Nonconforming methods for PDEs with diffusion

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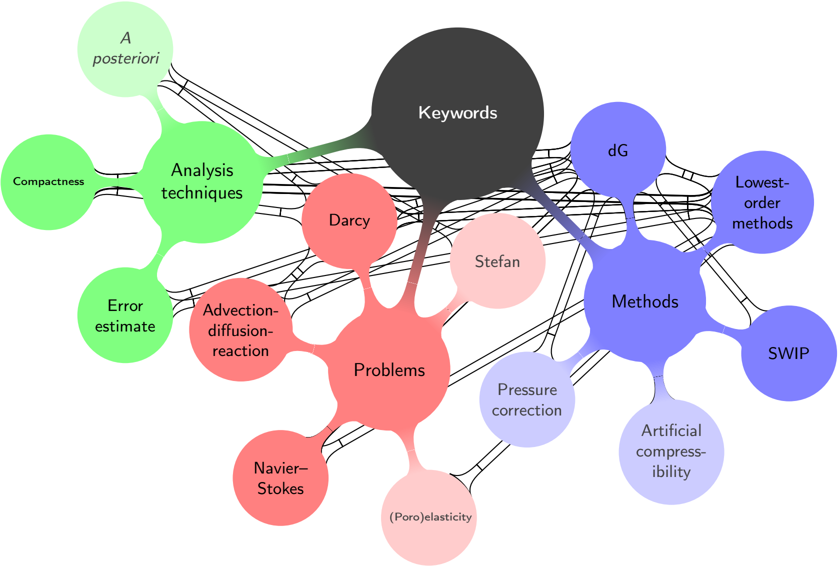
University of Sussex, May 6th, 2011

Overview



Keywords
A
posteriori
Compactness
Analysis
techniques
Error
estimate
Darcy
Stefan
Advection-
diffusion-
reaction
Problems
Navier–
Stokes
(Poro)elasticity
dG
Lowest-
order
methods
Methods
SWIP
Pressure
correction
Artificial
compress-
ibility

Outline

Broken polynomial spaces on general meshes

Highlights on discontinuous Galerkin methods

Functional analysis results on broken polynomial spaces

Cell centered Galerkin methods

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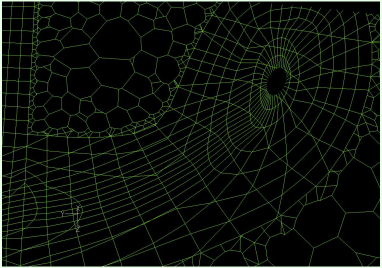
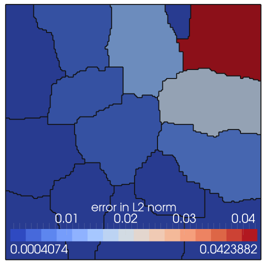
Motivations and goals

- Avoid remeshing (e.g. in subsoil modeling)
- Improve domain/solution fitting
- Improve performance (fewer DOFs, reduced fill-in)
- Nonconforming/aggregative mesh adaptivity
- See [Di Pietro & Ern, 2011b, Ch. 1]

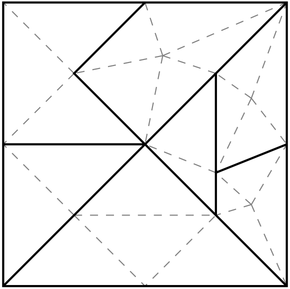
Example





Admissible mesh sequences I

- Let $\Omega \subset \mathbb{R}^d$ be an open connected polyhedral domain
- Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a sequence of refined meshes of Ω
- All $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{S}_h = \{S\}$



Admissible mesh sequences II

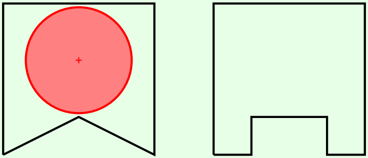
Trace and inverse inequalities

- $(\mathfrak{S}_h)_{h\in\mathcal{H}}$ is shape-regular in the sense of Ciarlet
- Every simplex $S \subset T$ is s.t. $h_S \approx h_T$

Optimal polynomial approximation

Every element T is star-shaped w.r.t. a ball of diameter $\delta_T \approx h_T$

Example (Admissible and non-admissible element)



Broken spaces

- For $1 \leqslant p < \infty$ we introduce the broken Sobolev spaces

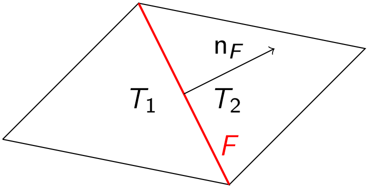
$$H^p(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v|_T \in H^p(T)\}$$

- We define the broken polynomial spaces of total degree $\leqslant k$

$$\mathbb{P}_d^k(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v|_T \in \mathbb{P}_d^k(T)\}$$

- We denote by ∇_h the broken gradient operator

Trace operators and dG-norm



- For all $F \subset \partial T_1 \cap \partial T_2$ and all $v \in H^{\frac{1}{2}}(\mathcal{T}_h)$ we let

$$[\![v]\!] := v|_{T_1} - v|_{T_2}, \quad \{v\} := \frac{1}{2}\left(v|_{T_1} + v|_{T_2}\right)$$

- We introduce the dG-norm on $H_0^1(\mathcal{T}_h)$

$$\|v\|_{\mathrm{dG}} := \left(\|\nabla_h v\|^2_{L^2(\Omega)^d} + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|^2_{L^2(F)} \right)^{1/2}$$

Outline

Broken polynomial spaces on general meshes

Highlights on discontinuous Galerkin methods

- Heterogeneous diffusion
- Vanishing diffusion with advection

Functional analysis results on broken polynomial spaces

Cell centered Galerkin methods

Motivations and goals

- Darcy flow through highly heterogeneous anisotropic media
- Reactive transport problems with singular interfaces
- Convergence to nonsmooth solutions
- See [Di Pietro et al., 2008, Di Pietro & Ern, 2011a]
- Important references
  - [Gastaldi & Quarteroni, 1989]
  - [Houston, Schwab, & Süli, 2002]
  - [Burman & Zunino, 2006]

The SWIP method I

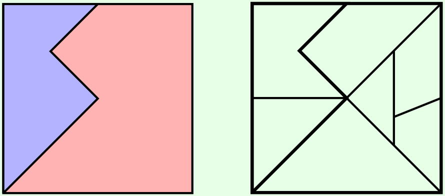
$$-\nabla\cdot(\kappa\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- There is a partition P_Ω s.t.

$$\kappa \in \mathbb{P}^0_d(P_\Omega) \text{ with } 0 < \underline{\lambda} \leqslant \kappa \leqslant \overline{\lambda}$$

- For all $h \in \mathcal{H}$, $\mathcal{T}_h$ is compatible with P_Ω

Example (Partition P_Ω and compatible mesh)



The SWIP method II

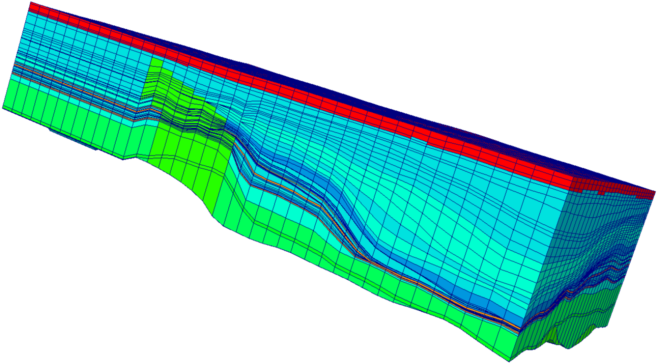


Figure: Example of partition P_{Ω}: stratigraphy of a sedimentary bassin

The SWIP method III

$$a_h^{\text{sip}}(w, v_h) := \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h + \sum_{F \in \mathcal{F}_h} \int_F \frac{\eta}{h_F} [\![w]\!][\![v_h]\!] - \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h w\} \cdot \mathrm{n}_F [\![v_h]\!] - \sum_{F \in \mathcal{F}_h} \int_F [\![w]\!] \{\kappa \nabla_h v_h\} \cdot \mathrm{n}_F$$

Error estimate (SIP method of [Arnold, 1982])

Assume $u \in V_* := H_0^1(\Omega) \cap H^2(P_\Omega)$ and let $u_h \in V_h := \mathbb{P}_d^k(\mathcal{T}_h)$ discrete solution with $k \geqslant 1$. Then, $\exists C \neq C(h, \kappa)$ s.t.

$$\|u - u_h\|_{\text{dG}} \lesssim \left(1 + C \frac{\overline{\lambda}}{\underline{\lambda}}\right) \inf_{v_h \in V_h} \|u - v_h\|_{\text{dG},*}$$

Estimate not robust w.r.t. the heterogeneity of $\kappa \implies$ SWIP method

The SWIP method IV

$$a_h^{\mathrm{swip}}(w, v_h) := \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h + \sum_{F \in \mathcal{F}_h} \int_F \eta \frac{\gamma_{\kappa,F}}{h_F} [\![w]\!] [\![v_h]\!]$$

$$- \sum_{F \in \mathcal{F}_h} \int_F [\{\kappa \nabla_h w\}_\omega \cdot \mathrm{n}_F [\![v_h]\!] + [\![w]\!] \{\kappa \nabla_h v_h\}_\omega \cdot \mathrm{n}_F]$$

- Weighted averages + harmonic mean in penalty

$$\{v\}_\omega := \tfrac{\kappa_2}{\kappa_1+\kappa_2} v_1 + \tfrac{\kappa_1}{\kappa_1+\kappa_2} v_2, \qquad \gamma_{\kappa,F} := 2 \tfrac{\kappa_1 \kappa_2}{\kappa_1+\kappa_2}$$

- Data-dependent energy norm on $H^1(\mathcal{T}_h)$

$$|\!|\!| v |\!|\!|_{\mathrm{swip}}^2 := \|\kappa^{\frac{1}{2}} \nabla_h v\|_{L^2(\Omega)^d}^2 + \sum_{F \in \mathcal{F}_h} \frac{\gamma_{\kappa,F}}{h_F} \|[\![v]\!]\|_{L^2(F)}^2$$

The SWIP method V

Error estimate (SWIP)

Assume $u \in V_*$. Then, $\exists C \neq C(h,\kappa)$ s.t.

$$|||u - u_h|||_{\text{swip}} \leqslant C \inf_{v_h \in V_h} |||u - v_h|||_{\text{swip},*}$$

Corollary (Convergence rate)

If, moreover $u \in H^{k+1}(P_\Omega)$, $\exists C \neq C(h,\kappa)$ s.t.

$$|||u - u_h|||_{swip} \lesssim C\overline{\lambda}^{-1/2} h^k \|u\|_{H^{k+1}(P_\Omega)}.$$

- The error estimate is robust w.r.t. κ
- Yields FV 2pt method for suitable $\mathcal{T}_h$ and κ
- Minor modifications allow to treat the case $u \in H^{3/2+\epsilon}(P_\Omega)$

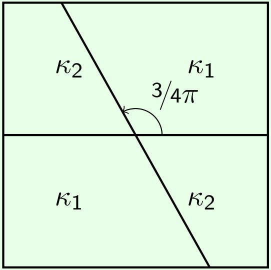
Convergence of the SWIP method to nonsmooth solutions

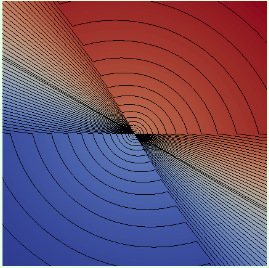
- However, in general we only have [Nicaise & Sändig, 1994]

$$u \in W^{2,p}(P_\Omega) \Rightarrow u \in H^{1+\alpha}(P_\Omega), \quad \alpha = 1 + 2\,(1 - 1/p) > 0$$

- Also in this case optimal convergence estimates can be proven for $d = 2$ [Di Pietro & Ern, 2011a]

Example ($u \in H^{1.29}(P_\Omega)$, $\kappa_1/\kappa_2 = 30$)





Application to vanishing diffusion with advection I

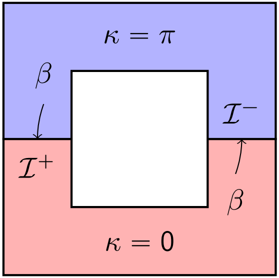
$$\nabla\cdot(-\kappa\nabla u + \beta u) + \mu u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

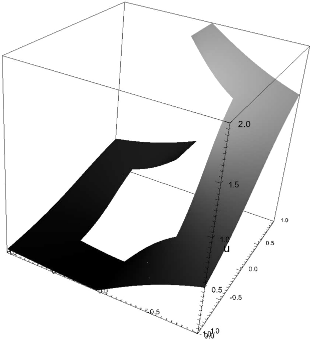
- Let $\beta \in [W^{1,\infty}(\Omega)]^d$, $\mu > 0$ with $0 < \mu_0 \leqslant \mu - 1/2\nabla\cdot\beta$ and

$$0 \leqslant \underline{\lambda} \leqslant \kappa \leqslant \overline{\lambda},$$

- u may have discontinuities, c.f. [Di Pietro et al., 2008]

Application to vanishing diffusion with advection II





- Flux continuity $[\![-\kappa\nabla u + \beta u]\!]\cdot \mathrm{n}_F = 0$ on $\mathcal{I}^\pm$
- Potential continuity $[\![u]\!] = 0$ on $\mathcal{I}^+$

Goal: Automatic detection of singular interfaces

Application to vanishing diffusion with advection III

$$a_h^{\text{dar}}(w, v_h) := a_h^{\text{swip}}(w, v_h) + a_h^{\text{upw}}(w, v_h) + \int_\Omega \mu w v_h$$

Energy norm error estimate

Using SWIP diffusion + upwind advection, $\exists C \neq C(h, \kappa)$ s.t.

$$|||u - u_h|||_{\text{dar}} \lesssim C \inf_{w_h \in V_h} |||u - w_h|||_{\text{dar},*},$$

with $|||\cdot|||_{\text{dar}}$ inf-sup norm and $|||\cdot|||_{\text{dar},*}$ continuity norm.

- $\kappa \equiv 0 \implies$ [Johnson & Pitkäranta, 1986]
- $\beta \equiv 0$, $\kappa > 0 \implies$ [Arnold, Brezzi, Cockburn, & Marini, 2002]

Outline

Motivations and goals

- Discrete Sobolev embeddings for broken polynomial spaces
- Discrete compactness for diffusive problems
- Convergence to minimal regularity solutions
- See [Di Pietro & Ern, 2010, Agélas, Di Pietro et al., 2010]
- Important references
  - [Karakashian & Jureidini, 1998]
  - [Lasis & Süli, 2003]
  - [Girault, Rivière, & Wheeler, 2005]
  - [Eymard, Gallouët, & Herbin, 2008]
  - [Brenner, 2009]
  - [Buffa & Ortner, 2009]

Sobolev embeddings I

- Define the broken Sobolev spaces

$$W^{s,p}(\mathcal{T}_h) := \{v \in L^p(\Omega) \mid \forall T \in \mathcal{T}_h,\ v|_T \in W^{s,p}(T)\}$$

- We introduce the broken $W^{1,p}(\mathcal{T}_h)$-norms

$$\|v\|_{\mathrm{dG},p} := \left(\|\nabla_h v\|_{L^p(\Omega)^d}^p + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \|[\![v]\!]\|_{L^p(F)}^p \right)^{1/p}$$

Sobolev embeddings II

Sobolev embeddings in broken polynomial spaces

For all q such that

- $1 \leqslant q \leqslant p^* := \frac{pd}{d-p}$ if $1 \leqslant p < d$
- $1 \leqslant q < \infty$ if $d \leqslant p < \infty$

there exists $\sigma_{p,q}$ such that

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h), \qquad \|v_h\|_{L^q(\Omega)} \leqslant \sigma_{p,q}\|v_h\|_{\mathrm{dG},p}$$

Proof.

- For $p = 1$ use $\|v_h\|_{L^{1*}(\Omega)} \lesssim \|v_h\|_{\mathrm{BV}} \lesssim \|v_h\|_{\mathrm{dG},1}$
- For $1 < p < d$ use L^{1*}-estimate for $|v_h|^\alpha$, Hölder's and trace inequalities
- For $d \leqslant p < \infty$ use the previous point together with the comparison of broken $W^{1,p}(\mathcal{T}_h)$-norms $\square$

Compactness I

- Fix a polynomial degree $l \geqslant 0$ and let $v \in H^1(\mathcal{T}_h)$
- For $F \in \mathcal{F}_h$ the local lifting $\mathrm{r}_F^l([\![v]\!]) \in \mathbb{P}_d^l(\mathcal{T}_h)^d$ solves

$$\int_\Omega \mathrm{r}_F^l([\![v]\!]) \cdot \tau_h = \int_F [\![v]\!] \{\tau_h\}_\omega \cdot \mathrm{n}_F \quad \forall \tau_h \in \mathbb{P}_d^l(\mathcal{T}_h)^d$$

- We define the discrete gradient

$$G_h^l(v) := \nabla_h v - \sum_{F \in \mathcal{F}_h} \mathrm{r}_F^l([\![v]\!])$$

- For smooth functions φ, $\lim_{h \to 0} G_h^l(\pi_h \varphi) = \nabla \varphi$

Compactness II

Compactness

Let $(v_h)_{h\in\mathcal{H}}$ be a sequence in $(\mathbb{P}^k_d(\mathcal{T}_h))_{h\in\mathcal{H}}$ bounded in the $\|\cdot\|_{\mathrm{dG}}$-norm. Then, $\exists v \in H^1_0(\Omega)$ s.t., as $h \to 0$, up to a subsequence

$$\begin{aligned} v_h &\to v && \text{in } L^2(\Omega) \\ G^l_h(v_h) &\rightharpoonup \nabla v && \text{for all } l \geqslant 0 \text{ weakly in } L^2(\Omega)^d \end{aligned}$$

Proof.

- $\exists v \in L^2(\Omega)$ s.t. $v_h \to v$ in $L^2(\Omega)$ up to a subsequence
- $\exists w \in L^2(\mathbb{R}^d)^d$ s.t. $G^l_h(v_h) \rightharpoonup w$ in $L^2(\mathbb{R}^d)^d$
- Using the definition of G^l_h yields for all $\Phi \in C^\infty_0(\mathbb{R}^d)^d$

$$\int_{\mathbb{R}^d} w\cdot\Phi = \lim_{h\to 0}\int_\Omega G^l_h(v_h)\cdot\Phi = -\int_{\mathbb{R}^d} v(\nabla\cdot\Phi)$$

hence $w = \nabla v$ □

Convergence by compactness

Convergence

The sequence of SWIP solutions $(u_h)_{h\in\mathcal{H}}$ is s.t., as $h \to 0$,

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega), \\ \nabla_h u_h &\to \nabla u && \text{strongly in } L^2(\Omega)^d, \\ |u_h|_J &\to 0. \end{aligned}$$

Proof.

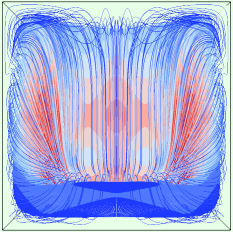
Based on $a_h^{\text{swip}}(w_h, v_h) = \int_\Omega \kappa G_h^l(w_h)\cdot G_h^l(v_h) + j_h(w_h, v_h)$. □

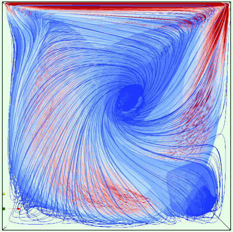
- No need to plugin the exact solution in a_h (no dG paradox)
- SWIP variants in [Agélas, Di Pietro, et al., 2010]

Applications

- Steady Navier–Stokes equations [Di Pietro & Ern, 2010]
- Unsteady Navier–Stokes equations (with pressure correction) [Botti & Di Pietro, 2011]
- Nonlinear Laplacian [Burman & Ern, 2008]
- Cell centered Galerkin methods [Di Pietro, 2010, Di Pietro, 2011a, Di Pietro, 2011b]

Example (3D lid-driven cavity flow, Re $= 1000$)





Outline

Motivations and goals

- Handle general polyhedral meshes
- Improve efficiency and robustness while preserving accuracy
- `FreeFEM`-like platform for lowest-order methods
- Consistent dG methods with 1 DOF per element
- See [Di Pietro, 2010, Di Pietro, 2011a, Di Pietro, 2011b]
- Important references
  - [Aavatsmark et al., 1994–11]
  - [Edwards et al., 1994–11]
  - [Eymard, Gallouët, Herbin et al., 2000–11]
  - [Brezzi, Lipnikov, Shashkov et al., 2005–11]

Discrete space

- Fix the vector space of DOFs, e.g.,

$$\mathbb{V}_h = \mathbb{R}^{\mathcal{T}_h}, \qquad \mathbf{v}_h = (v_T)_{T\in\mathcal{T}_h} \in \mathbb{R}^{\mathcal{T}_h}$$

- Reconstruct a piecewise constant gradient

$$\mathfrak{G}_h : \mathbb{V}_h \to \mathbb{P}_d^0(\mathcal{T}_h)^d$$

- Reconstruct a broken affine function (x_T = cell center)

$$\forall T \in \mathcal{T}_h, \quad \mathfrak{R}_h(\mathbf{v}_h)|_T(x) = v_T + \mathfrak{G}_h(\mathbf{v}_h)|_T \cdot (x - x_T) \in \mathbb{P}_d^1(\mathcal{T}_h)$$

Use as a discrete space in dG methods

$$V_h^{\text{ccg}} := \mathfrak{R}_h(\mathbb{V}_h) \subset \mathbb{P}_d^1(\mathcal{T}_h)$$

A gradient reconstruction based on Green's formula

$$\forall T \in \mathcal{T}_h, \qquad \mathfrak{G}_h(\mathsf{v}_h)|_T = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} \left(v_F - v_T \right) \mathsf{n}_{T,F}$$

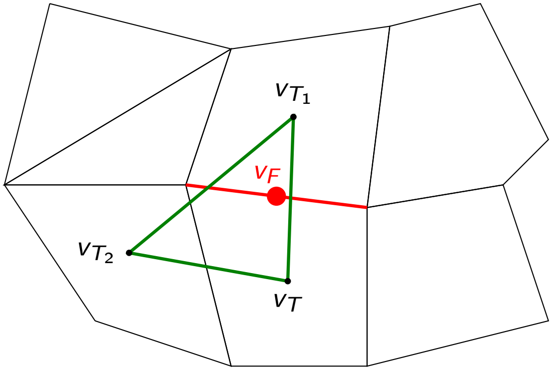


Figure: Barycentric trace reconstruction

Application to the incompressible Navier–Stokes equations I

- We consider the space couple

$$U_h := [V_h^{\text{ccg}}]^d, \quad P_h := \mathbb{P}_d^0(\mathcal{T}_h)/\mathbb{R}$$

- The discrete problem reads

$$\begin{aligned} a_h^{\text{sip}}(u_h, v_h) + t_h(u_h, u_h, v_h) + b_h(v_h, p_h) &= \int_\Omega f \cdot v_h \quad && \forall v_h \in U_h \\ -b_h(u_h, q_h) + s_h(p_h, q_h) &= 0 && \forall q_h \in P_h \end{aligned}$$

- Method inspired by the dG work [Di Pietro & Ern, 2010]
- Fewer DOFs than dG but comparable precision

Application to the incompressible Navier–Stokes equations II

- The pressure-velocity coupling is realized by the bilinear form

$$b_h(v_h, q_h) := -\int_\Omega D_h(v_h) q_h, \qquad D_h(v_h) := \mathrm{tr}(G_h(v_h))$$

- Pressure stabilization required for stability

$$s_h(p_h, q_h) := \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F \frac{h_F}{\nu} [\![p_h]\!] [\![q_h]\!], \qquad |q_h|_p^2 = s_h(q_h, q_h)$$

Lemma (Stability of the pressure-velocity coupling)

There exists $\beta > 0$ independent of the meshsize h s.t.

$$\forall q_h \in P_h, \qquad \beta \|q_h\|_{L^2(\Omega)} \leqslant \sup_{v_h \in U_h \setminus \{0\}} \frac{b_h(v_h, q_h)}{|\!|\!| v_h |\!|\!|} + |q_h|_p.$$

Application to the incompressible Navier–Stokes equations III

- Temam's device for discontinuous approximations
- Non-dissipative formulation
- Asymptotic consistency for smooth/discrete test functions

$$
\begin{aligned}
t_h(w,u,v) := & \int_\Omega (w \cdot \nabla_h u_i) v_i - \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F \{w\} \cdot \mathrm{n}_F [\![u]\!] \cdot \{v\} \\
& + \frac{1}{2} \int_\Omega (\nabla_h \cdot w)(u \cdot v) - \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F [\![w]\!] \cdot \mathrm{n}_F \{u \cdot v\}
\end{aligned}
$$

Application to the incompressible Navier–Stokes equations IV

Lemma (Existence of a discrete solution)

There exists at least one discrete solution $(u_h, p_h) \in X_h$.

Convergence

Let $((u_h, p_h))_{h\in\mathcal{H}}$ be a sequence of approximate solutions on $(\mathcal{T}_h)_{h\in\mathcal{H}}$. Then, as $h \to 0$, up to a subsequence,

$$
\begin{aligned}
u_h &\to u, && \text{in } [L^2(\Omega)]^d,\\
\nabla_h u_h &\to \nabla u, && \text{in } [L^2(\Omega)]^{d,d},\\
|u_h|_{\mathrm{J}} &\to 0, &&\\
p_h &\to p, && \text{in } L^2(\Omega),\\
|p_h|_p &\to 0. &&
\end{aligned}
$$

If (u, p) is unique, the whole sequence converges.

Application to the incompressible Navier–Stokes equations V

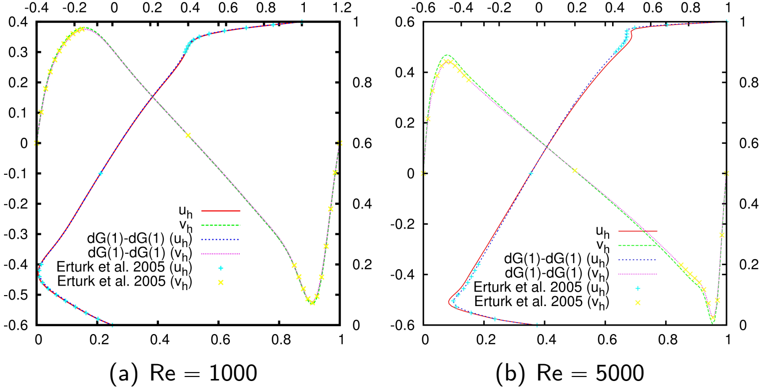


Figure: Lid-driven cavity problem

Thank you for your attention!

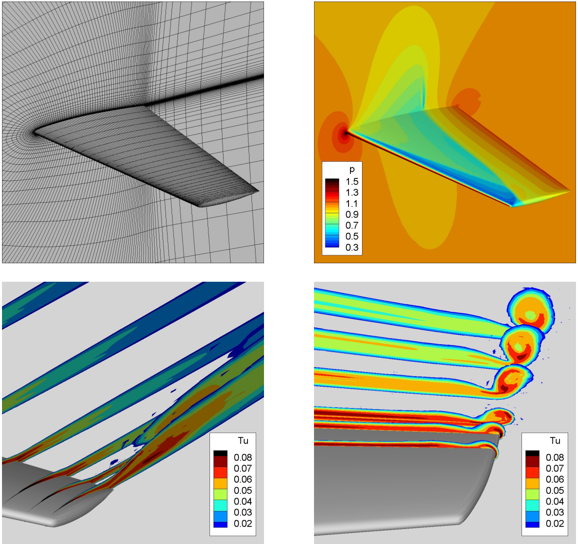


Figure: Onera M6 [Bassi, Crivellini, Di Pietro, & Rebay, 2006]

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Outline

Implementation

Consider, for instance,

$$a_h(w,v) = \int_\Omega \kappa \nabla_h w \cdot \nabla_h v + \sum_{F \in \mathcal{F}_h} \frac{\gamma_F}{h_F} \eta \int_F [\![w]\!][\![v]\!]$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \left[\{\kappa \nabla_h w\}_\omega \cdot \mathrm{n}_F [\![v]\!] + [\![w]\!] \{\kappa \nabla_h v\}_\omega \cdot \mathrm{n}_F \right]$$

- All the integrals in a_h can be evaluated using barycenters
- The general bilinear term $\mathfrak{T}$ can be recast into the form

$$\mathfrak{T} = \chi \mathcal{A}_{\mathfrak{T}}(\mathbf{u}_h) \mathcal{L}_{\mathfrak{T}}(\mathbf{v}_h), \qquad \mathbf{u}_h, \mathbf{v}_h \in \mathbb{V}_h,$$

  where χ is a real coefficient and

$$\mathcal{A}_{\mathfrak{T}}(\mathbf{u}_h) = \alpha_0 + \sum_{T \in \mathcal{T}_{\mathcal{A}}} \alpha_T u_T, \qquad \mathcal{L}_{\mathfrak{T}}(\mathbf{v}_h) = \sum_{T \in \mathcal{T}_{\mathcal{L}}} \lambda_T v_T$$

- The corresponding local contribution is

$$\mathbf{M}_{\mathfrak{T}} = \chi \left[\lambda_T \alpha_{T'} \right]_{T \in \mathcal{T}_{\mathcal{L}},\, T' \in \mathcal{T}_{\mathcal{A}}}, \qquad \mathbf{r}_{\mathfrak{T}} = \chi \left(\lambda_T \alpha_0 \right)_{T \in \mathcal{T}_{\mathcal{L}}}$$

FreeFEM-like implementation of ccG methods I

```
Form2 ah =    // Diffusion
  integrate(All<Cell>::items(𝒯_h),
            sum(1 ⩽ i ⩽ d)(dot(grad(u(i)),grad(v(i))))
            ) +
  integrate(Internal<Face>::items(𝒯_h),
            sum(1 ⩽ i ⩽ d)(−dot(N(),avg(grad(u(i))))*jump(v(i))
                      −jump(u(i))*dot(N(),avg(grad(v(i))))
                      +η/H()*jump(u(i))*jump(v(i)))
            );

Form2 bh =    // Pressure gradient
  integrate(All<Cell>::items(𝒯_h),−id(p)*div(v)) +
  integrate(All<Face>::items(𝒯_h),avg(p)*dot(N(),jump(v)));

Form2 bth =   // Velocity divergence
  integrate(All<Cell>::items(𝒯_h),div(u)*id(q)) +
  integrate(All<Face>::items(𝒯_h),−dot(N(),jump(u))*avg(q));

Form2 sh =    // Pressure stabilization
  integrate(Internal<Face>::items(𝒯_h),H()*jump(p)*jump(q));
```

FreeFEM-like implementation of ccG methods II

```
// Definition of U_h = [V_h^ccg]^d and P_h := P_d^0(T_h)
typedef
FunctionSpace<Span<Polynomial<DIM,1>,
                   GreenFormulaGradient<LInterpolator>
                   >::type CCGSpace;
typedef
FunctionSpace<Span<Polynomial<DIM,0> >::type P0Space;

CCGSpaceType * Uh = CCGSpaceType::create(T_h);
P0SpaceType  * Ph = P0SpaceType::create(T_h);

// Test and trial functions
CCGSpace::TrialVectorFunction u(Uh,"u");
CCGSpace::TestVectorFunction  v(Uh,"v");
P0Space::TrialFunction        p(Ph,"p");
P0Space::TestFunction         q(Ph,"q");
```

See [Di Pietro & Gratien, 2011]

Outline

A density result

Test space $\mathcal{Q}_{\mathcal{T}_h,\kappa}$

Let $\mathcal{Q}_{\mathcal{T}_h,\kappa}$ be the space of functions $\varphi : \overline{\Omega} \to \mathbb{R}$ s.t.

(i) φ is globally continuous and locally smooth

$$\varphi \in C_0(\overline{\Omega}) \cap C^2(\mathcal{T}_h)$$

(ii) the tangential derivatives of φ are continuous across $F \in \mathcal{F}_h^{\mathrm{i}}$

(iii) the diffusive flux of φ is continuous across every $F \in \mathcal{F}_h^{\mathrm{i}}$, i.e.

$$\forall F \subset \partial T_1 \cap \partial T_2, \qquad (\kappa \nabla \varphi)|_{T_1} \cdot \mathrm{n}_F = (\kappa \nabla \varphi)|_{T_2} \cdot \mathrm{n}_F .$$

Then, $\mathcal{Q}_{\mathcal{T}_h,\kappa}$ is dense in $H_0^1(\Omega)$.

Outline

MPFA finite volume methods I

- Consider again the problem

$$-\nabla\cdot(\kappa\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

- Define the space of cell centered DOFs

$$\mathbb{V}_h := \mathbb{R}^{\mathcal{T}_h}, \qquad \mathbf{v}_h = (v_T)_{T\in\mathcal{T}_h} \in \mathbb{V}_h$$

- For $\Phi_F : \mathbb{V}_h \to \mathbb{R}$ numerical flux and $\epsilon_{T,F} := \mathrm{n}_T\cdot\mathrm{n}_F$,

$$\text{Find } \mathbf{u}_h \in \mathbb{V}_h \text{ s.t. for all } T \in \mathcal{T}_h, \sum_{F\in\mathcal{F}_T} \epsilon_{T,F}\Phi_F(\mathbf{u}_h) = \langle f\rangle_T$$

Goal: Consistent + stable + inexpensive Φ_F on general meshes

MPFA finite volume methods II

An abstract convergence result

Assume

- $\mathcal{Q}$ is a test space dense in $H_0^1(\Omega)$ s.t. $\mathcal{Q} \subset C_0(\overline{\Omega}) \cap C^2(\mathcal{T}_h)$
- Coercivity. The FV bilinear form is coercive uniformly in h
- Consistency. For all $\varphi \in \mathcal{Q}$ with $\boldsymbol{\varphi}_h := (\varphi(x_T))_{T\in\mathcal{T}_h}$

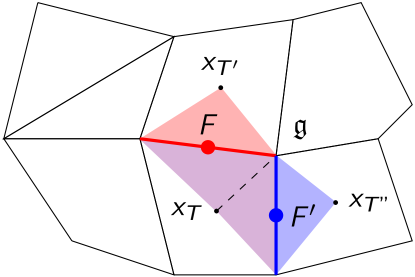
$$\lim_{h\to 0} \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \frac{d_{T,F}}{|F|_{d-1}} \left| \Phi_F(\boldsymbol{\varphi}_h) - |F|_{d-1} \langle \kappa \nabla\varphi \rangle_T \cdot \mathrm{n}_{T,F} \right|^2 = 0$$

Then,

$$u_h \to u \text{ in } L^2(\Omega)$$

with, for all $T \in \mathcal{T}_h$, $u_h|_T = u_T$. Under mild assumptions we can also reconstruct a strongly convergent gradient.

The L-construction



For a group $\mathfrak{g} = \{F, F'\}$ and a $\mathbf{v}_h \in \mathbb{V}_h$ we construct the function $\xi^{\mathfrak{g}}_{\mathbf{v}_h}$ s.t.

- $\xi^{\mathfrak{g}}_{\mathbf{v}_h}$ is affine in each coloured patch
- $\xi^{\mathfrak{g}}_{\mathbf{v}_h}(x_K) = v_K$ for all $K \in \{T, T', T''\}$
- $\xi^{\mathfrak{g}}_{\mathbf{v}_h}$ is continuous and has continuous diffusive flux across F and F'

The G-method I

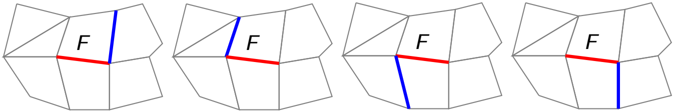


Figure: $\mathcal{G}_F$ = Faces sharing an element and a node with F

The flux F is a convex linear combination of subfluxes

$$\forall \mathbf{v}_h \in \mathbb{V}_h, \qquad \Phi_F(\mathbf{v}_h) := \sum_{\mathfrak{g}\in\mathcal{G}_F} \theta_F^{\mathfrak{g}} (\kappa \nabla \xi_{\mathbf{v}_h}^{\mathfrak{g}})|_T \cdot \mathrm{n}_F$$

The G-method II

Convergence

Let $(\mathbf{u}_h)_{h\in\mathcal{H}}$ be the sequence of approximate solutions on $(\mathcal{T}_h)_{h\in\mathcal{H}}$ and let u_h be s.t. $u_h|_T = u_T$ for all $T \in \mathcal{T}_h$. Then, as $h \to 0$

$$u_h \to u \text{ strongly in } L^2(\Omega) \ .$$

- First analysis of L-type methods
- The L-method of [Aavatsmark et al., 2008] is obtained as a special case
- The analysis does not rely on the analogy with mixed FEs!

Outline

Table: Convergence results for Kovasznay's problem

$\text{card}(\mathcal{T}_h)$	$\|u - u_h\|_{[L^2(\Omega)]^d}$	ord	$\|p - p_h\|_{L^2(\Omega)}$	ord
224	1.5288e-01	–	2.5693e-01	–
896	4.1691e-02	1.87	1.0847e-01	1.24
3584	1.1115e-02	1.91	4.0251e-02	1.43
14336	2.9261e-03	1.93	1.7487e-02	1.20
57344	7.6622e-04	**1.93**	8.7005e-03	**1.01**

$\text{card}(\mathcal{T}_h)$	$\|\|\|(u - u_h, p - p_h)\|\|\|_{\text{sto}}$	ord
224	4.5730e-01	–
896	2.1185e-01	1.11
3584	1.0319e-01	1.04
14336	5.1495e-02	1.00
57344	2.6540e-02	**0.96**

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