Hybrid High-Order methods for poroelasticity

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Features of HHO methods

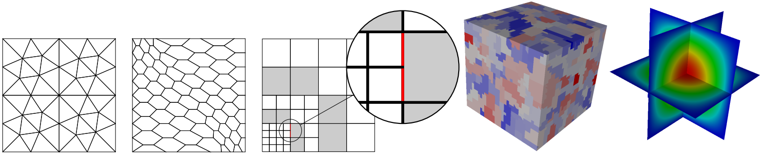

Figure: Examples of supported meshes $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ in 2d and 3d

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Physical fidelity leading to robustness in singular limits
- Natural extension to nonlinear problems
- Reduced computational cost after static condensation

Outline

References

- Linear elasticity, $k \geq 1$ [DP and Ern, 2015]
- Nonlinear elasticity [Botti,DP, Sochala, 2017]
- Linear elasticity, $k = 0$ [Botti, DP, Guglielmana, 2019]

New book!

D. A. Di Pietro and J. Droniou
The Hybrid High-Order Method for Polytopal Meshes
Design, Analysis, and Applications
528 pages, http://hal.archives-ouvertes.fr/hal-02151813v2

Model problem I

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a bounded, connected polyhedral domain
- For $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^d)$, we consider the elasticity problem

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u})) &= \boldsymbol{f} \qquad &&\text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \qquad &&\text{on } \partial\Omega, \end{aligned}$$

  with $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ possibly nonlinear strain-stress law
- In weak form: Find $u \in \boldsymbol{U} := H^1_0(\Omega)^d$ s.t.

$$a(u, v) := \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) : \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall v \in \boldsymbol{U}$$

- From here on, the dependence of $\boldsymbol{\sigma}$ on $\boldsymbol{x}$ will not be made explicit

Model problem II

Example (Linear elasticity)

Given a uniformly elliptic fourth-order tensor-valued function $\boldsymbol{C} : \Omega \to \mathbb{R}^{d\times d\times d\times d}$, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = \boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau}.$$

For uniform isotropic materials, the expression simplifies to

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d \quad \text{with} \quad 2\mu - d\lambda^- \geq \alpha > 0.$$

Example (Hencky–Mises model)

Given $\lambda : \mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu(\mathrm{dev}(\boldsymbol{\tau}))\boldsymbol{\tau} + \lambda(\mathrm{dev}(\boldsymbol{\tau}))\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d,$$

where $\mathrm{dev}(\boldsymbol{\tau}) := \mathrm{tr}(\boldsymbol{\tau}^2) - d^{-1}\,\mathrm{tr}(\boldsymbol{\tau})^2$.

Model problem III

Example (Isotropic damage model)

Given the damage function $D : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ and $\boldsymbol{C}$ as above, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = (1 - D(\boldsymbol{\tau}))\, \boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau}.$$

Example (Second-order model)

Given Lamé parameters $\mu, \lambda \in \mathbb{R}$ and second-order moduli $A, B, C \in \mathbb{R}$, for all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_d + A\boldsymbol{\tau}^2 + B \operatorname{tr}(\boldsymbol{\tau}^2)\boldsymbol{I}_d + 2B \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{\tau} + C \operatorname{tr}(\boldsymbol{\tau})^2 \boldsymbol{I}_d.$$

Projectors on local polynomial spaces

- Let $l \geq 0$, $X \in \mathcal{T}_h \cup \mathcal{F}_h$. The L^2-projector $\pi_X^{0,l} : L^2(X) \longrightarrow \mathbb{P}^l(X)$ is s.t.

$$\pi_X^{0,l} v = \arg \min_{w \in \mathbb{P}^l(X)} \|w - v\|^2_{L^2(X;\mathbb{R})}$$

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- The vector version $\boldsymbol{\pi}_X^{0,l}$ is obtained component-wise
- Let $l \geq 1$, $T \in \mathcal{T}_h$. The strain projector $\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l} : H^1(T)^d \to \mathbb{P}^l(T)^d$ is s.t.

$$\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l} \boldsymbol{v} = \arg \min_{\boldsymbol{w} \in \mathbb{P}^l(T)^d,\ \int_T (\boldsymbol{w}-\boldsymbol{v}) = \boldsymbol{0},\ \int_T \boldsymbol{\nabla}_{\rm ss}(\boldsymbol{w}-\boldsymbol{v}) = \boldsymbol{0}} \|\boldsymbol{\nabla}_{\rm s}(\boldsymbol{w}-\boldsymbol{v})\|_{L^2(T;\mathbb{R}^{d\times d})}^2$$

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$$\int_T \boldsymbol{\nabla}_{\rm s}(\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l} \boldsymbol{v} - \boldsymbol{v}) : \boldsymbol{\nabla}_{\rm s} \boldsymbol{w} = 0 \quad \forall \boldsymbol{w} \in \mathbb{P}^l(T; \mathbb{R}^d)$$

and

$$\int_T \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l} \boldsymbol{v} = \int_T \boldsymbol{v}, \qquad \int_T \boldsymbol{\nabla}_{\rm ss} \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l} \boldsymbol{v} = \int_T \boldsymbol{\nabla}_{\rm ss} \boldsymbol{v}$$

Projectors on local polynomial spaces

- Let $l \geq 0$, $X \in \mathcal{T}_h \cup \mathcal{F}_h$. The L^2-projector $\pi_X^{0,l} : L^2(X) \to \mathbb{P}^l(X)$ is s.t.

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- $\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},1}$ coincides with the elliptic projector of [DP and Droniou, 2017b]

Approximation properties for the strain projector I

Theorem (Optimal approximation properties of the strain projector)

Denote by $(\mathcal{M}_h)_{h\in\mathcal{H}} = (\mathcal{T}_h, \mathcal{F}_h)_{h\in\mathcal{H}}$ *a regular mesh sequence with star-shaped elements. Let an integer* $s \in \{1,\ldots,l+1\}$ *be given. Then, for all* $T \in \mathcal{T}_h$*, all* $\boldsymbol{v} \in H^s(T)^d$*, and all* $m \in \{0,\ldots,s\}$*,*

$$|\boldsymbol{v} - \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}|_{H^m(T;\mathbb{R}^d)} \lesssim h_T^{s-m}|\boldsymbol{v}|_{H^s(T;\mathbb{R}^d)}.$$

Moreover, if $m \le s-1$*, then, for all* $F \in \mathcal{F}_T$*,*

$$|\boldsymbol{v} - \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}|_{H^m(F,\mathbb{R}^d)} \lesssim h_T^{s-m-\frac{1}{2}}|\boldsymbol{v}|_{H^s(T;\mathbb{R}^d)}.$$

Hidden constants depend only on d*,* l*,* s*,* m*, and the mesh regularity.*

Approximation properties for the strain projector II

- It suffices to prove (cf. [DP and Droniou, 2017b]): For all $T \in \mathcal{T}_h$

$$
\begin{aligned}
\|\nabla \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^{d\times d})} &\lesssim |\boldsymbol{v}|_{H^1(T;\mathbb{R}^d)}, && \text{if } m \geq 1,\\
\|\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^d)} &\lesssim \|\boldsymbol{v}\|_{L^2(T;\mathbb{R}^d)} + h_T|\boldsymbol{v}|_{H^1(T;\mathbb{R}^d)} && \text{if } m = 0
\end{aligned}
$$

- To prove the first relation, we insert $\pm\boldsymbol{\pi}_T^{0,0}(\nabla_{\rm ss}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v})$ and bound

$$
\begin{aligned}
&\|\nabla \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^{d\times d})}\\
&\qquad \leq \|\nabla \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v} - \boldsymbol{\pi}_T^{0,0}(\nabla_{\rm ss}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v})\|_{L^2(T;\mathbb{R}^{d\times d})} + \|\boldsymbol{\pi}_T^{0,0}(\nabla_{\rm ss}\boldsymbol{v})\|_{L^2(T;\mathbb{R}^{d\times d})}
\end{aligned}
$$

- For the term in red, we need local Korn inequalities to write

$$
\|\nabla \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v} - \boldsymbol{\pi}_T^{0,0}(\nabla_{\rm ss}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v})\|_{L^2(T;\mathbb{R}^{d\times d})} \lesssim \|\nabla_{\rm s}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^{d\times d})},
$$

  where the hidden constant should be independent of T

Approximation properties for the strain projector III

Lemma (Uniform local Korn inequalities)

Denoting by $(\mathcal{M}_h)_{h\in\mathcal{H}}$ a regular mesh sequence with star-shaped elements it holds, for all $h \in \mathcal{H}$ and all $T \in \mathcal{T}_h$,

$$\|\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\pi}_T^{0,0}(\boldsymbol{\nabla}_{\rm ss}\boldsymbol{u})\|_T \lesssim \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_T \qquad \forall \boldsymbol{u} \in H^1(T)^d,$$

with hidden constant depending only on d and the mesh regularity (and independent of h and T).

Proof.

See [Botti, DP, and Droniou, 2018]. □

Computing displacement projections from L^2-projections

- For all $\boldsymbol{v} \in H^1(T;\mathbb{R}^d)$ and all $\boldsymbol{\tau} \in C^\infty(\overline{T};\mathbb{R}^{d\times d}_{\rm sym})$, it holds

$$\int_T \nabla_{\rm s}\boldsymbol{v}:\boldsymbol{\tau} = -\int_T \boldsymbol{v}\cdot(\nabla\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{v}\cdot\boldsymbol{\tau}\boldsymbol{n}_{TF}$$

- Specialising to $\boldsymbol{\tau} = \nabla_{\rm s}\boldsymbol{w}$ with $\boldsymbol{w} \in \mathbb{P}^{k+1}(T)^d$, $k \geq 0$, gives

$$\int_T \nabla_{\rm s}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v}:\nabla_{\rm s}\boldsymbol{w} = -\int_T \boldsymbol{\pi}_T^{0,k}\boldsymbol{v}\cdot(\nabla\cdot\nabla_{\rm s}\boldsymbol{w}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\cdot\nabla_{\rm s}\boldsymbol{w}\,\boldsymbol{n}_{TF}$$

- Moreover, we have

$$\int_T \boldsymbol{v} = \int_T \boldsymbol{\pi}_T^{0,k}\boldsymbol{v},\quad \int_T \nabla_{\rm ss}\boldsymbol{v} = \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F\left(\boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\otimes\boldsymbol{n}_{TF} - \boldsymbol{n}_{TF}\otimes\boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\right)$$

- **Hence, $\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v}$ can be computed from $\boldsymbol{\pi}_T^{0,k}\boldsymbol{v}$ and $(\boldsymbol{\pi}_F^{0,k}\boldsymbol{v})_{F\in\mathcal{F}_T}$!**

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- Moreover, we have

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- **Hence, $\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v}$ can be computed from $\boldsymbol{\pi}_T^{0,k}\boldsymbol{v}$ and $(\boldsymbol{\pi}_F^{0,k}\boldsymbol{v})_{F\in\mathcal{F}_T}$!**
- The same holds for $\boldsymbol{\pi}_T^{0,k}(\boldsymbol{\nabla}_{\rm s}\boldsymbol{v})$ (specialise to $\boldsymbol{\tau} \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\rm sym})$)

Discrete unknowns

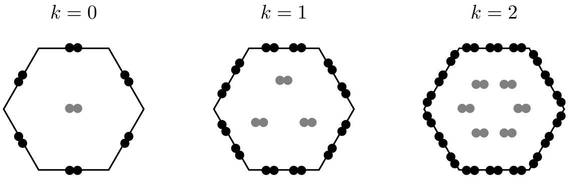


Figure: $\underline{\boldsymbol{U}}_T^k$ for $k \in \{0, 1, 2\}$

- Let a polynomial degree $k \geq 0$ be fixed
- For all $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$\underline{\boldsymbol{U}}_T^k := \Big\{ \underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}) \ : \\ \boldsymbol{v}_T \in \mathbb{P}^k(T; \mathbb{R}^d) \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F; \mathbb{R}^d) \quad \forall F \in \mathcal{F}_T \Big\}$$

- The local interpolator $\underline{\boldsymbol{I}}_T^k : H^1(T; \mathbb{R}^d) \to \underline{\boldsymbol{U}}_T^k$ is s.t.

$$\underline{\boldsymbol{I}}_T^k \boldsymbol{v} := (\boldsymbol{\pi}_T^{0,k} \boldsymbol{v}, (\boldsymbol{\pi}_F^{0,k} \boldsymbol{v})_{F \in \mathcal{F}_T}) \quad \forall \boldsymbol{v} \in H^1(T)^d$$

Local displacement and strain reconstructions I

- We introduce the displacement reconstruction operator

$$\mathbf{p}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T;\mathbb{R}^d)$$

s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$ and all $\boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$,

$$\int_T \boldsymbol{\nabla}_{\mathrm{s}} \mathbf{p}_T^{k+1} \underline{\boldsymbol{v}}_T : \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w} = -\int_T \boldsymbol{v}_T \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w}) + \sum_{F \in \mathcal{F}_T} \int_F \boldsymbol{v}_F \cdot \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w} \, \boldsymbol{n}_{TF}$$

and

$$\int_T \mathbf{p}_T^{k+1} \underline{\boldsymbol{v}}_T = \int_T \boldsymbol{v}_T, \quad \int_T \boldsymbol{\nabla}_{\mathrm{ss}} \mathbf{p}_T^{k+1} \underline{\boldsymbol{v}}_T = \frac{1}{2} \sum_{F \in \mathcal{F}_T} \int_F (\boldsymbol{v}_F \otimes \boldsymbol{n}_{TF} - \boldsymbol{n}_{TF} \otimes \boldsymbol{v}_F)$$

- By construction, the following commutation property holds:

$$\boxed{\mathbf{p}_T^{k+1} \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1} \boldsymbol{v} \qquad \forall \boldsymbol{v} \in H^1(T;\mathbb{R}^d)}$$

Local displacement and strain reconstructions II

- For nonlinear problems, $\nabla_{\mathrm{s}}\mathbf{p}_T^{k+1}$ is not sufficiently rich
- We therefore also define the strain reconstruction operator

$$\mathbf{G}_{\mathrm{s},T}^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T;\mathbb{R}_{\mathrm{sym}}^{d\times d})$$

such that, for all $\boldsymbol{\tau} \in \mathbb{P}^k(T;\mathbb{R}_{\mathrm{sym}}^{d\times d})$,

$$\int_T \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T \cdot (\nabla\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T} \int_F \boldsymbol{v}_F \cdot \boldsymbol{\tau}\boldsymbol{n}_{TF}$$

- By construction, it holds

$$\boxed{\mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \boldsymbol{\pi}_T^{0,k}(\nabla_{\mathrm{s}}\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in H^1(T;\mathbb{R}^d)}$$

Local contribution I

$$a_{|T}(\boldsymbol{u}, \boldsymbol{v}) \approx \mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := \int_T \boldsymbol{\sigma}(\mathbf{G}^k_{\mathrm{s},T}\underline{\boldsymbol{u}}_T):\mathbf{G}^k_{\mathrm{s},T}\underline{\boldsymbol{v}}_T + \mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

Assumption (Stabilization bilinear form)

The bilinear form $\mathrm{s}_T : \underline{\boldsymbol{U}}^k_T \times \underline{\boldsymbol{U}}^k_T \to \mathbb{R}$ satisfies the following properties:

- Symmetry and positivity. s_T is symmetric and positive semidefinite.
- Stability. It holds, with hidden constant independent of h and T and $\|\cdot\|_{\boldsymbol{\varepsilon},h}$ natural DOF strain seminorm: For all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}^k_T$,

$$\|\mathbf{G}^k_{\mathrm{s},T}\underline{\boldsymbol{v}}_T\|^2_{L^2(T;\mathbb{R}^{d\times d})} + \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \simeq \|\underline{\boldsymbol{v}}_T\|^2_{\boldsymbol{\varepsilon},T}.$$

- Polynomial consistency. For all $w \in \mathbb{P}^{k+1}(T)$ and all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}^k_T$,

$$\mathrm{s}_T(\underline{\boldsymbol{I}}^k_T \boldsymbol{w}, \underline{\boldsymbol{v}}_T) = 0.$$

Local contribution II

Remark (Polynomial degree)

Stability and polynomial consistency are incompatible for $k = 0$.

Remark (Dependency)

s_T satisfies polynomial consistency if and only if it depends on its arguments via the difference operators s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\boldsymbol{\delta}_T^k \underline{\boldsymbol{v}}_T \coloneqq \boldsymbol{\pi}_T^{0,k}(\mathbf{p}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_T),$$
$$\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{v}}_T \coloneqq \boldsymbol{\pi}_F^{0,k}(\mathbf{p}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_F) \quad \forall F \in \mathcal{F}_T.$$

Example (Classical HHO stabilisation)

$$\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) \coloneqq \sum_{F \in \mathcal{F}_T} \frac{\gamma}{h_F} \int_F \left(\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{u}}_T - \boldsymbol{\delta}_T^k \underline{\boldsymbol{u}}_T\right) \cdot \left(\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{v}}_T - \boldsymbol{\delta}_T^k \underline{\boldsymbol{v}}_T\right).$$

Discrete problem

- Define the global space with single-valued interface unknowns

$$\underline{\boldsymbol{U}}_h^k := \Big\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}) \ : \\ \boldsymbol{v}_T \in \mathbb{P}^k(T;\mathbb{R}^d) \quad \forall T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F;\mathbb{R}^d) \quad \forall F \in \mathcal{F}_h \Big\}$$

and its subspace with strongly enforced boundary conditions

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ s.t.

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) = \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{f}\cdot\boldsymbol{v}_h \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$$

Global discrete Korn inequalities

Lemma (Global Korn inequality on broken polynomial spaces)

Let an integer $l \geq 1$ *be fixed and, given* $\boldsymbol{v}_h \in \mathbb{P}^l(\mathcal{T}_h; \mathbb{R}^d)$*, set*

$$\|\boldsymbol{v}_h\|_{\mathrm{dG},h}^2 := \|\boldsymbol{\nabla}_{\mathrm{s},h}\boldsymbol{v}_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})}^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \|[\boldsymbol{v}_h]_F\|_{L^2(F;\mathbb{R}^d)}^2.$$

Then it holds, with hidden constant depending only on Ω, d, l, *and* ϱ,

$$\|\boldsymbol{\nabla}_h\boldsymbol{v}_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \lesssim \|\boldsymbol{v}_h\|_{\mathrm{dG},h}.$$

Corollary (Global Korn inequality on HHO spaces)

Assume $k \geq 1$*. Then it holds, for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$*, letting* $\boldsymbol{v}_h \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}^d)$ *be s.t.* $(\boldsymbol{v}_h)_{|T} := \boldsymbol{v}_T$ *for all* $T \in \mathcal{T}_h$ *and with hidden constant as above,*

$$\|\boldsymbol{v}_h\|_{L^2(\Omega;\mathbb{R}^d)} + \|\boldsymbol{\nabla}_h\boldsymbol{v}_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \lesssim \|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},h}.$$

Existence and uniqueness I

Assumption (Strain-stress law/1)

The strain-stress law is a Carathéodory function s.t. $\boldsymbol{\sigma}(\cdot, \mathbf{0}) = \mathbf{0}$ and there exist $0 < \underline{\sigma} \leq \overline{\sigma}$ s.t., for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$,

$$\begin{aligned}
\|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau})\|_{\mathbb{R}^{d\times d}} &\leq \overline{\sigma} \|\boldsymbol{\tau}\|_{\mathbb{R}^{d\times d}}, && \text{(growth)} \\
\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) : \boldsymbol{\tau} &\geq \underline{\sigma} \|\boldsymbol{\tau}\|^2_{\mathbb{R}^{d\times d}}, && \text{(coercivity)} \\
(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) &\geq 0. && \text{(monotonicity)}
\end{aligned}$$

Remark (Choice of the penalty parameter)

A natural choice is to take the penalty parameter s.t.

$$\gamma \in [\underline{\sigma}, \overline{\sigma}].$$

Existence and uniqueness II

Theorem (Discrete existence and uniqueness)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ *denote a regular mesh sequence with star-shaped elements and assume* $k \geq 1$*. Then, for all* $h \in \mathcal{H}$*, there exist a solution* $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ *to the discrete problem, which satisfies*

$$\|\underline{\boldsymbol{u}}_h\|_{\boldsymbol{\varepsilon},h} \lesssim \|\boldsymbol{f}\|_{L^2(\Omega;\mathbb{R}^d)},$$

with hidden constant only depending on Ω, $\underline{\sigma}$, γ, ϱ, *and* k.
Moreover, if $\boldsymbol{\sigma}$ *is strictly monotone, then the solution is unique.*

Convergence and error estimate

Theorem (Convergence)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ denote a regular mesh sequence with star-shaped elements and assume $k \geq 1$. Then, for all $q \in [1, +\infty)$ if $d = 2$ and $q \in [1, 6)$ if $d = 3$, as $h \to 0$ it holds, up to a subsequence, that

$$\begin{aligned} \boldsymbol{u}_h &\to \boldsymbol{u} && \text{strongly in } L^q(\Omega; \mathbb{R}^d),\\ \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{u}}_h &\rightharpoonup \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u} && \text{weakly in } L^2(\Omega; \mathbb{R}^{d\times d}). \end{aligned}$$

If, additionally, $\boldsymbol{\sigma}$ is strictly monotone,

$$\mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{u}}_h \to \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u} \quad \text{strongly in } L^2(\Omega; \mathbb{R}^{d\times d})$$

and, the continuous solution being unique, the whole sequence converges.

Error estimate

Assumption (Strain-stress law/2)

There exists $\sigma_*, \sigma^* \in (0, +\infty)$ s.t., for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\rm sym}$,

$$\|\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})\|_{\mathbb{R}^{d\times d}} \le \sigma^* \|\boldsymbol{\tau} - \boldsymbol{\eta}\|_{\mathbb{R}^{d\times d}}, \qquad \text{(Lipschitz continuity)}$$

$$(\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})):(\boldsymbol{\tau} - \boldsymbol{\eta}) \ge \sigma_* \|\boldsymbol{\tau} - \boldsymbol{\eta}\|^2_{\mathbb{R}^{d\times d}}. \qquad \text{(strong monotonicity)}$$

Theorem (Error estimate)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ denote a regular mesh sequence with star-shaped elements and $k \ge 1$. Then, if $\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)$ and $\boldsymbol{\sigma}(\cdot,\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}) \in H^{k+1}(\mathcal{T}_h;\mathbb{R}^{d\times d})$,

$$\|\mathbf{G}^k_{{\rm s},h}\underline{\boldsymbol{u}}_h - \boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + |\underline{\boldsymbol{u}}_h|_{{\rm s},h} \lesssim h^{k+1}\left(|\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)} + |\boldsymbol{\sigma}(\cdot,\boldsymbol{\nabla}_{\rm s}\boldsymbol{u})|_{H^{k+1}(\mathcal{T}_h;\mathbb{R}^{d\times d})}\right),$$

with hidden constant only depending on Ω, k, $\overline{\sigma}$, $\underline{\sigma}$, σ^*, σ_*, γ, the mesh regularity and an upper bound of $\|\boldsymbol{f}\|_{L^2(\Omega;\mathbb{R}^d)}$.

The lowest-order case I

- **For $k = 0$, stability cannot be enforced through local terms**
- We therefore consider $\mathrm{a}_h^{\mathrm{lo}} : \underline{\boldsymbol{U}}_h^0 \times \underline{\boldsymbol{U}}_h^0$ s.t.

$$\mathrm{a}_h^{\mathrm{lo}}(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) + \mathrm{j}_h(\mathbf{p}_h^1 \underline{\boldsymbol{u}}_h, \mathbf{p}_h^1 \underline{\boldsymbol{v}}_h),$$

with jump penalisation bilinear form

$$\mathrm{j}_h(\boldsymbol{u}, \boldsymbol{v}) := \sum_{F \in \mathcal{F}_h} h_F^{-1}([\boldsymbol{u}]_F, [\boldsymbol{v}]_F)_F$$

The lowest-order case II

- Consider, e.g., isotropic homogeneous linear elasticity, that is

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d \quad \text{with} \quad 2\mu - d\lambda^- \geq \alpha > 0$$

- Coercivity is ensured by Korn's inequality in broken spaces:

$$\alpha |||\underline{\boldsymbol{v}}_h|||_{\boldsymbol{\varepsilon},h}^2 \lesssim \mathrm{a}_h^{\mathrm{lo}}(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^0,$$

where

$$|||\underline{\boldsymbol{v}}_h|||_{\boldsymbol{\varepsilon},h} := \left(\|\boldsymbol{v}_h\|_{\mathrm{dG},h}^2 + |\underline{\boldsymbol{v}}_h|_{\mathrm{s},h}^2 \right)^{\frac{1}{2}}, \quad |\underline{\boldsymbol{v}}_h|_{\mathrm{s},h} := \left(\sum_{T \in \mathcal{T}_h} \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \right)^{\frac{1}{2}}$$

Error estimates I

Theorem (Energy error estimate, $k = 0$)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ denote a regular mesh sequence. Then, if $\boldsymbol{u} \in H^2(\mathcal{T}_h;\mathbb{R}^d)$,

$$\|\boldsymbol{\nabla}_h \mathrm{p}_h^1 \underline{\boldsymbol{u}}_h - \boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + |\underline{\boldsymbol{u}}_h|_{\mathrm{s},h} \lesssim h\alpha^{-1}\left(|\boldsymbol{u}|_{H^2(\mathcal{T}_h;\mathbb{R}^d)} + |\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u})|_{H^1(\mathcal{T}_h;\mathbb{R}^{d\times d})}\right),$$

with hidden constant independent of h, $\boldsymbol{u}$, of the Lamé parameters and of $\boldsymbol{f}$. This estimate can be proved to be uniform in λ.

Remark (Star-shaped assumption)

We do not need the star-shaped assumption for $k = 0$, since the strain projector coincides with the elliptic projector, whose approximation properties do not require local Korn inequalities.

Error estimates II

Theorem (L^2-error estimate)

Under the assumptions of the above theorem, and further assuming $\lambda \geq 0$, elliptic regularity, and $\boldsymbol{f} \in H^1(\mathcal{T}_h; \mathbb{R}^d)$, it holds that

$$\|\mathbf{p}_h^1 \underline{\boldsymbol{u}}_h - \boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^d)} \lesssim h^2 \|\boldsymbol{f}\|_{H^1(\mathcal{T}_h;\mathbb{R}^d)},$$

with hidden constant independent of both h and λ.

Outline

References

- Linear poroelasticity [Boffi, Botti, DP, 2016]
- Nonlinear poroelasticity [Botti, DP, Sochala, 2019]
- Random coefficients [Botti, DP, Le Maître, Sochala, 2019]
- Abstract analysis [Botti, Botti, DP, 2019a] (in preparation)
- Multi-network [Botti, Botti, DP, 2019b] (in preparation)

The poroelasticity problem I

- Momentum balance: For any control volume $V \subset \Omega$, enforce

$$\int_V \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \int_{\partial V} \tilde{\boldsymbol{\sigma}} \boldsymbol{n} + \int_V \boldsymbol{f},$$

with $\tilde{\boldsymbol{\sigma}} \coloneqq \boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}) - p \boldsymbol{I}_d$. Under the quasi-static assumption,

$$\boxed{-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{f} \qquad \text{in } \Omega \times (0, t_{\mathrm{F}})}$$

- Mass conservation: For any control volume $V \subset \Omega$, enforce

$$\int_V \frac{\partial \phi}{\partial t} + \int_{\partial V} \boldsymbol{\Phi}\cdot\boldsymbol{n} = \int_V g,$$

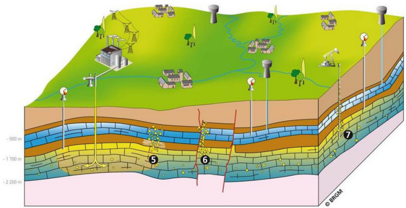
with porosity $\phi = C_0 p + \boldsymbol{\nabla}\cdot\boldsymbol{u}$ and flux $\boldsymbol{\Phi} = -\boldsymbol{\kappa}\boldsymbol{\nabla} p$. Substituting,

$$\boxed{\partial_t (C_0 p + \boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\boldsymbol{\nabla} p) = g \qquad \text{in } \Omega \times (0, t_{\mathrm{F}})}$$

- IC, BC, and, if $C_0 = 0$, compatibility conditions not detailed

The poroelasticity problem II

$$
\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}(\nabla_{\mathrm{s}}\boldsymbol{u}) + \nabla p &= \boldsymbol{f} && \text{in } \Omega\times(0,t_{\mathrm{F}})\\
\partial_t\left(C_0 p + \nabla\cdot\boldsymbol{u}\right) - \nabla\cdot(\boldsymbol{\kappa}\nabla p) &= g && \text{in } \Omega\times(0,t_{\mathrm{F}})
\end{aligned}
$$



- Presence of different layers and, possibly, fractures
- Strongly heterogeneous and anisotropic permeability tensor $\boldsymbol{\kappa}$
- General stress-strain relations $\boldsymbol{\sigma}$ (nonlinear, $\lambda \to +\infty, \ldots$)
- Singular limit $C_0 = 0$ (incompressible grains)

Weak formulation

- Let $\boldsymbol{f} \in L^2(0, t_{\mathrm{F}}; L^2(\Omega; \mathbb{R}^d))$, $g \in L^2(0, t_{\mathrm{F}}; L^2(\Omega; \mathbb{R}))$, $\phi^0 \in L^2(\Omega; \mathbb{R})$,

$$P := H^1(\Omega; \mathbb{R}) \text{ if } C_0 > 0, \; P := \left\{ q \in H^1(\Omega; \mathbb{R}) \; : \; \int_\Omega q = 0 \right\} \text{ if } C_0 = 0$$

- Define the bilinear forms $b : \boldsymbol{U} \times P \to \mathbb{R}$ and $c : P \times P \to \mathbb{R}$ s.t.

$$b(\boldsymbol{v}, q) := -\int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{v} \; q, \qquad c(r, q) := \int_\Omega \boldsymbol{\kappa}\boldsymbol{\nabla} r\cdot\boldsymbol{\nabla} q$$

- We seek $(\boldsymbol{u}, p) \in L^2(0, t_{\mathrm{F}}; \boldsymbol{U} \times P)$ s.t., $\forall(\boldsymbol{v}, q, \varphi) \in \boldsymbol{U} \times P \times C_{\mathrm{c}}^\infty((0, t_{\mathrm{F}}))$,

$$\boxed{\begin{gathered}\int_0^{t_{\mathrm{F}}} a(\boldsymbol{u}(t), \boldsymbol{v})\varphi(t)\,\mathrm{d}t + \int_0^{t_{\mathrm{F}}} b(\boldsymbol{v}, p(t))\varphi(t)\,\mathrm{d}t = \int_0^{t_{\mathrm{F}}}\int_\Omega (\boldsymbol{f}(t)\cdot\boldsymbol{v})\,\varphi(t)\,\mathrm{d}t, \\ \int_0^{t_{\mathrm{F}}}\int_\Omega \phi(t) d_t\varphi(t)\,\mathrm{d}t + \int_0^{t_{\mathrm{F}}} c(p, q)\varphi(t)\,\mathrm{d}t = \int_0^{t_{\mathrm{F}}}\int_\Omega g(t) q \varphi(t)\,\mathrm{d}t, \\ \int_\Omega (C_0 p(0) + \boldsymbol{\nabla}\cdot\boldsymbol{u}(0))q = \int_\Omega \phi^0 q\end{gathered}}$$

Features of the HHO method

- High-order method on general polyhedral meshes
- Inf-sup-stable hydro-mechanical coupling
- Robustness with respect to heterogeneous-anisotropic permeability
- Seamless treatment of incompressible grains ($C_0 = 0$)
- Locally equilibrated tractions and fluxes
- Numerically robust with respect to spurious pressure oscillations

- Mimicking the IBP formula: $\forall(\boldsymbol{v}, q) \in H^1(T;\mathbb{R}^d) \times C^\infty(\overline{T};\mathbb{R})$,

$$\int_T (\boldsymbol{\nabla}\cdot\boldsymbol{v})\ q = -\int_T \boldsymbol{v}\cdot\boldsymbol{\nabla} q + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}\cdot\boldsymbol{n}_{TF})\ q,$$

we introduce divergence reconstruction $\mathrm{D}_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^\ell(T)$ s.t.

$$\boxed{\int_T \mathrm{D}_T^k \underline{\boldsymbol{v}}_T\ q = -\int_T \boldsymbol{v}_T\cdot\boldsymbol{\nabla} q + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}_F\cdot\boldsymbol{n}_{TF})\ q \quad \forall q \in \mathbb{P}^k(T)}$$

- By construction, it holds, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\mathrm{D}_T^k \underline{\boldsymbol{v}}_T = \mathrm{tr}(\mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T),$$

hence, for all $\boldsymbol{v} \in H^1(T;\mathbb{R}^d)$,

$$\boxed{\mathrm{D}_T^k \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \pi_T^{0,k}(\boldsymbol{\nabla}\cdot\boldsymbol{v})}$$

Discrete divergence and hydro-mechanical coupling II

- The hydro-mechanical coupling is realised by the bilinear form

$$\boxed{\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) := -\sum_{T\in\mathcal{T}_h}\int_T \mathrm{D}_T^k \underline{\boldsymbol{v}}_T\ q_T}$$

- Inf-sup stability: There is $\beta > 0$ independent of h s.t.

$$\forall q_h \in P_h^k, \quad \beta\|q_h\|_{L^2(\Omega;\mathbb{R})} \le \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_{h,0}^k,\ \|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},h}=1} \mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h)$$

- **Result valid on general meshes and for any $k \ge 0$**

Darcy term

- For all $F \in \mathcal{F}_h^{\mathrm{i}}$ s.t. $F \subset \partial T_1 \cap \partial T_2$ and all $q_h \in \mathbb{P}^k(\mathcal{T}_h)$,

$$[q_h]_F := (q_h)_{|T_1} - (q_h)_{|T_2}, \quad \{q_h\}_F := \tfrac{\kappa_2}{\kappa_1+\kappa_2}(q_h)_{|T_1} + \tfrac{\kappa_1}{\kappa_1+\kappa_2}(q_h)_{|T_2}$$

  where $\boldsymbol{n}_F$ points out of T_1 and, for $i \in \{1,2\}$, $\kappa_i := \boldsymbol{n}_F^t \boldsymbol{\kappa}_{|T_i} \boldsymbol{n}_F$
- Applied to vector functions, $[\cdot]_F$ and $\{\cdot\}_F$ act component-wise
- The Darcy bilinear form is s.t.

$$\boxed{\begin{aligned} \mathrm{c}_h(r_h, q_h) := \int_\Omega \boldsymbol{\kappa} \boldsymbol{\nabla}_h r_h \cdot \boldsymbol{\nabla}_h q_h + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \frac{\varsigma \lambda_{\boldsymbol{\kappa},F}}{h_F} \int_F [r_h]_F [q_h]_F \\ - \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F \left([q_h]_F \{\boldsymbol{\kappa} \boldsymbol{\nabla}_h r_h\}_F + [r_h]_F \{\boldsymbol{\kappa} \boldsymbol{\nabla}_h q_h\}_F \right) \cdot \boldsymbol{n}_F, \end{aligned}}$$

  where $\varsigma > 0$ is a penalty parameter assumed large enough and

$$\lambda_{\boldsymbol{\kappa},F} := \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$$

Discrete problem I

- Let $\underline{\boldsymbol{U}}_{h,0}^k$ as for the elasticity problem and set

$$P_h^k := \mathbb{P}^k(\mathcal{T}_h) \text{ if } C_0 > 0,\ P_h^k := \left\{ q_h \in \mathbb{P}^k(\mathcal{T}_h) : \int_\Omega q_h = 0 \right\} \text{ if } C_0 = 0$$

- Let $N \in \mathbb{N}^*$, $\tau := t_{\mathrm{F}}/N$, and $\mathcal{T}_\tau := (t^n := n\tau)_{n=0,\ldots,N}$
- Let V denote a vector space and, for all $\varphi_\tau := (\varphi^i)_{0 \le i \le N} \in V^{N+1}$,

$$\delta_t^n \varphi_\tau := \frac{\varphi^n - \varphi^{n-1}}{\tau} \in V \quad \forall 1 \le n \le N$$

be the discrete backward derivative operator

Discrete problem II

We let $(\underline{\boldsymbol{u}}_{h\tau}, p_{h\tau}) \in \left[\underline{\boldsymbol{U}}_{h,0}^k\right]^{N+1} \times \left[P_h^k\right]^{N+1}$ satisfy, for $n = 1, \ldots, N$,

$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{u}}_h^n, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h^n) &= \int_\Omega \overline{\boldsymbol{f}}^n \cdot \boldsymbol{v}_h, \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ \int_\Omega C_0 \delta_t^n p_{h\tau}\ q_h - \mathrm{b}_h(\delta_t^n \underline{\boldsymbol{u}}_{h\tau}, q_h) + \mathrm{c}_h(p_h^n, q_h) &= \int_\Omega \overline{g}^n q_h \qquad \forall q_h \in P_h^k, \end{aligned}$$

with

$$\overline{\boldsymbol{f}}^n \coloneqq \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \boldsymbol{f}(t)\,\mathrm{d}t \in L^2(\Omega)^d, \qquad \overline{g}^n \coloneqq \frac{1}{\tau} \int_{t^{n-1}}^{t^n} g(t)\,\mathrm{d}t \in L^2(\Omega).$$

The initial condition is accounted for by enforcing

$$\int_\Omega C_0 p_h^0\ q_h - \mathrm{b}_h(\underline{\boldsymbol{u}}_h^0, q_h) = \int_\Omega \phi^0 q_h \qquad \forall q_h \in P_h^k$$

Theorem (Error estimate)

Set, for any $0 \leq n \leq N$, $\underline{\boldsymbol{e}}_h := \underline{\boldsymbol{u}}_h^n - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}^n$ *and* $\epsilon_h := p_h^n - \pi_h^{0,k} p^n$*. Assume* Ω *convex,* $\boldsymbol{\kappa} \in \mathbb{P}^0(\Omega; \mathbb{R}^{d\times d})$*, as well as*

$$\boldsymbol{u} \in H^1(\mathcal{T}_\tau; \boldsymbol{U}) \cap L^2(0, t_{\mathrm{F}}; H^{k+1}(\mathcal{T}_h; \mathbb{R}^d)), \quad \boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}) \in L^2(0, t_{\mathrm{F}}; H^{k+1}(\mathcal{T}_h; \mathbb{R}^{d\times d})),$$
$$p \in L^2(0, t_{\mathrm{F}}; P \cap H^{k+1}(\mathcal{T}_h; \mathbb{R})), \qquad \phi \in H^1(\mathcal{T}_\tau; L^2(\Omega; \mathbb{R})),$$

with $\phi = C_0 p + \boldsymbol{\nabla}\cdot\boldsymbol{u}$*. If* $C_0 > 0$*, we further assume* $\pi_\Omega^{0,0} p \in H^1(\mathcal{T}_\tau; \mathbb{P}^0(\Omega))$*. Then,*

$$\sum_{n=1}^N \tau \left(\|\underline{\boldsymbol{e}}_h^n\|_{\boldsymbol{\varepsilon},h}^2 + \|\epsilon_h^n - \pi_\Omega^{0,0}\epsilon_h^n\|_{L^2(\Omega)}^2 + C_0 \|\epsilon_h^n\|_{L^2(\Omega)}^2 \right) + \|z_h^N\|_{\mathrm{c},h}^2 \lesssim \left(h^{2k+2} C_1 + \tau^2 C_2 \right),$$

with hidden constant independent of h, τ, C_0, $\boldsymbol{\kappa}$, *and* t_{F}, $z_h^N := \sum_{n=1}^N \tau \epsilon_h^n$*, and*

$$C_1 := \|\boldsymbol{u}\|_{L^2(0,t_{\mathrm{F}};H^{k+2}(\mathcal{T}_h;\mathbb{R}^d))}^2 + \|\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u})\|_{L^2(0,t_{\mathrm{F}};H^{k+1}(\mathcal{T}_h;\mathbb{R}^{d\times d}))}^2 + (1 + C_0)\frac{\overline{K}}{\underline{K}} \|p\|_{L^2(0,t_{\mathrm{F}};H^{k+1}(\mathcal{T}_h;\mathbb{R}))}^2,$$
$$C_2 := \|\boldsymbol{u}\|_{H^1(\mathcal{T}_\tau;H^1(\Omega;\mathbb{R})^d)}^2 + \|\phi\|_{H^1(\mathcal{T}_\tau;L^2(\Omega;\mathbb{R}))}^2 + C_0 \|\pi_\Omega^{0,0} p\|_{H^1(\mathcal{T}_\tau)}^2.$$

Convergence (linear case) I

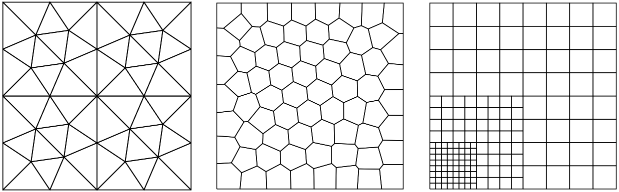


Figure: Meshes for the convergence test

In $\Omega = (0,1)^2 \times [0, t_{\mathrm{F}} = 1]$, we consider linear poroelasticity with $\mu = 1$, $\lambda = 1$, $\boldsymbol{\kappa} = \boldsymbol{I}_d$, $C_0 = 0$, and exact solution

$$\boldsymbol{u}(\boldsymbol{x},t) = \sin(\pi t)\begin{pmatrix} -\cos(\pi x_1)\cos(\pi x_2) \\ \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix},$$

$$p(\boldsymbol{x},t) = -\cos(\pi t)\sin(\pi x_1)\cos(\pi x_2),$$

$$(\boldsymbol{f},\, g) \text{ inferred from } \boldsymbol{u}, p$$

Convergence (linear case) II

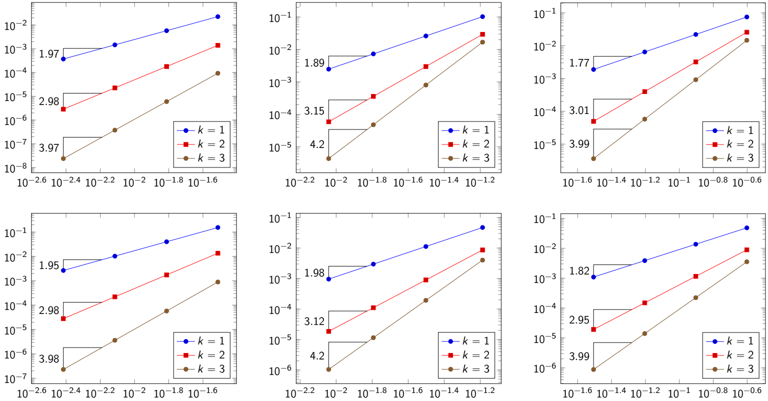


Figure: L^2-error on the pressure (top) and H^1-error on the displacement (bottom) vs. h for (from left to right) the triangular, Voronoi, and locally refined meshes

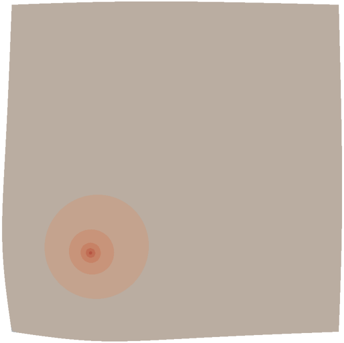
Barry and Mercer I

- $\Omega = (0, 1)^2$
- $C_0 = 0$, $\boldsymbol{\kappa} = \boldsymbol{I}_d$,
- On $\partial\Omega$, we enforce

$$\boldsymbol{u}\cdot\boldsymbol{\tau} = 0,\ \boldsymbol{n}^{\mathrm{T}}\boldsymbol{\nabla u n} = 0,\ p = 0$$

- Source term periodic in time

$$g(\boldsymbol{x}, t) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\sin(t)$$



Barry and Mercer II

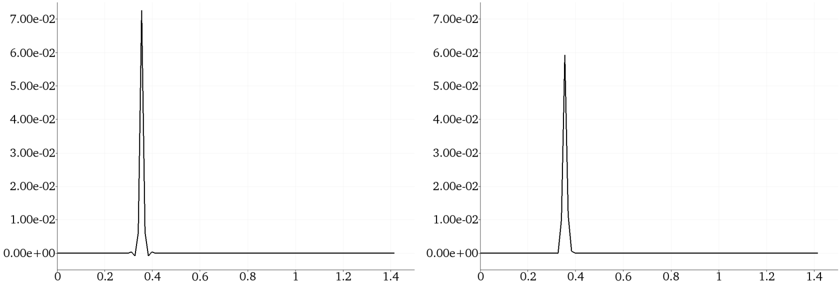


Figure: Pressure profiles along $(0,0)$–$(1,1)$ for $\boldsymbol{\kappa} = 1 \cdot 10^{-6}\boldsymbol{I}_d$ and $\boldsymbol{\tau} = 1 \cdot 10^{-4}$: (*left*) Small oscillations on the Cartesian mesh, card$(\mathcal{T}_h) = 4028$; (*right*) No oscillations is present on the Voronoi mesh, card$(\mathcal{T}_h) = 4192$

Convergence (nonlinear case) I

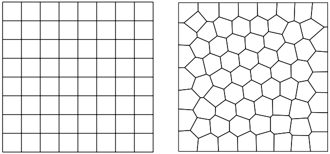


Figure: Meshes for the convergence test

In $\Omega = (0,1)^2 \times [0, t_{\mathrm{F}} = 1]$, we consider nonlinear poroelasticity with $\mu = 1$, $\lambda = 1$, $\boldsymbol{\kappa} = \boldsymbol{I}_d$, $C_0 = 0$, strain-stress law

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = (1 + \exp(-\operatorname{dev}\boldsymbol{\tau}))\operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_d + (4 - 2\exp(-\operatorname{dev}\boldsymbol{\tau}))\boldsymbol{\tau},$$

and exact solution

$$\boldsymbol{u}(\boldsymbol{x}, t) = t^2 \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix},$$

$$p(\boldsymbol{x}, t) = -\pi^{-1}\left(\sin(\pi x_1)\cos(\pi x_2) + \cos(\pi x_1)\sin(\pi x_2)\right),$$

$(\boldsymbol{f},\ g)$ inferred from $\boldsymbol{u}, p$

Convergence (nonlinear case) II

h	$\left(\sum_{n=1}^{N} \tau \Vert \underline{\boldsymbol{e}}_h^n \Vert_{\boldsymbol{\varepsilon},h}^2\right)^{\frac{1}{2}}$	OCV	$\left(\sum_{n=1}^{N} \tau \Vert \epsilon_h^n \Vert_{\Omega}^2\right)^{\frac{1}{2}}$	OCV
Cartesian mesh family				
$6.25 \cdot 10^{-2}$	$3.10 \cdot 10^{-2}$	—	0.39	—
$3.12 \cdot 10^{-2}$	$8.52 \cdot 10^{-3}$	1.86	$9.65 \cdot 10^{-2}$	2.00
$1.56 \cdot 10^{-2}$	$2.22 \cdot 10^{-3}$	1.94	$2.44 \cdot 10^{-2}$	1.98
$7.81 \cdot 10^{-3}$	$5.61 \cdot 10^{-4}$	1.99	$6.18 \cdot 10^{-3}$	1.99
$3.91 \cdot 10^{-3}$	$1.41 \cdot 10^{-4}$	2.00	$1.56 \cdot 10^{-3}$	1.99
Voronoi mesh family				
$6.50 \cdot 10^{-2}$	$3.28 \cdot 10^{-2}$	—	0.27	—
$3.15 \cdot 10^{-2}$	$8.48 \cdot 10^{-3}$	1.87	$6.58 \cdot 10^{-2}$	1.96
$1.61 \cdot 10^{-2}$	$2.20 \cdot 10^{-3}$	2.01	$1.63 \cdot 10^{-2}$	2.08
$9.09 \cdot 10^{-3}$	$5.72 \cdot 10^{-4}$	2.36	$4.24 \cdot 10^{-3}$	2.36
$4.26 \cdot 10^{-3}$	$1.42 \cdot 10^{-4}$	1.83	$1.05 \cdot 10^{-3}$	1.84

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