^k(F)$$

- The **tangent trace** $\gamma_{t,F}^k : \underline{\mathbf{X}}_{\text{curl},F}^k \rightarrow \mathcal{P}^k(F)^2$ is s.t.,

$$\begin{aligned} \int_F \gamma_{t,F}^k \underline{\mathbf{v}}_F \cdot (\text{rot}_F r + \mathbf{w}) \\ = \int_F C_F^k \underline{\mathbf{v}}_F r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{v}_E r + \int_F \mathbf{v}_F \cdot \mathbf{w} \\ \forall (r, \mathbf{w}) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{c,k}(F) \end{aligned}$$

- We have the following **polynomial consistency**:

$$C_F^k(\underline{\mathbf{I}}_{\text{rot},F}^k \mathbf{v}) = \text{rot}_F \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{N}^{k+1}(F) \quad \text{and} \quad \gamma_{t,F}^k(\underline{\mathbf{I}}_{\text{rot},F}^k \mathbf{v}) = \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{P}^k(F)$$



Two-dimensional DDR complex

Space	V (vertex)	E (edge)	F (face)
$\underline{X}_{\text{grad},F}^k$	\mathbb{R}	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{X}_{\text{curl},F}^k$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

- Define the **discrete gradient**

$$\underline{G}_F^k q_F := (\pi_{\mathcal{RT},F}^k \mathbf{G}_F^k q_F, (G_E^k q_E)_{E \in \mathcal{E}_F})$$

- The **two-dimensional DDR complex** reads

$$\underline{X}_{\text{grad},F}^k \xrightarrow{\underline{G}_F^k} \underline{X}_{\text{curl},F}^k \xrightarrow{C_F^k} \mathcal{P}^k(F) \longrightarrow \{0\}$$

- If F is simply connected, this complex is **exact**

A glance at the general case I

- $\underline{X}_h^{k,r}$ spanned by vectors of polynomial components
- Recursive and hierarchical construction on d -cells, $d = r + 1, \dots, n$, of
 - A **discrete exterior derivative**

$$d_f^{k,r} : \underline{X}_f^{k,r} \rightarrow \mathcal{P}^k \Lambda^{r+1}(f)$$

- Based on it, an associated **discrete potential** ($\simeq k$ -form inside f)

$$P_f^{k,r} : \underline{X}_f^{k,r} \rightarrow \mathcal{P}^k \Lambda^r(f)$$

- Reconstructions mimic the **Stokes formula**: $\forall(\omega, \mu) \in \Lambda^\ell(f) \times \Lambda^{n-\ell-1}(f)$,

$$\int_f d^\ell \omega \wedge \mu = (-1)^{\ell+1} \int_f \omega \wedge d^{n-\ell-1} \mu + \int_{\partial f} \text{tr}_{\partial f} \omega \wedge \text{tr}_{\partial f} \mu$$

A glance at the general case II

- For a polytopal domain $\Omega \subset \mathbb{R}^n$ and a form degree r , the DDR space is

$$\underline{X}_h^{k,r} := \bigtimes_{d=r}^n \bigtimes_{f \in \Delta_d(\mathcal{T}_h)} \mathcal{P}^{-,k} \Lambda^{d-r}(f) \text{ with } \Delta_d(\mathcal{T}_h) := \{d\text{-faces of } \mathcal{T}_h\}$$

- We recursively define, for $f \in \Delta_d(\mathcal{T}_h)$, $d = r, \dots, n$,

- If $r = d$,

$$P_f^{k,d} \underline{\omega}_f := \star^{-1} \omega_f \in \mathcal{P}^k \Lambda^d(f)$$

- If $r+1 \leq d \leq n$, we first let, for all $\underline{\omega}_f \in \underline{X}_f^{k,r}$ and all $\mu \in \mathcal{P}^k \Lambda^{d-r-1}(f)$,

$$\int_f d_f^{k,r} \underline{\omega}_f \wedge \mu = (-1)^{r+1} \int_f \star^{-1} \omega_f \wedge d\mu + \int_{\partial f} P_{\partial f}^{k,r} \underline{\omega}_f \wedge \text{tr}_{\partial f} \mu$$

then, for all $(\mu, \nu) \in \kappa \mathcal{P}^{k,d-r}(f) \times \kappa \mathcal{P}^{k-1,d-r+1}(f)$,

$$\begin{aligned} (-1)^{r+1} \int_f P_f^{k,r} \underline{\omega}_f \wedge (d\mu + \nu) &= \int_f d_f^{k,r} \underline{\omega}_f \wedge \mu \\ &\quad - \int_{\partial f} P_{\partial f}^{k,r} \underline{\omega}_f \wedge \text{tr}_{\partial f} \mu + (-1)^{r+1} \int_f \star^{-1} \omega_f \wedge \nu \end{aligned}$$

A glance at the general case III

- The following **polynomial consistency properties** hold:

$$\begin{aligned} P_f^{k,r} I_f^{k,r} \omega &= \omega \quad \forall \omega \in \mathcal{P}^k \Lambda^r(f), \\ d_f^{k,r} I_f^{k,r} \omega &= d\omega \quad \forall \omega \in \mathcal{P}^{-,k+1} \Lambda^r(f) \end{aligned}$$

- Setting

$$\underline{d}_h^{k,r} \underline{\omega}_h := (\pi_f^{-,k,d-r-1}(\star d_f^{k,r} \underline{\omega}_f))_{f \in \Delta_d(\mathcal{T}_h), d \in [k+1, n]},$$

the **global DDR complex of differential forms** reads

$$\underline{X}_h^{k,0} \xrightarrow{\underline{d}_h^{k,0}} \underline{X}_h^{k,1} \longrightarrow \dots \longrightarrow \underline{X}_h^{k,n-1} \xrightarrow{\underline{d}_h^{k,n-1}} \underline{X}_h^{k,n} \longrightarrow \{0\}$$

A glance at the general case IV

For $n = 3$, vector proxies yield the DDR complex of [DP and Droniou, 2023a]:

$$\underline{X}_{\text{grad},T}^k \xrightarrow{\underline{G}_T^k} \underline{X}_{\text{curl},T}^k \xrightarrow{\underline{C}_T^k} \underline{X}_{\text{div},T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \longrightarrow \{0\}$$

Space	V	E	F	T (element)
$\underline{X}_T^{k,0} \cong \underline{X}_{\text{grad},T}^k$	\mathbb{R}	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}_T^{k,1} \cong \underline{X}_{\text{curl},T}^k$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$	$\mathcal{RT}^k(T)$
$\underline{X}_T^{k,2} \cong \underline{X}_{\text{div},T}^k$			$\mathcal{P}^k(F)$	$\mathcal{N}^k(T)$
$\underline{X}_T^{k,3} \cong \mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Commutation with the interpolators

Lemma (Local commutation properties)

The following diagrams commute:

$$\begin{array}{ccccccc}
 C^\infty(\bar{T}) & \xrightarrow{\text{grad}} & C^\infty(\bar{T})^3 & \xrightarrow{\text{curl}} & C^\infty(\bar{T})^3 & \xrightarrow{\text{div}} & C^\infty(\bar{T}) \longrightarrow \{0\} \\
 \downarrow I_{\text{grad},T}^k & & \downarrow I_{\text{curl},T}^k & & \downarrow I_{\text{div},T}^k & & \downarrow i_T \\
 \underline{X}_{\text{grad},T}^k & \xrightarrow{\underline{G}_T^k} & \underline{X}_{\text{curl},T}^k & \xrightarrow{\underline{C}_T^k} & \underline{X}_{\text{div},T}^k & \xrightarrow{D_T^k} & \mathcal{P}^k(T) \longrightarrow \{0\}
 \end{array}$$

- Crucial for both algebraic and analytical properties
- Compatibility of projections with **Helmholtz–Hodge decompositions**
 \implies robustness of DDR schemes with respect to the physics, e.g.:
 - Stokes [Beirão da Veiga, Dassi, DP, Droniou, 2022]
 - Navier–Stokes [DP, Droniou, Qian, 2023]
 - Reissner–Mindlin [DP and Droniou, 2023b]

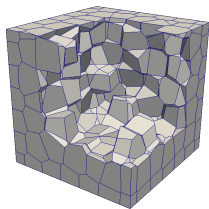
Local discrete L^2 -products

- Based on the element potentials, we construct **local discrete L^2 -products**

$$(\underline{x}_T, \underline{y}_T)_{\bullet, T} = \underbrace{\int_T P_{\bullet, T} \underline{x}_T \cdot P_{\bullet, T} \underline{y}_T}_{\text{consistency}} + \underbrace{s_{\bullet, T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\text{grad, curl, div}\}$$

- The L^2 -products are built to be **polynomially consistent**

Global DDR complex



$$\underline{X}_{\text{grad},h}^k \xrightarrow{\underline{G}_h^k} \underline{X}_{\text{curl},h}^k \xrightarrow{\underline{C}_h^k} \underline{X}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \longrightarrow \{0\}$$

- **Global DDR spaces** on a mesh \mathcal{T}_h are defined gluing boundary components
- **Global operators** are obtained collecting local components
- **Global L^2 -products** $(\cdot, \cdot)_{\bullet,h}$ are obtained assembling element-wise

Cohomology of the global three-dimensional DDR complex

$$\underline{X}_{\text{grad},h}^k \xrightarrow{\underline{G}_h^k} \underline{X}_{\text{curl},h}^k \xrightarrow{\underline{C}_h^k} \underline{X}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \longrightarrow \{0\}$$

Theorem (Cohomology of the 3D DDR complex [DP, Droniou, Pitassi, 2023])

For any $k \geq 0$, the DDR sequence forms a complex whose *cohomology spaces are isomorphic to those of the continuous de Rham complex*. In particular, if Ω has a trivial topology (i.e., $b_1 = b_2 = 0$), the DDR complex is *exact*, i.e.,

$$\text{Im } \underline{G}_h^k = \text{Ker } \underline{C}_h^k, \quad \text{Im } \underline{C}_h^k = \text{Ker } D_h^k, \quad \text{Im } D_h^k = \mathcal{P}^k(\mathcal{T}_h).$$

Remark (Extension to PEC [Bonaldi, DP, Droniou, Hu, 2023])

The above result extends to the de Rham complex of differential forms.

- 1 Three model problems and their well-posedness
- 2 Discrete de Rham (DDR) complexes
- 3 Application to magnetostatics**

Uniform discrete Poincaré inequality for the curl

Theorem (Poincaré inequality for the curl [DP and Hanot, 2024])

For all $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k$, it holds

$$\inf_{\underline{\mathbf{z}}_h \in \text{Ker } \underline{\mathbf{C}}_h^k} \|\underline{\mathbf{v}}_h - \underline{\mathbf{z}}_h\|_{\text{curl},h} \lesssim \|\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h\|_{\text{div},h},$$

with hidden constant only depending on Ω , mesh regularity, and k .

This results holds for domains of general topology!

Adjoint consistency of the discrete curl

Adjoint consistency measures the failure to satisfy a global IBP. For the curl,

$$\int_{\Omega} \mathbf{w} \cdot \mathbf{curl} \mathbf{v} - \int_{\Omega} \mathbf{curl} \mathbf{w} \cdot \mathbf{v} = 0 \text{ if } \mathbf{w} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega$$

Theorem (Adjoint consistency for the curl)

Let $\mathcal{E}_{\text{curl},h} : (C^0(\overline{\Omega})^3 \cap \mathbf{H}_0(\mathbf{curl}; \Omega)) \times \underline{\mathbf{X}}_{\text{curl},h}^k \rightarrow \mathbb{R}$ be s.t.

$$\mathcal{E}_{\text{curl},h}(\mathbf{w}, \underline{\mathbf{v}}_h) := (\underline{\mathbf{I}}_{\text{div},h}^k \mathbf{w}, \underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h)_{\text{div},h} - \int_{\Omega} \mathbf{curl} \mathbf{w} \cdot \mathbf{P}_{\text{curl},h}^k \underline{\mathbf{v}}_h.$$

Then, for all $\mathbf{w} \in C^0(\overline{\Omega})^3 \cap \mathbf{H}_0(\mathbf{curl}; \Omega)$ s.t. $\mathbf{w} \in H^{k+2}(\mathcal{T}_h)^3: \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k$,

$$|\mathcal{E}_{\text{curl},h}(\mathbf{w}, \underline{\mathbf{v}}_h)| \lesssim h^{k+1} \left(\|\underline{\mathbf{v}}_h\|_{\text{curl},h} + \|\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h\|_{\text{div},h} \right).$$

DDR scheme

- Assume $b_2 = 0$. We seek $(\mathbf{H}, \mathbf{A}) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}(\text{div}; \Omega)$ s.t.

$$\int_{\Omega} \mu \mathbf{H} \cdot \boldsymbol{\tau} - \int_{\Omega} \mathbf{A} \cdot \text{curl } \boldsymbol{\tau} = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\text{curl}; \Omega),$$

$$\int_{\Omega} \text{curl } \mathbf{H} \cdot \mathbf{v} + \int_{\Omega} \text{div } \mathbf{A} \text{ div } \mathbf{v} = \int_{\Omega} \mathbf{J} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{H}(\text{div}; \Omega)$$

- With obvious substitutions: Find $(\underline{\mathbf{H}}_h, \underline{\mathbf{A}}_h) \in \underline{\mathbf{X}}_{\text{curl},h}^k \times \underline{\mathbf{X}}_{\text{div},h}^k$ s.t.

$$(\mu \underline{\mathbf{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\text{curl},h} - (\underline{\mathbf{A}}_h, \underline{\mathbf{C}}_h^k \underline{\boldsymbol{\tau}}_h)_{\text{div},h} = 0 \quad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k,$$

$$(\underline{\mathbf{C}}_h^k \underline{\mathbf{H}}_h, \underline{\mathbf{v}}_h)_{\text{div},h} + \int_{\Omega} D_h^k \underline{\mathbf{A}}_h D_h^k \underline{\mathbf{v}}_h = l_h(\underline{\mathbf{v}}_h) \quad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{div},h}^k$$

- If $b_2 \neq 0$, we need to add **orthogonality of $\underline{\mathbf{A}}_h$ to harmonic forms**

$$\underline{\mathfrak{H}}_{\text{div},h}^k := \left\{ \underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\text{div},h}^k : D_h^k \underline{\mathbf{w}}_h = 0 \text{ and } (\underline{\mathbf{w}}_h, \underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h)_{\text{div},h} = 0 \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k \right\}$$

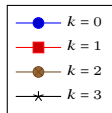
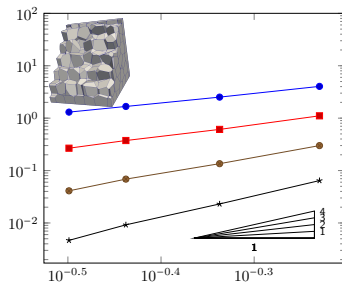
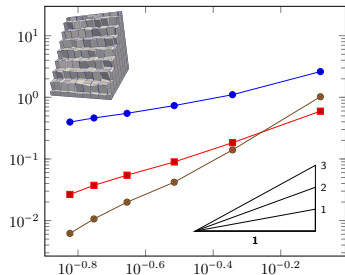
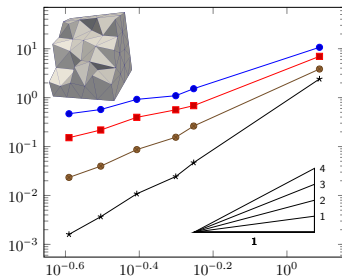
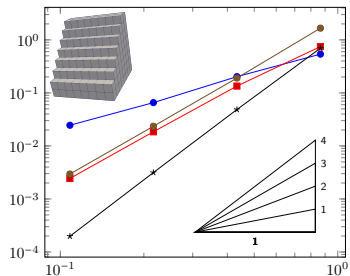
- **Inf-sup stability** is proved as in the continuous case for the norm

$$\|(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)\|_h := \left(\|\underline{\boldsymbol{\tau}}_h\|_{\text{curl},h}^2 + \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h\|_{\text{div},h}^2 + \|\underline{\boldsymbol{v}}_h\|_{\text{div},h}^2 + \|D_h^k \underline{\boldsymbol{v}}_h\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

- Crucial points are the **isomorphism in cohomology** and **Poincaré inequality**
- Assuming $\boldsymbol{H} \in C^0(\overline{\Omega})^3 \cap H^{k+2}(\mathcal{T}_h)^3$ and $\boldsymbol{A} \in C^0(\overline{\Omega})^3 \times H^{k+2}(\mathcal{T}_h)^3$, it holds

$$\|(\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}_{\text{curl},h}^k \boldsymbol{H}, \underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}_{\text{div},h}^k \boldsymbol{A})\|_h \lesssim h^{k+1}$$

Numerical examples (energy error vs. meshsize)



Conclusions and perspectives

- **Fully discrete approach** for PDEs relating to the de Rham complex
- **Key features:** support of general polyhedral meshes and high-order
- **Novel computational strategies** made possible
- Natural extensions to **differential forms**

- Unified proof of **analytical properties** using differential forms
- Development of **novel complexes** (e.g., elasticity, Hessian, . . .)
- Applications (possibly beyond continuum mechanics)

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