Discontinuous Galerkin methods and applications

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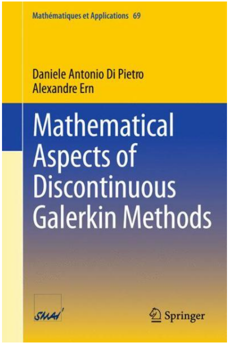
Reference for this course



D. A. Di Pietro and A. Ern,
Mathematical Aspects of Discontinuous Galerkin Methods,
Number 69 in Mathématiques & Applications, Springer, Berlin, 2011

Introduction I

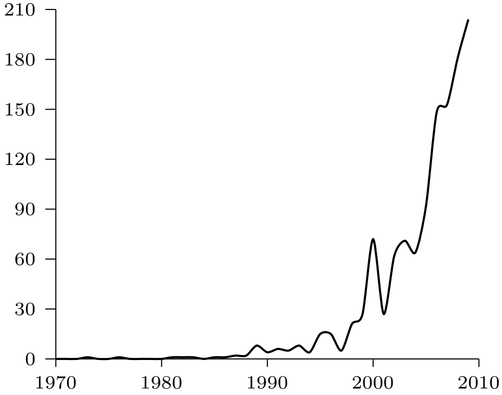
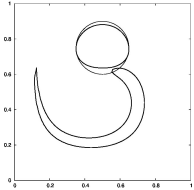
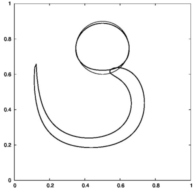


Figure: Entries with the keyword "discontinuous Galerkin" in MathSciNet

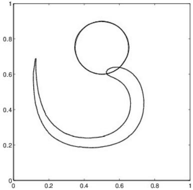
Introduction II



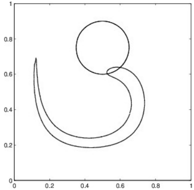
(a) SUPG (4800)



(b) SUPG (13300)



(c) dG-P3 (5120)



(d) dG-P3 (13520)

Figure: Accuracy in advective problems [DP et al., 2006]

Introduction III

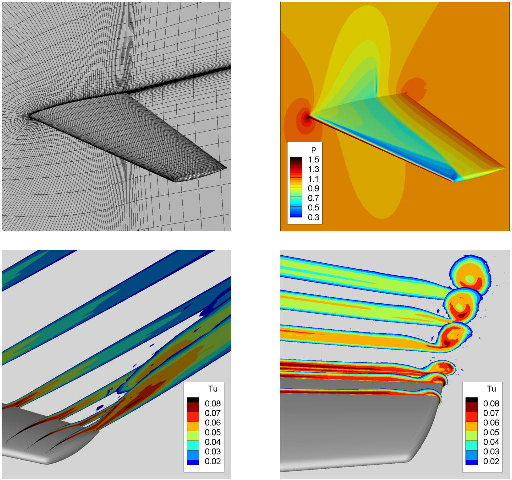


Figure: Unsteady compressible Navier–Stokes, Onera M6 wing [Bassi, Crivellini, DP, & Rebay, 2006]

Introduction IV

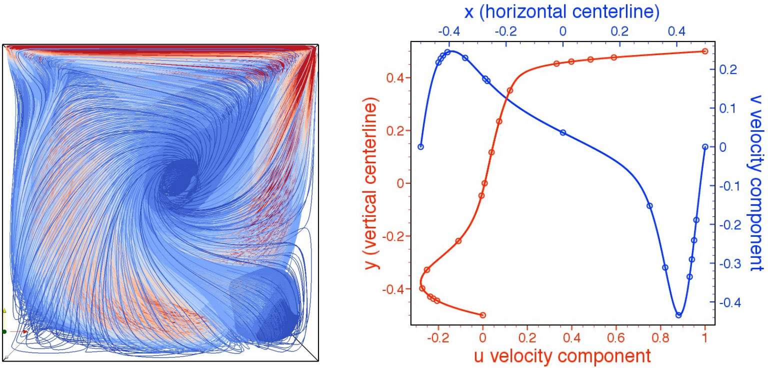


Figure: High-order accuracy in convection-dominated flows (3d lid-driven cavity, [Botti and DP, 2011])

Introduction V

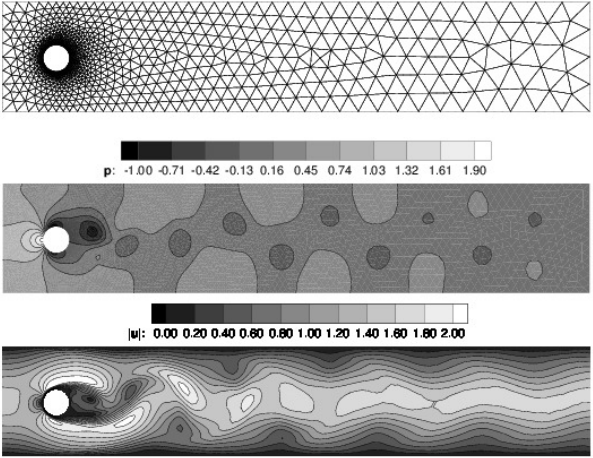
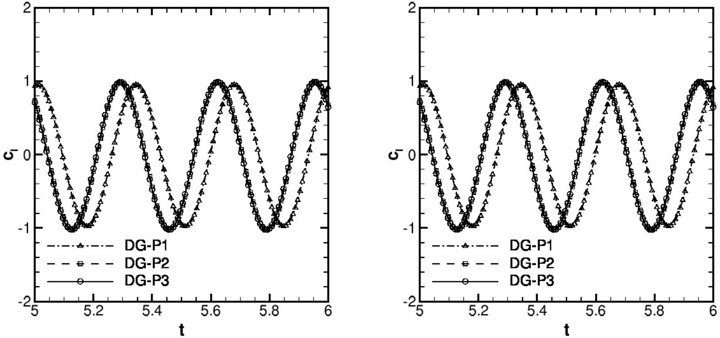


Figure: Unsteady incompressible Navier–Stokes, Turek cylinder [Bassi, Crivellini, DP, & Rebay, 2007]

Introduction VI



(a) Lift coefficient (b) Drag coefficient

Figure: High-order in space-time

Introduction VII

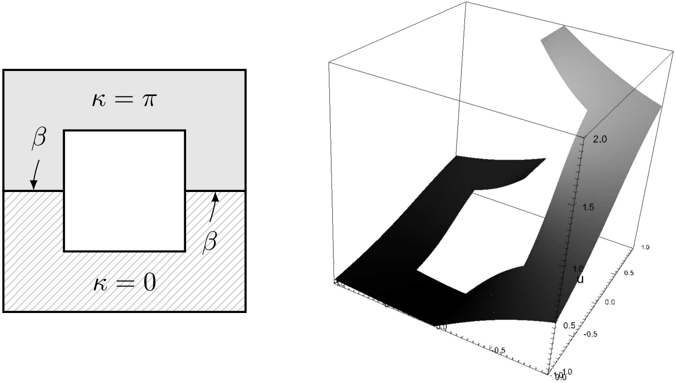
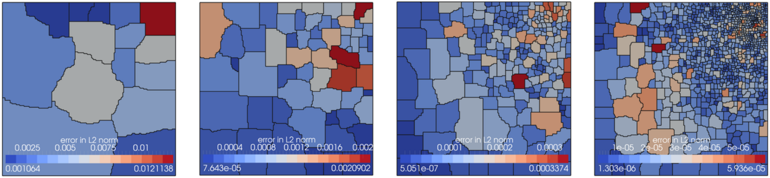


Figure: Degenerate advection-diffusion [DP et al., 2008]



(a) 15 el. (b) 63 el. (c) 250 el. (d) 1024 el.

Figure: Adaptive derefinement [Bassi, Botti, Colombo, DP, Tesini, 2012]

The origins: First-order PDEs

- [Reed and Hill, 1973], dG for steady neutron transport
- [Lesaint and Raviart, 1974], first error estimate
- [Johnson and Pitkäranta, 1986], improved estimate
- [Cockburn and Shu, 1989], explicit Runge–Kutta dG methods

The origins: Second-order PDES

- [Nitsche, 1971], boundary penalty methods
- [Babuška and Zlámal, 1973], Interior Penalty for bcs
- [Arnold, 1982], Symmetric Interior Penalty (SIP) dG method
- [Bassi and Rebay, 1997], compressible Navier–Stokes equations
- [Arnold et al., 2002], unified analysis

Part I

Basic concepts

Outline

Faces, averages, and jumps I

Definition (Mesh)

A mesh $\mathcal{T}$ of Ω is a finite collection of disjoint open polyhedra $\mathcal{T} = \{T\}$ s.t. $\bigcup_{T\in\mathcal{T}} \overline{T} = \overline{\Omega}$. Each $T \in \mathcal{T}$ is called a mesh element.

Definition (Element diameter, meshsize)

Let $\mathcal{T}$ be a mesh of Ω. For all $T \in \mathcal{T}$, h_T denotes the diameter T, and the meshsize is defined as

$$h := \max_{T\in\mathcal{T}} h_T.$$

We use the notation $\mathcal{T}_h$ for a mesh $\mathcal{T}$ with meshsize h.

Faces, averages, and jumps II

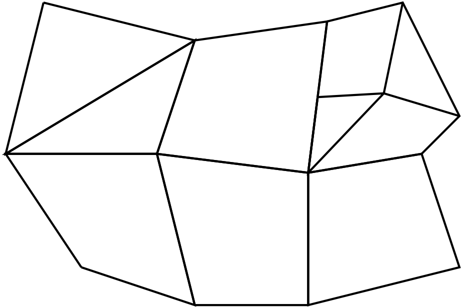


Figure: Example of mesh

Faces, averages, and jumps III

Definition (Mesh faces)

Let $\mathcal{T}_h$ be a mesh of the domain Ω. A closed subset F of $\overline{\Omega}$ is a mesh face if $|F|_{d-1} > 0$ and either one of the two following conditions holds:

- $\exists T_1, T_2 \in \mathcal{T}_h$, $T_1 \neq T_2$, s.t. $F = \partial T_1 \cap \partial T_2$ (interface);
- $\exists T \in \mathcal{T}_h$ s.t. $F = \partial T \cap \partial \Omega$ (boundary face).

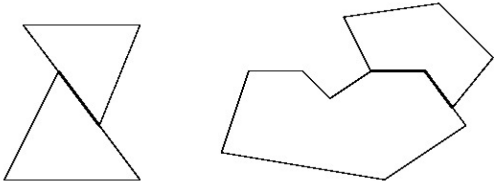


Figure: Examples of interfaces

Faces, averages, and jumps IV

- Interfaces are collected in $\mathcal{F}_h^i$, boundary faces in $\mathcal{F}_h^b$, and

$$\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^b.$$

- For all $T \in \mathcal{T}_h$ we let

$$\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\},$$

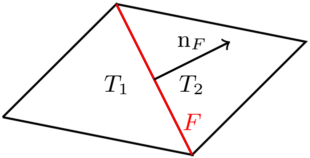
  and we set

$$N_\partial := \max_{T \in \mathcal{T}_h} \operatorname{card}(\mathcal{F}_T)$$

- Symmetrically, for all $F \in \mathcal{F}_h$, we let

$$\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$$

Faces, averages, and jumps V



Definition (Interface averages and jumps)

Assume $v : \Omega \to \mathbb{R}$ smooth enough to admit a possibly two-valued trace on all interfaces. Then, for all $F \in \mathcal{F}_h^i$ we let

$$\{\!\{v\}\!\} := \frac{1}{2}(v|_{T_1} + v|_{T_2}), \quad [\![v]\!] := v|_{T_1} - v|_{T_2}.$$

For all $F \in \mathcal{F}_h^b$ with $F \subset \partial T$ we conventionally set $\{\!\{v\}\!\} = [\![v]\!] = v|_T$.

Broken polynomial spaces I

k	$d = 1$	$d = 2$	$d = 3$
0	1	1	1
1	2	3	4
2	3	6	10
3	4	10	20

Table: Dimension of $\mathbb{P}_d^k$ for $1 \leq d \leq 3$ and $0 \leq k \leq 3$

Discontinuous Galerkin methods hinge on broken polynomial spaces,

$$\boxed{\mathbb{P}_d^k(\mathcal{T}_h) := \left\{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v|_T \in \mathbb{P}_d^k(T)\right\}}$$

Hence, the number of DOFs is

$$\dim(\mathbb{P}_d^k(\mathcal{T}_h)) = \operatorname{card}(\mathcal{T}_h) \times \operatorname{card}(\mathbb{P}_d^k) = \operatorname{card}(\mathcal{T}_h) \times \frac{(k+d)!}{k!d!}$$

Broken polynomial spaces II

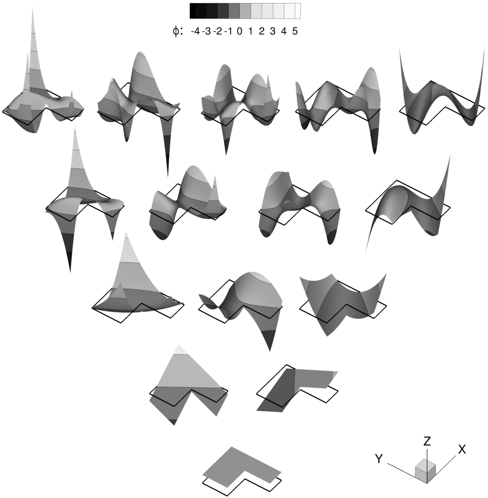


Figure: Orthonormal polynomial basis functions for an L-shaped element

Basic facts on Lebesgue and Sobolev spaces I

- Let $v : \Omega \to \mathbb{R}$ be Lebesgue measurable
- Let $1 \leq p \leq \infty$ be a real number. We set

$$\|v\|_{L^p(\Omega)} := \left(\int_\Omega |v|^p \right)^{1/p} \qquad 1 \leq p < \infty,$$

  and

$$\|v\|_{L^\infty(\Omega)} := \inf\{M > 0 \mid |v(x)| \leq M \text{ a.e. } x \in \Omega\}$$

- In either case, we define the Lebesgue space

$$\boxed{L^p(\Omega) := \{v \text{ Lebesgue measurable} \mid \|v\|_{L^p(\Omega)} < \infty\}}$$

Basic facts on Lebesgue and Sobolev spaces II

- Equipped with $\|\cdot\|_{L^p(\Omega)}$, $L^p(\Omega)$ is a Banach space for all p
- $L^2(\Omega)$ is a Hilbert space when equipped with the scalar product

$$(v, w)_{L^2(\Omega)} := \int_\Omega vw$$

- We record the Cauchy–Schwarz inequality: For all $v, w \in L^2(\Omega)$,

$$(v, w)_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}$$

Basic facts on Lebesgue and Sobolev spaces III

- Let ∂_i denote the distributional partial derivative with respect to x_i
- For a d-uple $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ we note

$$\partial^\alpha v := \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d} v$$

- For an integer $m \geq 0$ we define the Sobolev space

$$\boxed{H^m(\Omega) = \left\{ v \in L^2(\Omega) \mid \forall \alpha \in A_d^m,\ \partial^\alpha v \in L^2(\Omega) \right\}}$$

Basic facts on Lebesgue and Sobolev spaces IV

- $H^m(\Omega)$ is a Hilbert space when equipped with the scalar product

$$(v, w)_{H^m(\Omega)} := \sum_{\alpha \in A_d^m} (\partial^\alpha v, \partial^\alpha w)_{L^2(\Omega)},$$

  leading to (with $A_d^k := \{\alpha \in \mathbb{N}^d \mid |\alpha|_{\ell^1} \leq k\}$),

$$\|v\|_{H^m(\Omega)} := \left(\sum_{\alpha \in A_d^m} \|\partial^\alpha v\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad |v|_{H^m(\Omega)} := \left(\sum_{\alpha \in \overline{A}_d^m} \|\partial^\alpha v\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

- For $m = 1$, letting $\nabla v = (\partial_1 v, \ldots, \partial_d v)^t$ yields

$$(v, w)_{H^1(\Omega)} = (v, w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{[L^2(\Omega)]^d}$$

Basic facts on Lebesgue and Sobolev spaces V

- It is useful to record the following trace inequality:

$$\|v\|_{L^2(\partial\mathcal{D})} \leq C\|v\|_{L^2(\mathcal{D})}^{1/2}\|v\|_{H^1(\mathcal{D})}^{1/2},$$

which implies that functions in $H^1(\mathcal{D})$ have traces in $L^2(\partial\mathcal{D})$

Broken Sobolev spaces and broken gradient I

- In the analysis we need to formulate local regularity requirements for the exact solution
- To this purpose we introduce the broken Sobolev spaces

$$\boxed{H^m(\mathcal{T}_h) := \left\{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v|_T \in H^m(T)\right\}}$$

- Clearly, $H^m(\Omega) \subset H^m(\mathcal{T}_h)$
- Owing to the trace inequality,

functions in $H^1(\mathcal{T}_h)$ have trace in $L^2(\partial T)$ for all $T \in \mathcal{T}_h$

Broken Sobolev spaces and broken gradient II

Definition (Broken gradient)

The broken gradient $\nabla_h : H^1(\mathcal{T}_h) \to [L^2(\Omega)]^d$ is defined s.t.

$$\forall v \in H^1(\mathcal{T}_h), \qquad (\nabla_h v)|_T := \nabla(v|_T) \qquad \forall T \in \mathcal{T}_h.$$

Broken Sobolev spaces and broken gradient III

Lemma (Characterization of $H^1(\Omega)$)

A function $v \in H^1(\mathcal{T}_h)$ belongs to $H^1(\Omega)$ if and only if

$$[\![v]\!] = 0 \qquad \forall F \in \mathcal{F}_h^i.$$

Moreover there holds, for all $v \in H^1(\Omega)$,

$$\nabla_h v = \nabla v \text{ in } [L^2(\Omega)]^d.$$

Abstract nonconforming error analysis I

- Let X be a function space s.t.

$$X \hookrightarrow L^2(\Omega) \equiv L^2(\Omega)' \hookrightarrow X'$$

with dense and continuous injection

Abstract nonconforming error analysis II

- We consider the model linear problem

$$\boxed{\text{Find } u \in X \text{ s.t. } a(u,w) = \langle f, w\rangle_{X',X} \text{ for all } w \in X} \qquad (\Pi)$$

with a bounded bilinear form in $X \times X$ and $f \in X'$

- For $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$ the dG problem reads

$$\boxed{\text{Find } u_h \in V_h \text{ s.t. } a_h(u_h,w_h) = l_h(w_h) \text{ for all } w_h \in V_h} \qquad (\Pi_h)$$

with a_h bilinear form on $V_h \times V_h$ and l_h linear form on V_h

- In general dG methods are nonconforming, i.e.,

$$\boxed{V_h = \mathbb{P}_d^k(\mathcal{T}_h) \not\subset X}$$

Abstract nonconforming error analysis III

- We formulate general conditions to bound the error

$$\|u - u_h\|$$

in terms of the approximation properties of V_h,

$$\inf_{y_h \in V_h} \|u - y_h\|_*$$

- In the analysis of dG methods we often have

$$\|\cdot\| \neq \|\cdot\|_*$$

Abstract nonconforming error analysis IV

Definition (Discrete stability)

We say that the discrete bilinear form a_h enjoys discrete stability on V_h if there is $C_{\mathrm{sta}} > 0$ independent of h s.t.

$$\forall v_h \in V_h, \qquad C_{\mathrm{sta}} \|v_h\| \le \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{\|w_h\|}, \qquad \text{(inf-sup)}$$

or, equivalently,

$$C_{\mathrm{sta}} \le \inf_{v_h \in V_h \setminus \{0\}} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{\|v_h\| \|w_h\|}.$$

Stability is a purely discrete property which is intimately linked with the well-posedness of the discrete problem

Abstract nonconforming error analysis V

- A sufficient condition for discrete stability is coercivity,

$$\boxed{\forall v_h \in V_h, \qquad C_{\mathrm{sta}} \| v_h \|^2 \leq a_h(v_h, v_h)}$$

- Discrete coercivity implies (inf-sup) since, for all $v_h \in V_h \setminus \{0\}$,

$$C_{\mathrm{sta}} \| v_h \| \leq \frac{a_h(v_h, v_h)}{\| v_h \|} \leq \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{\| w_h \|}$$

Abstract nonconforming error analysis VI

- For consistency we need to plug u into the first argument of a_h
- However, in most cases a_h cannot be extended to $X \times V_h$

Assumption (Regularity of the exact solution)

We assume that there is $X_* \subset X$ s.t.

- a_h *can be extended to* $X_* \times V_h$ *and*
- *the exact solution* u *is s.t.* $u \in X_*$.

Abstract nonconforming error analysis VII

Definition (Consistency)

The discrete problem (Π_h) is consistent if for the exact solution $u \in X_*$,

$$a_h(u, w_h) = l_h(w_h) \qquad \forall w_h \in V_h. \qquad \text{(cons.)}$$

Lemma (Galerkin orthogonality)

If $u \in X_*$ and a_h is consistent, Galerkin orthogonality holds, i.e.,

$$a_h(u - u_h, w_h) = 0 \qquad \forall w_h \in V_h.$$

Abstract nonconforming error analysis VIII

$$\boxed{X_{*h} := X_* + V_h}$$

- The error $u - u_h$ belongs to X_{*h}
- It is often not possible to express boundedness in terms of the $\|\cdot\|$ norm, so we introduce a second norm $\|\cdot\|_*$ s.t.

$$\forall v \in X_{*h}, \qquad \|v\| \leq \|v\|_*$$

Definition (Boundedness)

We say that the discrete bilinear form a_h is bounded in $X_{*h} \times V_h$ if there is C_{bnd} independent of h s.t.

$$\forall (v, w_h) \in X_{*h} \times V_h, \qquad |a_h(v, w_h)| \leq C_{\text{bnd}} \|v\|_* \|w_h\|.$$

Abstract nonconforming error analysis IX

Theorem (Abstract error estimate)

Let u solve (Π) and assume $u \in X_*$. Then, assuming discrete stability, consistency, and boundedness, there holds

$$\|u - u_h\| \leq \left(1 + \frac{C_{\rm bnd}}{C_{\rm sta}}\right) \inf_{y_h \in V_h} \|u - y_h\|_*. \qquad \text{(est.)}$$

Abstract nonconforming error analysis X

$$\boxed{\inf_{y_h \in V_h} \|u - y_h\| \le \|u - u_h\| \le C \inf_{y_h \in V_h} \|u - y_h\|_*}$$

Definition (Optimal, quasi-optimal, and suboptimal error estimate)

We say that the above error estimate is

- optimal if $\|\cdot\| = \|\cdot\|_*$
- quasi-optimal if $\|\cdot\| \neq \|\cdot\|_*$, but the lower and upper bounds converge, for smooth u, at the same convergence rate as $h \to 0$
- suboptimal if the upper bound converges more slowly

Abstract nonconforming error analysis XI

Proof.

- Let $y_h \in V_h$. Owing to discrete stability and consistency,

$$\|u_h - y_h\| \leq C_{\mathrm{sta}}^{-1} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(u_h - y_h, w_h)}{\|w_h\|}$$

$$= C_{\mathrm{sta}}^{-1} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(u - y_h, w_h) + \cancel{a_h(u_h - u, w_h)}}{\|w_h\|}$$

- Hence, using boundedness,

$$\|u_h - y_h\| \leq C_{\mathrm{sta}}^{-1} C_{\mathrm{bnd}} \|u - y_h\|_*$$

- Estimate (est.) then results from the triangle inequality, the fact that $\|u - y_h\| \leq \|u - y_h\|_*$, and that y_h is arbitrary in V_h

□

Roadmap for the design of dG methods

1. Extend the continuous bilinear form to $X_{*h} \times X_h$ by replacing

$$\nabla \leftarrow \nabla_h$$

2. Check for stability
   - remove bothering terms in a consistent way
   - if necessary, tighten stability by penalizing jumps
3. If things have been properly done, consistency is preserved
4. Prove boundedness by appropriately selecting $|||\cdot|||_*$

Mesh regularity I

- To prove discrete stability, consistency, and boundedness we need basic results such as trace and inverse inequalities
- To assert the convergence of a method, the discrete space must enjoy approximation properties of the form

$$\inf_{y_h \in V_h} \|u - y_h\|_* \le C_u h^l$$

This requires regularity assumptions on the mesh sequence

$$\mathcal{T}_{\mathcal{H}} := (\mathcal{T}_h)_{h \in \mathcal{H}}$$

Mesh regularity II

Definition (Shape and contact regularity)

The mesh sequence $\mathcal{T}_{\mathcal{H}}$ is shape- and contact-regular if for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{S}_h$ s.t.

(i) There is a $\varrho_1 > 0$, independent of h, s.t.

$$\forall T' \in \mathfrak{S}_h, \qquad \varrho_1 h_{T'} \le r_{T'},$$

with $r_{T'}$ radius of the largest ball inscribed in T';

(ii) there is $\varrho_2 > 0$, independent of h s.t.

$$\forall T \in \mathcal{T}_h, \forall T' \in \mathfrak{S}_T, \quad \varrho_2 h_T \le h_{T'}.$$

If $\mathcal{T}_h$ is itself matching and simplicial, the only requirement is shape-regularity with parameter $\varrho_1 > 0$ independent of h.

Mesh regularity III

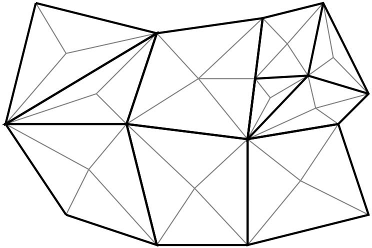


Figure: Mesh $\mathcal{T}_h$ and matching simplicial submesh $\mathfrak{S}_h$

Mesh regularity IV

Lemma (Discrete inverse and trace inequalities)

Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact-regular mesh sequence. Then, for all $h \in \mathcal{H}$, all $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$, and all $T \in \mathcal{T}_h$,

$$\|\nabla v_h\|_{[L^2(T)]^d} \le C_{\rm inv} h_T^{-1} \|v_h\|_{L^2(T)},$$

$$\|v_h\|_{L^2(F)} \le C_{\rm tr} h_T^{-1/2} \|v_h\|_{L^2(T)} \qquad \forall F \in \mathcal{F}_T$$

where $C_{\rm inv}$ and $C_{\rm tr}$ only depend on ϱ, d, and k.

Lemma (Continuous trace inequality)

Moreover, for all $h \in \mathcal{H}$, all $v \in H^1(\mathcal{T}_h)$, all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$,

$$\|v\|_{L^2(F)}^2 \le C_{\rm cti} \left(2\|\nabla v\|_{[L^2(T)]^d} + d h_T^{-1} \|v\|_{L^2(T)}\right) \|v\|_{L^2(T)},$$

with $C_{\rm cti}$ only depending on ϱ and d.

Mesh regularity V

- The last requirement is that the spaces

$$(\mathbb{P}_d^k(\mathcal{T}_h))_{h\in\mathcal{H}},$$

  enjoy optimal approximation properties
- Since we consider continuous problems posed in a space X s.t.

$$X \hookrightarrow L^2(\Omega) \equiv L^2(\Omega)' \hookrightarrow X',$$

  it is natural to focus on the L^2-orthogonal projector π_h^k
- This also allows to deal naturally with polyhedral elements

Mesh regularity VI

Lemma (Optimal polynomial approximation)

Let $\mathcal{T}_{\mathcal{H}}$ denote a shape- and contact-regular mesh sequence. Then, for all $h \in \mathcal{H}$, all $T \in \mathcal{T}_h$, and all polynomial degree k, there holds

$$\forall s \in \{0, \ldots, k+1\}, \forall m \in \{0, \ldots, s\}, \forall v \in H^s(T),$$
$$|v - \pi_h^k v|_{H^m(T)} \leq C_{\text{app}} h_T^{s-m} |v|_{H^s(T)},$$

where C_{app} is independent of both T and h.

Proof.

Follows from [Dupont and Scott, 1980] □

Part II

Scalar first-order PDES

Outline

The continuous problem I

- We consider the following steady advection-reaction problem:

$$\boxed{\begin{aligned} \beta\cdot\nabla u + \mu u = f &\quad \text{in } \Omega, \\ u = 0 &\quad \text{on } \partial\Omega^-, \end{aligned}}$$

where $f \in L^2(\Omega)$ and

$$\partial\Omega^{\pm} := \{x \in \partial\Omega \mid \pm\beta(x)\cdot\mathrm{n}(x) > 0\}$$

- We further assume

$$\mu \in L^\infty(\Omega), \quad \beta \in [\mathrm{Lip}(\Omega)]^d, \quad \Lambda := \mu - \frac{1}{2}\nabla\cdot\beta \geq \mu_0$$

- This implies, in particular, $\beta \in [W^{1,\infty}(\Omega)]^d$

Traces in the graph space I

- To follow the roadmap, we first rework the continuous problem to enforce BCs weakly
- The natural space to look for the solution is the graph space

$$V := \left\{ v \in L^2(\Omega) \mid \beta\cdot\nabla v \in L^2(\Omega) \right\},$$

equipped with the inner product

$$(v, w)_V := (v, w)_{L^2(\Omega)} + (\beta\cdot\nabla v, \beta\cdot\nabla w)_{L^2(\Omega)}$$

- It can be proved that V is a Hilbert space

Traces in the graph space II

- To formulate BCs, we investigate the traces on $\partial\Omega$ of functions in V
- Our aim is to give a meaning to such traces in the space

$$L^2(|\beta{\cdot}\mathrm{n}|;\partial\Omega) := \left\{ v \text{ is measurable on } \partial\Omega \;\middle|\; \int_{\partial\Omega} |\beta{\cdot}\mathrm{n}| v^2 < \infty \right\}$$

- We assume henceforth inflow/outflow separation,

$$\mathrm{dist}(\partial\Omega^-, \partial\Omega^+) := \min_{(x,y)\in\partial\Omega^- \times \partial\Omega^+} |x-y| > 0$$

Traces in the graph space III

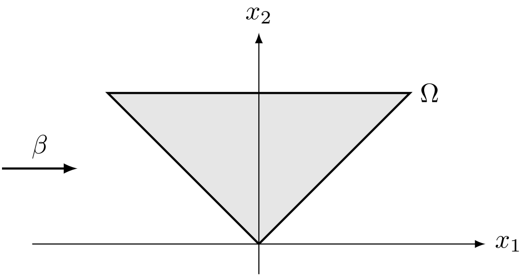


Figure: Counter-example for inflow/outflow separation

Traces in the graph space IV

Lemma (Traces and integration by parts)

In the above framework, the trace operator

$$\gamma : C^0(\overline{\Omega}) \ni v \longmapsto \gamma(v) := v|_{\partial\Omega} \in L^2(|\beta{\cdot}\mathrm{n}|;\partial\Omega)$$

extends continuously to V, i.e., there is C_γ s.t., for all $v \in V$,

$$\|\gamma(v)\|_{L^2(|\beta\cdot\mathrm{n}|;\partial\Omega)} \le C_\gamma \|v\|_V.$$

Moreover, the following IBP formula holds true: For all $v, w \in V$,

$$\boxed{\int_\Omega [(\beta{\cdot}\nabla v)w + (\beta{\cdot}\nabla w)v + (\nabla{\cdot}\beta)vw] = \int_{\partial\Omega} (\beta{\cdot}\mathrm{n})\gamma(v)\gamma(w).}$$

Weak formulation and well-posedness I

- We introduce the following bilinear form:

$$\boxed{a(v,w) := \int_\Omega \mu v w + \int_\Omega (\beta\cdot\nabla v) w + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus v w,}$$

  where

$$x^\oplus := \frac{1}{2}\left(|x| + x\right), \qquad x^\ominus := \frac{1}{2}\left(|x| - x\right)$$

- For all $v, w \in V$, the Cauchy–Schwarz inequality together with the bound $\|\gamma(v)\|_{L^2(|\beta\cdot\mathrm{n}|;\partial\Omega)} \le C_\gamma \|v\|_V$ yield

$$|a(v,w)| \le \left(1 + \|\mu\|^2_{L^\infty(\Omega)}\right)^{\frac{1}{2}} \|v\|_V \|w\|_{L^2(\Omega)} + C_\gamma^2 \|v\|_V \|w\|_V,$$

  i.e., a is bounded in $V \times V$

Weak formulation and well-posedness II

Lemma (L^2-coercivity of a)

The bilinear form a is L^2-coercive on V, namely,

$$\forall v \in V, \qquad a(v,v) \geq \mu_0 \|v\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \frac{1}{2} |\beta{\cdot}\mathrm{n}| v^2.$$

Weak formulation and well-posedness III

$$\boxed{a(v,w) := \int_\Omega \mu v w + \int_\Omega (\beta\cdot\nabla v) w + \int_{\partial\Omega} (\beta\cdot \mathrm{n})^\ominus v w,}$$

Proof.

For all $v \in V$, IBP yields

$$\begin{aligned} a(v,v) &= \int_\Omega \left(\mu - \frac{1}{2}\nabla\cdot\beta\right) v^2 + \int_{\partial\Omega} \frac{1}{2}(\beta\cdot\mathrm{n}) v^2 + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus v^2 \\ &= \int_\Omega \Lambda v^2 + \int_{\partial\Omega} \frac{1}{2}|\beta\cdot\mathrm{n}| v^2 \\ &\geq \mu_0 \|v\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \frac{1}{2}|\beta\cdot\mathrm{n}| v^2, \end{aligned}$$

where we have used the assumption $\Lambda \geq \mu_0 > 0$ to conclude. □

Weak formulation and well-posedness IV

$$\boxed{\text{Find } u \in V \text{ s.t. } a(u,w) = \int_\Omega fw \text{ for all } w \in V} \qquad (\Pi)$$

Lemma (Well-posedness and characterization of (Π))

Problem (Π) is well-posed and its solution $u \in V$ is s.t.

$$\begin{aligned} \beta\cdot\nabla u + \mu u &= f \quad \text{a.e. in } \Omega, \\ u &= 0 \quad \text{a.e. in } \partial\Omega^-. \end{aligned}$$

- We have devised a weak formulation with weakly enforced homogeneous inflow BCs
- The ideas can be extended to inhomogeneous BCs and systems of equations [Ern et al., 2007]

Roadmap for the design of dG methods

1. Extend the continuous bilinear form to $X_{*h} \times X_h$ by replacing

$$\nabla \leftarrow \nabla_h$$

2. Check for stability
   - remove bothering terms in a consistent way
   - if necessary, tighten stability by penalizing jumps
3. If things have been properly done, consistency is preserved
4. Prove boundedness by appropriately selecting $|||\cdot|||_*$

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Heuristic derivation I

Assumption (Regularity of exact solution and space V_*)

We assume that there is a partition $P_\Omega = \{\Omega_i\}_{1 \le i \le N_\Omega}$ of Ω into disjoint polyhedra s.t., for the exact solution u,

$$u \in V_* := V \cap H^1(P_\Omega).$$

Additionally, we set $V_{*h} := V_* + V_h$.

Lemma (Jumps of u across interfaces)

If $u \in V_*$, then, for all $F \in \mathcal{F}_h^i$,

$$(\beta{\cdot}\mathrm{n}_F)[\![u]\!]_F(x) = 0 \qquad \text{for a.e. } x \in F.$$

Heuristic derivation II

- Let $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$, $k \geq 1$
- Our starting point is the (consistent) extension of a to $V_{*h} \times V_h$,

$$a_h^{(0)}(v, w_h) := \int_\Omega \left\{ \mu v w_h + (\beta \cdot \nabla_h v) w_h \right\} + \int_{\partial\Omega} (\beta \cdot \mathrm{n})^{\ominus} v w_h$$

We mimic L^2-coercivity at the discrete level by introducing additional consistent terms that vanish when we plug u into the first argument

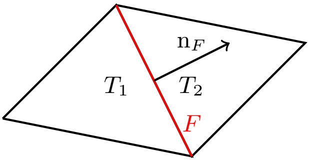
Heuristic derivation III

- Element-by-element IBP yields for all $v_h \in V_h$,

$$
\begin{aligned}
a_h^{(0)}(v_h, v_h) &= \int_\Omega \left\{ \mu v_h^2 + (\beta\cdot\nabla_h v_h) v_h \right\} + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus v_h^2 \\
&= \int_\Omega \mu v_h^2 + \sum_{T\in\mathcal{T}_h} \int_T (\beta\cdot\nabla v_h) v_h + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus v_h^2 \\
&= \int_\Omega \mu v_h^2 + \sum_{T\in\mathcal{T}_h} \int_T \frac{1}{2}(\beta\cdot\nabla v_h^2) + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus v_h^2 \\
&= \int_\Omega \Lambda v_h^2 + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{1}{2}(\beta\cdot\mathrm{n}_T) v_h^2 + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus v_h^2,
\end{aligned}
$$

  where we have used $\Lambda := \mu - \frac{1}{2}\nabla\cdot\beta$
- Let us focus on the boundary terms

Heuristic derivation IV



- Using the continuity of $(\beta\cdot\mathrm{n}_F)$ across all $F \in \mathcal{F}_h^i$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{1}{2}(\beta\cdot\mathrm{n}_T)v_h^2 = \sum_{F\in\mathcal{F}_h^i} \int_F \frac{1}{2}(\beta\cdot\mathrm{n}_F)[\![v_h^2]\!] + \sum_{F\in\mathcal{F}_h^b} \int_F \frac{1}{2}(\beta\cdot\mathrm{n})v_h^2$$

- For all $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$, $v_i = v_h|_{T_i}$, $i \in \{1,2\}$, there holds

$$\frac{1}{2}[\![v_h^2]\!] = \frac{1}{2}(v_1^2 - v_2^2) = \frac{1}{2}(v_1 - v_2)(v_1 + v_2) = [\![v_h]\!]\{\!\{v_h\}\!\}$$

Heuristic derivation V

- As a result,

$$a_h^{(0)}(v_h, v_h) = \int_\Omega \Lambda v_h^2 + \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot \mathrm{n}_F) [\![v_h]\!] \{\!\{ v_h \}\!\} + \sum_{F \in \mathcal{F}_h^b} \int_F \frac{1}{2} (\beta \cdot \mathrm{n}) v_h^2 + \int_{\partial\Omega} (\beta \cdot \mathrm{n})^\ominus v_h^2,$$

- Combining the two rightmost terms, we arrive at

$$a_h^{(0)}(v_h, v_h) = \int_\Omega \Lambda v_h^2 + \boxed{\sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot \mathrm{n}_F) [\![v_h]\!] \{\!\{ v_h \}\!\}} + \int_{\partial\Omega} \frac{1}{2} |\beta \cdot \mathrm{n}| v_h^2$$

- The boxed term is nondefinite

Heuristic derivation VI

- A natural idea is to modify $a_h^{(0)}$ as follows:

$$a_h^{\mathrm{cf}}(v, w_h) := \int_\Omega \Big\{ \mu v w_h + (\beta\cdot\nabla_h v) w_h \Big\} + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\ominus} v w_h$$
$$- \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)[\![v]\!]\{\!\{w_h\}\!\}$$

- The highlighted term is consistent since $u \in V_*$ implies

$$(\beta\cdot\mathrm{n}_F)[\![u]\!]_F(x) = 0 \qquad \text{for a.e. } x \in F$$

- Moreover, it ensures L^2-coercivity since, this time,

$$a_h^{\mathrm{cf}}(v_h, v_h) = \int_\Omega \Lambda v_h^2 + \int_{\partial\Omega} \frac{1}{2}|\beta\cdot\mathrm{n}| v_h^2 \qquad \forall v_h \in V_h$$

Heuristic derivation VII

$$\int_\Omega \left\{\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h\right\},\ \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\ominus} v_h w_h$$

$$\sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)[\![v_h]\!]\{\!\{w_h\}\!\}$$

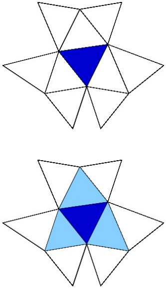


Figure: Stencil of the different terms

Heuristic derivation VIII

$$\|v\|_{\mathrm{cf}}^2 := \tau_{\mathrm{c}}^{-1}\|v\|_{L^2(\Omega)}^2 + \int_{\partial\Omega} \frac{1}{2}|\beta{\cdot}\mathrm{n}|v^2, \quad \tau_{\mathrm{c}} := \{\max(\|\mu\|_{L^\infty(\Omega)}, L_\beta)\}^{-1}$$

Lemma (Consistency and discrete coercivity)

The discrete bilinear form a_h^{cf} satisfies the following properties:

(i) *Consistency, i.e., assuming $u \in V_*$,*

$$a_h^{\mathrm{cf}}(u, v_h) = \int_\Omega f v_h \qquad \forall v_h \in V_h;$$

(ii) *Coercivity on V_h with $C_{\mathrm{sta}} := \min(1, \tau_{\mathrm{c}}\mu_0)$,*

$$\forall v_h \in V_h, \qquad a_h^{\mathrm{cf}}(v_h, v_h) \geq C_{\mathrm{sta}}\|v_h\|_{\mathrm{cf}}^2.$$

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Error estimate I

Lemma (Boundedness)

There holds

$$\forall (v, w_h) \in V_{*h} \times V_h, \qquad a_h^{\rm cf}(v, w_h) \le C_{\rm bnd} \| v \|_{{\rm cf},*} \| w_h \|_{\rm cf},$$

with $C_{\rm bnd}$ *independent of* h *and of* μ *and* β, *and with* $\beta_{\rm c} := \|\beta\|_{[L^\infty(\Omega)]^d}$,

$$\boxed{\| v \|_{{\rm cf},*}^2 := \| v \|_{\rm cf}^2 + \sum_{T \in \mathcal{T}_h} \tau_{\rm c} \| \beta \cdot \nabla v \|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_h} \tau_{\rm c} \beta_{\rm c}^2 h_T^{-1} \| v \|_{L^2(\partial T)}^2 .}$$

Error estimate II

$$\boxed{\text{Find } u_h \in V_h \text{ s.t. } a_h^{\rm cf}(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h} \qquad (\Pi_h^{\rm cf})$$

Theorem (Error estimate)

Let u solve (Π) and let u_h solve $(\Pi_h^{\rm cf})$ where $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \geq 1$. Then, there holds

$$\|u - u_h\|_{\rm cf} \leq C \inf_{y_h \in V_h} \|u - y_h\|_{{\rm cf},*},$$

with C independent of h and depending on the data only through the factor

$$C_{\rm sta}^{-1} = \{\min(1, \tau_{\rm c}\mu_0)\}^{-1}.$$

Error estimate III

Corollary (Convergence rate for smooth solutions)

Assume $u \in H^{k+1}(\Omega)$*. Then, there holds*

$$|||u - u_h|||_{\mathrm{cf}} \leq C_u h^k,$$

with $C_u = C\|u\|_{H^{k+1}(\Omega)}$ *and* C *independent of* h *and depending on the data only through the factor* $\{\min(1, \tau_{\mathrm{c}}\mu_0)\}^{-1}$*.*

Proof.

Let $y_h = \pi_h^k u$ in the error estimate and use the approximation properties of the sequence of discrete spaces $(V_h)_{h\in\mathcal{H}}$. □

Error estimate IV

- This estimate is suboptimal by $\frac{1}{2}$ power of h
- Indeed, in the inequalities

$$\inf_{y_h \in V_h} \|u - y_h\|_{\mathrm{cf}} \leq \|u - u_h\|_{\mathrm{cf}} \leq C \inf_{y_h \in V_h} \|u - y_h\|_{\mathrm{cf},*},$$

the upper bound converges more slowly than the lower bound

$$\|v\|_{\mathrm{cf}}^2 := \tau_{\mathrm{c}}^{-1} \|v\|_{L^2(\Omega)}^2 + \int_{\partial\Omega} \frac{1}{2} |\beta \cdot \mathrm{n}| v^2,$$

$$\|v\|_{\mathrm{cf},*}^2 := \|v\|_{\mathrm{cf}}^2 + \sum_{T \in \mathcal{T}_h} \tau_{\mathrm{c}} \|\beta \cdot \nabla v\|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_h} \tau_{\mathrm{c}} \beta_{\mathrm{c}}^2 h_T^{-1} \|v\|_{L^2(\partial T)}^2.$$

Numerical fluxes I

$$a_h^{\mathrm{cf}}(v,w_h) := \int_\Omega \Big\{\mu v w_h + (\beta\cdot\nabla_h v) w_h\Big\} + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\ominus} v w_h - \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)[\![v]\!]\{\!\{w_h\}\!\}$$

Lemma (Equivalent expression for a_h^{cf})

For all $(v,w_h)\in V_{*h}\times V_h$, *there holds*

$$a_h^{\mathrm{cf}}(v,w_h) = \int_\Omega \Big\{(\mu - \nabla\cdot\beta) v w_h - v(\beta\cdot\nabla_h w_h)\Big\} + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\oplus} v w_h + \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)\{\!\{v\}\!\}[\![w_h]\!].$$

Numerical fluxes II

- IBP of the advective term leads to

$$a_h^{\mathrm{cf}}(v, w_h) = \int_\Omega \Big\{ (\mu - \nabla\cdot\beta) v w_h - v(\beta\cdot\nabla_h w_h) \Big\} + \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\beta\cdot\mathrm{n}_T) v w_h + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\ominus} v w_h - \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F) [\![v]\!] \{\!\{w_h\}\!\}$$

- Exploiting the continuity of $\beta\cdot\mathrm{n}_F$ we obtain

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} (\beta\cdot\mathrm{n}_T) v w_h = \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F) [\![v w_h]\!] + \sum_{F\in\mathcal{F}_h^b} \int_F (\beta\cdot\mathrm{n}) v w_h$$

Numerical fluxes III

- To conclude we use the magic formula

$$\begin{aligned}[\![vw_h]\!] &= v_1w_1 - v_2w_2 \\ &= \frac{1}{2}(v_1 - v_2)(w_1 + w_2) + \frac{1}{2}(v_1 + v_2)(w_1 - w_2) \\ &= [\![v]\!]\{\!\{w_h\}\!\} + \{\!\{v\}\!\}[\![w_h]\!],\end{aligned}$$

where $v_i := v|_{T_i}$ and $w_i := w_h|_{T_i}$ for $i \in \{1, 2\}$

Numerical fluxes IV

- We now consider a point of view closer to finite volumes
- Let $T \in \mathcal{T}_h$ and $\xi \in \mathbb{P}_d^k(T)$
- For a set $S \subset \Omega$, denote by χ_S the characteristic function of S s.t.

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

- With the goal of setting $v_h = \xi\chi_T$ in (Π_h^{cf}) observe that

$$[\![\xi\chi_T]\!] = \epsilon_{T,F}\xi \qquad \text{with} \qquad \epsilon_{T,F} := \mathrm{n}_T{\cdot}\mathrm{n}_F$$

Numerical fluxes V

$$a_h^{\mathrm{cf}}(u_h, v_h) = \int_\Omega \Big\{(\mu - \nabla\cdot\beta) u_h v_h - u_h(\beta\cdot\nabla_h v_h)\Big\} + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\oplus} u_h v_h + \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)\{\!\{u_h\}\!\}[\![v_h]\!].$$

- Letting $v_h = \xi\chi_T$ in the alternative form for a_h (cf. above) we infer

$$a_h(u_h, \xi\chi_T) = \int_T \Big\{(\mu - \nabla\cdot\beta) u_h \xi - u_h(\beta\cdot\nabla\xi)\Big\} + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h)\xi = \int_T f\xi,$$

where the numerical fluxes $\phi_F(u_h)$ given by

$$\phi_F(u_h) := \begin{cases} (\beta\cdot\mathrm{n}_F)\{\!\{u_h\}\!\} & \text{if } F \in \mathcal{F}_h^i, \\ (\beta\cdot\mathrm{n})^{\oplus} u_h & \text{if } F \in \mathcal{F}_h^b \end{cases}$$

Numerical fluxes VI

- For $\xi|_T \equiv 1$ we recover the FV local conservation,

$$\forall T \in \mathcal{T}_h \quad \int_T (\mu - \nabla{\cdot}\beta) u_h + \sum_{F \in \mathcal{F}_T} \int_F \phi_{T,F}(u_h) = \int_T f,$$

  where $\phi_{T,F}(u_h) := \epsilon_{T,F} \phi_F(u_h)$
- We next modify the numerical flux to recover quasi-optimality

Upwinding I

- The error estimate for centered fluxes is suboptimal
- This can be improved by tightening stability with a least-square penalization of interface jumps
- In terms of fluxes this approach amounts to upwinding
- As a side benefit, we can estimate the advective derivative error

Upwinding II

- We consider the new bilinear form

$$\boxed{a_h^{\mathrm{upw}}(v_h, w_h) := a_h^{\mathrm{cf}}(v_h, w_h) + s_h(v_h, w_h),}$$

where, for $\eta > 0$,

$$s_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot \mathrm{n}_F| [\![v_h]\!] [\![w_h]\!]$$

- This term is consistent under the regularity assumption

Upwinding III

- Specifically,

$$a_h^{\text{upw}}(v_h, w_h) := \int_\Omega \left\{ \mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h \right\} + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\ominus} v_h w_h - \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F) [\![v_h]\!] \{\!\{ w_h \}\!\} + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta\cdot\mathrm{n}_F| [\![v_h]\!] [\![w_h]\!]$$

- Or, after element-by-element IBP,

$$a_h^{\text{upw}}(v_h, w_h) = \int_\Omega \left\{ (\mu - \nabla\cdot\beta) v_h w_h - v_h(\beta\cdot\nabla_h w_h) \right\} + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^{\oplus} v_h w_h + \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F) \{\!\{ v_h \}\!\} [\![w_h]\!] + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta\cdot\mathrm{n}_F| [\![v_h]\!] [\![w_h]\!]$$

Upwinding IV

$$\int_\Omega \left\{\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h\right\}, \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus v_h w_h$$

$$\sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)[\![v_h]\!]\{\!\{w_h\}\!\},$$

$$\sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2}|\beta\cdot\mathrm{n}_F|[\![v_h]\!][\![w_h]\!]$$

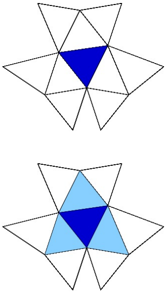


Figure: Stencil of the different terms

Upwinding V

$$\boxed{\text{Find } u_h \in V_h \text{ s.t. } a_h^{\text{upw}}(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h} \qquad (\Pi_h^{\text{upw}})$$

Upwinding VI

$$\|v\|_{\mathrm{uw}\flat}^2 := \|v\|_{\mathrm{cf}}^2 + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2}|\beta\cdot\mathrm{n}_F|[\![v]\!]^2$$

Lemma (Consistency and discrete coercivity)

The discrete bilinear form a_h^{upw} satisfies the following properties:

(i) *Consistency, i.e., assuming $u \in V_*$,*

$$a_h^{\mathrm{upw}}(u, v_h) = \int_\Omega f v_h \qquad \forall v_h \in V_h,$$

(ii) *Coercivity on V_h with $C_{\mathrm{sta}} = \min(1, \tau_c\mu_0)$,*

$$\forall v_h \in V_h, \qquad a_h^{\mathrm{upw}}(v_h, v_h) \geq C_{\mathrm{sta}}\|v_h\|_{\mathrm{uw}\flat}^2.$$

Numerical fluxes

- Proceeding as for a_h^{cf} we infer for all $T \in \mathcal{T}_h$,

$$a_h(u_h, \xi\chi_T) = \int_T \Big\{(\mu - \nabla{\cdot}\beta)u_h\xi - u_h(\beta{\cdot}\nabla\xi)\Big\} + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h)\xi = \int_T f\xi,$$

  where, this time,

$$\phi_F(u_h) = \begin{cases} \beta{\cdot}\mathrm{n}_F\{\!\{u_h\}\!\} + \frac{\eta}{2}|\beta{\cdot}\mathrm{n}_F|[\![u_h]\!] & \text{if } F \in \mathcal{F}_h^i, \\ (\beta{\cdot}\mathrm{n})^{\oplus}u_h & \text{if } F \in \mathcal{F}_h^b \end{cases}$$

- The choice $\eta = 1$ leads to the classical upwind fluxes

$$\phi_F(u_h) = \begin{cases} \beta{\cdot}\mathrm{n}_F u_h^{\uparrow} & \text{if } F \in \mathcal{F}_h^i, \\ (\beta{\cdot}\mathrm{n})^{\oplus}u_h & \text{if } F \in \mathcal{F}_h^b \end{cases}$$

Error estimates based on inf-sup stability I

- We define the stronger norm ($\beta_c := \|\beta\|_{[L^\infty(\Omega)]^d}$)

$$\|v\|_{\mathrm{uw}\sharp}^2 := \|v\|_{\mathrm{uw}\flat}^2 + \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v\|_{L^2(T)}^2$$

- We assume in what follows that the model is well-resolved and reaction is not dominant,

$$h \le \beta_c \tau_c$$

Error estimates based on inf-sup stability II

Lemma (Discrete inf-sup condition for a_h^{upw})

There is $C'_{\mathrm{sta}} > 0$, *independent of* h, μ, *and* β, *s.t.*

$$\forall v_h \in V_h, \qquad C'_{\mathrm{sta}} C_{\mathrm{sta}} |||v_h|||_{\mathrm{uw}\sharp} \leq \mathbb{S} := \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h^{\mathrm{upw}}(v_h, w_h)}{|||w_h|||_{\mathrm{uw}\sharp}},$$

with $\boxed{C_{\mathrm{sta}} = \min(1, \tau_{\mathrm{c}}\mu_0) \leq 1}$ L^2*-coercivity constant.*

Error estimates based on inf-sup stability III

Lemma (Boundedness)

There holds

$$\forall (v, w_h) \in V_{*h} \times V_h, \qquad |a_h^{\text{upw}}(v, w_h)| \leq C_{\text{bnd}} \|v\|_{\text{uw}\sharp,*} \|w_h\|_{\text{uw}\sharp},$$

with C_{bnd} *independent of* h, μ, *and* β *and*

$$\boxed{\|v\|_{\text{uw}\sharp,*}^2 := \|v\|_{\text{uw}\sharp}^2 + \sum_{T \in \mathcal{T}_h} \beta_{\text{c}} \left(h_T^{-1} \|v\|_{L^2(T)}^2 + \|v\|_{L^2(\partial T)}^2 \right).}$$

Error estimates based on inf-sup stability IV

Theorem (Error estimate)

Let u solve (Π) and let u_h solve (Π_h^{upw}) where $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \geq 0$. Then, there holds

$$\|u - u_h\|_{\mathrm{uw}\sharp} \leq C \inf_{y_h \in V_h} \|u - y_h\|_{\mathrm{uw}\sharp,*},$$

with C independent of h and depending on the data only through the factor $\{\min(1, \tau_{\mathrm{c}}\mu_0)\}^{-1}$.

Corollary (Convergence rate for smooth solutions)

Assume $u \in H^{k+1}(\Omega)$. Then, there holds

$$\|u - u_h\|_{\mathrm{uw}\sharp} \leq C_u h^{k+1/2},$$

with $C_u = C\|u\|_{H^{k+1}(\Omega)}$ and C independent of h and depending on the data only through the factor $\{\min(1, \tau_{\mathrm{c}}\mu_0)\}^{-1}$.

The unsteady case I

$$\boxed{\begin{aligned} \partial_t u + \beta\cdot\nabla u + \mu u &= f && \text{in } \Omega \times (0, t_{\mathrm{F}}), \\ u &= 0 && \text{on } \partial\Omega^- \times (0, t_{\mathrm{F}}), \\ u(\cdot, t = 0) &= u_0 && \text{in } \Omega \end{aligned}} \qquad (\Pi(t))$$

The unsteady case II

- We define $A_h^{\text{upw}} : V_{*h} \to V_h$ s.t. with $\eta = 1$ (upwind),

$$\boxed{\forall (v, w_h) \in V_{*h} \times V_h, \qquad (A_h^{\text{upw}} v, w_h)_{L^2(\Omega)} = a_h^{\text{upw}}(v, w_h)}$$

- The space semidiscrete problem reads

$$\boxed{d_t u_h(t) + A_h^{\text{upw}} u_h(t) = f_h(t) \qquad \forall t \in [0, t_{\mathrm{F}}]} \qquad (\Pi_h(t))$$

with initial condition $u_h(0) = \pi_h u_0$ and source term

$$f_h(t) = \pi_h f(t) \qquad \forall t \in [0, t_{\mathrm{F}}],$$

- $(\Pi_h(t))$ is a system of coupled ODEs

The unsteady case III

Lemma (Consistency and discrete dissipation for A_h^{upw})

The discrete operator A_h^{upw} satisfies the following properties:

- *Consistency: For the exact solution $u \in C^0(H^1(\Omega)) \cap C^1(L^2(\Omega))$,*

$$\pi_h d_t u(t) + A_h^{\mathrm{upw}} u(t) = f_h(t) \qquad \forall t \in [0, t_{\mathrm{F}}].$$

- *Discrete dissipation: For all $v_h \in V_h$,*

$$(A_h^{\mathrm{upw}} v_h, v_h)_{L^2(\Omega)} = |v_h|_\beta^2 + (\Lambda v_h, v_h)_{L^2(\Omega)},$$

where we have defined on V_{*h} the seminorm

$$|v|_\beta^2 := \int_{\partial\Omega} \frac{1}{2} |\beta{\cdot}\mathrm{n}| v^2 + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{1}{2} |\beta{\cdot}\mathrm{n}_F| [\![v]\!]^2.$$

Time discretization I

- Let δt be the (constant) time step s.t.

$$t^n := n\delta t, \quad \forall 0 \leq n \leq N, \qquad t_{\mathrm{F}} = N\delta t$$

- We assume that the time step resolves the reference time τ_{c}

$$\delta t \leq \tau_{\mathrm{c}} \text{ and } \delta t \leq t_{\mathrm{F}}$$

- For a function of time $\varphi \in C^0(V)$ we set

$$\varphi^n := \varphi(t^n)$$

UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

Time discretization II

- The simplest time marching scheme is the forward Euler scheme,

$$u_h^{n+1} = u_h^n - \delta t A_h^{\mathrm{upw}} u_h^n + \delta t f_h^n$$

- Equivalently,

$$\frac{u_h^{n+1} - u_h^n}{\delta t} + A_h^{\mathrm{upw}} u_h^n = f_h^n$$

Time discretization III

- To improve the accuracy of time discretization, one possibility is to consider explicit Runge–Kutta (RK) schemes
- Such schemes are one-step methods where, at each time step, starting from u_h^n, s stages, $s \geq 1$, are performed to compute u_h^{n+1}
- Explicit RK schemes can be formulated in various forms

Time discretization IV

- Herein we focus on the increment form

$$k_i = -A_h^{\mathrm{upw}}\left(u_h^n + \delta t \sum_{j=1}^{s} a_{ij} k_j\right) + f_h(t^n + c_i \delta t) \qquad \forall i \in \{1, \dots, s\},$$

$$u_h^{n+1} = u_h^n + \delta t \sum_{i=1}^{s} b_i k_i. \tag{RK$_s$}$$

  where

  - $(a_{ij})_{1 \le i,j \le s}$ are real numbers
  - $(b_i)_{1 \le i \le s}$ are real numbers s.t. $\sum_{i=1}^{s} b_i = 1$
  - $(c_i)_{1 \le i \le s}$ are real numbers in $[0, 1]$ s.t. $c_i = \sum_{j=1}^{s} a_{ij}$ $\forall 1 \le i \le s$

- The k_i can be interpreted as intermediate increments

Time discretization V

- These quantities are usually collected in the so-called Butcher's array

$$\left[\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}\right]$$

- The scheme is explicit whenever

$$a_{ij} = 0 \text{ for all } j \geq i$$

- Explicit schemes require to invert the mass matrix at each stage
- For dG method, the mass matrix is (block) diagonal

Time discretization VI

- The forward Euler scheme is actually a one-stage RK method with

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}\right] \qquad \begin{cases} k_1 = -A_h^{\mathrm{upw}} u_h^n + f_h^n \\ u_h^{n+1} = u_h^n + \delta t k_1 \end{cases}$$

Time discretization VII

- Two examples of two-stage RK schemes are the improved Euler

$$
\left[\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}\right]
\qquad
\begin{cases} k_1 = -A_h^{\mathrm{upw}} u_h^n + f_h^n \\ k_2 = -A_h^{\mathrm{upw}} (u_h^n + \frac{1}{2}\delta t k_1) + f_h^{n+1/2} \\ u_h^{n+1} = u_h^n + \delta t k_2 \end{cases}
$$

with $f_h^{n+1/2} = f_h(t^n + \frac{1}{2}\delta t)$ and Heun schemes

$$
\left[\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}\right]
\qquad
\begin{cases} k_1 = -A_h^{\mathrm{upw}} u_h^n + f_h^n \\ k_2 = -A_h^{\mathrm{upw}} (u_h^n + \delta t k_1) + f_h^{n+1} \\ u_h^{n+1} = u_h^n + \delta t \frac{1}{2}(k_1 + k_2) \end{cases}
$$

Time discretization VIII

- For $f = 0$, since A_h^{upw} is linear, both schemes can be written

$$u_h^{n+1} = u_h^n - \delta t A_h^{\text{upw}} u_h^n + \frac{1}{2}\delta t^2 (A_h^{\text{upw}})^2 u_h^n.$$

- On the right-hand side, we recognize a second-order Taylor expansion in time at t^n where the time derivatives have been substituted using

$$d_t u(t^n) = -A_h^{\text{upw}} u(t^n),$$

and replacing $u \leftarrow u_h$

Time discretization IX

- An example of three-stage RK scheme is the three-stage Heun scheme for which

$$
\left[\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/3 & 1/3 & 0 & 0 \\
2/3 & 0 & 2/3 & 0 \\
\hline
 & 1/4 & 0 & 3/4
\end{array}\right]
\qquad
\begin{cases}
k_1 = -A_h^{\mathrm{upw}} u_h^n + f_h^n, \\
k_2 = -A_h^{\mathrm{upw}} (u_h^n + \frac{1}{3}\delta t k_1) + f_h^{n+1/3} \\
k_3 = -A_h^{\mathrm{upw}} (u_h^n + \frac{2}{3}\delta t k_2) + f_h^{n+2/3} \\
u_h^{n+1} = u_h^n + \frac{1}{4}\delta t (k_1 + 3k_3)
\end{cases}
$$

- Straightforward algebra shows

$$
u_h^{n+1} = u_h^n - \delta t A_h^{\mathrm{upw}} u_h^n + \frac{1}{2}\delta t^2 (A_h^{\mathrm{upw}})^2 u_h^n - \frac{1}{6}\delta t^3 (A_h^{\mathrm{upw}})^3 u_h^n
$$

- We recognize now a third-order Taylor expansion in time

Time discretization X

- Finally, an example of four-stage RK scheme is

$$\left[\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}\right] \qquad \left\{\begin{aligned} k_1 &= -A_h^{\mathrm{upw}} u_h^n + f_h^n, \\ k_2 &= -A_h^{\mathrm{upw}} (u_h^n + \tfrac{1}{2}\delta t k_1) + f_h^{n+1/2} \\ k_3 &= -A_h^{\mathrm{upw}} (u_h^n + \tfrac{1}{2}\delta t k_2) + f_h^{n+1/2} \\ k_4 &= -A_h^{\mathrm{upw}} (u_h^n + \delta t k_3) + f_h^{n+1} \\ u_h^{n+1} &= u_h^n + \tfrac{1}{6}\delta t (k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}\right.$$

Time discretization XI

- An alternative formulation of RK schemes consists in introducing intermediate stages for the discrete solution instead of the intermediate increments k_i
- When A_h^{upw} is linear, the two formulations are equivalent in the absence of external forcing
- In the nonlinear case, the form based on intermediate stages for the discrete solution is more appropriate

Main convergence results I

- We next state some error estimates under CFL conditions of the form

$$\boxed{\delta t \leq \varrho \frac{h}{\beta_c}, \quad \varrho > 0} \qquad \text{(CFL)}$$

- For the forward Euler scheme, we only consider the case $k = 0$ since the CFL to achieve stability is too stringent for $k \geq 1$
- For explicit RK2 and RK3 schemes, we consider dG schemes with polynomial degree $k \geq 0$ for space semidiscretization

Main convergence results II

Theorem (Convergence for forward Euler)

Set $V_h = \mathbb{P}^0_d(\mathcal{T}_h)$, *assume* $u \in C^0(H^1(\Omega)) \cap C^2(L^2(\Omega))$ *and* (CFL) *with* $\varrho \le \varrho^{\mathrm{Eul}}$ *for* ϱ^{Eul} *independent of* h, δt, f, μ, *and* β. *Then, there holds*

$$\|u^N - u_h^N\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1} \delta t |u^m - u_h^m|_\beta^2\right)^{\frac{1}{2}} \lesssim e^{C_{\mathrm{sta}} \frac{t_{\mathrm{F}}}{\tau_*}} (\chi_1 \delta t + \chi_2 h^{\frac{1}{2}}),$$

where $\chi_1 = t_{\mathrm{F}}^{\frac{1}{2}} \tau_*^{\frac{1}{2}} \|d_t^2 u\|_{C^0(L^2(\Omega))}$ *and* $\chi_2 = t_{\mathrm{F}}^{\frac{1}{2}} \beta_{\mathrm{c}}^{\frac{1}{2}} \|u\|_{C^0(H^1(\Omega))}$, *and* C_{sta} *is independent of* h, δt, *and the data* f, μ, *and* β.

Main convergence results III

- We reformulate the RK2 scheme as

$$\begin{aligned} w_h^n &= u_h^n - \delta t A_h^{\text{upw}} u_h^n + \delta t f_h^n, \\ u_h^{n+1} &= \tfrac{1}{2}(u_h^n + w_h^n) - \tfrac{1}{2}\delta t A_h^{\text{upw}} w_h^n + \tfrac{1}{2}\delta t \psi_h^n, \end{aligned}$$

  with initial condition $u_h^0 = \pi_h u_0$.
- We assume $f \in C^2(L^2(\Omega))$ and

$$\|\psi_h^n - f_h^n - \delta t d_t f_h^n\|_{L^2(\Omega)} \lesssim \delta t^2 \|d_t^2 f(t)\|_{C^0(L^2(\Omega))}.$$

Main convergence results IV

Theorem (Convergence for RK2)

Assume $u \in C^3(L^2(\Omega)) \cap C^0(H^1(\Omega))$. *Set* $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$ *with* $k \geq 1$.

- *In the case* $k \geq 2$, *assume the 4/3-CFL condition*

$$\delta t \leq \varrho' \tau_*^{-\frac{1}{3}} \left(\frac{h}{\beta_c}\right)^{\frac{4}{3}}, \qquad \varrho' > 0;$$

- *In the case* $k = 1$, *assume the CFL condition* (CFL), *that is,*

$$\delta t \leq \varrho^{\mathrm{RK2}} \frac{h}{\beta_c},$$

  with ϱ^{RK2} *independent of* h, δt, f, μ, *and* β.

Finally, assume $d_t^s u \in C^0(H^{k+1-s}(\Omega))$ *for* $s \in \{0, 1\}$. *Then,*

$$\|u^N - u_h^N\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1} \delta t |u^m - u_h^m|_\beta^2\right)^{\frac{1}{2}} \lesssim e^{C_{\mathrm{sta}} \frac{t_F}{\tau_*}} (\chi_1 \delta t^2 + \chi_2 h^{k+\frac{1}{2}}),$$

where C_{sta} *is independent of* h, δt, *and the data* f, μ, *and* β, *and* χ_1 *and* χ_2 *depend only on* t_F, τ_*, β_c, *and bounded norms of* f *and* u.

Main convergence results V

- We reformulate the RK3 scheme as

$$\begin{aligned} w_h^n &= u_h^n - \delta t A_h^{\mathrm{upw}} u_h^n + \delta t f_h^n, \\ y_h^n &= \tfrac{1}{2}(u_h^n + w_h^n) - \tfrac{1}{2}\delta t A_h^{\mathrm{upw}} w_h^n + \tfrac{1}{2}\delta t (f_h^n + \delta t d_t f_h^n), \\ u_h^{n+1} &= \tfrac{1}{3}(u_h^n + w_h^n + y_h^n) - \tfrac{1}{3}\delta t A_h^{\mathrm{upw}} y_h^n + \tfrac{1}{3}\delta t \psi_h^n, \end{aligned}$$

  with initial condition $u_h^0 = \pi_h u_0$.
- We assume $f \in C^3(L^2(\Omega))$ and

$$\|\psi_h^n - f_h^n - \delta t d_t f_h^n - \tfrac{1}{2}\delta t^2 d_t^2 f_h^n\|_{L^2(\Omega)} \lesssim \delta t^3 \|d_t^3 f\|_{C^0(L^2(\Omega))}.$$

Main convergence results VI

Theorem (Convergence for RK3)

Assume $u \in C^4(L^2(\Omega)) \cap C^0(H^1(\Omega))$. *Set* $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$ *for* $k \geq 1$. *Assume*

$$\delta t \leq \varrho^{\mathrm{RK3}} \frac{h}{\beta_{\mathrm{c}}},$$

for ϱ^{RK3} *independent of* h, δt, f, μ, *and* β. *Finally, assume* $d_t^s u \in C^0(H^{k+1-s}(\Omega))$ *for* $s \in \{0,1,2\}$. *Then,*

$$\|u^N - u_h^N\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1} \delta t |u^m - u_h^m|_\beta^2 \right)^{\frac{1}{2}} \lesssim e^{C_{\mathrm{sta}} \frac{t_{\mathrm{F}}}{\tau_*}} (\chi_1 \delta t^3 + \chi_2 h^{k+\frac{1}{2}}),$$

where C_{sta} *is independent of* h, δt, *and the data* f, μ, *and* β, *and* χ_1 *and* χ_2 *depend only on* t_{F}, τ_*, β_{c}, *and bounded norms of* f *and* u.

Part III

Scalar second-order PDEs

Outline

Setting I

- For $f \in L^2(\Omega)$ we consider the model problem

$$\boxed{\begin{aligned} -\triangle u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}}$$

- The weak formulation reads with $V := H_0^1(\Omega)$,

$$\boxed{\text{Find } u \in V \text{ s.t. } a(u,v) = \int_\Omega fv \text{ for all } v \in V,} \tag{Π}$$

where

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v$$

Setting II

- The well-posedness of (Π) hinges on Poincaré's inequality,

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{[L^2(\Omega)]^d}$$

- Indeed, a classical result is the coercivity of a,

$$\forall v \in H_0^1(\Omega), \quad a(v,v) \geq \frac{1}{1+C_\Omega^2} \|v\|_{H^1(\Omega)}^2$$

Lemma (Continuity of the potential and of the diffusive flux)

Letting $[\![v]\!]_F = \{\!\{v\}\!\}_F = v$ *for all* $F \in \mathcal{F}_h^b$*, there holds*

$$\begin{aligned} [\![u]\!] &= 0 \qquad \forall F \in \mathcal{F}_h, \\ [\![\nabla u]\!]\cdot \mathrm{n}_F &= 0 \qquad \forall F \in \mathcal{F}_h^i. \end{aligned}$$

Setting III

Assumption (Regularity of exact solution and space V_*)

We assume that the exact solution u is s.t.

$$u \in V_* := V \cap H^2(\Omega).$$

We set $V_{*h} := V_* + V_h$. This implies, in particular, that the traces of both u and $\nabla u \cdot \mathrm{n}_F$ are square-integrable.

Roadmap for the design of dG methods

1. Extend the continuous bilinear form to $X_{*h} \times X_h$ by replacing
$$\nabla \leftarrow \nabla_h$$
2. Check for stability
   - remove bothering terms in a consistent way
   - if necessary, tighten stability by penalizing jumps
3. If things have been properly done, consistency is preserved
4. Prove boundedness by appropriately selecting $|||\cdot|||_*$

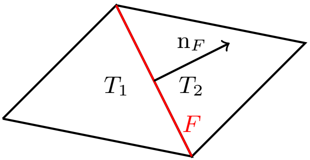
Symmetric Interior Penalty: Heuristic derivation I

$$\boxed{V_h := \mathbb{P}_d^k(\mathcal{T}_h), \quad k \geq 1}$$

- We derive a dG method for (Π) based on a bilinear form a_h
- For all $(v, w_h) \in V_{*h} \times V_h$ we set

$$\boxed{a_h^{(0)}(v, w_h) := \int_\Omega \nabla_h v \cdot \nabla_h w_h = \sum_{T \in \mathcal{T}_h} \int_T \nabla v \cdot \nabla w_h}$$

Consistency I



- Integrating by parts element-by-element we arrive at

$$a_h^{(0)}(v, w_h) = -\sum_{T\in\mathcal{T}_h}\int_T (\triangle v) w_h + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla v\cdot n_T) w_h$$

- The second term in the RHS can be reformulated as follows:

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla v\cdot n_T) w_h = \sum_{F\in\mathcal{F}_h^i}\int_F [\![(\nabla_h v) w_h]\!]\cdot n_F + \sum_{F\in\mathcal{F}_h^b}\int_F (\nabla v\cdot n_F) w_h$$

Consistency II

- Moreover,

$$[\![(\nabla_h v) w_h]\!] = \{\!\{\nabla_h v\}\!\}[\![w_h]\!] + [\![\nabla_h v]\!]\{\!\{w_h\}\!\},$$

since letting $a_i = (\nabla v)|_{T_i}$, $b_i = w_h|_{T_i}$, $i \in \{1, 2\}$, yields

$$\begin{aligned} [\![(\nabla_h v) w_h]\!] &= a_1 b_1 - a_2 b_2 \\ &= \tfrac{1}{2}(a_1 + a_2)(b_1 - b_2) + (a_1 - a_2)\tfrac{1}{2}(b_1 + b_2) \\ &= \{\!\{\nabla_h v\}\!\}[\![w_h]\!] + [\![\nabla_h v]\!]\{\!\{w_h\}\!\}. \end{aligned}$$

- As a result, and accounting also for boundary faces,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} (\nabla v \cdot \mathrm{n}_T) w_h = \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot \mathrm{n}_F [\![w_h]\!] + \sum_{F \in \mathcal{F}_h^i} \int_F [\![\nabla_h v]\!] \cdot \mathrm{n}_F \{\!\{w_h\}\!\}$$

Consistency III

- In conclusion,

$$a_h^{(0)}(v, w_h) = -\sum_{T\in\mathcal{T}_h}\int_T(\triangle v)w_h + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla_h v\}\!\}\cdot \mathrm{n}_F [\![w_h]\!]$$
$$+ \sum_{F\in\mathcal{F}_h^i}\int_F [\![\nabla_h v]\!]\cdot \mathrm{n}_F \{\!\{w_h\}\!\}$$

- To check consistency, set $v = u$. For all $w_h \in V_h$,

$$a_h^{(0)}(u, w_h) = \int_\Omega f w_h + \sum_{F\in\mathcal{F}_h}\int_F (\nabla u\cdot \mathrm{n}_F)[\![w_h]\!]$$

- Hence, we modify $a_h^{(0)}$ as follows:

$$\boxed{a_h^{(1)}(v, w_h) := \int_\Omega \nabla_h v\cdot\nabla_h w_h - \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla_h v\}\!\}\cdot \mathrm{n}_F [\![w_h]\!]}$$

Symmetry I

- A desirable property is symmetry since
  - it simplifies the solution of the linear system
  - it is used to prove optimal L^2 error estimates
- We consider the following modification of $a_h^{(1)}$:

$$a_h^{\mathrm{cs}}(v, w_h) := \int_\Omega \nabla_h v \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{\nabla_h v\}\!\} \cdot \mathrm{n}_F [\![w_h]\!] + [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot \mathrm{n}_F \right)$$

Symmetry II

- Element-by-element integration by parts yields

$$a_h^{\mathrm{cs}}(v, w_h) = -\sum_{T\in\mathcal{T}_h}\int_T (\triangle v) w_h + \sum_{F\in\mathcal{F}_h^i}\int_F [\![\nabla_h v]\!]\cdot \mathrm{n}_F \{\!\{w_h\}\!\} - \sum_{F\in\mathcal{F}_h}\int_F [\![v]\!]\{\!\{\nabla_h w_h\}\!\}\cdot \mathrm{n}_F$$

- This shows that a_h^{cs} retains consistency since

$$[\![\nabla_h u]\!]_F\cdot \mathrm{n}_F = 0 \qquad \text{for all } F\in\mathcal{F}_h^i,$$
$$[\![u]\!]_F = 0 \qquad \text{for all } F\in\mathcal{F}_h$$

Coercivity I

- For all $v_h \in V_h$ there holds

$$a_h^{\text{cs}}(v_h, v_h) = \|\nabla_h v_h\|^2_{[L^2(\Omega)]^d} \boxed{-2 \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot \mathrm{n}_F [\![v_h]\!]}$$

- The boxed term is nondefinite
- We further modify a_h^{cs} as follows: For all $(v, w_h) \in V_{*h} \times V_h$,

$$\boxed{a_h^{\text{sip}}(v, w_h) := a_h^{\text{cs}}(v, w_h) + s_h(v, w_h),}$$

with the stabilization bilinear form

$$s_h(v, w_h) := \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!]$$

Coercivity II

- We aim at asserting coercivity in the norm

$$\forall v \in V_{*h}, \qquad \|v\|_{\mathrm{sip}} := \left(\|\nabla_h v\|^2_{[L^2(\Omega)]^d} + |v|^2_{\mathrm{J}} \right)^{\frac{1}{2}},$$

  with jump seminorm

$$|v|_{\mathrm{J}} := (\eta^{-1} s_h(v,v))^{\frac{1}{2}} = \left(\sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|^2_{L^2(F)} \right)^{\frac{1}{2}}$$

- We anticipate the following discrete Poincaré's inequality:

$$\boxed{\forall v_h \in V_h, \quad \|v_h\|_{L^2(\Omega)} \leq \sigma_2 \|v_h\|_{\mathrm{sip}},}$$

  with $\sigma_2 > 0$ is independent of h

Coercivity III

The choice for s_h is justified by the following result.

Lemma (Bound on consistency and symmetry terms)

For all $(v, w_h) \in V_{*h} \times V_h$,

$$\left| \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot \mathrm{n}_F [\![w_h]\!] \right| \leq \left(\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\nabla v|_T \cdot \mathrm{n}_F\|^2_{L^2(F)} \right)^{\frac{1}{2}} |w_h|_{\mathrm{J}}.$$

Moreover, if $v = v_h \in V_h$,

$$\left| \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot \mathrm{n}_F [\![w_h]\!] \right| \leq C_{\mathrm{tr}} N_\partial^{\frac{1}{2}} \|\nabla_h v_h\|_{[L^2(\Omega)]^d} |v_h|_{\mathrm{J}}.$$

Coercivity IV

Lemma (Discrete coercivity)

For all $\eta > \underline{\eta} := C_{\mathrm{tr}}^2 N_\partial$ *there holds*

$$\forall v_h \in V_h, \qquad a_h^{\mathrm{sip}}(v_h, v_h) \geq C_\eta \|\!|v_h|\!\|_{\mathrm{sip}}^2,$$

with $C_\eta := (\eta - C_{\mathrm{tr}}^2 N_\partial)(1+\eta)^{-1}$.

Coercivity V

$$a_h^{\text{sip}}(v, w_h) = \int_\Omega \nabla_h v \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{\nabla_h v\}\!\} \cdot \mathrm{n}_F [\![w_h]\!] + [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot \mathrm{n}_F \right) + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!],$$

- Using the bound on consistency and symmetry terms,

$$a_h^{\text{sip}}(v_h, v_h) \geq \|\nabla_h v_h\|^2_{[L^2(\Omega)]^d} - 2C_{\text{tr}} N_\partial^{1/2} \|\nabla_h v_h\|_{[L^2(\Omega)]^d} |v_h|_{\mathrm{J}} + \eta |v_h|^2_{\mathrm{J}}$$

- For all $\beta \in \mathbb{R}^+$, $\eta > \beta^2$, $x, y \in \mathbb{R}$, there holds

$$x^2 - 2\beta xy + \eta y^2 \geq \frac{\eta - \beta^2}{1 + \eta} (x^2 + y^2)$$

- Let $\beta = C_{\text{tr}} N_\partial^{1/2}$, $x = \|\nabla_h v_h\|_{[L^2(\Omega)]^d}$, $y = |v_h|_{\mathrm{J}}$ to conclude

Coercivity VI

Lemma (Boundedness)

There is C_{bnd}, independent of h, s.t.

$$\forall (v, w_h) \in V_{*h} \times V_h, \qquad a_h^{\mathrm{sip}}(v, w_h) \leq C_{\mathrm{bnd}} \|v\|_{\mathrm{sip},*} \|w_h\|_{\mathrm{sip}}.$$

where

$$\boxed{\|v\|_{\mathrm{sip},*} := \left(\|v\|_{\mathrm{sip}}^2 + \sum_{T \in \mathcal{T}_h} h_T \|\nabla v|_T \cdot \mathrm{n}_T\|_{L^2(\partial T)}^2 \right)^{\frac{1}{2}}}$$

Basic energy error estimate I

$$\boxed{\text{Find } u_h \in V_h \text{ s.t. } a_h^{\text{sip}}(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h}$$

Theorem (Energy error estimate)

Assume $u \in V_*$ and $\eta > \underline{\eta}$. Then, there is C, independent of h, s.t.

$$|||u - u_h|||_{\text{sip}} \leq C \inf_{v_h \in V_h} |||u - v_h|||_{\text{sip},*}.$$

Basic energy error estimate II

Corollary (Convergence rate in $\|\!|\cdot|\!\|_{\mathrm{sip}}$-norm)

Additionally assume $u \in H^{k+1}(\Omega)$*. Then, there holds*

$$\|\!|u - u_h|\!\|_{\mathrm{sip}} \leq C_u h^k,$$

with $C_u = C\|u\|_{H^{k+1}(\Omega)}$ *and* C *independent of* h*.*

- The above estimate shows that convergence requires $k \geq 1$, i.e., we cannot take $k = 0$
- For an extension to the lowest-order case, cf. [DP, 2012]

L^2-norm error estimate I

- Using the broken Poincaré inequality of [Brenner, 2004] one can infer

$$\|u - u_h\|_{L^2(\Omega)} \leq \sigma_2' C_u h^k$$

- This estimate is suboptimal by one power in h
- An optimal estimate can be recovered exploiting symmetry
- Further regularity for the problem needs to be assumed

L^2-norm error estimate II

Definition (Elliptic regularity)

Elliptic regularity holds true for the model problem (Π) if there is C_{ell}, only depending on Ω, s.t., for all $\psi \in L^2(\Omega)$, the solution to the problem,

$$\text{Find } \zeta \in H_0^1(\Omega) \text{ s.t. } a(\zeta, v) = \int_\Omega \psi v \text{ for all } v \in H_0^1(\Omega),$$

is in V_* and satisfies

$$\|\zeta\|_{H^2(\Omega)} \leq C_{\mathrm{ell}} \|\psi\|_{L^2(\Omega)}.$$

Elliptic regularity holds, e.g., if the domain Ω is convex [Grisvard, 1992]

L^2-norm error estimate III

Theorem (L^2-norm error estimate)

Let $u \in V_*$ solve (Π) and assume elliptic regularity. Then, there is C, independent of h, s.t.

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch \|u - u_h\|_{\mathrm{sip},*}.$$

Corollary (Convergence rate in $\|\cdot\|_{L^2(\Omega)}$-norm)

Additionally assume $u \in H^{k+1}(\Omega)$. Then, there holds

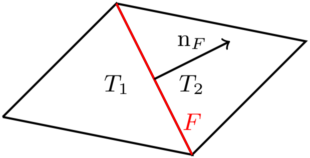
$$\|u - u_h\|_{L^2(\Omega)} \leq C_u h^{k+1}.$$

with $C_u = C\|u\|_{H^{k+1}(\Omega)}$ and C independent of h.

Liftings I

- Liftings map jumps onto vector-valued functions defined on elements
- Liftings play a key role in several developments
  - Flux and mixed formulations
  - Computable lower bound for η
  - Convergence to minimal regularity solutions
- The theoretical developments will eventually allow us to analyze dG methods for nonlinear problems such as the Navier–Stokes equations

Liftings II



- For an integer $l \geq 0$, we define the (local) lifting operator

$$\mathrm{r}_F^l : L^2(F) \longrightarrow [\mathbb{P}_d^l(\mathcal{T}_h)]^d,$$

as follows: For all $\varphi \in L^2(F)$,

$$\boxed{\int_\Omega \mathrm{r}_F^l(\varphi)\cdot\tau_h = \int_F \{\!\{\tau_h\}\!\}\cdot \mathrm{n}_F \varphi \qquad \forall \tau_h \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d}$$

- We observe that $\mathrm{supp}(\mathrm{r}_F^l) = \bigcup_{T \in \mathcal{T}_F} \overline{T}$

Liftings III

- For all $l \geq 0$ and $v \in H^1(\mathcal{T}_h)$, we define the (global) lifting

$$\boxed{\mathrm{R}_h^l([\![v]\!]) := \sum_{F \in \mathcal{F}_h} \mathrm{r}_F^l([\![v]\!]) \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d}$$

- $\mathrm{R}_h^l([\![v]\!])$ maps the jumps of v into a global, vector-valued volumic contribution which is homogeneous to a gradient

Liftings IV

Lemma (Bound on local lifting)

Let $F \in \mathcal{F}_h$ *and let* $l \geq 0$*. For all* $v \in H^1(\mathcal{T}_h)$*, there holds*

$$\| \mathrm{r}_F^l([\![v]\!]) \|_{[L^2(\Omega)]^d} \leq C_{\mathrm{tr}} h_F^{-\frac{1}{2}} \| [\![v]\!] \|_{L^2(F)}.$$

Lemma (Bound on global lifting)

Let $l \geq 0$*. For all* $v \in H^1(\mathcal{T}_h)$*, there holds*

$$\| \mathrm{R}_h^l([\![v]\!]) \|_{[L^2(\Omega)]^d} \leq N_\partial^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}_h} \| \mathrm{r}_F^l([\![v]\!]) \|_{[L^2(\Omega)]^d}^2 \right)^{\frac{1}{2}} \leq C_{\mathrm{tr}} N_\partial^{\frac{1}{2}} |v|_{\mathrm{J}}.$$

Discrete gradients I

- For $l \geq 0$, we define the discrete gradient operator

$$G_h^l : H^1(\mathcal{T}_h) \longrightarrow [L^2(\Omega)]^d,$$

as follows: For all $v \in H^1(\mathcal{T}_h)$,

$$\boxed{G_h^l(v) := \nabla_h v - \mathrm{R}_h^l([\![v]\!])}$$

- The discrete gradient accounts for inter-element and boundary jumps

Lemma (Bound on discrete gradient)

Let $l \geq 0$. For all $v \in H^1(\mathcal{T}_h)$, there holds

$$\|G_h^l(v)\|_{[L^2(\Omega)]^d} \leq (1 + C_{\mathrm{tr}}^2 N_\partial)^{\frac{1}{2}} \|v\|_{\mathrm{sip}}.$$

Reformulation of a_h^{sip} I

- Let $l \in \{k-1, k\}$ and set $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \geq 1$
- There holds for all $v_h, w_h \in V_h$,

$$a_h^{\mathrm{cs}}(v_h, w_h) = \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \int_\Omega \nabla_h v_h \cdot \mathrm{R}_h^l([\![w_h]\!]) - \int_\Omega \nabla_h w_h \cdot \mathrm{R}_h^l([\![v_h]\!])$$

- Indeed $\nabla_h v_h \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d$ with $l \geq k-1$,

$$\forall F \in \mathcal{F}_h, \quad \int_F \{\!\{\nabla_h v_h\}\!\} \cdot \mathrm{n}_F [\![w_h]\!] = \int_\Omega \nabla_h v_h \cdot \mathrm{r}_F^l([\![w_h]\!])$$

- Using the definition of discrete gradients,

$$\boxed{a_h^{\mathrm{cs}}(v_h, w_h) = \int_\Omega G_h^l(v_h) \cdot G_h^l(w_h) - \int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \mathrm{R}_h^l([\![w_h]\!])}$$

Reformulation of a_h^{sip} II

- Plugging the above expression into a_h^{sip},

$$\boxed{a_h^{\mathrm{sip}}(v_h, w_h) = \int_\Omega G_h^l(v_h)\cdot G_h^l(w_h) + \hat{s}_h^{\mathrm{sip}}(v_h, w_h),}$$

with

$$\hat{s}_h^{\mathrm{sip}}(v_h, w_h) := \sum_{F\in\mathcal{F}_h} \frac{\eta}{h_F}\int_F [\![v_h]\!][\![w_h]\!] - \int_\Omega \mathrm{R}_h^l([\![v_h]\!])\cdot \mathrm{R}_h^l([\![w_h]\!])$$

- Dropping the negative term in $\hat{s}_h^{\mathrm{sip}}$ leads to the Local Discontinuous Galerkin (LDG) method of [Cockburn and Shu, 1998]
- This method has the drawback of having a significantly larger stencil

Reformulation of a_h^{sip} III

$$\int_\Omega \nabla_h v_h \cdot \nabla_h w_h$$

$$\int_\Omega \left(\nabla_h v_h \cdot \mathrm{R}_h^l([\![w_h]\!]) + \nabla_h w_h \cdot \mathrm{R}_h^l([\![v_h]\!]) \right),$$

$$\sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

$$\int_\Omega \mathrm{R}_h^l([\![u_h]\!]) \cdot \mathrm{R}_h^l([\![v_h]\!]), \quad \int_\Omega G_h^l(v_h) \cdot G_h^l(w_h)$$

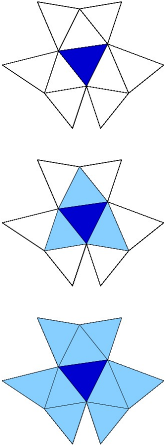


Figure: Stencil of the different terms

Reformulation of a_h^{sip} IV

Lemma (Coercivity (alternative form))

For all $v_h \in V_h$,

$$\|G_h(v_h)\|^2_{[L^2(\Omega)]^d} + (\eta - C_{\mathrm{tr}}^2 N_\partial)|v_h|^2_{\mathrm{J}} \leq a_h(v_h, v_h).$$

Proof.

Observe that

$$a_h(v_h, v_h) = \|G_h(v_h)\|^2_{[L^2(\Omega)]^d} + \eta|v_h|^2_{\mathrm{J}} - \|R_h([\![v_h]\!])\|^2_{[L^2(\Omega)]^d},$$

and use the L^2-stability of R_h to conclude. □

Numerical fluxes I

- Let $T \in \mathcal{T}_h$, $\xi \in \mathbb{P}_d^k(T)$. Element-by-element IBP yields

$$\int_T f\xi = -\int_T (\triangle u)\xi = \int_T \nabla u \cdot \nabla \xi - \int_{\partial T} (\nabla u \cdot \mathrm{n}_T)\xi.$$

- Hence, letting $\Phi_F(u) := -\nabla u \cdot \mathrm{n}_F$ and $\epsilon_{T,F} = \mathrm{n}_T \cdot \mathrm{n}_F$,

$$\boxed{\int_T \nabla u \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \Phi_F(u)\xi = \int_T f\xi.}$$

- Our goal is to identify a similar local conservation property for u_h

Numerical fluxes II

- Using $v_h = \xi\chi_T$ as test function we obtain

$$\int_T f\xi = a_h^{\text{sip}}(u_h, \xi\chi_T) = \int_T \nabla u_h\cdot\nabla\xi - \sum_{F\in\mathcal{F}_T}\int_F \{\!\{(\nabla\xi)\chi_T\}\!\}\cdot\mathrm{n}_F[\![u_h]\!]$$
$$- \sum_{F\in\mathcal{F}_T}\int_F \{\!\{\nabla_h u_h\}\!\}\cdot\mathrm{n}_F[\![\xi\chi_T]\!] + \sum_{F\in\mathcal{F}_T}\int_F \frac{\eta}{h_F}[\![u_h]\!][\![\xi\chi_T]\!]$$

- Let $l \in \{k-1, k\}$. For all $T \in \mathcal{T}_h$ and all $\xi \in \mathbb{P}_d^k(T)$,

$$\int_T G_h^l(u_h)\cdot\nabla\xi + \sum_{F\in\mathcal{F}_T}\epsilon_{T,F}\int_F \phi_F(u_h)\xi = \int_T f\xi,$$

with

$$\phi_F(u_h) := \underbrace{-\{\!\{\nabla_h u_h\}\!\}\cdot\mathrm{n}_F}_{\text{consistency}} + \underbrace{\frac{\eta}{h_F}[\![u_h]\!]}_{\text{penalty}}$$

UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

Numerical fluxes III

- Taking $\xi \equiv 1$ we infer the FV flux conservation property,

$$\sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) = \int_T f$$

Also in the elliptic case local conservation holds on the computational mesh (as opposed to vertex- or face-centered dual mesh)

Part IV

Applications in fluid dynamics

Outline

The Stokes problem I

- We consider the flow of a highly viscous fluid
- The governing Stokes equations read

$$\boxed{\begin{aligned} -\triangle u + \nabla p &= f && \text{in } \Omega, \\ \nabla\cdot u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \langle p \rangle_\Omega &= 0 \end{aligned}}$$

The Stokes problem II

- Let $L_0^2(\Omega) := \left\{v \in L^2(\Omega) \,|\, \langle v \rangle_\Omega = 0\right\}$ and set

$$U := [H_0^1(\Omega)]^d, \quad P := L_0^2(\Omega), \quad X := U \times P$$

- The spaces U, P, and X are Hilbert spaces when equipped with the inner products inducing the norms

$$\begin{aligned}
\|v\|_U &:= \|v\|_{[H^1(\Omega)]^d} := \left(\sum_{i=1}^{d} \|v_i\|_{H^1(\Omega)}^2\right)^{1/2} \\
\|q\|_P &:= \|q\|_{L^2(\Omega)}, \\
\|(v,q)\|_X &:= \left(\|v\|_U^2 + \|q\|_P^2\right)^{1/2}
\end{aligned}$$

The Stokes problem III

- For all $(u, p), (v, q) \in X$ let

$$a(u, v) := \int_\Omega \nabla u{:}\nabla v, \quad b(v, q) := -\int_\Omega q \nabla\cdot v, \quad B(v) := \int_\Omega f\cdot v,$$

- The weak formulation reads: Find $(u, p) \in X$ s.t.

$$\boxed{\begin{aligned} a(u, v) + b(v, p) &= B(v) & \forall v \in U, \\ -b(u, q) &= 0 & \forall q \in P \end{aligned}} \qquad (\Pi_\mathrm{S})$$

- (Π_S) is a constrained minimization problem with the pressure acting as the Lagrange multiplier of the incompressibility constraint

The Stokes problem IV

- Equivalently, letting

$$S((u,p),(v,q)) := a(u,v) + b(v,p) - b(u,q),$$

we can formulate the problem as

$$\boxed{\text{Find } (u,p) \in X \text{ s.t. } S((u,p),(v,q)) = B(v) \text{ for all } (v,q) \in X}$$

The Stokes problem V

- Well-posedness hinges on the coercivity of a and on the inf-sup condition

$$\inf_{q \in P \setminus \{0\}} \sup_{v \in U \setminus \{0\}} \frac{b(v,q)}{\|v\|_U \|q\|_P} \geq \beta_\Omega > 0$$

- Equivalently,

$$\boxed{\forall q \in P, \quad \beta_\Omega \|q\|_P \leq \sup_{v \in U \setminus \{0\}} \frac{b(v,q)}{\|v\|_U}}$$

The Stokes problem VI

Lemma (Surjectivity of the divergence operator from U to P)

Let $\Omega \in \mathbb{R}^d$, $d \geq 1$, be a connected domain. Then, there exists $\beta_\Omega > 0$ s.t. for all $q \in P$, there is $v \in U$ satisfying

$$q = \nabla \cdot v \quad \text{and} \quad \beta_\Omega \|v\|_U \leq \|q\|_P.$$

Proof.

See, e.g., [Girault and Raviart, 1986]. □

The Stokes problem VII

Proof of the continuous inf-sup condition

Let $q \in P$ and let $v \in U$ denote its velocity lifting. The case $v = 0$ is trivial, so let us suppose $v \neq 0$:

$$\begin{aligned}
\|q\|_P^2 = \int_\Omega q \nabla\cdot v = -b(v,q) \\
&\leq \sup_{w \in U\setminus\{0\}} \frac{b(w,q)}{\|w\|_U} \|v\|_U \\
&\leq \beta_\Omega^{-1} \sup_{w \in U\setminus\{0\}} \frac{b(w,q)}{\|w\|_U} \|q\|_P,
\end{aligned}$$

and the conclusion follows.

Equal-order discretization I

- For an integer $k \geq 1$ define the following spaces:

$$U_h := [\mathbb{P}_d^k(\mathcal{T}_h)]^d, \quad P_h := \mathbb{P}_d^k(\mathcal{T}_h) \cap L_0^2(\Omega), \quad X_h := U_h \times P_h$$

- Discrete pressure-velocity coupling: For all $(v_h, q_h) \in X_h$, set

$$\begin{aligned} b_h(v_h, q_h) &:= -\int_\Omega (\nabla_h \cdot v_h) q_h + \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \cdot \mathrm{n}_F \{\!\{ q_h \}\!\} = -\int_\Omega D_h^l(v_h) q_h \\ &= \int_\Omega v_h \cdot \nabla q_h - \sum_{F \in \mathcal{F}_h^i} \int_F \{\!\{ v_h \}\!\} \cdot \mathrm{n}_F [\![q_h]\!], \end{aligned}$$

with $l = k$ and

$$\boxed{D_h^l(v_h) := \mathrm{tr}(G_h^l(v_h)) = \nabla_h \cdot v_h - \mathrm{tr}(R_h^l([\![v_h]\!]))}$$

Equal-order discretization II

- Extending the domain of b_h to $[H^1(\mathcal{T}_h)]^d \times H^1(\mathcal{T}_h)$, we obtain the consistency properties

$$\forall (v, q_h) \in U \times P_h, \qquad b_h(v, q_h) = -\int_\Omega q_h \nabla\cdot v,$$
$$\forall (v_h, q) \in U_h \times H^1(\Omega), \qquad b_h(v_h, q) = \int_\Omega v_h \cdot \nabla q,$$

since, for all $v \in U$ and all $q \in H^1(\Omega)$,

$$[\![v]\!] = 0 \qquad \forall F \in \mathcal{F}_h$$
$$[\![q]\!] = 0 \qquad \forall F \in \mathcal{F}_h^i$$

Equal-order discretization III

Lemma (Discrete inf-sup condition)

There is $\beta > 0$ *independent of* h *s.t. s.t.*

$$\forall q_h \in P_h, \quad \beta \|q_h\|_P \le \sup_{v_h \in U_h \setminus \{0\}} \frac{b_h(v_h, q_h)}{\|v_h\|_{\mathrm{dG}}} + |q_h|_p,$$

where

$$|q_h|_p^2 := \sum_{F \in \mathcal{F}_h^i} h_F \|[\![q_h]\!]\|_{L^2(F)}^2.$$

Equal-order discretization IV

- We stabilize the pressure-velocity coupling using the bilinear form

$$\forall (p_h, q_h) \in P_h, \qquad s_h(p_h, r_h) := \sum_{F \in \mathcal{F}_h^i} h_F \int_F [\![p_h]\!][\![q_h]\!]$$

- We consider the bilinear form

$$S_h((u_h, p_h), (v_h, q_h)) := \\ a_h(u_h, v_h) + b_h(v_h, p_h) - b_h(u_h, q_h) + s_h(p_h, q_h),$$

where

$$a_h(w, v) := \sum_{i=1}^{d} a_h^{\mathrm{sip}}(w_i, v_i)$$

Equal-order discretization V

- The discrete problem reads: Find $(u_h, p_h) \in X_h$ s.t.

$$\boxed{S_h((u_h, p_h), (v_h, q_h)) = B(v_h) \qquad \forall (v_h, q_h) \in X_h} \qquad (\Pi_{\mathrm{S},h})$$

- Equivalently: Find $(u_h, p_h) \in X_h$ s.t.

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) &= B(v_h) && \forall v_h \in U_h, \\ -b_h(u_h, q_h) + s_h(p_h, q_h) &= 0 && \forall q_h \in P_h \end{aligned}$$

- This corresponds to a linear system of the form

$$\begin{bmatrix} \mathbf{A}_h & \mathbf{B}_h \\ -\mathbf{B}_h^t & \mathbf{C}_h \end{bmatrix} \begin{bmatrix} \mathbf{U}_h \\ \mathbf{P}_h \end{bmatrix} = \begin{bmatrix} \mathbf{F}_h \\ \mathbf{0} \end{bmatrix}$$

Stability I

- Equip X_h with the the following norm:

$$\boxed{\|(v_h, q_h)\|_{\mathrm{S}}^2 := \|v_h\|_{\mathrm{vel}}^2 + \|q_h\|_P^2 + |q_h|_p^2,}$$

  where

$$\|v\|_{\mathrm{vel}}^2 := \sum_{i=1}^{d} \|v_i\|_{\mathrm{sip}}^2$$

- Owing to partial coercivity,

$$\forall (v_h, q_h) \in X_h, \quad \alpha \|v_h\|_{\mathrm{vel}}^2 + |q_h|_p^2 \leq S_h((v_h, q_h), (v_h, q_h))$$

Stability II

Lemma (Discrete inf-sup for S_h)

There is $c_S > 0$ *independent of* h *s.t., for all* $(v_h, q_h) \in X_h$,

$$c_S \|(v_h, q_h)\|_\mathrm{S} \le \sup_{(w_h, r_h) \in X_h \setminus \{0\}} \frac{S_h((v_h, q_h), (w_h, r_h))}{\|(w_h, r_h)\|_\mathrm{S}}.$$

Proof.

Consequence of the coercivity of a_h and the discrete inf-sup on b_h. □

Convergence to smooth solutions I

Assumption (Regularity of the exact solution and space X_*)

We assume that the exact solution (u, p) *is in* $X_* := U_* \times P_*$ *where*

$$U_* := U \cap [H^2(\Omega)]^d, \qquad P_* := P \cap H^1(\Omega).$$

We set

$$U_{*h} := U_* + U_h, \qquad P_{*h} := P_* + P_h, \qquad X_{*h} := X_* + X_h.$$

Lemma (Jumps of ∇u and p across interfaces)

Assume $(u, p) \in X_*$. *Then,*

$$[\![\nabla u]\!] \cdot \mathrm{n}_F = 0 \quad \text{and} \quad [\![p]\!] = 0 \qquad \forall F \in \mathcal{F}_h^i.$$

Convergence to smooth solutions II

Lemma (Consistency)

Assume that $(u,p) \in X_*$. *Then,*

$$S_h((u,p),(v_h,q_h)) = \int_\Omega f \cdot v_h \qquad \forall (v_h,q_h) \in X_h.$$

Convergence to smooth solutions III

- We have proved an inf-sup condition for S_h
- It remains to investigate the boundedness of S_h
- Letting

$$\|(v,q)\|_{\mathrm{sto},*}^2 := \|(v,q)\|_{\mathrm{sto}}^2 + \sum_{T\in\mathcal{T}_h} h_T \|\nabla v|_T \cdot \mathrm{n}_T\|_{L^2(\partial T)}^2 + \sum_{T\in\mathcal{T}_h} h_T \|q\|_{L^2(\partial T)}^2,$$

there holds for all $(v,q) \in X_{*h}$ and all $(w_h, r_h) \in X_h$,

$$S_h((v,q),(w_h,r_h)) \leq C_{\mathrm{bnd}} \|(v,q)\|_{\mathrm{sto},*} \|(w_h,r_h)\|_{\mathrm{sto}},$$

with C_{bnd} independent of the meshsize

Convergence to smooth solutions IV

Theorem ($|||\cdot|||_{\mathrm{sto}}$-norm error estimate and convergence rate)

Let $(u, p) \in X_*$ *denote the unique solution of problem* (Π_{S})*. Let* $(u_h, p_h) \in X_h$ *solve* $(\Pi_{\mathrm{S},h})$*. Then, there is* C*, independent of* h*, such that*

$$|||(u - u_h, p - p_h)|||_{\mathrm{sto}} \le C \inf_{(v_h, q_h) \in X_h} |||(u - v_h, p - q_h)|||_{\mathrm{sto},*}.$$

Moreover, if $(u, p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$*,*

$$|||(u - u_h, p - p_h)|||_{\mathrm{sto}} \le C_{u,p} h^k,$$

with $C_{u,p} = C \left(\|u\|_{[H^{k+1}(\Omega)]^d} + \|p\|_{H^k(\Omega)} \right)$*.*

Numerical fluxes I

- Define the inviscid fluxes

$$\hat{p} := \begin{cases} \{\!\{p_h\}\!\} & \text{if } F \in \mathcal{F}_h^i, \\ p_h & \text{if } F \in \mathcal{F}_h^b, \end{cases}$$

$$\hat{u} := \begin{cases} \{\!\{u_h\}\!\} + h_F [\![p_h]\!] \mathrm{n}_F & \text{if } F \in \mathcal{F}_h^i, \\ 0 & \text{if } F \in \mathcal{F}_h^b, \end{cases}$$

- Additionally, we consider here the vector-valued viscous flux

$$\phi_F^{\mathrm{diff}}(u_h) = -\{\!\{\nabla_h u_h\}\!\} \cdot \mathrm{n}_F + \frac{\eta}{h_F} [\![u_h]\!]$$

Numerical fluxes II

- Let $T \in \mathcal{T}_h$ and let $\xi \in [\mathbb{P}_d^k(T)]^d$ with $\xi = (\xi_i)_{1 \le i \le d}$
- Setting $v_h = \xi \chi_T$ in the discrete momentum conservation equation, we obtain for $l \in \{k-1, k\}$,

$$\int_T \sum_{i=1}^{d} G_h^l(u_{h,i}) \cdot \nabla \xi_i - \int_T p_h \nabla \cdot \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \left[\phi_F^{\mathrm{diff}}(u_h) + \hat{p} \mathrm{n}_F \right] \cdot \xi = \int_T f \cdot \xi$$

Numerical fluxes III

- Similarly, let $\zeta \in \mathbb{P}_d^k(T)$
- Setting $q_h = \zeta\chi_T - \langle \zeta\chi_T \rangle_\Omega$ in the discrete mass conservation equation, we obtain

$$-\int_T u_h \cdot \nabla\zeta + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \hat{u} \cdot \mathrm{n}_F \zeta = 0$$

Convergence to minimal regularity solutions I

Theorem (Convergence to minimal regularity solutions)

Let $(u_{\mathcal{H}}, p_{\mathcal{H}}) := ((u_h, p_h))_{h\in\mathcal{H}}$ *solve* $(\Pi_{\mathrm{S},h})$ *on the admissible mesh sequence* $\mathcal{T}_{\mathcal{H}}$*. Then, as* $h \to 0$,

$$\begin{aligned}
u_h &\to u && \textit{strongly in } [L^2(\Omega)]^d,\\
G_h(u_h) &\to \nabla u && \textit{strongly in } [L^2(\Omega)]^{d,d},\\
\nabla_h u_h &\to \nabla u && \textit{strongly in } [L^2(\Omega)]^{d,d},\\
|u_h|_{\mathrm{J}} &\to 0,\\
p_h &\to p && \textit{strongly in } L^2(\Omega),\\
|p_h|_p &\to 0,
\end{aligned}$$

where $(u, p) \in X$ *is the unique solution to* (Π_{S}).

Convergence to minimal regularity solutions II

Lemma (A priori estimate)

The problem $(\Pi_{\mathrm{S},h})$ *is well-posed with the following a priori estimate:*

$$\|(u_h, p_h)\|_{\mathrm{S}} \leq \frac{\sigma_2}{c_S} \|f\|_{[L^2(\Omega)]^d}.$$

- A priori estimate + discrete Rellich theorem [DP and Ern, 2010]: convergence of $(u_{\mathcal{H}}, p_{\mathcal{H}})$ up to a subsequence
- Test using regular functions and conclude using density that the limit solves (Π_{S})
- Use continuous uniqueness to infer that the whole sequence converges
- Use partial coercivity to prove convergence of the gradients

The incompressible Navier–Stokes problem I

- The Navier–Stokes problem reads

$$\boxed{\begin{aligned} -\nu\triangle u + (u{\cdot}\nabla)u + \nabla p &= f && \text{in } \Omega, \\ \nabla{\cdot}u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \langle p\rangle_\Omega &= 0 \end{aligned}}$$

- The nonlinear advection term is the physical source of turbulence
- Uniqueness holds only under a suitable small data assumption

The incompressible Navier–Stokes problem II

- We introduce the trilinear form $t \in \mathcal{L}(U \times U \times U, \mathbb{R})$ is such that

$$t(w, u, v) := \int_\Omega (w\cdot\nabla u)\cdot v = \int_\Omega \sum_{i,j=1}^{d} w_j(\partial_j u_i)v_i.$$

- The weak formulation reads: Find $(u, p) \in X$ s.t., for all $(v, q) \in X$,

$$\boxed{\nu a(u, v) + b(v, p) + t(u, u, v) - b(u, q) = B(v)} \qquad (\Pi_{\mathrm{NS}})$$

The incompressible Navier–Stokes problem III

Lemma (Skew-symmetry of trilinear form)

Letting

$$t'(w,u,v) := t(w,u,v) + \frac{1}{2}\int_\Omega (\nabla\cdot w) u\cdot v,$$

there holds, for all $w \in U$,

$$\boxed{\forall v \in U, \qquad t'(w,v,v) = 0.}$$

Moreover, if $w \in V := \{v \in U \mid \nabla\cdot v = 0\}$,

$$\forall v \in U, \qquad t(w,v,v) = 0.$$

The incompressible Navier–Stokes problem IV

- Let $w \in U$. We observe that, for all $v \in U$,

$$t(w,v,v)+\frac{1}{2}\int_\Omega (\nabla\cdot w)|v|^2 = \int_\Omega \frac{1}{2} w\cdot\nabla|v|^2 + \frac{1}{2}\int_\Omega (\nabla\cdot w)|v|^2 = \int_\Omega \frac{1}{2}\nabla\cdot(w|v|^2),$$

- The divergence theorem yields

$$t(w,v,v) + \frac{1}{2}\int_\Omega (\nabla\cdot w)|v|^2 = \frac{1}{2}\int_{\partial\Omega} (w\cdot \mathrm{n})|v|^2 = 0,$$

  since $(w\cdot\mathrm{n})$ vanishes on $\partial\Omega$ thus proving the first point

- The second point is an immediate consequence of the first

The incompressible Navier–Stokes problem V

- As a consequence, letting $(v, q) = (u, p)$ in (Π_{NS}),

$$\nu \|\nabla u\|^2_{[L^2(\Omega)]^{d,d}} = \int_\Omega f \cdot u,$$

  where we have used $\nabla \cdot u = 0$
- This shows that convection does not influence energy balance

Design of the discrete trilinear form I

- Our starting point is, for $w_h, u_h, v_h \in U_h$,

$$t_h^{(0)}(w_h, u_h, v_h) := \int_\Omega (w_h \cdot \nabla_h u_h) \cdot v_h + \frac{1}{2} \int_\Omega (\nabla_h \cdot w_h) u_h \cdot v_h$$

- Skew-symmetry: For all $w_h, v_h \in U_h$, element-wise IBP yields,

$$t_h^{(0)}(w_h, v_h, v_h) = \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F [\![w_h]\!] \cdot \mathrm{n}_F \{\!\{ v_h \cdot v_h \}\!\} + \sum_{F \in \mathcal{F}_h^i} \int_F \{\!\{ w_h \}\!\} \cdot \mathrm{n}_F [\![v_h]\!] \cdot \{\!\{ v_h \}\!\}$$

- We modify $t_h^{(0)}$ as

$$\begin{aligned} t_h(w_h, u_h, v_h) := & \int_\Omega (w_h \cdot \nabla_h u_h) \cdot v_h - \sum_{F \in \mathcal{F}_h^i} \int_F \{\!\{ w_h \}\!\} \cdot \mathrm{n}_F [\![u_h]\!] \cdot \{\!\{ v_h \}\!\} \\ & + \frac{1}{2} \int_\Omega (\nabla_h \cdot w_h)(u_h \cdot v_h) - \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F [\![w_h]\!] \cdot \mathrm{n}_F \{\!\{ u_h \cdot v_h \}\!\} \end{aligned}$$

Design of the discrete trilinear form II

Lemma (Skew-symmetry of discrete trilinear form)

For all $w_h \in U_h$*, there holds*

$$\forall v_h \in U_h, \qquad t_h(w_h, v_h, v_h) = 0.$$

Design of the discrete trilinear form III

- Let

$$N_h((u_h,p_h),(v_h,q_h)) := \nu a_h(u_h,v_h) + b_h(v_h,p_h) - b_h(u_h,q_h) + t_h(u_h,u_h,v_h)$$

- The discrete problem reads: Find $(u_h, p_h) \in X_h$ s.t.

$$\boxed{N_h((u_h,p_h),(v_h,q_h)) = B(v_h) \qquad \forall (v_h,q_h) \in X_h} \qquad (\Pi_{\mathrm{NS},h})$$

- The existence of a solution to $(\Pi_{\mathrm{NS},h})$ can be proved by a topological degree argument

A priori estimate

Lemma (A priori estimate)

There are c_1, c_2 *independent of* h *such that*

$$\|(u_h, p_h)\|_{\mathrm{S}} \leq c_1 \|f\|_{[L^2(\Omega)]^d} + c_2 \|f\|^2_{[L^2(\Omega)]^d}.$$

Also in this case, this a priori estimate is instrumental to apply the discrete Rellich theorem of [DP and Ern, 2010]

Convergence to minimal regularity solutions

Theorem (Convergence to minimal regularity solutions)

Let $(u_{\mathcal{H}}, p_{\mathcal{H}}) := ((u_h, p_h))_{h\in\mathcal{H}}$ *solve* $(\Pi_{\mathrm{NS},h})$ *on the admissible mesh sequence* $\mathcal{T}_{\mathcal{H}}$*. Then, as* $h \to 0$ *and up to a subsequence,*

$$\begin{aligned}
u_h &\to u && \textit{strongly in } [L^2(\Omega)]^d,\\
G_h(u_h) &\to \nabla u && \textit{strongly in } [L^2(\Omega)]^{d,d},\\
\nabla_h u_h &\to \nabla u && \textit{strongly in } [L^2(\Omega)]^{d,d},\\
|u_h|_{\mathrm{J}} &\to 0, &&\\
p_h &\rightharpoonup p && \textit{weakly in } L^2(\Omega),\\
|p_h|_p &\to 0. &&
\end{aligned}$$

Moreover, under the small data condition, the whole sequence converges.

Numerical validation I

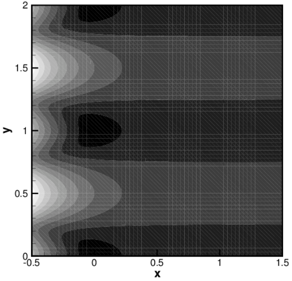
- Let $\Omega = (-0.5, 1.5) \times (0, 2)$
- We consider Kovasznay's solution

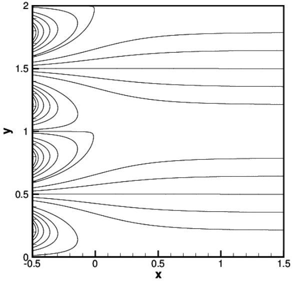
$$
\begin{aligned}
u_1 &= 1 - e^{-\pi x_2} \cos(2\pi x_2), \\
u_2 &= -\frac{1}{2} e^{\pi x_1} \sin(2\pi x_2), \\
p &= -\frac{1}{2} e^{\pi x_1} \cos(2\pi x_2) - \widetilde{p},
\end{aligned}
$$

with $\widetilde{p} \simeq -0.920735694$, $\nu = \frac{1}{3\pi}$ and $f = 0$

- $\mathcal{T}_{\mathcal{H}}$ is a family of uniformly refined triuangular meshes, with h ranging from 0.5 down to 0.03125

Numerical validation II





h	$\lVert e_{h,u} \rVert_{[L^2(\Omega)]^d}$	order	$\lVert e_{h,p} \rVert_{L^2(\Omega)}$	order	$\lVert e_h \rVert_S$	order
h_0	$8.87e-01$	–	$1.62e+00$	–	$1.19e+01$	–
$h_0/2$	$2.39e-01$	1.89	$6.11e-01$	1.41	$7.26e+00$	0.71
$h_0/4$	$5.94e-02$	2.01	$2.01e-01$	1.60	$3.68e+00$	0.98
$h_0/8$	$1.59e-02$	1.90	$7.40e-02$	1.44	$1.85e+00$	0.99
$h_0/16$	$4.17e-03$	**1.93**	$3.14e-02$	**1.23**	$9.25e-01$	**1.00**

A variation with a simple physical interpretation I

$$
\boxed{
\begin{aligned}
\partial_t u + \nabla\cdot(-\nu\nabla u + F(u,p)) &= f, && \text{in } \Omega,\\
\nabla\cdot u &= 0, && \text{in } \Omega,\\
u &= 0, && \text{on } \partial\Omega,\\
\int_\Omega p &= 0
\end{aligned}
}
$$

$$F_{ij}(u,p) := u_i u_j + p\delta_{ij}$$

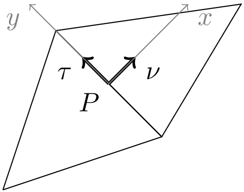
A variation with a simple physical interpretation II

- Let $F \in \mathcal{F}_h^i$, $P \in F$ and define

$$u_\nu := u{\cdot}\mathrm{n}_F, \quad u_\tau := u{\cdot}\tau_F$$

- Restricting the problem to the normal direction we have

$$\begin{aligned}\frac{h_F^2}{c^2}\partial_t p + \partial_x u_\nu &= 0,\\ \partial_t u_\nu + \partial_x(u_\nu^2 + p) &= 0,\\ \partial_t u_\tau + \partial_x(u_\nu u_\tau) &= 0\end{aligned}$$



- To recover a hyperbolic problem we add an artificial compressibility term
- The inviscid flux can be obtained as the solution associated Riemann problem with initial datum (u_h^+, p_h^+), (u_h^-, p_h^-) at P

A variation with a simple physical interpretation III

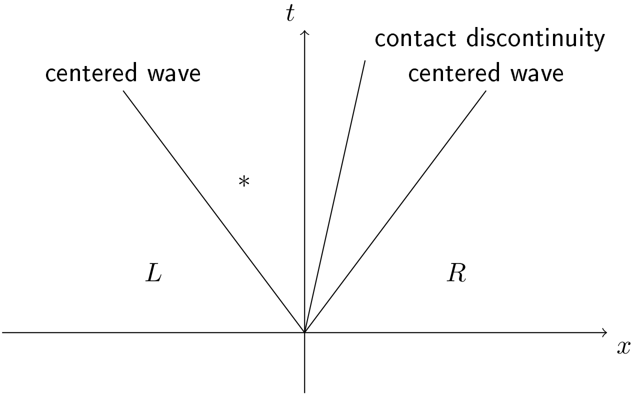


Figure: Structure of the Riemann problem.

A variation with a simple physical interpretation IV

- The exact solution can be found using the Riemann invariants (*rarefactions*) and the Rankine-Hugoniot jump conditions (*shocks*)
- Following a similar procedure, it is possible to write the Riemann problem associated to the Stokes equations
- Let (u^*, p^*) be the solution We define the inviscid flux as

$$\hat{F}(u_h^+, p_h^+; u_h^-, p_h^-) := F(u^*, p^*) = u_i^* u_j^* + p^* \delta_{ij},$$
$$\hat{u}(u_h^+, p_h^+; u_h^-, p_h^-) := u^*.$$

- In the Stokes case, an explicit expression is available for the fluxes

Numerical Fluxes for the Linearized Problems

- We introduce the pressure flux $\hat{p} = p^*$ so that $(\hat{u}, \hat{p}) = (u^*, p^*)$
- In the Stokes case we obtain

$$\hat{u} := \{\!\{u_h\}\!\} + \frac{h_F}{2c} [\![p_h]\!] \mathrm{n}_F,$$
$$\hat{p} := \{\!\{p_h\}\!\} + \frac{c}{2h_F} [\![u_h]\!] \cdot \mathrm{n}_F$$

- Take $c = 2$ and compare with the numerical fluxes for the method we have analyzed!

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