Hybrid High-Order methods for nonlinear problems

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Hybrid High-Order (HHO) methods

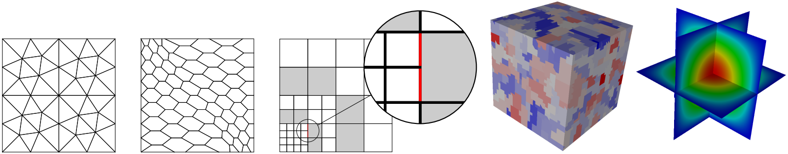


Figure: Examples of supported meshes $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ in 2d and 3d

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Natural extension to nonlinear problems
- Reduced computational cost after static condensation
- **Key idea:** replace spaces and operators with discrete counterparts

References for this presentation

- HHO for Leray–Lions problems
  - Analysis tools and convergence [DP and Droniou, 2017a]
  - Basic error estimates [DP and Droniou, 2017b]
  - Stabilization-free [DP, Droniou, Manzini, 2018]
  - Improved estimates (general meshes) [DP, Droniou, Harnist, 2021]
  - Improved estimates (standard meshes) [Carstensen and Tran, 2020]
- Applications
  - Nonlinear elasticity [Botti, DP, Sochala, 2017]
  - Nonlinear poroelasticity [Botti, DP, Sochala, 2018]
  - Non-Newtonian fluids [Botti, Castanon Quiroz, DP, Harnist, 2020]
- General introduction to HHO methods:

Di Pietro, D. A. and Droniou, J. (2020).
The Hybrid High-Order method for polytopal meshes. Design, analysis, and applications, volume 19 of *Modeling, Simulation and Application*.
Springer International Publishing.

Outline

1. Leray–Lions problems
2. Nonlinear elasticity

Model problem

- Let $\Omega \subset \mathbb{R}^d$ denote a bounded connected polyhedral domain
- Let $p \in (0, +\infty)$ and $p' := \frac{p}{p-1}$
- Consider the problem: Given $f \in L^{p'}(\Omega)$, find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\nabla}u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

- In weak formulation: Find $u \in W_0^{1,p}(\Omega)$ s.t.

$$\int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}u) \cdot \boldsymbol{\nabla}v = \int_\Omega fv \qquad \forall v \in W_0^{1,p}(\Omega).$$

- The key differential operator is the gradient

Flux function

Assumption (Flux function I)

Given $p \in (0, +\infty)$, the Carathéodory function[1] $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is s.t., for for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^d$,

- Growth. There exists a real number $\overline{\sigma} > 0$ s.t.

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \mathbf{0})| \leq \overline{\sigma} |\boldsymbol{\eta}|^{p-1}.$$

- Coercivity. There is a real number $\underline{\sigma} > 0$ s.t.,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq \underline{\sigma} |\boldsymbol{\eta}|^p.$$

- Monotonicity. It holds

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\xi})) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) \geq 0.$$

[1] $\boldsymbol{\sigma}(\boldsymbol{x}, \cdot)$ continuous, $\boldsymbol{\sigma}(\cdot, \boldsymbol{\eta})$ measurable

L^2-orthogonal projectors on local polynomial spaces

- Let a polynomial degree $k \geq 0$ and a mesh element or face X be fixed
- Define the polynomial space

$$\mathbb{P}^k(X) := \{\text{restriction to } X \text{ of } d\text{-variate polynomials of total degree} \leq k\}$$

- The L^2-orthogonal projector $\pi_X^k : L^2(X) \to \mathbb{P}^k(X)$ is s.t.

$$\int_X (\pi_X^k v - v) w = 0 \text{ for all } w \in \mathbb{P}^k(X)$$

- Optimal approximation properties hold [DP and Droniou, 2020]

A key remark

- Let a polytopal mesh element $T \in \mathcal{T}_h$ be fixed
- Recall the following IBP formula, valid for all $(v, \boldsymbol{\tau}) \in H^1(T) \times C^\infty(\overline{T})^d$:

$$\int_T \nabla v \cdot \boldsymbol{\tau} = -\int_T v\ (\nabla \cdot \boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F v\ (\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF})$$

- Given an integer $k \geq 0$, taking $\boldsymbol{\tau} \in \mathbb{P}^k(T)^d$ we can write

$$\int_T \boldsymbol{\pi}_T^k(\nabla v) \cdot \boldsymbol{\tau} = -\int_T \pi_T^k v\ (\nabla \cdot \boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^k v_{|F}\ (\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF})$$

- **Hence, $\boldsymbol{\pi}_T^k(\nabla v)$ can be computed from $\pi_T^k v$ and $(\pi_F^k v_{|F})_{F \in \mathcal{F}_T}$!**

Local HHO space and interpolator

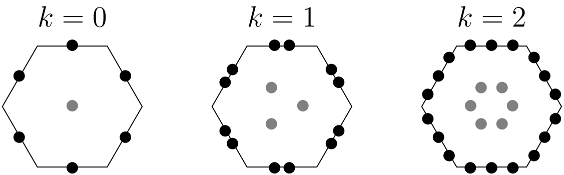


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$ and $d = 2$

- For $k \geq 0$ and $T \in \mathcal{T}_h$, define the local HHO space

$$\underline{U}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \, : \, v_T \in \mathbb{P}^k(T) \text{ and } v_F \in \mathbb{P}^k(F) \text{ for all } F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ is s.t., for all $v \in H^1(T)$,

$$\underline{I}_T^k v := \left(\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T} \right)$$

Gradient reconstruction

- Let $T \in \mathcal{T}_h$. We define the local gradient reconstruction

$$\boldsymbol{G}_T^k : \underline{U}_T^k \to \mathbb{P}^k(T)^d$$

s.t., for all $\underline{v}_T \in \underline{U}_T^k$,

$$\int_T \boldsymbol{G}_T^k \underline{v}_T \cdot \boldsymbol{\tau} = -\int_T v_T \; (\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F v_F \; (\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T)^d$$

- By construction, we have,

$$\boldsymbol{G}_T^k(\underline{I}_T^k v) = \boldsymbol{\pi}_T^k(\boldsymbol{\nabla} v) \quad \forall v \in H^1(T)$$

- $(\boldsymbol{G}_T^k \circ \underline{I}_T^k)$ therefore has optimal approximation properties in $\mathbb{P}^k(T)^d$

Global HHO space and gradient reconstruction

- The global HHO space is obtained patching interface unknowns:

$$\underline{U}_h^k := \Big\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) \ : \\ v_T \in \mathbb{P}^k(T) \text{ for all } T \in \mathcal{T}_h \text{ and } v_F \in \mathbb{P}^k(F) \text{ for all } F \in \mathcal{F}_h \Big\}$$

- The global gradient $\boldsymbol{G}_h^k : \underline{U}_h^k \to \mathbb{P}^k(\mathcal{T}_h)^d$ is s.t.

$$\forall \underline{v}_h \in \underline{U}_h^k, \quad (\boldsymbol{G}_h^k \underline{v}_h)_{|T} := \boldsymbol{G}_T^k \underline{v}_T \quad \forall T \in \mathcal{T}_h$$

- Accounting for boundary conditions, we set

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \ : \ v_F = 0 \text{ for all } F \in \mathcal{F}_h \text{ s.t. } F \subset \partial\Omega \right\}$$

Discrete Sobolev norms

- We need to endow $\underline{U}_h^k$ with a Sobolev structure
- We define the discrete Sobolev seminorms s.t., for all $\underline{v}_h \in \underline{U}_h^k$,

$$\|\underline{v}_h\|_{1,p,h} \coloneqq \left(\sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,p,T}^p \right)^{\frac{1}{p}}$$

where, for all $T \in \mathcal{T}_h$,

$$\|\underline{v}_T\|_{1,p,T} \coloneqq \left(\|\nabla v_T\|_{L^p(T)^d}^p + \sum_{F \in \mathcal{F}_T} h_F^{1-p} \|v_F - v_T\|_{L^p(F)}^p \right)^{\frac{1}{p}}$$

Remark (Scaling and asymptotically small faces)

The factor h_F^{1-p} ensures appropriate scaling. Replacing h_F^{1-p} with h_T^{1-p} enables asymptotically small faces [Droniou and Yemm, 2021].

Discrete functional analysis results I

Theorem (Discrete Sobolev–Poincaré inequalities)

Let

$$1 \le q \le \frac{dp}{d-p} \text{ if } 1 \le p < d \text{ and } 1 \le q < +\infty \text{ if } p \ge d.$$

Then, for all $\underline{v}_h \in \underline{U}_{h,0}^k$*, letting* $v_h \in \mathbb{P}^k(\mathcal{T}_h)$ *be s.t.*

$$(v_h)_{|T} := v_T \qquad \forall T \in \mathcal{T}_h,$$

it holds, with $C > 0$ *depending only on* Ω*,* k*,* p*,* q*, and mesh regularity,*

$$\|v_h\|_{L^q(\Omega)} \le C \|\underline{v}_h\|_{1,p,h}.$$

Corollary (Discrete Sobolev norms)

The mapping $\|\cdot\|_{1,p,h}$ *is a norm on* $\underline{U}_{h,0}^k$*.*

Discrete functional analysis results II

Theorem (Discrete compactness)

Let $(\mathcal{M}_h)_{h>0}$ be a regular mesh sequence and $(\underline{v}_h)_{h>0} \in (\underline{U}_{h,0}^k)_{h>0}$ a sequence such that

$$\|\underline{v}_h\|_{1,p,h} \leq C \text{ for all } h > 0.$$

Then, there exists $v \in W_0^{1,p}(\Omega)$ s.t., up to a subsequence as $h \to 0$,

- $v_h \to v$ *strongly in* $L^q(\Omega)$ *for all* $1 \leq q < \begin{cases} \frac{dp}{d-p} & \text{if } p < d, \\ +\infty & \text{otherwise;} \end{cases}$
- $\boldsymbol{G}_h^k \underline{v}_h \rightharpoonup \boldsymbol{\nabla} v$ *weakly in* $L^p(\Omega)^d$.

Proposition (Strong convergence of the discrete gradient for smooth functions)

With $(\mathcal{M}_h)_{h>0}$ as before it holds, for all $\varphi \in W^{1,p}(\Omega)$,

$$\boldsymbol{G}_h^k(\underline{I}_h^k \varphi) \to \boldsymbol{\nabla}\varphi \text{ strongly in } L^p(\Omega)^d \text{ as } h \to 0.$$

Discrete problem I

- Define the function $\mathrm{a}_h : \underline{U}_h^k \times \underline{U}_h^k \to \mathbb{R}$ s.t.

$$\mathrm{a}_h(\underline{w}_h, \underline{v}_h) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{G}_h^k \underline{w}_h) \cdot \boldsymbol{G}_h^k \underline{v}_h + \sum_{T \in \mathcal{T}_h} \mathrm{s}_T(\underline{w}_T, \underline{v}_T)$$

- Above, s_T is a stabilization obtained penalizing face residuals s.t.
  - $\|\boldsymbol{G}_T^k \underline{v}_T\|_{L^p(T)^d}^p + \mathrm{s}_T(\underline{v}_T, \underline{v}_T) \simeq \|\underline{v}_T\|_{1,p,T}^p$ uniformly in h
  - $\mathrm{s}_T(\underline{I}_T^k w, \underline{v}_T) = 0$ for all $(w, \underline{v}_T) \in \mathbb{P}^{k+1}(T) \times \underline{U}_T^k$
  - Hölder continuity and strong monotonicity hold

Discrete problem II

The discrete Leray–Lions problem reads:

$$\text{Find } \underline{u}_h \in \underline{U}_{h,0}^k \text{ s.t. } \mathrm{a}_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k \qquad (\Pi_{\mathrm{ll}})$$

Lemma (Existence and a priori bound)

Problem (Π_{ll}) *admits at least one solution, and any solution* $\underline{u}_h \in \underline{U}_{h,0}^k$ *to this problem satisfies the a priori bound*

$$\|\underline{u}_h\|_{1,p,h} \le C \|f\|_{L^{p'}(\Omega)}^{\frac{1}{p-1}},$$

with real number $C > 0$ *independent of* h*.*

Remark (Uniqueness)

Uniqueness can be proved replacing monotonicity with strict monotonicity.

Convergence I

Theorem (Convergence)

Let $(\mathcal{M}_h)_{h>0}$ be a regular mesh sequence and let $(\underline{u}_h)_{h>0}$ be the corresponding sequence of discrete solutions. Then, as $h \to 0$, up to a subsequence,

- $u_h \to u$ *strongly in* $L^q(\Omega)$ *with* $1 \le q < \begin{cases} \frac{dp}{d-p} & \text{if } p < d, \\ +\infty & \text{otherwise,} \end{cases}$
- $\boldsymbol{G}_h^k \underline{u}_h \rightharpoonup \boldsymbol{\nabla} u$ *weakly in* $L^p(\Omega)^d$,

with $u \in W_0^{1,p}(\Omega)$ solution to the continuous problem. If, additionally, $\boldsymbol{\sigma}$ is strictly monotone, then u is unique and $\boldsymbol{G}_h^k \underline{u}_h$ converges strongly.

Convergence II

Proof.

- Combining the a priori bound with discrete compactness, we infer the existence of $u \in W_0^{1,p}(\Omega)$ s.t. the above convergences hold
- Taking $\underline{v}_h = \underline{I}_h^k \varphi$ as test function with $\varphi \in C_c^\infty(\Omega)$ and using Minty's trick, we infer that u solves the continuous problem
- Using Vitali's theorem, we prove strong convergence of $\boldsymbol{G}_h^k \underline{u}_h$ under strict monotonicity of $\boldsymbol{\sigma}$

□

Error estimates I

Assumption (Flux function II)

In addition to Assumption I, it holds, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^d$,

- Hölder continuity. There exists a real number $\sigma^* > 0$ s.t.

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\xi})| \leq \sigma^* |\boldsymbol{\eta} - \boldsymbol{\xi}| \left(|\boldsymbol{\eta}|^{p-2} + |\boldsymbol{\xi}|^{p-2}\right).$$

- Strong monotonicity. There exists a real number $\sigma_* > 0$ s.t.

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\xi})) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) \geq \sigma_* |\boldsymbol{\eta} - \boldsymbol{\xi}|^2 \left(|\boldsymbol{\eta}| + \boldsymbol{\xi}\right)^{p-2}.$$

Remark (p-Laplacian)

The above assumptions are verified by the p-Laplace flux function

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) = |\boldsymbol{\eta}|^{p-2} \boldsymbol{\eta}.$$

Error estimates II

Theorem (Basic error estimate)

Assume $u \in W^{k+2,p}(\mathcal{T}_h)$ *and* $\boldsymbol{\sigma}(\cdot, \nabla u) \in W^{k+1,p'}(\mathcal{T}_h)^d$ *and let*

- *if* $p \geq 2$,

$$\mathcal{E}_h(u) := h^{k+1}|u|_{W^{k+2,p}(\mathcal{T}_h)} + h^{\frac{k+1}{p-1}}\left(|u|^{\frac{1}{p-1}}_{W^{k+2,p}(\mathcal{T}_h)} + |\boldsymbol{\sigma}(\cdot, \nabla u)|^{\frac{1}{p-1}}_{W^{k+1,p'}(\mathcal{T}_h)^d}\right);$$

- *if* $p < 2$,

$$\mathcal{E}_h(u) := h^{(k+1)(p-1)}|u|^{p-1}_{W^{k+2,p}(\mathcal{T}_h)} + h^{k+1}|\boldsymbol{\sigma}(\cdot, \nabla u)|_{W^{k+1,p'}(\mathcal{T}_h)^d}.$$

Then, it holds

$$\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h} \leq C\mathcal{E}_h(u),$$

with $C > 0$ *depending only on* Ω, k, p, $\underline{\sigma}$, $\overline{\sigma}$, σ_*, σ^*, *and mesh regularity.*

Improved error estimates

- The above estimate gives the following asymptotic convergence rates:

$$\begin{cases} h^{\frac{k+1}{p-1}} & \text{if } p \geq 2, \\ h^{(k+1)(p-1)} & \text{if } 1 < p < 2 \end{cases}$$

- Successively [DP, Droniou, Harnist, 2021] proved

$$h^{k+1} \text{ in the non-degenerate case for } 1 < p \leq 2,$$

with intermediate rates depending on a degeneracy parameter

- Very recently, [Carstensen and Tran, 2020] proved convergence in

$$h^{\frac{k+1}{3-p}} \text{ for } 1 < p \leq 2$$

for a variation of the HHO method on conforming simplicial meshes based on a stable gradient inspired by [DP, Droniou, Manzini, 2018]

Numerical example

Convergence for $p = 3$

h	$\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$	EOC
	$k = 1$ (1)	
$3.07 \cdot 10^{-2}$	$1.71 \cdot 10^{-2}$	—
$1.54 \cdot 10^{-2}$	$4.72 \cdot 10^{-3}$	1.87
$7.68 \cdot 10^{-3}$	$1.16 \cdot 10^{-3}$	2.02
$3.84 \cdot 10^{-3}$	$2.96 \cdot 10^{-4}$	1.97
$1.92 \cdot 10^{-3}$	$7.77 \cdot 10^{-5}$	**1.93**
	$k = 2$ ($\frac{3}{2}$)	
$3.07 \cdot 10^{-2}$	$2.72 \cdot 10^{-3}$	—
$1.54 \cdot 10^{-2}$	$2.32 \cdot 10^{-4}$	3.57
$7.68 \cdot 10^{-3}$	$3.32 \cdot 10^{-5}$	2.79
$3.84 \cdot 10^{-3}$	$7.25 \cdot 10^{-6}$	2.2
$1.92 \cdot 10^{-3}$	$1.81 \cdot 10^{-6}$	**2.00**
	$k = 3$ (2)	
$3.07 \cdot 10^{-2}$	$3.1 \cdot 10^{-4}$	—
$1.54 \cdot 10^{-2}$	$2.97 \cdot 10^{-5}$	3.4
$7.68 \cdot 10^{-3}$	$4.4 \cdot 10^{-6}$	2.74
$3.84 \cdot 10^{-3}$	$9.76 \cdot 10^{-7}$	2.17
$1.92 \cdot 10^{-3}$	$2.41 \cdot 10^{-7}$	**2.02**

Table: Triangular mesh family

h	$\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$	EOC
	$k = 1$ (1)	
$6.5 \cdot 10^{-2}$	$3.06 \cdot 10^{-2}$	—
$3.15 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	1.41
$1.61 \cdot 10^{-2}$	$3.35 \cdot 10^{-3}$	1.77
$9.09 \cdot 10^{-3}$	$1.25 \cdot 10^{-3}$	1.72
$4.26 \cdot 10^{-3}$	$3.58 \cdot 10^{-4}$	**1.65**
	$k = 2$ ($\frac{3}{2}$)	
$6.5 \cdot 10^{-2}$	$1.18 \cdot 10^{-2}$	—
$3.15 \cdot 10^{-2}$	$2.33 \cdot 10^{-3}$	2.24
$1.61 \cdot 10^{-2}$	$4.4 \cdot 10^{-4}$	2.48
$9.09 \cdot 10^{-3}$	$1.02 \cdot 10^{-4}$	2.56
$4.26 \cdot 10^{-3}$	$1.42 \cdot 10^{-5}$	**2.60**
	$k = 3$ (2)	
$6.5 \cdot 10^{-2}$	$2.75 \cdot 10^{-3}$	—
$3.15 \cdot 10^{-2}$	$2.69 \cdot 10^{-4}$	3.21
$1.61 \cdot 10^{-2}$	$4.01 \cdot 10^{-5}$	2.84
$9.09 \cdot 10^{-3}$	$1.31 \cdot 10^{-5}$	1.96
$4.26 \cdot 10^{-3}$	$2.21 \cdot 10^{-6}$	**2.35**

Table: Voronoi mesh family

Outline

Model problem I

- Let $d \in \{2,3\}$. Given $\boldsymbol{f} \in L^2(\Omega)^d$, the nonlinear elasticity problem reads: Find $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ s.t.

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) &= \boldsymbol{f} \quad \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega,
\end{aligned}
$$

  with $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ strain-stress law and strain operator

$$
\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u} := \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^\top\right)
$$

- In weak formulation: Find $\boldsymbol{u} \in H^1_0(\Omega)^d$ s.t.

$$
\int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) : \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \quad \forall \boldsymbol{v} \in H^1_0(\Omega)^d
$$

- The extension of stability results is non-trivial

Model problem II

Example (Linear elasticity)

Given a uniformly elliptic fourth-order tensor-valued function $\boldsymbol{C} : \Omega \to \mathbb{R}^{d^4}$, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = \boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau}.$$

For homogeneous isotropic media, $\boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau} = 2\mu\boldsymbol{\tau} + \lambda \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_d$.

Example (Hencky–Mises model)

Given $\lambda : \mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu(\operatorname{dev}(\boldsymbol{\tau}))\boldsymbol{\tau}\boldsymbol{I}_d + \lambda(\operatorname{dev}(\boldsymbol{\tau})) \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_d,$$

where $\operatorname{dev}(\boldsymbol{\tau}) \coloneqq \operatorname{tr}(\boldsymbol{\tau}^2) - d^{-1} \operatorname{tr}(\boldsymbol{\tau})^2$.

Model problem III

Example (Isotropic damage model)

Given the damage function $D : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to (0,1)$ and $\boldsymbol{C}$ as above, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = (1 - D(\boldsymbol{\tau}))\, \boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau}.$$

Example (Second-order model)

Given Lamé parameters μ, λ and second-order moduli A, B, C, for all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d + A\boldsymbol{\tau}^2 + B\,\mathrm{tr}(\boldsymbol{\tau}^2)\boldsymbol{I}_d + 2B\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{\tau} + C\,\mathrm{tr}(\boldsymbol{\tau})^2\boldsymbol{I}_d.$$

Strain-stress law

For the sake of simplicity, we focus on the Hilbertian case:

Assumption (Strain-stress law I)

The Carathéodory function $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is s.t., for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau}, \boldsymbol{\upsilon} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$,

- Growth. There exists a real number $\overline{\sigma} > 0$ s.t.

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \mathbf{0})| \leq \overline{\sigma}|\boldsymbol{\tau}|.$$

- Coercivity. There is a real number $\underline{\sigma} > 0$ s.t.,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) : \boldsymbol{\tau} \geq \underline{\sigma}|\boldsymbol{\tau}|^2.$$

- Monotonicity. It holds

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\upsilon})) : (\boldsymbol{\tau} - \boldsymbol{\upsilon}) \geq 0.$$

Local HHO space and strain reconstruction

- Given $T \in \mathcal{T}_h$, the vector version of the local HHO space is

$$\underline{\boldsymbol{U}}_T^k := \left\{ \underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}) \ : \ \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \text{ for all } F \in \mathcal{F}_T \right\}$$

furnished with the strain seminorm

$$\|\underline{\boldsymbol{v}}_T\|_{\boldsymbol{\varepsilon},T}^2 := \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{v}_T\|_{L^2(T)^{d\times d}}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^2(F)^d}^2$$

- By similar principles as before, we define the strain reconstruction

$$\boldsymbol{G}_{{\rm s},T}^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T; \mathbb{R}_{\rm sym}^{d\times d})$$

s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$ and all $\boldsymbol{\tau} \in \mathbb{P}^k(T; \mathbb{R}_{\rm sym}^{d\times d})$,

$$\int_T \boldsymbol{G}_{{\rm s},T}^k \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T \cdot (\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F \boldsymbol{v}_F \cdot (\boldsymbol{\tau}\boldsymbol{n}_{TF})$$

- With $\underline{\boldsymbol{I}}_T^k$ interpolator on $\underline{\boldsymbol{U}}_T^k$, $\boldsymbol{G}_{{\rm s},T}^k(\underline{\boldsymbol{I}}_T^k \boldsymbol{v}) = \boldsymbol{\pi}_T^k(\boldsymbol{\nabla}_{\rm s}\boldsymbol{v})$ for all $\boldsymbol{v} \in H^1(T)^d$

Global HHO space and strain reconstruction

- At the global level, we define the space

$$\underline{\boldsymbol{U}}_h^k := \Big\{\underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}) \;:\; \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \text{ for all } T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \text{ for all } F \in \mathcal{F}_h \Big\}$$

  along with its subspace with strongly enforced BC

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \;:\; \boldsymbol{v}_F = \boldsymbol{0} \text{ for all } F \in \mathcal{F}_h \text{ s.t. } F \subset \partial\Omega\right\}$$

- We denote by $\|{\cdot}\|_{\boldsymbol{\varepsilon},h}$ the global strain norm
- The global strain reconstruction $\boldsymbol{G}_{\mathrm{s},h}^k : \underline{\boldsymbol{U}}_h^k \to \mathbb{P}^k(\mathcal{T}_h;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is s.t.

$$\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k, \quad (\boldsymbol{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h)_{|T} := \boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T \quad \forall T \in \mathcal{T}_h$$

Local stabilization

As for the scalar case, we define the function $\mathrm{a}_h : \underline{\boldsymbol{U}}_h^k \times \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ s.t.

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) \coloneqq \int_\Omega \boldsymbol{\sigma}(\boldsymbol{G}_{\mathrm{s},h}^k \underline{\boldsymbol{u}}_h) : \boldsymbol{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h + \sum_{T \in \mathcal{T}_h} \mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

with bilinear forms $\mathrm{s}_T : \underline{\boldsymbol{U}}_T^k \times \underline{\boldsymbol{U}}_T^k \to \mathbb{R}$, $T \in \mathcal{T}_h$, satisfying:

Assumption (Stabilization bilinear form)

- Symmetry and positivity. s_T is symmetric and positive semidefinite.
- Stability. It holds, uniformly in h: For all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\|\boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T\|_{L^2(T)^{d\times d}}^2 + \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \simeq \|\underline{\boldsymbol{v}}_T\|_{\boldsymbol{\varepsilon},T}^2.$$

- Polynomial consistency. $\forall (\boldsymbol{w}, \underline{\boldsymbol{v}}_T) \in \mathbb{P}^{k+1}(T)^d \times \underline{\boldsymbol{U}}_T^k$, $\mathrm{s}_T(\underline{\boldsymbol{I}}_T^k \boldsymbol{w}, \underline{\boldsymbol{v}}_T) = 0$.

Remark (Polynomial degree)

Stability and polynomial consistency are incompatible for $k = 0$!

Discrete Korn inequality

Theorem (Discrete Korn inequality)

Assume $k \geq 1$. *Then, for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$, *letting* $\boldsymbol{v}_h \in \mathbb{P}^k(\mathcal{T}_h)^d$ *be s.t.* $(\boldsymbol{v}_h)_{|T} := \boldsymbol{v}_T$ *for all* $T \in \mathcal{T}_h$,

$$\|\boldsymbol{v}_h\|_{L^2(\Omega)^d} + \|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d\times d}} \lesssim \|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},h} \lesssim \underline{\sigma}^{-\frac{1}{2}} \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h)^{\frac{1}{2}}.$$

Proof.

- Prove a Korn inequality on broken polynomial spaces using the node-averaging operator on a simplicial submesh: For all $\boldsymbol{v}_h \in \mathbb{P}^k(\mathcal{T}_h)^d$,

$$\|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d\times d}}^2 \lesssim \|\boldsymbol{\nabla}_{\mathrm{s},h} \boldsymbol{v}_h\|_{L^2(\Omega)^{d\times d}}^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \|[\boldsymbol{v}_h]_F\|_{L^2(F)^d}^2$$

- Use $\|\cdot\|_{\boldsymbol{\varepsilon},h}$ to control the jumps via a triangle inequality
- Use the coercivity of $\boldsymbol{\sigma}$ to conclude □

Discrete problem and existence

The discrete elasticity problem reads:

$$\text{Find } \underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k \text{ s.t. } \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \text{ for all } \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k \qquad (\Pi_{\mathrm{el}})$$

Theorem (Existence and convergence)

Assume $k \geq 1$. Then, problem (Π_{el}) admits at least one solution, and any solution $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ to this problem satisfies the a priori bound

$$\|\underline{\boldsymbol{u}}_h\|_{\boldsymbol{\varepsilon},h} \leq C\|\boldsymbol{f}\|_{L^2(\Omega)^d},$$

with $C > 0$ depending only on Ω, $\underline{\sigma}$, $\overline{\sigma}$, k, and the mesh regularity parameter. Moreover, denoting by $(\underline{\boldsymbol{u}}_h)_{h>0}$ the sequence of discrete solutions on a regular mesh sequence $(\mathcal{M}_h)_{h>0}$, as $h \to 0$, up to a subsequence,

- $\boldsymbol{u}_h \to \boldsymbol{u}$ *strongly in* $L^q(\Omega)^d$ *with* $1 \leq q < \begin{cases} +\infty & \text{if } d = 2, \\ 6 & \text{if } d = 3, \end{cases}$
- $\boldsymbol{G}_{\mathrm{s},h}^k \underline{\boldsymbol{u}}_h \rightharpoonup \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}$ *weakly in* $L^2(\Omega)^{d\times d}$,

with $\boldsymbol{u} \in H_0^1(\Omega)^d$ solution to the continuous problem.

Error estimate

Assumption (Strain-stress law II)

In addition to Assumption I, it holds, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau}, \boldsymbol{\upsilon} \in \mathbb{R}^{d\times d}_{\rm sym}$,

- Hölder continuity. There exists a real number $\sigma^* > 0$ s.t.

$$|\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\upsilon})| \le \sigma^* |\boldsymbol{\tau} - \boldsymbol{\upsilon}|.$$

- Strong monotonicity. There exists a real number $\sigma_* > 0$ s.t.

$$(\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\upsilon})) : (\boldsymbol{\tau} - \boldsymbol{\upsilon}) \ge \sigma_* |\boldsymbol{\tau} - \boldsymbol{\upsilon}|^2.$$

Theorem (Error estimate)

Under Assumption II, and further assuming $k \ge 1$ and star-shaped elements. Then, if $\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h)^d$ and $\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\rm s}\boldsymbol{u}) \in H^{k+1}(\mathcal{T}_h)^{d\times d}$,

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{\boldsymbol{\varepsilon},h} \le C h^{k+1} \left(|\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^d} + |\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\rm s}\boldsymbol{u})|_{H^{k+1}(\mathcal{T}_h)^{d\times d}} \right),$$

with C depending only on Ω, $\overline{\sigma}$, $\underline{\sigma}$, σ^*, σ_*, k, the mesh regularity and an upper bound of $\|\boldsymbol{f}\|_{L^2(\Omega)^d}$.

- We let $\boldsymbol{\sigma}$ be given by the Hencky–Mises model
- We set $\Omega = (0,1)^2$ and consider the following displacement field

$$\boldsymbol{u}(\boldsymbol{x}) = \big(\sin(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\sin(\pi x_2)\big)$$

- $\boldsymbol{f}$ is inferred from the exact solution

Numerical examples II

Convergence

h	$\lVert \nabla_s \boldsymbol{u} - \boldsymbol{G}^k_{s,h} \underline{\boldsymbol{u}}_h \rVert$	EOC	$\lVert \boldsymbol{\pi}^k_h \boldsymbol{u} - \boldsymbol{u}_h \rVert$	EOC
		$k = 1$		
$3.07 \cdot 10^{-2}$	$5.59 \cdot 10^{-2}$	—	$7.32 \cdot 10^{-3}$	—
$1.54 \cdot 10^{-2}$	$1.51 \cdot 10^{-2}$	1.9	$1.05 \cdot 10^{-3}$	2.81
$7.68 \cdot 10^{-3}$	$3.86 \cdot 10^{-3}$	1.96	$1.34 \cdot 10^{-4}$	2.96
$3.84 \cdot 10^{-3}$	$1.01 \cdot 10^{-3}$	1.93	$1.7 \cdot 10^{-5}$	2.98
$1.92 \cdot 10^{-3}$	$2.59 \cdot 10^{-4}$	**1.96**	$2.15 \cdot 10^{-6}$	2.98
		$k = 2$		
$3.07 \cdot 10^{-2}$	$1.3 \cdot 10^{-2}$	—	$1.47 \cdot 10^{-3}$	—
$1.54 \cdot 10^{-2}$	$1.29 \cdot 10^{-3}$	3.35	$6.05 \cdot 10^{-5}$	4.62
$7.68 \cdot 10^{-3}$	$2.11 \cdot 10^{-4}$	2.6	$5.36 \cdot 10^{-6}$	3.48
$3.84 \cdot 10^{-3}$	$2.73 \cdot 10^{-5}$	2.95	$3.6 \cdot 10^{-7}$	3.9
$1.92 \cdot 10^{-3}$	$3.42 \cdot 10^{-6}$	**3.00**	$2.28 \cdot 10^{-8}$	3.98
		$k = 3$		
$3.07 \cdot 10^{-2}$	$2.81 \cdot 10^{-3}$	—	$2.39 \cdot 10^{-4}$	—
$1.54 \cdot 10^{-2}$	$3.72 \cdot 10^{-4}$	2.93	$1.95 \cdot 10^{-5}$	3.63
$7.68 \cdot 10^{-3}$	$2.16 \cdot 10^{-5}$	4.09	$5.47 \cdot 10^{-7}$	5.14
$3.84 \cdot 10^{-3}$	$1.43 \cdot 10^{-6}$	3.92	$1.66 \cdot 10^{-8}$	5.04
$1.92 \cdot 10^{-3}$	$9.51 \cdot 10^{-8}$	**3.91**	$5.34 \cdot 10^{-10}$	4.96

Table: Triangular mesh family

Numerical examples III

Convergence

h	$\lVert \nabla_{\mathrm{s}} \boldsymbol{u} - \boldsymbol{G}^k_{\mathrm{s},h} \underline{\boldsymbol{u}}_h \rVert$	EOC	$\lVert \boldsymbol{\pi}^k_h \boldsymbol{u} - \boldsymbol{u}_h \rVert$	EOC
		$k = 1$		
$6.3 \cdot 10^{-2}$	0.22	—	$2.75 \cdot 10^{-2}$	—
$3.42 \cdot 10^{-2}$	$3.72 \cdot 10^{-2}$	2.89	$3.73 \cdot 10^{-3}$	3.27
$1.72 \cdot 10^{-2}$	$7.17 \cdot 10^{-3}$	2.4	$4.83 \cdot 10^{-4}$	2.97
$8.59 \cdot 10^{-3}$	$1.44 \cdot 10^{-3}$	2.31	$6.14 \cdot 10^{-5}$	2.97
$4.3 \cdot 10^{-3}$	$2.4 \cdot 10^{-4}$	**2.59**	$7.7 \cdot 10^{-6}$	3.00
		$k = 2$		
$6.3 \cdot 10^{-2}$	$2.68 \cdot 10^{-2}$	—	$3.04 \cdot 10^{-3}$	—
$3.42 \cdot 10^{-2}$	$7.01 \cdot 10^{-3}$	2.2	$3.56 \cdot 10^{-4}$	3.51
$1.72 \cdot 10^{-2}$	$1.09 \cdot 10^{-3}$	2.71	$3.31 \cdot 10^{-5}$	3.46
$8.59 \cdot 10^{-3}$	$1.41 \cdot 10^{-4}$	2.95	$2.53 \cdot 10^{-6}$	3.7
$4.3 \cdot 10^{-3}$	$1.96 \cdot 10^{-5}$	**2.85**	$1.72 \cdot 10^{-7}$	3.89
		$k = 3$		
$6.3 \cdot 10^{-2}$	$1.11 \cdot 10^{-2}$	—	$1.08 \cdot 10^{-3}$	—
$3.42 \cdot 10^{-2}$	$1.92 \cdot 10^{-3}$	2.87	$9.29 \cdot 10^{-5}$	4.02
$1.72 \cdot 10^{-2}$	$2.79 \cdot 10^{-4}$	2.81	$6.13 \cdot 10^{-6}$	3.95
$8.59 \cdot 10^{-3}$	$2.54 \cdot 10^{-5}$	3.45	$2.88 \cdot 10^{-7}$	4.4
$4.3 \cdot 10^{-3}$	$1.61 \cdot 10^{-6}$	**3.99**	$1.24 \cdot 10^{-8}$	4.55

Table: Hexagonal mesh family

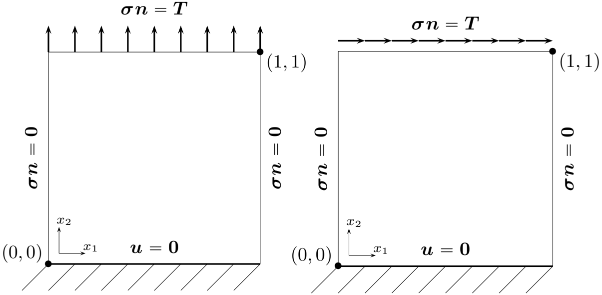
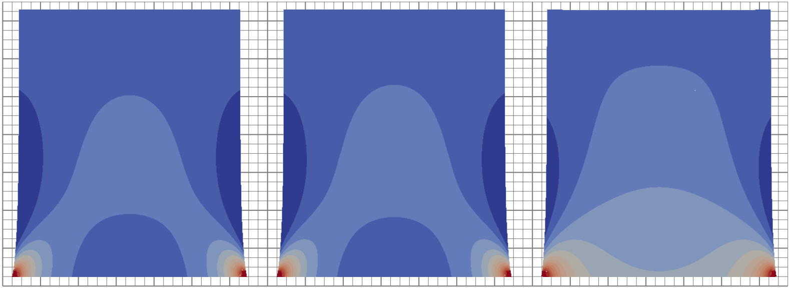


Figure: Shear and tensile test cases

Numerical examples II

Tensile and shear test cases

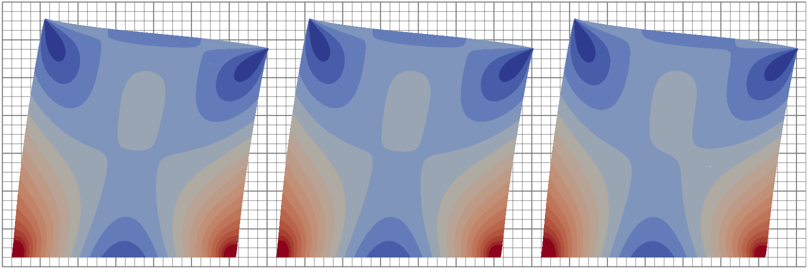


(a) Linear (b) Hencky–Mises (c) Second order

Figure: Tensile test case. Stress norm on the deformed domain. Values in $10^5 \mathrm{Pa}$

Numerical examples III

Tensile and shear test cases



(a) Linear (b) Hencky–Mises (c) Second order

Figure: Shear test case. Stress norm on the deformed domain. Values in $10^4 \mathrm{Pa}$

Numerical examples IV

Tensile and shear test cases

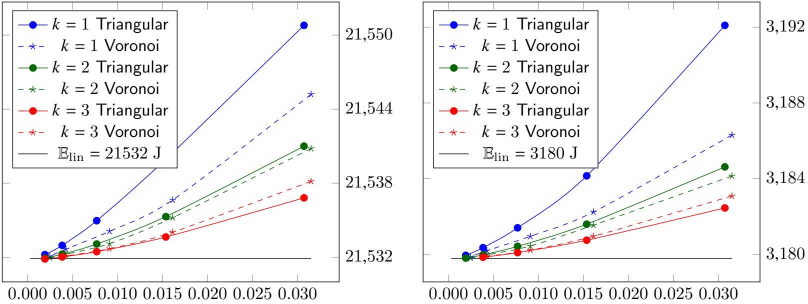


Figure: Energy vs h, tensile and shear test cases, linear model

Tensile and shear test cases

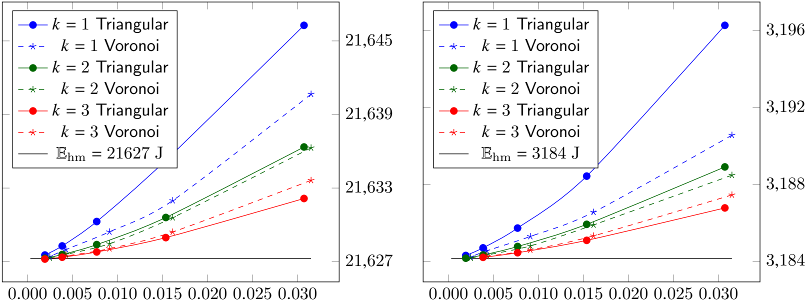


Figure: Energy vs h, tensile and shear test cases, Hencky–Mises model

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