Cell centered Galerkin methods for diffusive problems on general meshes

Daniele A. Di Pietro

IFP Energies nouvelles, dipietrd@ifpen.fr

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Outline

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General meshes I

- Avoid remeshing (e.g. in subsoil modeling)
- Improve domain/solution fitting
- Improve performance (fewer DOFs, reduced fill-in)
- Nonconforming/aggregative mesh adaptivity

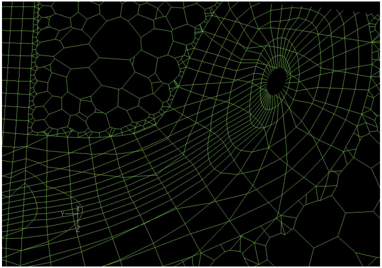


Figure: Near wellbore mesh

General meshes II

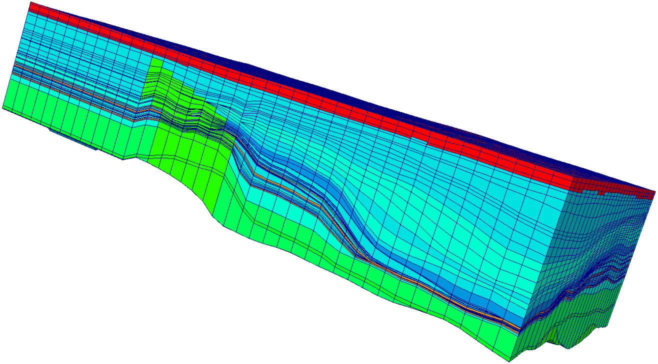


Figure: Stratigraphic mesh of a sedimentary bassin

Admissible mesh sequences for h-convergence I

- Let $\Omega \subset \mathbb{R}^d$ be an open connected bounded polyhedral domain
- Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a sequence of refined meshes of Ω
- For $k \geqslant 0$ we define the broken polynomial spaces

$$\mathbb{P}_d^k(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v_{|T} \in \mathbb{P}_d^k(T) \right\}$$

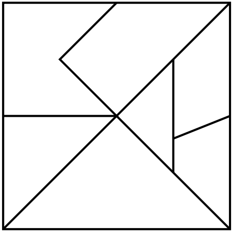


Figure: Mesh $\mathcal{T}_h$ with polygonal elements and nonmatching interfaces

Admissible mesh sequences for h-convergence II

Assumption (Trace and inverse inequalities)

- *Every* $\mathcal{T}_h$ *admits a simplicial submesh* $\mathfrak{S}_h$
- $(\mathfrak{S}_h)_{h \in \mathcal{H}}$ *is shape-regular in the sense of Ciarlet*
- *Every simplex* $S \subset T$ *is s.t.* $h_S \approx h_T$

Assumption (Optimal polynomial approximation)

Every element T *is star-shaped w.r.t. a ball of diameter* $\delta_T \approx h_T$

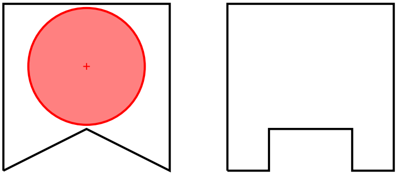


Figure: Admissible (*left*) and non-admissible (*right*) mesh elements

$\|\cdot\|_{dG,p}$-norms

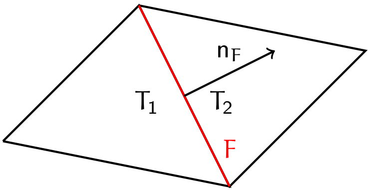


Figure: Notation for an interface $F \in \mathcal{F}_h^i$

- For $F \subset \partial T_1 \cap \partial T_2$ let

$$\{v\} := \frac{1}{2}\left(v_{|T_1} + v_{|T_2}\right), \qquad [\![v]\!] := v_{|T_1} - v_{|T_2}$$

- We introduce the discrete $W^{1,p}(\mathcal{T}_h)$-norms

$$\|v\|_{dG,p} := \left(\|\nabla_h v\|^p_{L^p(\Omega)^d} + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \|[\![v]\!]\|^p_{L^p(F)} \right)^{1/p}$$

Sobolev embeddings in $\mathbb{P}_d^k(\mathcal{T}_h)$-spaces I

Theorem (Discrete Sobolev embeddings [Di Pietro and Ern, 2010])

Let $k \geqslant 0$. *For all* q *such that*

- $1 \leqslant q \leqslant p^* := \frac{pd}{d-p}$ *if* $1 \leqslant p < d$
- $1 \leqslant q < \infty$ *if* $d \leqslant p < \infty$

there exists $\sigma_{p,q}$ *such that*

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h), \qquad \|v_h\|_{L^q(\Omega)} \leqslant \sigma_{p,q} \|v_h\|_{dG,p}$$

Proof.

- For $p = 1$ use $\|v_h\|_{L^{1*}(\Omega)} \lesssim \|v_h\|_{\mathrm{BV}} \lesssim \|v_h\|_{\mathrm{dG},1}$
- For $1 < p < d$ use L^{1*}-estimate for $|v_h|^\alpha$, Hölder's and trace inequalities
- For $d \leqslant p < \infty$ use the previous point together with the comparison of broken $W^{1,p}(\mathcal{T}_h)$-norms $\square$

Sobolev embeddings in $\mathbb{P}_d^k(\mathcal{T}_h)$-spaces II

- In the Hilbertian case $p = 2$ we have the usual

$$\|v\|_{\mathrm{dG}} := \left(\|\nabla_h v\|_{L^2(\Omega)^d}^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|_{L^2(F)}^2 \right)^{1/2}$$

- An important Sobolev embedding is the Poincaré inquality

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h) \qquad \|v_h\|_{L^2(\Omega)} \leqslant \sigma_{2,2} \|v_h\|_{\mathrm{dG}}.$$

Outline

Motivations and goals

- Darcy flow through heterogeneous anisotropic media
  - [Di Pietro and Ern, 2011a]
- Convergence to nonsmooth solutions in faulted media
  - [Di Pietro and Ern, 2011b]
- Darcy flow through deformable porous media (not detailed)
  - [Di Pietro, 2011]
- Reactive transport with singular interfaces (not detailed)
  - [Gastaldi and Quarteroni, 1989]
  - [Di Pietro et al., 2008]
- Important references for weighted averages
  - [Stenberg, 1998]
  - [Hansbo and Hansbo, 2002]
  - [Heinrich and Pietsch, 2002, Heinrich and Nicaise, 2003]
  - [Burman and Zunino, 2006]

The heterogeneous Darcy problem I

$$-\nabla\cdot(\kappa\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- There is a partition P_Ω s.t.

$$\kappa \in \mathbb{P}^0_d(P_\Omega) \text{ with } 0 < \underline{\kappa} \leqslant \kappa \leqslant \overline{\kappa}$$

- For all $h \in \mathcal{H}$, $\mathcal{T}_h$ is compatible with P_Ω
- We seek an approximate solution $u_h \in V_h$ with

$$V_h := \mathbb{P}^k_d(\mathcal{T}_h), \qquad k \geqslant 1$$

Find $u_h \in V_h$ s.t. $a_h(u_h, v_h) = \int_\Omega f v_h$ for all $v_h \in V_h$

The heterogeneous Darcy problem II

$$a_h^{\text{sip}}(w, v_h) := \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h w\} \cdot n_F [\![v_h]\!]$$

$$- \sum_{F \in \mathcal{F}_h} \int_F [\![w]\!] \{\kappa \nabla_h v_h\} \cdot n_F + \sum_{F \in \mathcal{F}_h} \int_F \frac{\eta}{h_F} [\![w]\!] [\![v_h]\!]$$

Theorem (Error estimate [Arnold, 1982])

Assume $u \in V_* := H_0^1(\Omega) \cap H^2(P_\Omega)$. *Then,* $\exists C \neq C(h, \kappa)$ *s.t.*

$$\|u - u_h\|_{dG} \leqslant C \max\left(1, \frac{\overline{\kappa}}{\underline{\kappa}}\right) \inf_{v_h \in V_h} \|u - v_h\|_{dG,*}$$

This estimate is not robust w.r.t. the heterogeneity of κ

The SWIP method I

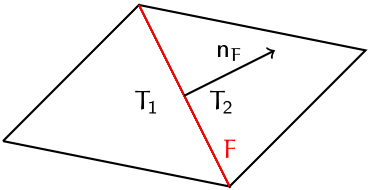


Figure: Notation for an interface $F \in \mathcal{F}_h^i$

- For $F \subset \partial T_1 \cap \partial T_2$ and $(\omega_1, \omega_2) > 0$, $\omega_1 + \omega_2 = 1$ let

$$\{v\}_\omega := \omega_1 v_{|T_1} + \omega_2 v_{|T_2}$$

- For $\omega_1 = \omega_2 = \frac{1}{2}$ we recover the standard average $\{v\}$

The SWIP method II

$$a_h^{\text{swip}}(w, v_h) := \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h w\}_{\omega_\kappa} \cdot n_F [\![v_h]\!]$$

$$- \sum_{F \in \mathcal{F}_h} \int_F [\![w]\!] \{\kappa \nabla_h v_h\}_{\omega_\kappa} \cdot n_F + \sum_{F \in \mathcal{F}_h} \int_F \eta \frac{\gamma_\kappa}{h_F} [\![w]\!] [\![v_h]\!]$$

- Weighted averages + harmonic mean in penalty

$$\{\Phi\}_{\omega_\kappa} := \frac{\kappa_2}{\kappa_1+\kappa_2} \Phi_{|T_1} + \frac{\kappa_1}{\kappa_1+\kappa_2} \Phi_{|T_2}, \quad \gamma_\kappa := 2 \frac{\kappa_1 \kappa_2}{\kappa_1+\kappa_2}$$

- Data-dependent energy norm on $H^1(\mathcal{T}_h)$

$$|\!|\!| v |\!|\!|_\kappa^2 := \| \kappa^{\frac{1}{2}} \nabla_h v \|_{L^2(\Omega)^d}^2 + \sum_{F \in \mathcal{F}_h} \frac{\gamma_\kappa}{h_F} \| [\![v]\!] \|_{L^2(F)}^2$$

The SWIP method III

Lemma (Properties of a_h^{swip} [Di Pietro and Ern, 2011b])

Let $V_{*h} := V_h + V_*$ *and assume* $u \in V_*$. *Then,*

- *Consistency. There holds*

$$\forall v_h \in V_h, \qquad a_h^{\text{swip}}(u, v_h) = \int_\Omega f v_h,$$

- *Coercivity. There exists* $C_{sta} \neq C_{sta}(h, \kappa)$ *s.t.*

$$\forall v_h \in V_h, \qquad a_h^{\text{swip}}(v_h, v_h) \geqslant C_{sta} |||v_h|||_\kappa^2$$

- *Boundedness. There exists* $C_{bnd} \neq C_{bnd}(h, \kappa)$ *s.t.*

$$\forall (w, v_h) \in V_{*h} \times V_h^{ccg}, \qquad a_h^{\text{swip}}(w, v_h) \leqslant C_{bnd} |||w|||_{\kappa,*} |||v_h|||_\kappa.$$

The SWIP method IV

Theorem (Error estimate [Di Pietro et al., 2008])

Assume $u \in V_* = H_0^1(\Omega) \cap H^2(P_\Omega)$. *Then,* $\exists C \neq C(h,\kappa)$ *s.t.*

$$|||u - u_h|||_\kappa \leqslant C \inf_{v_h \in V_h} |||u - v_h|||_{\kappa,*}$$

Corollary (Convergence rate)

If, moreover $u \in H^{k+1}(P_\Omega)$, $\exists C \neq C(h,\kappa)$ *s.t.*

$$|||u - u_h|||_\kappa \lesssim C\overline{\kappa}^{1/2} h^k \|u\|_{H^{k+1}(P_\Omega)}.$$

- Non convergent for $k = 0$ except on κ-orthogonal $\mathcal{T}_h$
- Minor modifications allow to treat the case

$$u \in H_0^1(\Omega) \cap H^{3/2+\epsilon}(P_\Omega)$$

Convergence of the SWIP method to nonsmooth solutions

- However, in general we only have [Nicaise and Sändig, 1994]

$$u \in W^{2,p}(P_\Omega) \Rightarrow u \in H^{1+\alpha}(P_\Omega), \quad \alpha = 1+d\left(\frac{1}{2}-\frac{1}{p}\right) > 0$$

- Optimal convergence rates for $d = 2$ [Di Pietro and Ern, 2011a]
- Convergence by compactness for $d > 2$

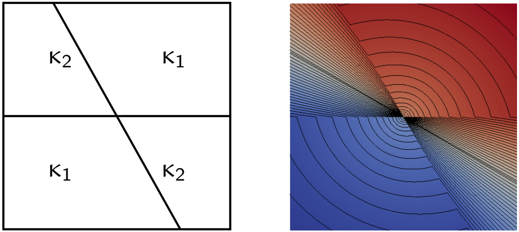


Figure: Faulted medium, $u \in H^{1.29}(P_\Omega)$, $\kappa_1/\kappa_2 = 30$

Discrete compactness I

- For $F \in \mathcal{F}_h$ and $l \geqslant 0$ the local lifting solves

$$\int_\Omega r^l_{\omega,F}([\![v]\!]) \cdot \tau_h = \int_F [\![v]\!] \{\tau_h\}_\omega \cdot n_F \qquad \forall \tau_h \in \mathbb{P}^l_d(\mathcal{T}_h)^d$$

- Following [Bassi and Rebay, 1997], we define

$$R^l_{h,\omega}(v) := \sum_{F \in \mathcal{F}_h} r^l_{\omega,F}([\![v]\!])$$

- For all $l \geqslant 0$ we define the gradient

$$G^l_{h,\omega}(v) := \nabla_h v - R^l_{h,\omega}(v)$$

- The subscript ω is omitted if $\omega_1 = \omega_2 = 1/2$

Discrete compactness II

Theorem (Compactness [Di Pietro and Ern, 2010])

Let $(v_h)_{h\in\mathcal{H}}$ *be a sequence in* $(\mathbb{P}_d^k(\mathcal{T}_h))_{h\in\mathcal{H}}$, $k \geqslant 0$

$$\forall h \in \mathcal{H}, \qquad \|v_h\|_{dG} \leqslant C \neq C(h).$$

Then, $\exists v \in H_0^1(\Omega)$ *s.t., as* $h \to 0$, *up to a subsequence*

$$\begin{aligned} v_h &\to v && \textit{in } L^2(\Omega), \\ G_h^l(v_h) &\rightharpoonup \nabla v && \textit{for all } l \geqslant 0 \textit{ weakly in } L^2(\Omega)^d. \end{aligned}$$

Proof.

- Kolmogorov criterion to prove compactness in $L^1(\Omega)$
- Sobolev embeddings to prove compactness in $L^2(\Omega)$
- Asymptotic consistency of G_h^l yields regularity of the limit □

Convergence to minimal regularity solutions

Theorem (Convergence [Di Pietro and Ern, 2011a])

Let $(u_h)_{h\in\mathcal{H}}$ *denote the sequence of discrete solutions on the admissible mesh family* $(\mathcal{T}_h)_{h\in\mathcal{H}}$*. Then,*

$$\begin{aligned} u_h &\to u && \textit{strongly in } L^2(\Omega), \\ \nabla_h u_h &\to \nabla u && \textit{strongly in } [L^2(\Omega)]^d, \\ |u_h|_J &\to 0. \end{aligned}$$

Proof.

Use the equivalent form for a_h^{swip}: For $l \in \{k-1, k\}$,

$$a_h^{\mathrm{swip}}(u_h, v_h) = \int_\Omega \kappa G_{h,\omega_\kappa}^l(u_h)\cdot G_{h,\omega_\kappa}^l(v_h) + s_h(u_h, v_h),$$

with $s_h(\cdot,\cdot) \geqslant 0$. □

Outline

Motivations and goals

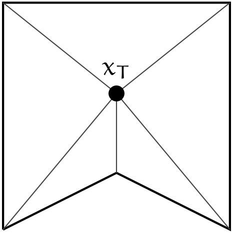
- Design consistent dG methods with 1 DOF per element
- Work on general polyhedral meshes as in dG methods
- Formulation of FV and lowest-order methods suitable for `FreeFEM`-like implementation
- See [Di Pietro, 2010, Di Pietro, 2012] and also [Botti and Di Pietro, 2011]
- Important references
  - [Aavatsmark *et al.*, 1994–11]
  - [Edwards *et al.*, 1994–11]
  - [Eymard, Gallouët, Herbin *et al.*, 2000–11]
  - [Brezzi, Lipnikov, Shashkov *et al.*, 2005–11]

Admissible mesh sequences

Cell centers

There exists a set of points $(x_T)_{T\in\mathcal{T}_h}$ s.t.

- all $T \in \mathcal{T}_h$ is star-shaped w.r.t. x_T
- for all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$, $\mathrm{dist}(x_T, F) \approx h_T$



Discrete space

1) Fix the vector space of DOFs, e.g.,

$$\mathbb{V}_h = \mathbb{R}^{\mathcal{T}_h}, \qquad \mathbf{v}_h = (v_T)_{T\in\mathcal{T}_h} \in \mathbb{R}^{\mathcal{T}_h}$$

2) Reconstruct an asymptotically consistent gradient

$$\mathfrak{G}_h : \mathbb{V}_h \to \mathbb{P}^0_d(\mathcal{T}_h)^d$$

3) Reconstruct a broken affine function

$$\forall T \in \mathcal{T}_h, \quad \mathfrak{R}_h(\mathbf{v}_h)_{|T}(x) = v_T + \mathfrak{G}_h(\mathbf{v}_h)_{|T}\cdot(x - x_T)$$

Use as a discrete space $V_h^{\mathrm{ccg}} := \mathfrak{R}_h(\mathbb{V}_h) \subset \mathbb{P}^1_d(\mathcal{T}_h)$

Application to heterogeneous diffusion

$$\text{Find } u_h \in V_h^{\text{ccg}} \text{ s.t. for all } v_h \in V_h^{\text{ccg}} \; a_h^{\text{swip}}(u_h, v_h) = \int_\Omega f v_h$$

- Consistency, coercivity, and boundedness hold *a fortiori* since

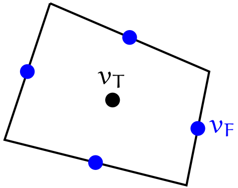
$$V_h^{\text{ccg}} \subset \mathbb{P}_d^1(\mathcal{T}_h)$$

- Fewer DOFs since

$$\dim(V_h^{\text{ccg}}) = \dim(\mathbb{P}_d^0(\mathcal{T}_h))$$

- Optimal convergence rate for $u \in H^2(P_\Omega)$
- Aubin–Nitsche trick yields optimal L^2-convergence

A gradient reconstruction based on Green's formula



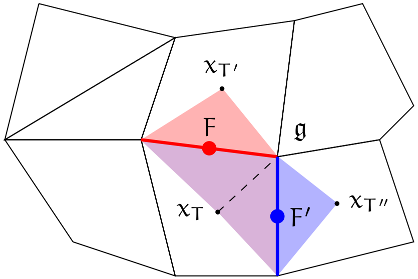
- Observe that, for all $v_h \in \mathbb{P}^0_d(\mathcal{T}_h)$ and all $T \in \mathcal{T}_h$,

$$G^0_h(v_h)_{|T} = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (\{v\} - v_T) n_{T,F}$$

- Let $(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}}) \in \mathbb{R}^{\mathcal{T}_h} \times \mathbb{R}^{\mathcal{F}_h}$. For all $T \in \mathcal{T}_h$ we set

$$\mathfrak{G}_h(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}})_{|T} = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (v_F - v_T) n_{T,F}$$

Trace interpolation: The L-construction I



For a group $\mathfrak{g} = \{F, F'\}$ and $\mathbf{v}_h \in \mathbb{V}_h$ we construct $\xi^{\mathfrak{g}}_{\mathbf{v}_h}$ s.t.

- $\xi^{\mathfrak{g}}_{\mathbf{v}_h}$ is affine in each coloured patch
- $\xi^{\mathfrak{g}}_{\mathbf{v}_h}(x_K) = v_K$ for all $K \in \{T, T', T''\}$
- $\xi^{\mathfrak{g}}_{\mathbf{v}_h}$ is continuous and has continuous flux across F and F′

Trace interpolation: The L-construction II

- The L-construction requires to solve a local system
- Examples of backup strategies if the system is not invertible
  - Barycentric interpolation
  - Full $\mathbb{P}^1_{\mathrm{d}}$ basis
- If more L-constructions are available, select the one best approximation properties

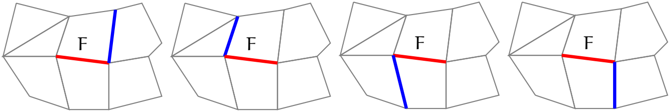


Figure: Groups $\mathfrak{g}$ containing F

Convergence to smooth solutions, heterogeneous case

Test space $\mathcal{Q}_{\mathcal{T}_h,\kappa}$ [Agélas, DP & Droniou, 2010]

Let $\mathcal{Q}_{\mathcal{T}_h,\kappa}$ be the space of functions $\varphi : \overline{\Omega} \to \mathbb{R}$ s.t.

(i) $\varphi \in C_0(\overline{\Omega}) \cap C^2(\mathcal{T}_h)$

(ii) the tangential derivatives of φ are continuous across $F \in \mathcal{F}_h^i$

(iii) the diffusive flux of φ is continuous across every $F \in \mathcal{F}_h^i$, i.e.

$$\forall F \subset \partial T_1 \cap \partial T_2, \qquad (\kappa \nabla \varphi)_{|T_1} \cdot n_F = (\kappa \nabla \varphi)_{|T_2} \cdot n_F.$$

Then, $\mathcal{Q}_{\mathcal{T}_h,\kappa}$ is dense in $H_0^1(\Omega)$.

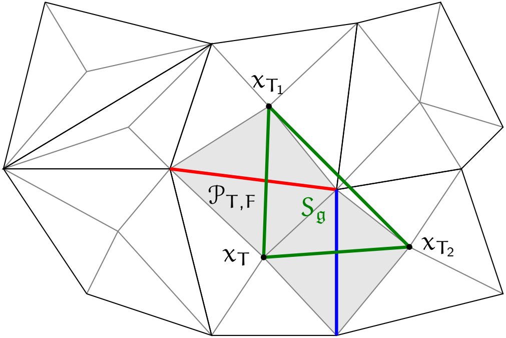
Corollary (Convergence rate, heterogeneous case)

Further assuming that $u \in \mathcal{Q}_{\mathcal{T}_h,\kappa}$, *there exists* $C \neq C(h)$ *s.t.*

$$|||u - u_h|||_\kappa \leqslant Ch \|u\|_{C^2(\mathcal{T}_h)}.$$

Convergence to smooth solutions, homogeneous case I

$$\textcolor{red}{\mathcal{P}_{\mathfrak{g}}} = \bigcup_{F\in\mathfrak{g},\, T\in\mathcal{T}_F} \mathcal{P}_{T,F} = \text{shaded region}$$



$\kappa = 1_d \Rightarrow \xi^{\mathfrak{g}_F}_{\mathbf{v}_h}$ is the Lagrange interpolate of v on $S_{\mathfrak{g}}$

Convergence to smooth solutions, homogeneous case II

Assumption (Approximation for the L-construction)

For $d \in \{2, 3\}$ *and* $\kappa = 1_d$, *there is* $C \neq C(h)$ h *s.t.* $\forall v \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\|v - \xi_{\mathbf{v}_h}^{\mathfrak{g}_F}\|_{L^2(\mathcal{P}_{\mathfrak{g}_F})} + h_{\mathcal{P}_{\mathfrak{g}_F}} |v - \xi_{\mathbf{v}_h}^{\mathfrak{g}_F}|_{H^1(\mathcal{P}_{\mathfrak{g}_F})} \leqslant C h^2_{\mathcal{P}_{\mathfrak{g}_F}} |v|_{H^2(\mathcal{P}_{\mathfrak{g}_F})}$$

Verified, e.g., by uniformly refined simplicial or semi-conformingly adapted Cartesian orthogonal meshes (use Deny–Lions Lemma)

Convergence to smooth solutions, homogeneous case III

Lemma (Approximation properties of V_h^{ccg}, $\kappa = 1_d$)

There exists $C \neq C(h)$ *s.t.*

$$\forall v \in H_0^1(\Omega) \cap H^2(\Omega), \qquad |||v - \mathfrak{R}_h(\mathbf{v}_h)|||_{\kappa,*} \leqslant Ch\|v\|_{H^2(\Omega)},$$

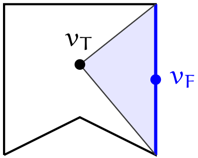
where $\mathbf{v}_h = \mathbf{I}_h(v) := (v(x_T))_{T \in \mathcal{T}_h} \in \mathbb{V}_h$.

Corollary (Convergence rates, $\kappa = 1_d$)

If the exact solution $u \in V_* := H_0^1(\Omega) \cap H^2(\Omega)$, *then*

$$\|u - u_h\|_{L^2(\Omega)} + h|||u - u_h||| \leqslant Ch^2\|u\|_{H^2(\Omega)}.$$

Stabilization using residuals



- Following [Eymard et al., 2010] define

$$\mathfrak{r}_h(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}})_{|\mathcal{P}_{T,F}} = \frac{\sqrt{d}}{d_{T,F}} \left[v_F - \left(v_T + \mathfrak{G}_h(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}}) \cdot (\overline{x}_F - \mathbf{x}_T) \right) \right] \mathsf{n}_{T,F}$$

- We introduce the stabilized gradient

$$\mathfrak{G}_h^{\mathsf{hyb}}(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}}) = \mathfrak{G}_h(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}}) + \mathfrak{r}_h(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}})$$

The L^2-norm of $\mathfrak{G}_h^{\mathsf{hyb}}$ is a norm on general polyhedral meshes

The SUSHI scheme with hybrid unknowns I

Find $u_h \in V_h^{\text{hyb}}$ with $V_h^{\text{hyb}} \subset \mathbb{P}_d^1(\mathcal{P}_h)$ defined from $\mathfrak{G}_h^{\text{hyb}}$ s.t.

$$\int_\Omega \kappa \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h \quad \forall v_h \in V_h^{\text{hyb}}$$

Theorem (Convergence [Eymard et al., 2010])

Let $(u_h)_{h\in\mathcal{H}}$ denote the sequence of discrete solutions on the admissible mesh family $(\mathcal{T}_h)_{h\in\mathcal{H}}$. Then, $P_0 u_h \to u$ in $L^2(\Omega)$ and $\nabla_h u_h \to u$ in $L^2(\Omega)^d$.

Generalization of the Crouzeix–Raviart FE to non-simplicial meshes

Pure diffusion I

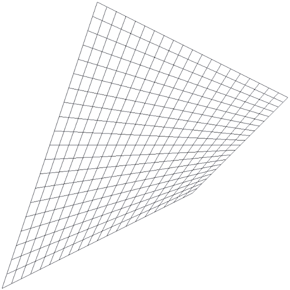
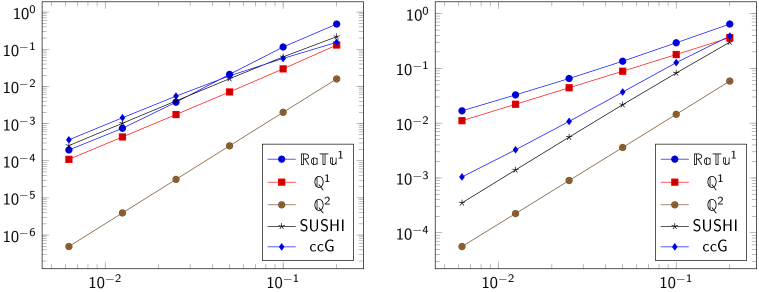


Figure: Skewed quadrangular mesh

$$u(\mathbf{x}) = \sin(\pi x_1)\cos(\pi x_2), \qquad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

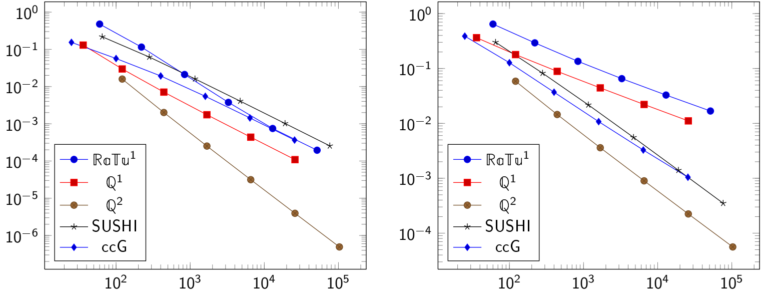
Pure diffusion II



(a) L^2-error vs. h (b) H^1-error vs. h

Figure: Error as a function of the meshsize

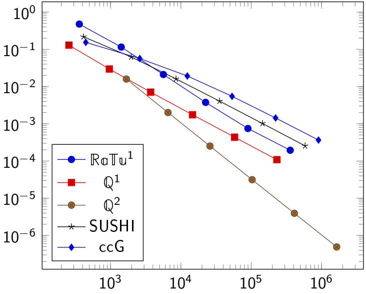
Pure diffusion III



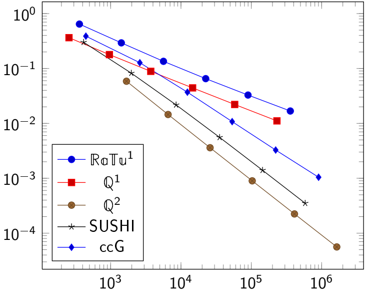
(a) L^2-error vs. N_{DOF} (b) H^1-error vs. N_{DOF}

Figure: Error as a function of the number of DOFs

Pure diffusion IV



(a) L^2-error vs. N_{nz}



(b) H^1-error vs. N_{nz}

Figure: Error as a function of matrix fill-in

Pure diffusion V

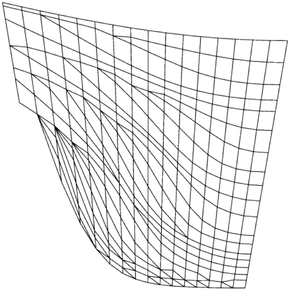
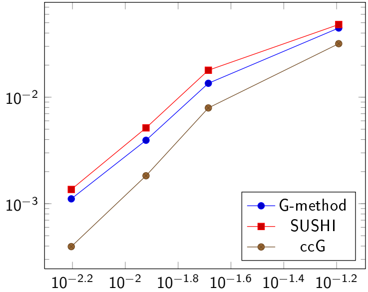


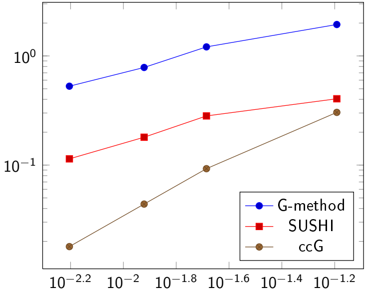
Figure: Stratigrafic mesh. Actual aspect ratio is 10:1

$$u(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2), \qquad \kappa = \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix}$$

Pure diffusion VI



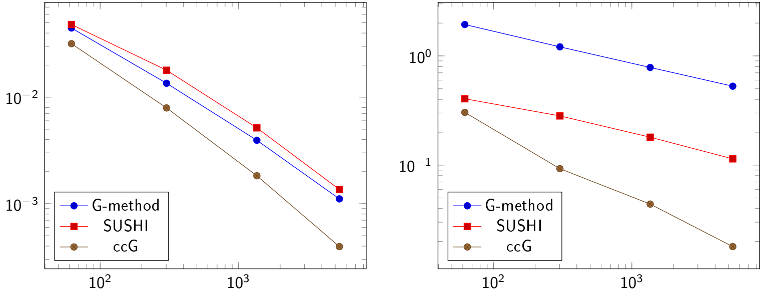
(a) L^2-error vs. h



(b) H^1-error vs. h

Figure: Error as a function of the meshsize

Pure diffusion VII



(a) L^2-error vs. N_{DOF} (b) H^1-error vs. N_{DOF}

Figure: Error as a function of the number of DOFs

Pure diffusion VIII

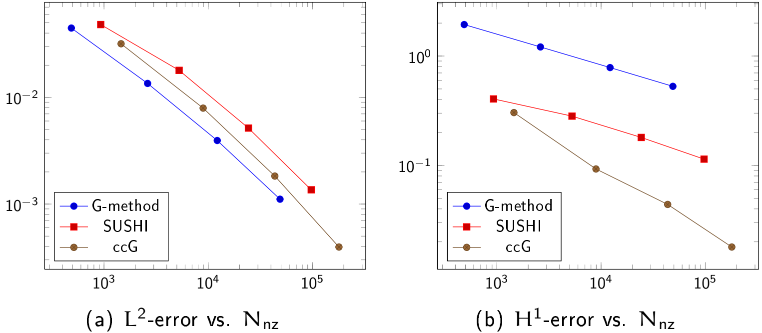


Figure: Error as a function of matrix fill-in

Incompressible Navier–Stokes equations I

$$\begin{aligned} -\nu\triangle u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \Omega, \\ \nabla\cdot u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \\ \langle p\rangle_\Omega &= 0. \end{aligned}$$

- We consider a discretization based on the following spaces:

$$U_h := [V_h^{\mathrm{ccg}}]^d, \quad P_h := \mathbb{P}_d^0(\mathcal{T}_h)/\mathbb{R}$$

- The discrete problem reads: For all $(v_h, q_h) \in U_h \times P_h$,

$$\begin{aligned} a_h^{\mathrm{swip}}(u_h, v_h) + t_h(u_h, u_h, v_h) + b_h(v_h, p_h) &= \int_\Omega f\cdot v_h \\ -b_h(u_h, q_h) + s_h(p_h, q_h) &= 0 \end{aligned}$$

Incompressible Navier–Stokes equations II

- The velocity-pressure coupling is realized by the bilinear form

$$b_h(v_h, q_h) := -\sum_{F\in\mathcal{F}_h^i}\int_\Omega \{v_h\}\cdot n_F[\![q_h]\!]$$

$$= -\int_\Omega q_h \nabla_h\cdot v_h + \sum_{F\in\mathcal{F}_h}\int_F [\![v_h]\!]\cdot n_F\{q_h\}$$

- Pressure stabilization ensures inf-sup stability

$$s_h(p_h, q_h) := \sum_{F\in\mathcal{F}_h^i}\int_F h_F[\![p_h]\!][\![q_h]\!], \qquad |q_h|_p^2 = s_h(q_h, q_h)$$

Incompressible Navier–Stokes equations III

$$t_h(w,u,v) := \int_\Omega (w\cdot\nabla_h u_i) v_i - \sum_{F\in\mathcal{F}_h^i} \int_F \{w\}\cdot n_F [\![u]\!]\cdot\{v\}$$
$$+\frac{1}{2}\int_\Omega (\nabla_h\cdot w)(u\cdot v) - \frac{1}{2}\sum_{F\in\mathcal{F}_h}\int_F [\![w]\!]\cdot n_F\{u\cdot v\}$$

- Extension of Temam's device to broken spaces
- Non-dissipative since

$$t_h(v_h,v_h,v_h) = 0 \qquad \forall v_h \in U_h$$

- Asymptotically consistent for smooth and discrete functions

Incompressible Navier–Stokes equations IV

Lemma (Alternative expression for t_h)

For all w_h, u_h, $v_h \in U_h$ *there holds*

$$t_h(w_h, u_h, v_h) = \int_\Omega w_h \cdot \mathcal{G}_h^2(u_{h,i}) v_{h,i} + \frac{1}{2} \int_\Omega D_h^2(w_h)(u_h \cdot v_h) + \frac{1}{4} \sum_{F \in \mathcal{F}_h^i} \int_F ([\![w_h]\!] \cdot n_F)([\![u_h]\!] \cdot [\![v_h]\!]).$$

Incompressible Navier–Stokes equations V

Lemma (Existence of a discrete solution)

There exists at least one discrete solution $(u_h, p_h) \in X_h$.

Theorem (Convergence)

Let $((u_h, p_h))_{h\in\mathcal{H}}$ *be a sequence of approximate solutions on* $(\mathcal{T}_h)_{h\in\mathcal{H}}$. *Then, as* $h \to 0$, *up to a subsequence,*

$$
\begin{aligned}
u_h &\to u, && \text{in } [L^2(\Omega)]^d, \\
\nabla_h u_h &\to \nabla u, && \text{in } [L^2(\Omega)]^{d,d}, \\
|u_h|_J &\to 0, && \\
p_h &\to p, && \text{in } L^2(\Omega), \\
|p_h|_p &\to 0. &&
\end{aligned}
$$

If (u, p) *is unique, the whole sequence converges.*

Navier–Stokes I

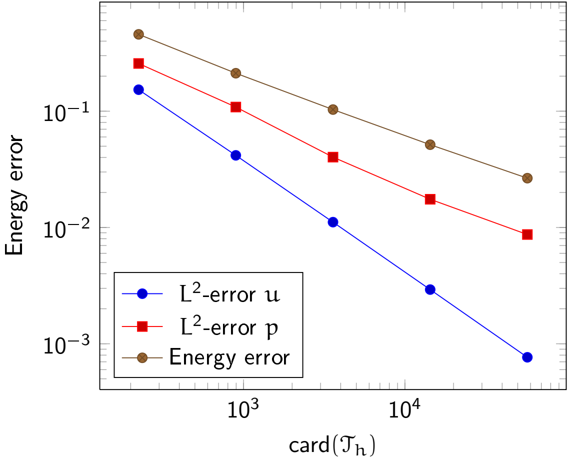


Figure: Convergence results for the Kovasznay problem

Navier–Stokes II

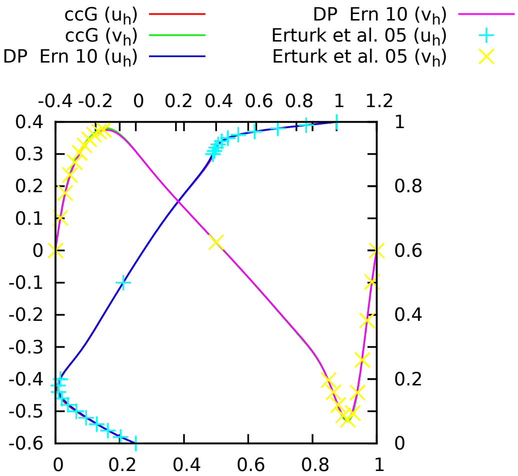


Figure: Lid-driven cavity problem in $d = 2$, Re $= 1000$

Navier–Stokes III

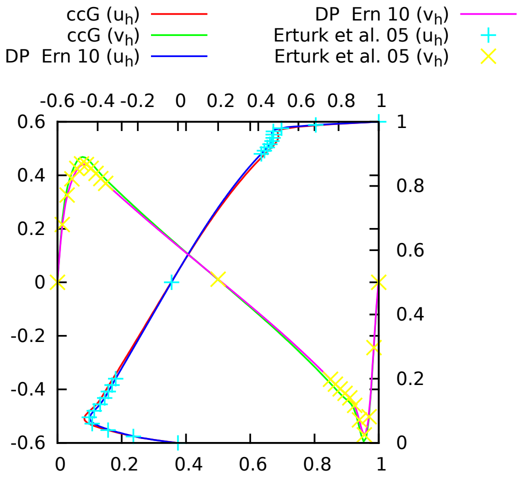


Figure: Lid-driven cavity problem in $d = 2$, $Re = 5000$

Outline

FreeFEM-like implementation in a nutshell I

```
// 1) Define the discrete space
typedef FunctionSpace<span<Polynomial<d, 1> >,
                      gradient<GreenFormula<LInterpolator> >
                      >::type CCGSpace;
CCGSpace Vh(𝒯h);

// 2) Create test and trial functions
CCGSpace::TrialFunction uh(Vh, "uh");
CCGSpace::TestFunction  vh(Vh, "vh");

// 3) Define the bilinear form
Form2 ah =
  integrate(All<Cell>(𝒯h),dot(grad(uh),grad(vh)))
 -integrate(All<Face>(𝒯h),dot(N(),avg(grad(uh)))*jump(vh)
                          +dot(N(),avg(grad(vh)))*jump(uh))
 +integrate(All<Face>(𝒯h),η/H()*jump(uh)*jump(vh));

// 4) Evaluate the bilinear form
MatrixContext context(A);
evaluate(ah, context);
```

FreeFEM-like implementation in a nutshell II

- Elements of arbitrary shape may be present
- The stencil of local contributions may vary from term to term
- The stencil may be data-dependent (cf. L-construction)
- The stencil may be non-local

- We cannot rely on reference element(s) + table of DOFs
- Instead, global DOF numbering + embedded stencil

Linear combination I

- Let $\mathbb{I} \subset \mathbb{V}_h$ denote the stencil of a discrete linear operator
- A `LinearCombination` $\mathtt{lc}^r = (I, \tau_I)_{I \in \mathbb{I}}$ implements

$$\mathtt{lc}^r(\mathbf{v}_h) = \sum_{I \in \mathbb{I}} \tau_I v_I + \tau_0 \in \mathbb{T}_r$$

- $0 \leqslant r \leqslant 2$ denotes the tensor rank of the result
- Algebraic composition of `LinearCombinations` is available

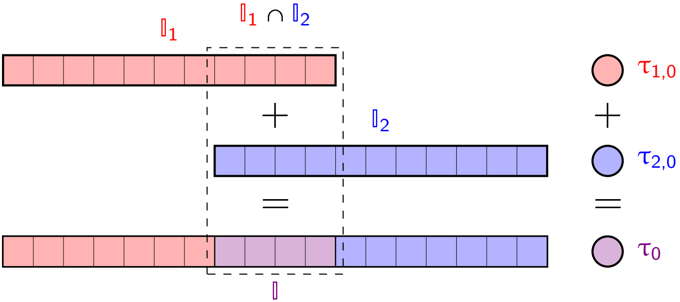
Linear combination II

```
// Cell unknown ν_T as a linear combination (I_T is the global DOF number)
LinearCombination<0> vT = Term(I_T,1.);

// Linear combination corresponding to 𝔊_{h|T}
LinearCombination<1> GT;
for(F ∈ ℱ_T)
{
  // Face unknown ν_F (possibly resulting from interpolation)
  const LinearCombination<0> & vF = T_h.eval(F);
  GT += |F|_{d-1}/|T|_d (vF-vT)n_{T,F};
}

// Actually perform algebraic operations on coefficients
GT.compact();
```

Figure: Implementation of the Green gradient $\mathfrak{G}_h$

Linear combination III

$$
\begin{aligned}
\mathtt{lc}^r &= \mathtt{lc}_1^r + \mathtt{lc}_2^r \\
&= \sum_{I \in \mathbb{I}_1} \tau_{1,I} v_I + \tau_{1,0} + \sum_{I \in \mathbb{I}_2} \tau_{2,I} v_I + \tau_{2,0} \\
&= \sum_{I \in \mathbb{I}} \tau_I v_I + \tau_0 \quad \text{(compaction)}
\end{aligned}
$$



FE-like assembly

- Let $u_h, v_h \in V_h^{ccg}$ and observe that

$$\int_T (\kappa\nabla_h u_h)_{|T}\cdot(\nabla_h v_h)_{|T} \rightsquigarrow |T|_d\, \mathtt{lc}_u\cdot\mathtt{lc}_v$$

$$\rightsquigarrow \mathbf{A}_T := [|T|_d\, \tau_{v,J}\cdot\tau_{u,I}]_{J\in\mathbb{J},I\in\mathbb{I}}$$

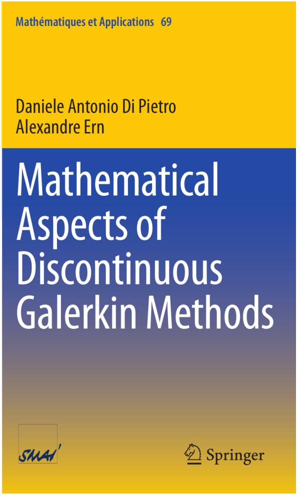
where $\mathtt{lc}_u = (J, \tau_{u,J})_{J\in\mathbb{J}}$ and $\mathtt{lc}_v = (I, \tau_{v,I})_{I\in\mathbb{I}}$

- The assembly step reads

$$\mathbf{A}(\mathbb{I},\mathbb{J}) \leftarrow \mathbf{A}(\mathbb{I},\mathbb{J}) + \mathbf{A}_T$$

The stencils $\mathbb{I}$ and $\mathbb{J}$ replace the table of DOFs!

Thank you for your attention!



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