An introduction to Discrete de Rham (DDR) methods

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References for this presentation

- FEEC [Arnold, Falk, Winther, 2006, Arnold, 2018]
- Introduction of DDR [DP, Droniou, Rapetti, 2020]
- DDR with Koszul complements [DP and Droniou, 2023a]
- Application to magnetostatics [DP and Droniou, 2021]
- Polytopal Exterior Calculus [Bonaldi, DP, Droniou, Hu, 2023]
- 2D div-div complex [DP and Droniou, 2023b]
- C++ open-source implementation available in HArDCore3D

Outline

Outline

Setting I

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain with Betti numbers b_i
- We have $b_0 = 1$ (number of connected components) and $b_3 = 0$
- b_1 accounts for the number of tunnels crossing Ω

$$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$$

- b_2, on the other hand, is the number of voids encapsulated by Ω

$$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$$

Setting II

- We consider PDE models that hinge on the vector calculus operators:

$$\mathbf{grad}\, q = \begin{pmatrix} \partial_1 q \\ \partial_2 q \\ \partial_3 q \end{pmatrix}, \ \mathbf{curl}\, \boldsymbol{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}, \ \mathrm{div}\, \boldsymbol{w} = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$$

  for smooth enough functions

$$q : \Omega \to \mathbb{R}, \qquad \boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \boldsymbol{w} : \Omega \to \mathbb{R}^3$$

- The corresponding L^2-domain spaces are

$$\begin{aligned} H^1(\Omega) &:= \left\{ q \in L^2(\Omega) \ : \ \mathbf{grad}\, q \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^3 \right\}, \\ \boldsymbol{H}(\mathbf{curl};\Omega) &:= \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\}, \\ \boldsymbol{H}(\mathrm{div};\Omega) &:= \left\{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \ : \ \mathrm{div}\, \boldsymbol{w} \in L^2(\Omega) \right\} \end{aligned}$$

Three model problems: Stokes

- Given $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)} \\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)} \\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)} \\
\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\
-\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

Three model problems: Magnetostatics

- For $\mu > 0$ and $\boldsymbol{J} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl};\Omega)$, the magnetostatics problem reads:
  Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$\begin{aligned} \mu \boldsymbol{H} - \mathbf{curl}\, \boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)} \\ \mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)} \\ \operatorname{div} \boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)} \\ \boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)} \end{aligned}$$

- Weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\operatorname{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \operatorname{div} \boldsymbol{A} \operatorname{div} \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\Omega) \end{aligned}$$

Three model problems: Darcy

- Given $\kappa > 0$ and $f \in L^2(\Omega)$, the Darcy problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} \kappa^{-1}\boldsymbol{u} - \mathbf{grad}\, p &= 0 && \text{in } \Omega, && \text{(Darcy's law)} \\ -\operatorname{div} \boldsymbol{u} &= f && \text{in } \Omega, && \text{(mass conservation)} \\ p &= 0 && \text{on } \partial\Omega && \text{(boundary condition)} \end{aligned}$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\operatorname{div}; \Omega) \times L^2(\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \kappa^{-1}\boldsymbol{u} \cdot \boldsymbol{v} + \int_\Omega p \operatorname{div} \boldsymbol{v} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}; \Omega), \\ -\int_\Omega \operatorname{div} \boldsymbol{u}\, q &= \int_\Omega f\, q && \forall q \in L^2(\Omega) \end{aligned}$$

A unified view

- The above problems are mixed formulations involving two fields
- They can be recast into the abstract setting: Find $(\sigma, u) \in \Sigma \times U$ s.t.

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= f(\tau) \quad \forall \tau \in \Sigma, \\ -b(\sigma, v) + c(u, v) &= g(v) \quad \forall v \in U, \end{aligned}$$

or, equivalently, in variational formulation,

$$\mathcal{A}((\sigma, u), (\tau, v)) = f(\tau) + g(v) \qquad \forall (\tau, v) \in \Sigma \times U$$

with

$$\mathcal{A}((\sigma, u), (\tau, v)) \coloneqq a(\sigma, \tau) + b(\tau, u) - b(\sigma, v) + c(u, v) = f(\tau) + g(v)$$

- Well-posedness holds under an inf-sup condition on $\mathcal{A}$

A unified tool for well-posedness: The de Rham complex

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- We have key properties depending on the topology of Ω:

$$\begin{aligned} \operatorname{Im}\mathbf{grad} &\subset \operatorname{Ker}\mathbf{curl}\,, \\ \operatorname{Im}\mathbf{curl} &\subset \operatorname{Ker}\operatorname{div}\,, \\ \Omega\subset\mathbb{R}^3\ (b_3=0) \implies \operatorname{Im}\operatorname{div} &= L^2(\Omega) \end{aligned}$$

(Darcy, magnetostatics)

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- We have key properties depending on the topology of Ω:

$$\text{no tunnels crossing } \Omega \ (b_1 = 0) \implies \operatorname{Im}\mathbf{grad} = \operatorname{Ker}\mathbf{curl} \quad \text{(Stokes)}$$

$$\text{no voids contained in } \Omega \ (b_2 = 0) \implies \operatorname{Im}\mathbf{curl} = \operatorname{Ker}\mathrm{div} \quad \text{(magnetostatics)}$$

$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \operatorname{Im}\mathrm{div} = L^2(\Omega) \quad \text{(Darcy, magnetostatics)}$$

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$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(Darcy, magnetostatics)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\mathbf{curl}\,/\mathrm{Im}\,\mathbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\mathbf{curl}$$

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- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\mathbf{curl}\,/\mathrm{Im}\,\mathbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\mathbf{curl}$$

- **Emulating these properties is key for stable discretizations**

Poincaré inequalities

- A consequence of the above facts are Poincaré-type inequalities
- It holds (see, e.g., [Arnold, 2018, Theorem 4.6])

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \lesssim \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \quad \forall \boldsymbol{v} \in (\mathrm{Ker}\,\mathbf{curl}\,)^\perp,$$
$$\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \lesssim \|\,\mathrm{div}\,\boldsymbol{w}\|_{L^2(\Omega)} \quad \forall \boldsymbol{w} \in (\mathrm{Ker}\,\mathrm{div}\,)^\perp,$$

with orthogonals taken w.r.t. the L^2-product

- By the properties of the de Rham complex,

$$\text{if } b_1 = 0,\ \boldsymbol{v} \in (\mathrm{Ker}\,\mathbf{curl}\,)^\perp \iff \int_\Omega \boldsymbol{v}\cdot\nabla q = 0 \text{ for all } q \in H^1(\Omega),$$
$$\text{if } b_2 = 0,\ \boldsymbol{w} \in (\mathrm{Ker}\,\mathrm{div}\,)^\perp \iff \int_\Omega \boldsymbol{w}\cdot\mathbf{curl}\,\boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$$

Well-posedness of the magnetostatics problem I

- Assume, for the sake of simplicity, $\mu = 1$ and set

$$\mathcal{A}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) \coloneqq a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}) - b(\boldsymbol{\sigma}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{v})$$

  with bilinear forms a, b, and c s.t.

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \coloneqq \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}, \quad b(\boldsymbol{\tau}, \boldsymbol{v}) \coloneqq -\int_\Omega \mathbf{curl}\, \boldsymbol{\tau} \cdot \boldsymbol{v}, \quad c(\boldsymbol{u}, \boldsymbol{v}) \coloneqq \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v}$$

- The variational formulation of magnetostatics reads:

  Find $(\boldsymbol{H}, \boldsymbol{A}) \in \mathcal{Z} \coloneqq \boldsymbol{H}(\mathbf{curl}; \Omega) \times \boldsymbol{H}(\mathrm{div}; \Omega)$ s.t.

$$\mathcal{A}((\boldsymbol{H}, \boldsymbol{A}), (\boldsymbol{\tau}, \boldsymbol{v})) = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathcal{Z}$$

- Define the norm s.t., $\forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathcal{Z}$,

$$\|(\boldsymbol{\tau}, \boldsymbol{v})\|_{\mathcal{Z}} \coloneqq \left(\|\boldsymbol{\tau}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div};\Omega)} \right)^{\frac{1}{2}}$$

Well-posedness of the magnetostatics problem II

Theorem (Well-posedness for magnetostatics)

Assume $b_2 = 0$. Then, it holds, for all $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathcal{Z}$,

$$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{\mathcal{Z}} \lesssim \$:= \sup_{(\boldsymbol{\tau},\boldsymbol{v}) \in \mathcal{Z}\setminus\{0\}} \frac{\mathcal{A}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v}))}{\|(\boldsymbol{\tau}, \boldsymbol{v})\|_{\mathcal{Z}}}.$$

Hence, the magnetostatics problem admits a unique solution that satisfies

$$\|(\boldsymbol{H}, \boldsymbol{A})\|_{\mathcal{Z}} \lesssim \|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}.$$

Well-posedness of the magnetostatics problem III

- Taking $(\boldsymbol{\tau}, \boldsymbol{v}) = (\boldsymbol{\sigma}, \boldsymbol{u} + \mathbf{curl}\,\boldsymbol{\sigma})$ and since $c(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v}$, we have

$$\begin{aligned}
&\mathcal{A}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\sigma}, \boldsymbol{u} + \mathbf{curl}\,\boldsymbol{\sigma})) \\
&\quad = a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) + \cancel{b(\boldsymbol{\sigma}, \boldsymbol{u})} - b(\boldsymbol{\sigma}, \cancel{\boldsymbol{u}} + \mathbf{curl}\,\boldsymbol{\sigma}) + c(\boldsymbol{u}, \boldsymbol{u} + \cancel{\mathbf{curl}\,\boldsymbol{\sigma}}) \qquad (\operatorname{div} \mathbf{curl} = 0) \\
&\quad = a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) - b(\boldsymbol{\sigma}, \mathbf{curl}\,\boldsymbol{\sigma}) + c(\boldsymbol{u}, \boldsymbol{u}) \\
&\quad = \|\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\,\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} + \|\operatorname{div} \boldsymbol{u}\|^2_{L^2(\Omega)} \\
&\quad = \|\boldsymbol{\sigma}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\operatorname{div} \boldsymbol{u}\|^2_{L^2(\Omega)}
\end{aligned}$$

- Hence,

$$\|\boldsymbol{\sigma}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\operatorname{div} \boldsymbol{u}\|^2_{L^2(\Omega)} \lesssim \$\|(\boldsymbol{\sigma}, \boldsymbol{u} + \mathbf{curl}\,\boldsymbol{\sigma})\|_{\mathcal{Z}} \lesssim \$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{\mathcal{Z}} \qquad (1)$$

- It only remains to estimate $\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}$. To this purpose, we write

$$\begin{aligned}
\boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{u}^\perp &\in \operatorname{Ker} \operatorname{div} \oplus (\operatorname{Ker} \operatorname{div})^\perp \\
&\overset{b_2=0}{=} \operatorname{Ker} \operatorname{div} \oplus (\operatorname{Im} \mathbf{curl})^\perp \qquad (\operatorname{Ker} \operatorname{div} = \operatorname{Im} \mathbf{curl})
\end{aligned}$$

Well-posedness of the magnetostatics problem IV

- By the Poincaré inequality for the divergence, we have

$$\|\boldsymbol{u}^\perp\|^2_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \lesssim \|\operatorname{div}\boldsymbol{u}^\perp\|^2_{L^2(\Omega)} = \|\operatorname{div}\boldsymbol{u}\|^2_{L^2(\Omega)} \lesssim \$\|(\boldsymbol{\sigma},\boldsymbol{u})\|_{\mathcal{Z}} \qquad (2)$$

- Since $b_2 = 0$, we can find $\boldsymbol{v} \in (\operatorname{Ker}\mathbf{curl})^\perp$ such that

$$\boldsymbol{u}^0 = -\mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \qquad (3)$$

Well-posedness of the magnetostatics problem V

- We then write

$$\$\|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \gtrsim \$\|(\boldsymbol{v},\boldsymbol{0})\|_{\mathcal{Z}} \gtrsim \mathcal{A}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{v},\boldsymbol{0})) = a(\boldsymbol{\sigma},\boldsymbol{v}) + b(\boldsymbol{v},\boldsymbol{u})$$
$$= \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{v} - \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{u}$$
$$= \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{u}^0\cdot\boldsymbol{u} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{v} + \|\boldsymbol{u}^0\|^2_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)},$$

- Rearranging the term and using a Cauchy–Schwarz inequality, we obtain

$$\|\boldsymbol{u}^0\|^2_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \lesssim \$\|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} + \|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}$$
$$\overset{(3)}{\lesssim} \left(\$ + \|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}\right)\|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)},$$

so that, simplifying, squaring both sides, and recalling (1),

$$\|\boldsymbol{u}^0\|^2_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)} \lesssim \$^2 + \$\|(\boldsymbol{\sigma},\boldsymbol{u})\|_{\mathcal{Z}} \tag{4}$$

Well-posedness of the magnetostatics problem VI

- Summing (1), (2), and (4), we get,

$$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{\mathcal{Z}}^2 \lesssim \$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{\mathcal{Z}} + \$^2,$$

  where we have additionally noticed that, by L^2-orthogonality,

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}^2 = \|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}^2 + \|\boldsymbol{u}^\perp\|_{\boldsymbol{L}^2(\Omega;\mathbb{R}^3)}^2$$

- Using Young's inequality we conclude the proof that

$$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{\mathcal{Z}} \lesssim \$$$

- The well-posedness of the magnetostatics problem readily follows

- Let $T \subset \mathbb{R}^3$ be a polyhedron and set, for any $r \geq -1$,

$$\mathcal{P}_r(T) \coloneqq \{\text{restrictions of 3-variate polynomials of degree} \leq r \text{ to } T\}$$

- Fix $r \geq 0$. Denoting by $\boldsymbol{x}_T$ a point inside T, it holds

$$\begin{aligned} \mathcal{P}_r(T)^3 &= \mathbf{grad}\,\mathcal{P}_{r+1}(T) \oplus (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}_{r-1}(T)^3 &&\eqqcolon \boldsymbol{\mathcal{G}}_r(T) \oplus \boldsymbol{\mathcal{G}}_r^{\rm c}(T) \\ &= \mathbf{curl}\,\mathcal{P}_{r+1}(T)^3 \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}_{r-1}(T) &&\eqqcolon \boldsymbol{\mathcal{R}}_r(T) \oplus \boldsymbol{\mathcal{R}}_r^{\rm c}(T) \end{aligned}$$

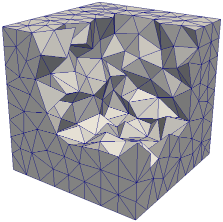
- Define the trimmed spaces that sit between $\mathcal{P}_r(T)^3$ and $\mathcal{P}_{r+1}(T)^3$:

$$\boldsymbol{\mathcal{N}}_{r+1}(T) \coloneqq \boldsymbol{\mathcal{G}}_r(T) \oplus \boldsymbol{\mathcal{G}}_{r+1}^{\rm c}(T) \qquad \text{[Nédélec, 1980]}$$

$$\boldsymbol{\mathcal{RT}}_{r+1}(T) \coloneqq \boldsymbol{\mathcal{R}}_r(T) \oplus \boldsymbol{\mathcal{R}}_{r+1}^{\rm c}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

The Finite Element way

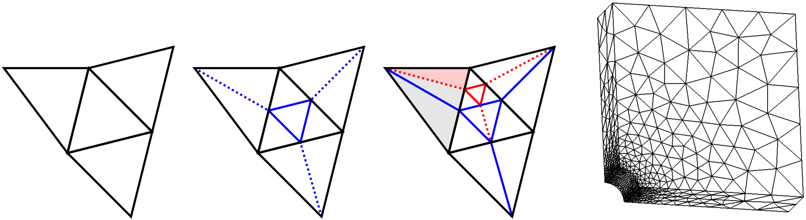
Global complex



- Let $\mathcal{T}_h$ be a conforming tetrahedral mesh of Ω and let $r \geq 0$
- Local spaces can be glued together to form a global FE complex:

$$
\begin{array}{ccccccccc}
\mathcal{P}_{r+1}^{\text{cont}}(\mathcal{T}_h) & \xrightarrow{\textbf{grad}} & \boldsymbol{\mathcal{N}}_{r+1}(\mathcal{T}_h) & \xrightarrow{\textbf{curl}} & \boldsymbol{\mathcal{RT}}_{r+1}(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}_r(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^1(\Omega) & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl};\Omega) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}
$$

- **The gluing only works on conforming meshes (simplicial complexes)!**



- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - ⟹ significant increase of DOFs on hexahedral elements

Outline

A higher-level view of vector calculus operators

- So far, we have treated $\mathbf{grad}$, $\mathbf{curl}$, and div as different operators
- A unified view is possible through exterior calculus
- This view can be exploited in the construction of numerical approximations

Alternating forms I

- Let $\mathrm{Alt}^k(\mathbb{R}^n)$ be the space of (multilinear) forms that are alternating, i.e.: For all $1 \leq i < j \leq k$ and all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$,

$$\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_k) = -\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_k)$$

- The exterior product of $\omega \in \mathrm{Alt}^i(\mathbb{R}^n)$ and $\mu \in \mathrm{Alt}^j(\mathbb{R}^n)$ is

$$\omega \wedge \mu \in \mathrm{Alt}^{i+j}(\mathbb{R}^n)$$

s.t., for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i+j}$ in $\mathbb{R}^n$,

$$(\omega \wedge \mu)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i+j}) := \sum_{\sigma \in \Sigma_{i,j}} \mathrm{sign}(\sigma)\, \omega(\boldsymbol{v}_{\sigma_1}, \ldots, \boldsymbol{v}_{\sigma_i})\, \mu(\boldsymbol{v}_{\sigma_{i+1}}, \ldots, \boldsymbol{v}_{\sigma_{i+j}}),$$

with

$$\Sigma_{i,j} := \left\{\text{permutations of } (1, \ldots, i+j) \ : \ \sigma_1 < \cdots < \sigma_i \text{ and } \sigma_{i+1} < \cdots < \sigma_{i+j}\right\}$$

Alternating forms II

Example (Exterior product of 1-forms)

Given $\omega, \mu \in \Lambda^1(\mathbb{R}^n)$, it holds, for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$,

$$(\omega \wedge \mu)(\boldsymbol{v}, \boldsymbol{w}) = \omega(\boldsymbol{v})\mu(\boldsymbol{w}) - \omega(\boldsymbol{w})\mu(\boldsymbol{v}),$$

so that, in particular, $\omega \wedge \omega = 0$.

Alternating forms III

- Let $\{\boldsymbol{e}_i\}_{1\leq i\leq n}$ denote the canonical basis of $\mathbb{R}^n$
- We consider the dual basis $\{\mathrm{d}x^i\}_{1\leq i\leq n}$ of $(\mathbb{R}^n)'$, characterised by

$$\mathrm{d}x^i(\boldsymbol{e}_j) = \delta_{ij} \qquad 1 \leq i, j \leq n$$

- Every $\omega \in \mathrm{Alt}^k(\mathbb{R}^n)$ can be expanded using this basis as

$$\omega = \sum_{1\leq\sigma_1<\cdots<\sigma_k\leq n} a_\sigma \, \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}, \quad a_\sigma \in \mathbb{R}$$

Inner product of alternating k-forms

- The scalar product in $\mathbb{R}^n$ induces an inner product $\langle\cdot,\cdot\rangle$ on $\mathrm{Alt}^\ell(\mathbb{R}^n)$
- If $\ell = 1$, $\langle\cdot,\cdot\rangle$ is simply the inner product of $(\mathbb{R}^n)'$
- For general ℓ, given two ℓ-forms expressed as exterior products of 1-forms

$$\omega = \omega^1 \wedge \cdots \wedge \omega^\ell, \qquad \mu = \mu^1 \wedge \cdots \wedge \mu^\ell,$$

  we set

$$\langle\omega,\mu\rangle := \det\left[\langle\omega^i,\mu^j\rangle\right]_{1\le i,j\le\ell}$$

Hodge star I

- The Hodge star operator $\star : \mathrm{Alt}^{\ell}(\mathbb{R}^n) \to \mathrm{Alt}^{n-\ell}(\mathbb{R}^n)$ is s.t.

$$\forall \omega \in \mathrm{Alt}^{\ell}(\mathbb{R}^n), \quad \langle \star\omega, \mu \rangle \mathrm{vol} = \omega \wedge \mu \quad \forall \mu \in \mathrm{Alt}^{n-\ell}(\mathbb{R}^n)$$

  where $\mathrm{vol} := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is the volume form
- It can be checked that $\star$ is an isomorphism
- In what follows, we will also need its inverse

$$\star^{-1} := (-1)^{\ell(n-\ell)} \star$$

Hodge star II

Example (Hodge star)

$n = 2$	$n = 3$
$\star 1 = \mathrm{d}x^1 \wedge \mathrm{d}x^2$	$\star 1 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$
$\star \mathrm{d}x^1 = \mathrm{d}x^2$	$\star \mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3$
$\star \mathrm{d}x^2 = -\mathrm{d}x^1$	$\star \mathrm{d}x^2 = -\mathrm{d}x^1 \wedge \mathrm{d}x^3$
	$\star \mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2$

Formulas for $\star$ applied to 2- and 3-forms (if $n = 3$) can be obtained taking the $\star^{-1}$ of the previous expressions, e.g., for $n = 3$,

$$\mathrm{d}x^1 = \star^{-1} \star \mathrm{d}x^1 = \star^{-1}(\mathrm{d}x^2 \wedge \mathrm{d}x^3) = (-1)^{2(3-2)} \star (\mathrm{d}x^2 \wedge \mathrm{d}x^3) = \star(\mathrm{d}x^2 \wedge \mathrm{d}x^3).$$

Vector proxies in dimension $n = 3$

For $n = 3$, we can identify vector proxies for all form degrees:

- $\mathrm{Alt}^0(\mathbb{R}^3) := \mathbb{R}$ by definition
- $\mathrm{Alt}^3(\mathbb{R}^3) = \star\mathrm{Alt}^0(\mathbb{R}^3) \cong \mathbb{R}$ since $\star$ is an isomorphism
- $\mathrm{Alt}^1(\mathbb{R}^3) = (\mathbb{R}^3)'$ and, for all $\omega \in \mathrm{Alt}^1(\mathbb{R}^3)$,

$$\omega = a\,\mathrm{d}x^1 + b\,\mathrm{d}x^2 + c\,\mathrm{d}x^3 \cong (a, b, c) \in \mathbb{R}^3$$

- $\mathrm{Alt}^2(\mathbb{R}^3) = \star\mathrm{Alt}^1(\mathbb{R}^3) \cong \mathbb{R}^n$ and, for all $\omega \in \mathrm{Alt}^2(\mathbb{R}^3)$,

$$\omega = a\,\underbrace{\mathrm{d}x^2 \wedge \mathrm{d}x^3}_{\star\mathrm{d}x^1} - b\,\underbrace{\mathrm{d}x^1 \wedge \mathrm{d}x^3}_{-\star\mathrm{d}x^2} + c\,\underbrace{\mathrm{d}x^1 \wedge \mathrm{d}x^2}_{\star\mathrm{d}x^3} \cong (a, b, c) \in \mathbb{R}^3$$

For general n, vector proxies are available for $\mathrm{Alt}^0(\mathbb{R}^n) \cong \mathrm{Alt}^n(\mathbb{R}^n)$ and $\mathrm{Alt}^1(\mathbb{R}^n) \cong \mathrm{Alt}^{n-1}(\mathbb{R}^n)$

Differential forms

- Let M denote an open set in an affine subspace of $\mathbb{R}^n$
- A (differential) k-form is given by

$$\omega = \sum_{1 \le \sigma_1 < \cdots < \sigma_k \le n} a_\sigma \, \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}, \quad a_\sigma : M \to \mathbb{R}$$

- The value of a k-form at $\boldsymbol{x} \in M$ is denoted $\omega_{\boldsymbol{x}}$:

$$\omega_{\boldsymbol{x}} = \sum_{1 \le \sigma_1 < \cdots < \sigma_k \le n} a_\sigma(\boldsymbol{x}) \, \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k} \in \mathrm{Alt}^k(\mathbb{R}^n)$$

- The space of k-forms (without regularity requirements on a_σ) is $\Lambda^k(M)$
- When regularity on the a_σ is required, we prepend it to $\Lambda^k(M)$, e.g.,

$L^2\Lambda^k(M)$ = space of k-forms with coefficients a_σ square-integrable on M,
$\mathcal{P}_r\Lambda^k(M)$ = space of k-forms with coefficients a_σ in $\mathcal{P}_r(M)$

Exterior derivative I

- The exterior derivative is the (unbounded) operator

$$\mathrm{d} : L^2\Lambda^k(M) \to L^2\Lambda^{k+1}(M)$$
$$\omega \mapsto \sum_{1\le\sigma_1<\cdots<\sigma_k\le n} \sum_{i=1}^{n} \frac{\partial a_\sigma}{\partial x_i} \, \mathrm{d}x^i \wedge \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$$

- In what follows, we define the domain of the exterior derivative

$$H\Lambda^k(M) \coloneqq \left\{\omega \in L^2\Lambda^k(M) \; : \; \mathrm{d}\omega \in L^2\Lambda^{k+1}(M)\right\}$$

- For $M = \Omega$ domain of $\mathbb{R}^3$,
  - d corresponds $\mathbf{grad}$, $\mathbf{curl}$, div regarded as unbounded operators
  - $H\Lambda^k(\Omega)$ to the usual spaces $H^1(\Omega)$, $\boldsymbol{H}(\mathbf{curl};\Omega)$, $\boldsymbol{H}(\mathrm{div};M)$, and $L^2(\Omega)$

Exterior derivative II

Example (Exterior derivative of a 0-form)

Let Ω be a domain of $\mathbb{R}^3$ and $\omega = \varphi \in C^1\Lambda^0(\overline{\Omega})$ a 0-form. Then

$$\mathrm{d}\omega = \frac{\partial \varphi}{\partial x_1}\mathrm{d}x^1 + \frac{\partial \varphi}{\partial x_2}\mathrm{d}x^2 + \frac{\partial \varphi}{\partial x_3}\mathrm{d}x^3 \cong \mathbf{grad}\,\varphi.$$

Exterior derivative III

Example (Exterior derivative of a 1-form)

Moving to a 1-form $C^1\Lambda^1(\overline{\Omega}) \ni \omega = a_1 \mathrm{d}x^1 + a_2 \mathrm{d}x^2 + a_3 \mathrm{d}x^3 \cong \boldsymbol{v}$, we have

$$
\begin{aligned}
\mathrm{d}\omega = {} & \frac{\partial a_1}{\partial x_1}\cancel{\mathrm{d}x^1 \wedge \mathrm{d}x^1} + \frac{\partial a_1}{\partial x_2}\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial a_1}{\partial x_3}\mathrm{d}x^3 \wedge \mathrm{d}x^1 \\
& + \frac{\partial a_2}{\partial x_1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{\partial a_2}{\partial x_2}\cancel{\mathrm{d}x^2 \wedge \mathrm{d}x^2} + \frac{\partial a_2}{\partial x_3}\mathrm{d}x^3 \wedge \mathrm{d}x^2 \\
& + \frac{\partial a_3}{\partial x_1}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\partial a_3}{\partial x_2}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial a_3}{\partial x_3}\cancel{\mathrm{d}x^3 \wedge \mathrm{d}x^3} \\
= {} & \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3 - \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
\cong {} & \mathbf{curl}\,\boldsymbol{v}.
\end{aligned}
$$

Exterior derivative IV

Example (Exterior derivative of a 2-form)

For a 2-form

$$C^1\Lambda^2(\overline{\Omega}) \ni \omega = a_1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 - a_2 \mathrm{d}x^1 \wedge \mathrm{d}x^3 + a_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cong \boldsymbol{w},$$

we have

$$\begin{aligned}
\mathrm{d}\omega = {} & \frac{\partial a_1}{\partial x_1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial a_1}{\partial x_2}\cancel{\mathrm{d}x^2 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3} + \frac{\partial a_1}{\partial x_3}\cancel{\mathrm{d}x^3 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3} \\
& - \frac{\partial a_2}{\partial x_1}\cancel{\mathrm{d}x^1 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3} - \frac{\partial a_2}{\partial x_2}\mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \frac{\partial a_2}{\partial x_3}\cancel{\mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3} \\
& + \frac{\partial a_3}{\partial x_1}\cancel{\mathrm{d}x^1 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2} + \frac{\partial a_3}{\partial x_2}\cancel{\mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2} + \frac{\partial a_3}{\partial x_3}\mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
= {} & \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\right)\mathrm{vol} \cong \mathrm{div}\,\boldsymbol{w}.
\end{aligned}$$

The continuous de Rham complex

- Let Ω denote a domain of $\mathbb{R}^n$
- In what follows, we will focus on the de Rham complex

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, we have the following interpretation in terms of vector proxies:

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

Stokes formula

- Let M denote an n-dimensional manifold and $\ell \in \mathbb{N}$ s.t. $0 \leq \ell \leq n$
- Let $\mathrm{tr}_{\partial M}$ be the trace operator (pullback of the inclusion $\partial M \hookrightarrow M$) s.t.

$$\mathrm{tr}_{\partial M} : \Lambda^k(M) \to \Lambda^k(\partial M)$$

- It holds, for all $(\omega, \mu) \in C^1\Lambda^\ell(\overline{M}) \times C^1\Lambda^{n-\ell-1}(\overline{M})$,

$$\int_M \mathrm{d}\omega \wedge \mu = (-1)^{\ell+1} \int_M \omega \wedge \mathrm{d}\mu + \int_{\partial M} \mathrm{tr}_{\partial M}\, \omega \wedge \mathrm{tr}_{\partial M}\, \mu$$

Outline

General ideas

- Discrete spaces with polynomial components attached to mesh entities
- For any form degree k, recursively on d-cells f, $d = k, \ldots, n$, construct
  - A local discrete potential

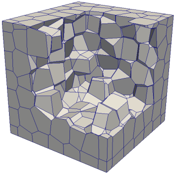
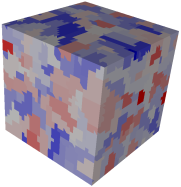
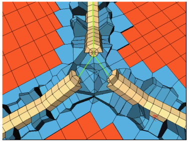
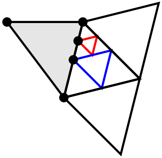
$$P^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^k(f)$$

  - If $d \geq k + 1$, a local discrete exterior derivative

$$\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^{k+1}(f)$$

- Connect the spaces through a global discrete exterior derivative

Domain and polytopal mesh

- Assume $\Omega \subset \mathbb{R}^n$ polytopal (polygon if $n = 2$, polyhedron if $n = 3, \ldots$)
- We consider a polytopal mesh $\mathcal{M}_h$ containing all (flat) d-cells, $0 \le d \le n$
- d-cells in $\mathcal{M}_h$ are collected in $\Delta_d(\mathcal{M}_h)$, so that, when $n = 3$,
    - $\Delta_0(\mathcal{M}_h) = \mathcal{V}_h$ is the set of vertices
    - $\Delta_1(\mathcal{M}_h) = \mathcal{E}_h$ is the set of edges
    - $\Delta_2(\mathcal{M}_h) = \mathcal{F}_h$ is the set of faces
    - $\Delta_3(\mathcal{M}_h) = \mathcal{T}_h$ is the set of elements

Local Koszul differential and complements I

- Let $f \in \Delta_d(\mathcal{M}_h)$, $d \in [0, n]$, and fix $\boldsymbol{x}_f \in f$
- We define the local Koszul differential $\kappa : \Lambda^{\ell+1}(f) \to \Lambda^{\ell}(f)$ s.t.

$$(\kappa\omega)_{\boldsymbol{x}}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell) = \omega_{\boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_f, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell)$$

  for all $\boldsymbol{x} \in f$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$ tangent vectors to f
- κ "binds" the first vector to $\boldsymbol{x} - \boldsymbol{x}_f$
- We define the Koszul complement space

$$\mathcal{K}_r^\ell(f) \coloneqq \kappa \mathcal{P}_{r-1} \Lambda^{\ell+1}(f)$$

Local Koszul differential and complements II

Example (Vector proxies for $\mathcal{K}_r^\ell(f_d)$)

ℓ \ d	0	1	2	3
0	$\{0\}$	$\mathcal{P}_\flat^r(f_1)$	$\mathcal{P}_\flat^r(f_2)$	$\mathcal{P}_\flat^r(f_3)$
1		$\{0\}$	$\boldsymbol{\mathcal{R}}_r^{\rm c}(f_2)$	$\boldsymbol{\mathcal{G}}_r^{\rm c}(f_3)$
2			$\{0\}$	$\boldsymbol{\mathcal{R}}_r^{\rm c}(f_3)$
3				$\{0\}$

$$\mathcal{K}_r^0(f_d) \cong \mathcal{P}_\flat^r(f_d) := (\boldsymbol{x} - \boldsymbol{x}_{f_d}) \cdot \boldsymbol{\mathcal{P}}_{r-1}(f_d) \quad \forall d \in \{1,2,3\},$$
$$\mathcal{K}_r^{d-1}(f_d) \cong \boldsymbol{\mathcal{R}}_r^{\rm c}(f_d) := (\boldsymbol{x} - \boldsymbol{x}_{f_d})\mathcal{P}_{r-1}(f_d) \quad \forall d \in \{2,3\},$$
$$\mathcal{K}_r^1(f_3) \cong \boldsymbol{\mathcal{G}}_r^{\rm c}(f_3) := (\boldsymbol{x} - \boldsymbol{x}_{f_3}) \times \boldsymbol{\mathcal{P}}_{r-1}(f_3)$$

Trimmed local polynomial spaces I

- Let $f \in \Delta_d(\mathcal{M}_h)$, $1 \le d \le n$, and integers $\ell \in [0, d]$ and $r \ge 0$ be fixed
- The following direct decompositions hold:

$$\mathcal{P}_r\Lambda^0(f) = \mathcal{P}_0\Lambda^0(f) \oplus \mathcal{K}_r^0(f),$$
$$\mathcal{P}_r\Lambda^\ell(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \quad \text{if } \ell \ge 1$$

- Lowering by one the polynomial degree of the first component for $\ell \ge 1$ yields trimmed polynomial spaces

$$\mathcal{P}_r^-\Lambda^0(f) := \mathcal{P}_r\Lambda^0(f),$$
$$\mathcal{P}_r^-\Lambda^\ell(f) := \mathrm{d}\mathcal{P}_r\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \quad \text{if } \ell \ge 1$$

Trimmed local polynomial spaces II

- Let $n = 3$ and $T = f_3 \in \Delta_3(\mathcal{M}_h) = \mathcal{T}_h$ be a mesh element
- The vector proxies for trimmed spaces are the Nédélec and Raviart–Thomas spaces

$$\mathcal{P}_r^-\Lambda^1(f_3) \cong \boldsymbol{\mathcal{N}}_r(T) := \boldsymbol{\mathcal{G}}_{r-1}(T) + \boldsymbol{\mathcal{G}}_r^{\mathrm{c}}(T)$$
$$\mathcal{P}_r^-\Lambda^2(f_3) \cong \boldsymbol{\mathcal{RT}}_r(T) := \boldsymbol{\mathcal{R}}_{r-1}(T) + \boldsymbol{\mathcal{R}}_r^{\mathrm{c}}(T)$$

- For $F = f_2 \in \Delta_2(\mathcal{M}_h)$, we have

$$\mathcal{P}_r^-\Lambda^1(f_2) \cong \boldsymbol{\mathcal{RT}}_r(F)$$

L^2-orthogonal projector onto $\mathcal{P}_r^- \Lambda^k(f)$

- We define the L^2-orthogonal projector $\pi_{r,f}^{-,k} : L^2\Lambda^k(f) \mapsto \mathcal{P}_r^-\Lambda^k(f)$ s.t.

$$\forall \omega \in L^2\Lambda^k(f), \quad \int_f \pi_{r,f}^{-,k}\omega \wedge \star\mu = \int_f \omega \wedge \star\mu \quad \forall \mu \in \mathcal{P}_r^-\Lambda^k(f)$$

- We note the following result: For all $(\omega, \mu) \in L^2\Lambda^k(f) \times \mathcal{P}_r^-\Lambda^{d-k}(f)$,

$$\int_f \star^{-1}\pi_{r,f}^{-,d-k}(\star\omega) \wedge \mu = \int_f \mu \wedge \star\pi_{r,f}^{-,d-k}(\star\omega) = \int_f \omega \wedge \mu$$

Discrete spaces and interpolators I

- The discrete $H\Lambda^k(\Omega)$ space, $0 \le k \le n$, is

$$\underline{X}_{r,h}^k \coloneqq \bigtimes_{d=k}^{n} \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \mathcal{P}_r^- \Lambda^{d-k}(f)$$

- Its restrictions to $f \in \Delta_d(\mathcal{M}_h)$, $k \le d \le n$, and ∂f are $\underline{X}_{r,f}^k$ and $\underline{X}_{r,\partial f}^k$
- The meaning of the polynomial components is provided by the interpolator

$$\begin{aligned} \underline{I}_{r,f}^k : C^0\Lambda^k(\overline{f}) &\to \underline{X}_{r,f}^k \\ \omega &\mapsto \left(\pi_{r,f'}^{-,d'-k}(\star \operatorname{tr}_{f'} \omega)\right)_{f' \in \Delta_{d'}(f),\, d' \in [k,d]} \end{aligned}$$

Discrete spaces and interpolators II

$k \backslash d$	0	1	2	3
0	$\mathbb{R} = \mathcal{P}_r\Lambda^0(f_0)$	$\mathcal{P}_{r-1}\Lambda^1(f_1)$	$\mathcal{P}_{r-1}\Lambda^2(f_2)$	$\mathcal{P}_{r-1}\Lambda^3(f_3)$
1		$\mathcal{P}_r\Lambda^0(f_1)$	$\mathcal{P}_r^-\Lambda^1(f_2)$	$\mathcal{P}_r^-\Lambda^2(f_3)$
2			$\mathcal{P}_r\Lambda^0(f_2)$	$\mathcal{P}_r^-\Lambda^1(f_3)$
3				$\mathcal{P}_r\Lambda^0(f_3)$

$k \backslash d$	0	1	2	3
0	$\mathbb{R} = \mathcal{P}_r(f_0)$	$\mathcal{P}_{r-1}(f_1)$	$\mathcal{P}_{r-1}(f_2)$	$\mathcal{P}_{r-1}(f_3)$
1		$\mathcal{P}_r(f_1)$	$\mathcal{RT}_r(f_2)$	$\mathcal{RT}_r(f_3)$
2			$\mathcal{P}_r(f_2)$	$\mathcal{N}_r(f_3)$
3				$\mathcal{P}_r(f_3)$

Discrete potential and exterior derivative I

- Let $d \in \mathbb{N}$ be s.t. $0 \le d \le n$ and $f \in \Delta_d(\mathcal{M}_h)$
- The Stokes formula on f reads: For all $(\omega, \mu) \in C^1\Lambda^k(\overline{f}) \times C^1\Lambda^{d-k-1}(\overline{f})$,

$$\int_f \mathrm{d}\omega \wedge \mu = (-1)^{k+1} \int_f \omega \wedge \mathrm{d}\mu + \int_{\partial f} \mathrm{tr}_{\partial f}\, \omega \wedge \mathrm{tr}_{\partial f}\, \mu$$

- Local reconstructions are obtained emulating this formula

Discrete potential and exterior derivative II

- If $d = k$,

$$P^k_{r,f}\underline{\omega}_f \coloneqq \star^{-1}\omega_f \in \mathcal{P}_r\Lambda^d(f)$$

- If $k+1 \le d \le n$, we first let, for all $\underline{\omega}_f \in \underline{X}^k_{r,f}$ and all $\mu \in \mathcal{P}_r\Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu = (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

then, for all $(\mu,\nu) \in \mathcal{K}^{d-k-1}_{r+1}(f) \times \mathcal{K}^{d-k}_r(f)$,

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge (\mathrm{d}\mu + \nu) = \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu + (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \nu$$

The case $n = 3$ and $k = 1$ I

- For $T = f_3 \in \Delta_3(\mathcal{M}_h) = \mathcal{T}_h$,

$$\underline{X}^1_{r,f} \cong \underline{\boldsymbol{X}}^r_{\mathbf{curl},T} := \bigtimes_{E \in \mathcal{E}_T} \mathcal{P}_r(E) \times \bigtimes_{F \in \mathcal{F}_T} \boldsymbol{\mathcal{RT}}_r(F) \times \boldsymbol{\mathcal{RT}}_r(T)$$

- Let

$$\underline{\boldsymbol{v}}_T = \big((v_E)_{E \in \mathcal{E}_T}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}, \boldsymbol{v}_T\big) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},T}$$

- The edge tangential trace is simply

$$\gamma^r_{\mathrm{t},E} \underline{\boldsymbol{v}}_E := v_E \quad \forall E \in \mathcal{E}_T$$

The case $n = 3$ and $k = 1$ II

- For all $F \in \mathcal{F}_T$, the face curl is given by: For all $q \in \mathcal{P}_r(F)$,

$$\int_F C_F^r \underline{\boldsymbol{v}}_F \; q = \int_F \boldsymbol{v}_F \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \varepsilon_{FE} \int_E \gamma_{\mathrm{t},E}^r \underline{\boldsymbol{v}}_E \; q$$

- The face tangential trace is such that, for all $(q, \boldsymbol{w}) \in \mathcal{P}_{r+1}^{\flat}(F) \times \boldsymbol{\mathcal{R}}_r^{\mathrm{c}}(F)$,

$$\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^r \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot}_F \, q + \boldsymbol{w}) = \int_F C_F^r \underline{\boldsymbol{v}}_F \; q + \sum_{E \in \mathcal{E}_F} \varepsilon_{FE} \int_E \gamma_{\mathrm{t},E}^r \underline{\boldsymbol{v}}_E \; q + \int_F \boldsymbol{v}_F \cdot \boldsymbol{w}$$

- The element curl satisfies, for all $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_r(T)$,

$$\int_T \boldsymbol{C}_T^r \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w} = \int_T \boldsymbol{v}_T \cdot \mathbf{curl}\, \boldsymbol{w} + \sum_{F \in \mathcal{F}_T} \varepsilon_{TF} \int_F \boldsymbol{\gamma}_{\mathrm{t},F}^r \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w} \times \boldsymbol{n}_F)$$

- Finally, by similar principles, we can construct $\boldsymbol{P}_{\mathbf{curl},T}^r : \underline{\boldsymbol{X}}_{\mathbf{curl},T}^r \to \boldsymbol{\mathcal{P}}_r(T)$

Complex property

Theorem (Complex property)

Let $\underline{\mathrm{d}}_{r,h}^{k} : \underline{X}_{r,h}^{k} \to \underline{X}_{r,h}^{k+1}$ *be s.t.*

$$\underline{\mathrm{d}}_{r,h}^{k}\underline{\omega}_h := \left(\pi_{r,f}^{-,d-k-1}(\star\mathrm{d}_{r,f}^{k}\underline{\omega}_f)\right)_{f\in\Delta_d(\mathcal{M}_h),\, d\in[k+1,n]}.$$

Then it holds, for all $0 \le k \le d \le n$*, all* $f \in \Delta_d(\mathcal{M}_h)$*, and all* $\underline{\omega}_f \in \underline{X}_{r,f}^{k-1}$*,*

$$P_{r,f}^{k}(\underline{\mathrm{d}}_{r,f}^{k-1}\underline{\omega}_f) = \mathrm{d}_{r,f}^{k-1}\underline{\omega}_f,$$

and, if $d \ge k+1$*,*

$$\mathrm{d}_{r,f}^{k}(\underline{\mathrm{d}}_{r,f}^{k-1}\underline{\omega}_f) = 0.$$

As a consequence, $\underline{\mathrm{d}}_{r,f}^{k}\underline{\mathrm{d}}_{r,f}^{k-1} = 0$ *and the DDR sequence defines a complex.*

Links between reconstructions and commutation I

Theorem (Commutation)

For all $0 \le k \le d-1 \le n-1$ and for all $f \in \Delta_d(\mathcal{M}_h)$, it holds

$$\underline{\mathrm{d}}_{r,f}^{k}(\underline{I}_{r,f}^{k}\omega) = \underline{I}_{r,f}^{k+1}(\mathrm{d}\omega) \qquad \forall \omega \in C^1\Lambda^k(\overline{f}),$$

expressing the commutativity of the following diagram:

$$\begin{array}{ccc}
C^1\Lambda^k(\overline{f}) & \xrightarrow{\ \mathrm{d}\ } & C^0\Lambda^{k+1}(\overline{f}) \\
\Big\downarrow \underline{I}_{r,f}^{k} & & \Big\downarrow \underline{I}_{r,f}^{k+1} \\
\underline{X}_{r,f}^{k} & \xrightarrow{\ \underline{\mathrm{d}}_{r,f}^{k}\ } & \underline{X}_{r,f}^{k+1}.
\end{array}$$

Polynomial consistency I

Theorem (Polynomial consistency)

For all integers $0 \le k \le d \le n$ *and all* $f \in \Delta_d(\mathcal{M}_h)$*, it holds*

$$P_{r,f}^k \underline{I}_{r,f}^k \omega = \omega \qquad \forall \omega \in \mathcal{P}_r \Lambda^k(f),$$

and, if $d \ge k+1$*,*

$$\mathrm{d}_{r,f}^k \underline{I}_{r,f}^k \omega = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}_{r+1}^- \Lambda^k(f).$$

Example (The case $(n, d, k) = (3, 3, 1)$)

The above properties translate as follows for $(n, d, k) = (3, 3, 1)$:

$$\begin{aligned} \boldsymbol{P}_{\mathbf{curl},T}^r \underline{\boldsymbol{I}}_{\mathbf{curl},T}^r \boldsymbol{v} &= \boldsymbol{v} & \qquad \forall \boldsymbol{v} &\in \boldsymbol{\mathcal{P}}_r(T), \\ \boldsymbol{C}_T^r \underline{\boldsymbol{I}}_{\mathbf{curl},T}^r \boldsymbol{v} &= \mathbf{curl}\, \boldsymbol{v} & \qquad \forall \boldsymbol{v} &\in \boldsymbol{\mathcal{N}}_{r+1}(T). \end{aligned}$$

Polynomial consistency II

- The proof is made by induction on $\rho := d - k$. If $\rho = 0$ (i.e., $d = k$), we have

$$P^k_{r,d}\underline{I}^k_{r,f}\omega = \star^{-1}\pi^{-,0}_{r,f}(\star\omega) = \star^{-1}\star\omega = \omega$$

- Assume that the lemma holds for a given $\rho \geq 0$, and consider d and k s.t.

$$d - k = \rho + 1$$

- By the link between potentials and differentials and the commutativity,

$$\mathrm{d}^k_{r,f}\underline{I}^k_{r,f}\omega = P^{k+1}_{r,f}(\underline{\mathrm{d}}^k_{r,f}\underline{I}^k_{r,f}\omega) = P^{k+1}_{r,f}\underline{I}^{k+1}_{r,f}(\mathrm{d}\omega) \quad \forall\omega \in \mathcal{P}^-_{r+1}\Lambda^k(f)$$

- Since $\mathrm{d}\omega \in \mathcal{P}_r\Lambda^{k+1}(f)$ and $d - (k+1) = \rho$, by the induction hypothesis

$$\mathrm{d}^k_{r,f}\underline{I}^k_{r,f}\omega = \mathrm{d}\omega$$

Polynomial consistency III

- For $\omega \in \mathcal{P}_r\Lambda^k(f)$, we write, for all $(\mu, \nu) \in \mathcal{K}_{r+1}^{d-k-1}(f) \times \mathcal{K}_r^{d-k}(f)$,

$$
\begin{aligned}
(-1)^{k+1} \int_f P_{r,f}^k \underline{I}_{r,f}^k \omega \wedge (\mathrm{d}\mu + \nu) &= \int_f \mathrm{d}\omega \wedge \mu \\
&\quad - \int_{\partial f} P_{r,\partial f}^k \underline{I}_{r,\partial f}^k \operatorname{tr}_{\partial f} \omega \wedge \operatorname{tr}_{\partial f} \mu + (-1)^{k+1} \int_f \cancelto{\omega}{(\star^{-1} \pi_{r,f}^{-,d-k} \star \omega)} \wedge \nu
\end{aligned}
$$

- Applying the polynomial consistency of $P_{r,\partial f}^k$ (valid by induction since $(d-1) - k = \rho$) and integrating by parts yields

$$
\int_f P_{r,f}^k \underline{I}_{r,f}^k \omega \wedge (\mathrm{d}\mu + \nu) = \int_f \omega \wedge (\mathrm{d}\mu + \nu),
$$

which, since $\mathrm{d}\mu + \nu$ spans $\mathcal{P}^r\Lambda^k(f)$, gives

$$
P_{r,f}^k \underline{I}_{r,f}^k \omega = \omega
$$

Global discrete exterior derivative and DDR complex

- Our next goal is to connect the spaces $\underline{X}_{r,h}^k$ to form a well-defined sequence
- We recall the global discrete exterior derivative $\underline{\mathrm{d}}_{r,h}^k : \underline{X}_{r,h}^k \to \underline{X}_{r,h}^{k+1}$ s.t.

$$\underline{\mathrm{d}}_{r,h}^k \underline{\omega}_h \coloneqq \left(\pi_{r,f}^{-,d-k-1}(\star \mathrm{d}_{r,f}^k \underline{\omega}_f)\right)_{f\in\Delta_d(\mathcal{M}_h),\, d\in[k+1,n]}$$

- The DDR sequence then reads

$$\underline{X}_{r,h}^0 \xrightarrow{\underline{\mathrm{d}}_{r,h}^0} \underline{X}_{r,h}^1 \longrightarrow \cdots \longrightarrow \underline{X}_{r,h}^{n-1} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{n-1}} \underline{X}_{r,h}^n \longrightarrow \{0\}$$

- Specifically, for $n=3$, we recover the complex of [DP and Droniou, 2023a]:

$$\underline{X}_{\mathbf{grad},h}^r \xrightarrow{\underline{\boldsymbol{G}}_h^r} \underline{\boldsymbol{X}}_{\mathbf{curl},h}^r \xrightarrow{\underline{\boldsymbol{C}}_h^r} \underline{\boldsymbol{X}}_{\mathrm{div},h}^r \xrightarrow{D_h^r} \mathcal{P}_r(\mathcal{T}_h) \longrightarrow \{0\}$$

Cohomology I

Theorem (Cohomology of the Discrete de Rham complex)

The cohomology of the DDR complex is isomorphic to that of the continuous de Rham complex.

Example (The case $n = 3$)

For $n = 3$, in terms of vector proxies, this implies

$$\text{no "tunnels" crossing } \Omega \ (b_1 = 0) \implies \operatorname{Im} \underline{\boldsymbol{G}}_h^r = \operatorname{Ker} \underline{\boldsymbol{C}}_h^r,$$
$$\text{no "voids" contained in } \Omega \ (b_2 = 0) \implies \operatorname{Im} \underline{\boldsymbol{C}}_h^r = \operatorname{Ker} D_h^r,$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \operatorname{Im} D_h^r = \mathcal{P}_k(\mathcal{T}_h)$$

Cohomology II

Proof.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Delta_k^*(\mathcal{M}_h) & \xrightarrow{\partial_k^*} & \Delta_{k+1}^*(\mathcal{M}_h) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \kappa_k} & & \uparrow{\scriptstyle \kappa_{k+1}} & & \\
\cdots & \longrightarrow & \underline{X}_{0,h}^k & \xrightarrow{\underline{\mathrm{d}}_{0,h}^k} & \underline{X}_{0,h}^{k+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle \underline{R}_h^k}\uparrow\ \downarrow{\scriptstyle \underline{E}_h^k} & & {\scriptstyle \underline{R}_h^{k+1}}\uparrow\ \downarrow{\scriptstyle \underline{E}_h^{k+1}} & & \\
\cdots & \longrightarrow & \underline{X}_{r,h}^k & \xrightarrow{\underline{\mathrm{d}}_{r,h}^k} & \underline{X}_{r,h}^{k+1} & \longrightarrow & \cdots
\end{array}$$

Key point: design of the extension cochain map □

Discrete L^2-product

- For all $0 \le k \le n$, we let $(\cdot,\cdot)_{k,h} : \underline{X}^k_{r,h} \times \underline{X}^k_{r,h} \to \mathbb{R}$ be s.t.

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} \coloneqq \sum_{f \in \Delta_n(\mathcal{M}_h)} (\underline{\omega}_f, \underline{\mu}_f)_{k,f}$$

with

$$(\underline{\omega}_f, \underline{\mu}_f)_{k,f} \coloneqq \int_f P^k_{r,f}\underline{\omega}_f \wedge \star P^k_{r,f}\underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) \quad \forall f \in \Delta_n(\mathcal{M}_h)$$

- Above, $s_{k,f}$ is a stabilization contribution s.t., with h_f diameter of f,

$$s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) = \sum_{d'=k}^{n-1} h_f^{n-d'} \sum_{f' \in \Delta_{d'}(f)} \int_{f'} (\mathrm{tr}_{f'} P^k_{r,f}\underline{\omega}_f - P^k_{r,f'}\underline{\omega}_{f'}) \wedge \star(\mathrm{tr}_{f'} P^k_{r,f}\underline{\mu}_f - P^k_{r,f'}\underline{\mu}_{f'})$$

Outline

Discrete problem

- We assume, from now on, $b_1 = b_2 = 0$ and $\mu \in \mathbb{R}$
- We seek $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\,\boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\,\boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{A}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- The discrete problem reads: Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}$ s.t.

$$\begin{aligned} (\mu \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h}, \\ (\underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^r_h \underline{\boldsymbol{A}}_h\, D^r_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^r_{\mathrm{div},h} \end{aligned}$$

- For $b_2 \neq 0$, we need to add orthogonality to harmonic forms

Stability

Theorem (Stability)

Define the bilinear form $\mathcal{A}_h : \left[\underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}\right]^2 \to \mathbb{R}$ *s.t.*

$$\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) \coloneqq$$
$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^r_h \underline{\boldsymbol{u}}_h \, D^r_h \underline{\boldsymbol{v}}_h.$$

Then, the following inf-sup condition holds: $\forall (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}$,

$$|||(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)|||_h \lesssim \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h} \setminus \{(\underline{\mathbf{0}}, \underline{\mathbf{0}})\}} \frac{\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h}$$

with $|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h^2 \coloneqq \|\underline{\boldsymbol{\tau}}_h\|^2_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\tau}}_h\|^2_{\mathrm{div},h} + \|\underline{\boldsymbol{v}}_h\|^2_{\mathrm{div},h} + \|D^r_h \underline{\boldsymbol{v}}_h\|^2_{L^2(\Omega)}$.

Proof.

Analogous to the continuous case! □

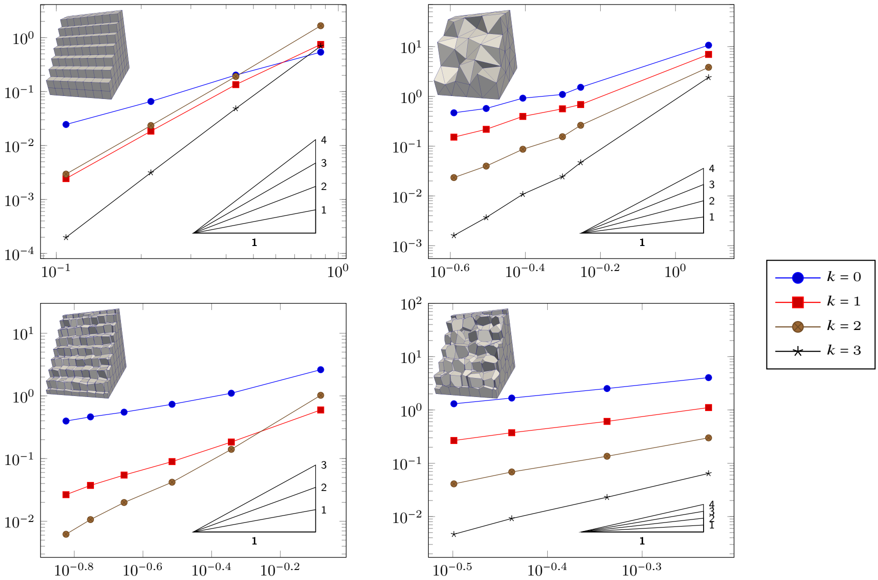
Error estimate

Theorem (Error estimate for the magnetostatics problem)

Assume $\boldsymbol{H} \in C^0(\overline{\Omega})^3 \cap H^{r+2}(\mathcal{T}_h)^3$ and $\boldsymbol{A} \in C^0(\overline{\Omega})^3 \cap H^{r+2}(\mathcal{T}_h)^3$. Then, we have the following error estimate:

$$|||(\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^r_{\mathbf{curl},h}\boldsymbol{H}, \underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}^r_{\mathrm{div},h}\boldsymbol{A})|||_h \lesssim h^{r+1}.$$

Convergence: Energy error vs. meshsize



Outline

Bases for local polynomial spaces I

- Let $T \in \mathcal{T}_h$ and $\ell \geq 0$, set $N_{\mathcal{P},T}^{\ell} := \dim\left(\mathcal{P}_\ell(T)\right) = \binom{\ell+3}{3}$, and denote by

$$\mathfrak{P}_{\ell,T} := \left\{\varphi_{\mathcal{P},T}^{i} \,:\, i \in [0, N_{\mathcal{P},T}^{\ell}[\right\}$$

  a basis for $\mathcal{P}_\ell(T)$ s.t. $\varphi_{\mathcal{P},T}^{0} \equiv C$ and and $\int_T \varphi_{\mathcal{P},T}^{i} = 0$ if $i \geq 1$
- For simplicity, we also assume that $\mathfrak{P}_{\ell,T} \subset \mathfrak{P}_{\ell+1,T}$ for all $\ell \geq 0$
- A basis $\boldsymbol{\mathfrak{P}}_{\ell,T}$ for $\boldsymbol{\mathcal{P}}_\ell(T)$ is obtained by tensorisation
- **The choice of $\mathfrak{P}_{\ell,T}$ has a sizeable impact on conditioning!**

Bases for local polynomial spaces II

- Let $N^{\ell}_{\boldsymbol{\mathcal{G}},T} \coloneqq \dim\left(\boldsymbol{\mathcal{G}}_{\ell}(T)\right) = N^{\ell+1}_{\mathcal{P},T} - 1$
- Bases $\mathfrak{G}^{\rm c}_{\ell,T}$, $\boldsymbol{\mathcal{G}}^{\rm c}_{\ell}(T)$ for $\mathfrak{R}^{\rm c}_{\ell,T}$, $\boldsymbol{\mathcal{R}}^{\rm c}_{\ell}(T)$ are obtained from their definitions
- $\mathbf{grad} : \mathcal{P}_{0,\ell+1}(T) \xrightarrow{\cong} \boldsymbol{\mathcal{G}}_{\ell}(T)$ being an isomorphism, a basis $\mathfrak{G}_{\ell,T}$ for $\boldsymbol{\mathcal{G}}_{\ell}(T)$ is

$$\mathfrak{G}_{\ell,T} \coloneqq \left\{\boldsymbol{\varphi}^{i}_{\boldsymbol{\mathcal{G}},T} \coloneqq \mathbf{grad}\,\varphi^{i+1}_{\mathcal{P},T} \;:\; i \in [0, N^{\ell}_{\boldsymbol{\mathcal{G}},T}[\right\}$$

- $\mathbf{curl} : \boldsymbol{\mathcal{G}}^{\rm c}_{\ell+1}(T) \xrightarrow{\cong} \boldsymbol{\mathcal{R}}_{\ell}(T)$ is an isomorphism, so a basis $\mathfrak{R}_{\ell,T}$ for $\boldsymbol{\mathcal{R}}_{\ell}(T)$ is

$$\mathfrak{R}_{\ell,T} \coloneqq \left\{\boldsymbol{\varphi}^{i}_{\boldsymbol{\mathcal{R}},T} \coloneqq \mathbf{curl}\,\boldsymbol{\varphi}^{i,\ell+1,\rm c}_{\boldsymbol{\mathcal{G}},T} \;:\; i \in [0, N^{\ell+1,\rm c}_{\boldsymbol{\mathcal{G}},T}[\right\}$$

- For spaces on faces, we proceed similarly using local orthogonal coordinates

Local reconstructions I

- A basis $\mathfrak{B}^r_{\mathrm{div},T}$ for $\underline{\boldsymbol{X}}^r_{\mathrm{div},T}$ is obtained setting

$$\mathfrak{B}^r_{\mathrm{div},T} := \mathfrak{G}_{r-1,T} \times \mathfrak{G}^{\mathrm{c}}_{r,T} \times \bigtimes_{F \in \mathcal{F}_T} \mathfrak{P}_{r,F}$$

- Let $\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_{\mathcal{G},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{G},T}, (v_F)_{F \in \mathcal{F}_T}) \in \underline{\boldsymbol{X}}^r_{\mathrm{div},T}$ with coefficients vector

$$\underline{\mathsf{V}}_T = \begin{bmatrix} \mathsf{V}_{\mathcal{G},T} \\ \mathsf{V}^{\mathrm{c}}_{\mathcal{G},T} \\ \mathsf{V}_{F_1} \\ \vdots \\ \mathsf{V}_{F_{\mathrm{card}(\mathcal{F}_T)}} \end{bmatrix} \in \mathbb{R}^{N^k_{\mathrm{div},T}}$$

Local reconstructions II

- The coefficient vector $\mathsf{D}_T \in \mathbb{R}^{N^r_{\mathcal{P},T}}$ of $D^r_T \underline{\boldsymbol{v}}_T$ solves

$$\mathbf{M}_{D,T}\mathsf{D}_T = -\mathbf{B}_{D,T}\mathsf{V}_{\mathcal{G},T} + \sum_{F\in\mathcal{F}_T} \omega_{TF}\mathbf{B}_{D,F}\mathsf{V}_F,$$

with

$$\mathbf{M}_{D,T} := \left[\int_T \varphi^i_{\mathcal{P},T}\varphi^j_{\mathcal{P},T}\right]_{(i,j)\in[0,N^r_{\mathcal{P},T}[^2},$$

$$\mathbf{B}_{D,T} := \left[\int_T \mathbf{grad}\,\varphi^i_{\mathcal{P},T}\cdot\boldsymbol{\varphi}^j_{\mathcal{G},T}\right]_{(i,j)\in[0,N^r_{\mathcal{P},T}[\times[0,N^r_{\mathcal{G},T}[},$$

$$\mathbf{B}_{D,F} := \left[\int_F \varphi^i_{\mathcal{P},T}\varphi^j_{\mathcal{P},F}\right]_{(i,j)\in[0,N^r_{\mathcal{P},T}[\times[0,N^r_{\mathcal{P},F}[}.$$

- $D^r_T : \underline{\boldsymbol{X}}^r_{\mathrm{div},T} \to \mathcal{P}_r(T)$ is represented by the matrix $\mathbf{D}_T \in \mathbb{R}^{N^r_{\mathcal{P},T}\times N^k_{\mathrm{div},T}}$ whose ith column is the solution of the above problem for $\underline{\mathsf{V}}_T = \mathsf{e}_i$
- $\boldsymbol{P}^r_{\mathrm{div},T} : \underline{\boldsymbol{X}}^r_{\mathrm{div},r} \to \boldsymbol{\mathcal{P}}_r(T)$ is represented by $\mathbf{P}_{\mathrm{div},T} \in \mathbb{R}^{3N^r_{\mathcal{P},T}\times N^k_{\mathrm{div},T}}$

Local L^2-product I

- The matrix representing the discrete L^2-product in $\underline{\boldsymbol{X}}^r_{\mathrm{div},r}$ is

$$\mathsf{L}_{\mathrm{div},T} := \mathsf{P}^\intercal_{\mathrm{div},T}\mathsf{M}_{\mathrm{div},T}\mathsf{P}_{\mathrm{div},T} + \mathsf{S}_{\mathrm{div},T} \in \mathbb{R}^{N^k_{\mathrm{div},T}\times N^k_{\mathrm{div},T}}$$

  where
  - $\mathsf{M}_{\mathrm{div},T} \in \mathbb{R}^{3N^{r+1}_{\mathcal{P},T}\times 3N^{r+1}_{\mathcal{P},T}}$ is the mass matrix of $\boldsymbol{\mathcal{P}}_r(T)$
  - $\mathsf{S}_{\mathrm{div},T}$ is the matrix representation of the stabilisation

Local L^2-product II

- The stabilisation bilinear form penalises in a least-square the difference

$$\boldsymbol{\Delta}_{\mathrm{div},T} \coloneqq \boldsymbol{\Pi}_{\mathcal{G},T}^{r-1}\mathbf{P}_{\mathrm{div},T} - \left[\mathbf{I}_{N_{\mathcal{G},T}^{r-1}}\mathbf{0}\cdots\mathbf{0}\right],$$
$$\boldsymbol{\Delta}_{\mathrm{div},F} \coloneqq \mathbf{T}_F\mathbf{P}_{\mathrm{div},T} - \left[\mathbf{0}\cdots\mathbf{I}_{N_{\mathcal{P},F}^{r}}\cdots\mathbf{0}\right],$$

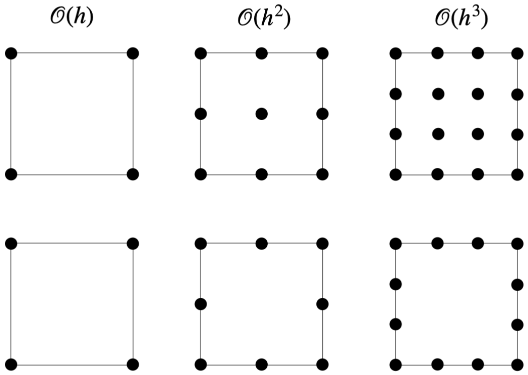
  where
  - $\boldsymbol{\Pi}_{\mathcal{G},T}^{r-1}$ represents $\boldsymbol{\pi}_{\mathcal{G},T}^{r-1}$ applied to $\boldsymbol{\mathcal{P}}_r(T)$
  - $\mathbf{T}_F$ represents the normal trace operator applied to $\boldsymbol{\mathcal{P}}_r(T)$
- Specifically, we can take

$$\mathbf{S}_{\mathrm{div},T} \coloneqq \boldsymbol{\Delta}_{\mathrm{div},T}^{\intercal}\mathbf{M}_{\mathcal{G},T}\boldsymbol{\Delta}_{\mathrm{div},T} + \sum_{F\in\mathcal{F}_T} h_F\boldsymbol{\Delta}_{\mathrm{div},F}^{\intercal}\mathbf{M}_{\mathcal{P},F}\boldsymbol{\Delta}_{\mathrm{div},F}$$

  with $\mathbf{M}_{\mathcal{G},T}$ and $\mathbf{M}_{\mathcal{P},F}$ mass matrices of $\boldsymbol{\mathcal{G}}_r(T)$ and $\mathcal{P}_r(F)$

Outline

Serendipity I



- Serendipity FEMs converge as standard FEM but with fewer DOFs
- It is possible to devise serendipity DDR sequences [DP and Droniou, 2023c]
- Ideas similar to [Beirão da Veiga et al., 2018]

Serendipity II

Definition (Boundaries selection)

For each $\tau \in \mathcal{T}_h \cup \mathcal{F}_h$, we select a set $\mathcal{B}_\tau$ of $\eta_\tau \geq 2$ faces/edges

- that are not pairwise aligned;
- s.t. τ lies on one side of the hyperplane H_σ spanned by each $\sigma \in \mathcal{B}_\tau$;
- are "uniformly far" from each other: $\mathrm{dist}_{\tau\sigma}(\boldsymbol{x}_{\sigma'}) \gtrsim 1$ for all $\sigma' \in \mathcal{B}_\tau \setminus \{\sigma\}$ with $\mathrm{dist}_{\tau\sigma}(\boldsymbol{x}) \coloneqq h_\tau^{-1}(\boldsymbol{x} - \boldsymbol{x}_\tau)\omega_{\tau\sigma} \cdot \boldsymbol{n}_\sigma$ scaled distance function to H_σ.

Serendipity III

- Setting

$$\ell_F := k + 1 - \eta_F \quad \forall F \in \mathcal{F}_h, \qquad \ell_T := k + 1 - \eta_T \quad \forall T \in \mathcal{T}_h,$$

the serendipity gradient and curl spaces are

$$\begin{aligned}
\underline{\widehat{X}}^r_{\mathbf{grad},h} := \Big\{ \underline{q}_T = \big((q_T)_{T\in\mathcal{T}_h}, (q_F)_{F\in\mathcal{F}_h}, (q_E)_{E\in\mathcal{E}_h}, (q_V)_{V\in\mathcal{V}_h} \big) \ : \\
q_T \in \mathcal{P}_{\ell_T}(T) \text{ for all } T \in \mathcal{T}_h,\ q_F \in \mathcal{P}_{\ell_F}(F) \text{ for all } F \in \mathcal{F}_h, \\
q_E \in \mathcal{P}_{r-1}(E) \text{ for all } E \in \mathcal{E}_h, \text{ and } q_V \in \mathbb{R} \text{ for all } V \in \mathcal{V}_h \Big\},
\end{aligned}$$

$$\begin{aligned}
\underline{\widehat{\boldsymbol{X}}}^r_{\mathbf{curl},h} := \Big\{ \underline{\boldsymbol{v}}_T = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h} \big) \ : \\
\boldsymbol{v}_T \in \mathcal{R}_{k-1}(T) \oplus \mathcal{R}^{\rm c}_{\ell_T+1}(T) \text{ for all } T \in \mathcal{T}_h, \\
\boldsymbol{v}_F \in \mathcal{R}_{k-1}(F) \oplus \mathcal{R}^{\rm c}_{\ell_F+1}(F) \text{ for all } F \in \mathcal{F}_h, \\
v_E \in \mathcal{P}_k(E) \text{ for all } E \in \mathcal{E}_h \Big\}
\end{aligned}$$

- Notice that, for $\eta_F = \eta_T = 2$, we recover the standard DDR spaces

Serendipity IV

- The serendipity DDR construction reads

$$
\begin{array}{ccccccc}
\underline{X}^r_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^r_h} & \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^r_h} & \underline{\boldsymbol{X}}^r_{\mathrm{div},h} & \xrightarrow{D^r_h} & \mathcal{P}_r(\mathcal{T}_h) \\
\underline{E}_{\mathbf{grad},h}\uparrow\ \ \downarrow \widehat{R}_{\mathbf{grad},h} & & \underline{\boldsymbol{E}}_{\mathbf{curl},h}\uparrow\ \ \downarrow \widehat{\underline{\boldsymbol{R}}}_{\mathbf{curl},h} & & \updownarrow & & \updownarrow \\
\underline{\widehat{X}}^r_{\mathbf{grad},h} & \xrightarrow{\widehat{\underline{\boldsymbol{G}}}^r_h} & \underline{\widehat{\boldsymbol{X}}}^r_{\mathbf{curl},h} & \xrightarrow{\widehat{\underline{\boldsymbol{C}}}^r_h} & \underline{\boldsymbol{X}}^r_{\mathrm{div},h} & \xrightarrow{D^r_T} & \mathcal{P}_r(\mathcal{T}_h)
\end{array}
$$

  with

$$
\widehat{\underline{\boldsymbol{G}}}^r_h := \widehat{\underline{\boldsymbol{R}}}_{\mathbf{curl},h}\underline{\boldsymbol{G}}^r_h\underline{E}_{\mathbf{grad},h}, \qquad \widehat{\underline{\boldsymbol{C}}}^r_h := \underline{\boldsymbol{C}}^r_h\underline{\boldsymbol{E}}_{\mathbf{curl},h}
$$

- Homological and analytical properties are inherited through extension and reduction cochain maps

Serendipity V

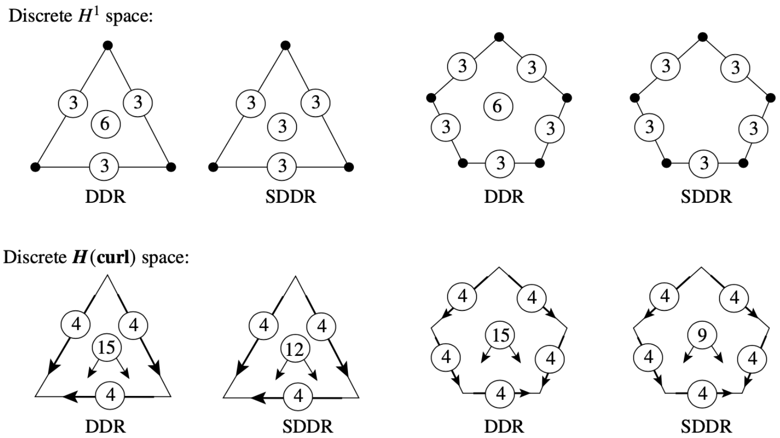


Figure: Comparison of local DDR and serendipity DDR (SDDR) spaces for $r = 3$

A serendipity scheme for magnetostatics

- We define the serendipity discrete L^2-product

$$[\cdot,\cdot]_{\mathbf{curl},h} \coloneqq (\underline{\boldsymbol{E}}_{\mathbf{curl},h}\cdot, \underline{\boldsymbol{E}}_{\mathbf{curl},h}\cdot)_{\mathbf{curl},h}$$

- The serendipity DDR scheme reads: Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\widehat{\boldsymbol{X}}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}$ s.t.

$$\begin{aligned}
[\mu\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h]_{\mathbf{curl},h} - (\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}^r_h\underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\widehat{\boldsymbol{X}}}^r_{\mathbf{curl},h},\\
(\underline{\boldsymbol{C}}^r_h\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^r_h\underline{\boldsymbol{A}}_h\, D^r_h\underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^r_{\mathrm{div},h}
\end{aligned}$$

- Analogous stability and convergence results as for the DDR scheme hold

Numerical tests: Magnetostatics

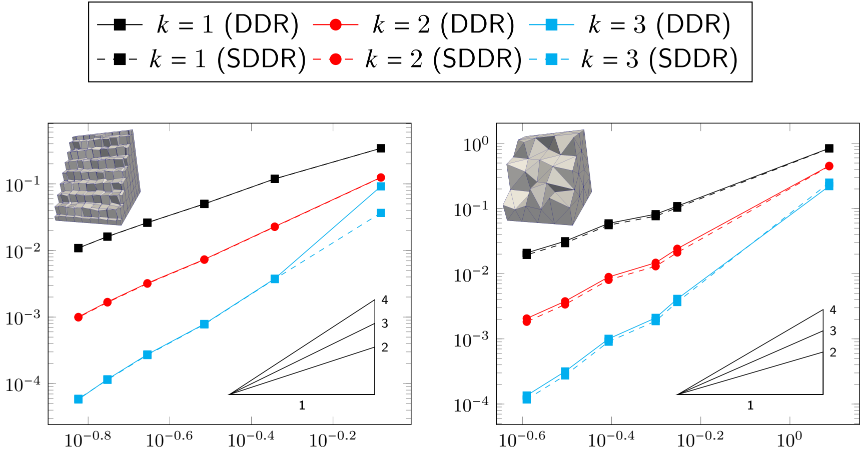


Figure: Relative errors in the discrete $\boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ norm vs. h, for the standard DDR scheme (continuous lines) and the SDDR scheme (dashed lines).

Numerical tests: Stokes

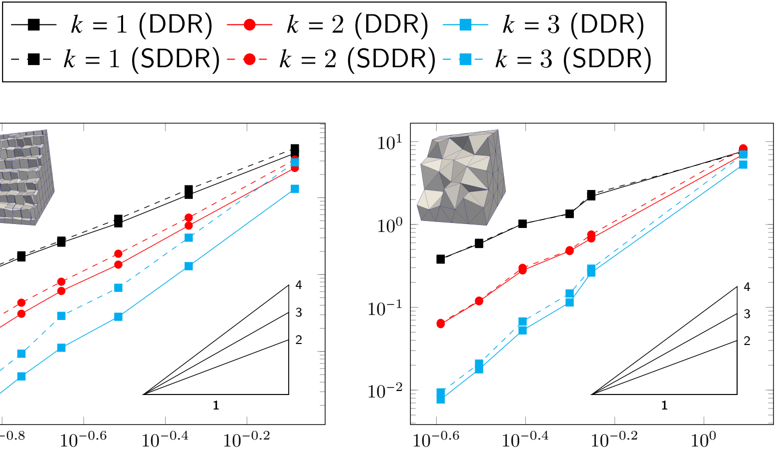


Figure: Relative errors in the discrete $\boldsymbol{H}(\mathbf{curl};\Omega) \times L^2(\Omega)^d$ norm (for the couple velocity–gradient of the pressure) vs. h, for the standard DDR SDDR schemes.

Outline

The two-dimensional div-div complex

$$\mathcal{RT}_1(\Omega) \hookrightarrow \boldsymbol{H}^1(\Omega;\mathbb{R}^2) \xrightarrow{\operatorname{sym}\mathbf{curl}} \boldsymbol{H}(\operatorname{div}\mathbf{div},\Omega;\mathbb{S}) \xrightarrow{\operatorname{div}\mathbf{div}} L^2(\Omega) \xrightarrow{0} 0$$

- This complex is relevant in solid mechanics (Kirchhoff–Love plates)
- For Ω contractible, it is exact, i.e.,

$$\operatorname{Ker}\operatorname{sym}\mathbf{curl} = \mathcal{RT}_1(\Omega), \quad \operatorname{Ker}\operatorname{div}\mathbf{div} = \operatorname{Im}\operatorname{sym}\mathbf{curl},$$
$$\operatorname{Im}\operatorname{div}\mathbf{div} = L^2(\Omega)$$

- Key novelty: algebraic constraint (symmetry) on spaces and operators

Mixed formulation for Kirchhoff–Love plates I

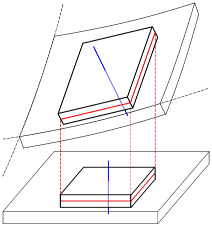


Figure: Image source: Wikipedia

With $\Omega \subset \mathbb{R}^2$ polygonal middleplane and orthogonal load $f : \Omega \to \mathbb{R}$:
Find the moment tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{S}$ and the deflection $u : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\boldsymbol{\sigma} + \mathbb{A}\, \mathbf{hess}\, u &= 0 \quad \text{in } \Omega,\\
-\operatorname{div} \mathbf{div}\, \boldsymbol{\sigma} &= f \quad \text{in } \Omega,\\
u = \partial_{\boldsymbol{n}} u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

with $\mathbb{A}\boldsymbol{\tau} = D\big[(1-\nu)\boldsymbol{\tau} + \nu \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_2\big]$ for all $\boldsymbol{\tau} \in \mathbb{S}$

Mixed formulation for Kirchhoff–Love plates II

- The DDR approximation is based on the weak formulation:
  Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S}) \times L^2(\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mathbb{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \mathrm{div}\,\mathbf{div}\,\boldsymbol{\tau}\, u &= 0 \qquad && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S}), \\
-\int_\Omega \mathrm{div}\,\mathbf{div}\,\boldsymbol{\sigma}\, v &= \int_\Omega f v && \forall v \in L^2(\Omega)
\end{aligned}
$$

- Well-posedness hinges on the inf-sup condition: For all $q \in L^2(\Omega)$,

$$
\|q\|_{L^2(\Omega)} \lesssim \sup_{\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S}) \setminus \{\mathbf{0}\}} \frac{\int_\Omega \mathrm{div}\,\mathbf{div}\,\boldsymbol{\tau}\ q}{\|\boldsymbol{\tau}\|_{\boldsymbol{L}^2(\Omega; \mathbb{R}^{2\times 2})}}
$$

  expressing the surjectivity of $\mathrm{div}\,\mathbf{div} : \boldsymbol{H}(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S}) \to L^2(\Omega)$
- **This corresponds to the exactness of the tail of the div-div complex**

A crucial remark I

- Let $(\mathcal{T}_h, \mathcal{F}_h, \mathcal{V}_h)$ denote a two-dimensional mesh of Ω
- The starting point is a local integration by parts formula for div-div
- For all $T \in \mathcal{T}_h$ and all $\boldsymbol{\tau} : T \to \mathbb{S}$ and $q : T \to \mathbb{R}$ smooth enough,

$$\begin{aligned}
\int_T \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}\ q = \int_T \boldsymbol{\tau} : \mathbf{hess}\, q &- \sum_{E \in \mathcal{E}_T} \omega_{TE} \sum_{V \in \mathcal{V}_E} \omega_{EV} \boldsymbol{\tau}(\boldsymbol{x}_V) \boldsymbol{n}_E \cdot \boldsymbol{t}_E\ q(\boldsymbol{x}_V) \\
&- \sum_{E \in \mathcal{E}_T} \omega_{TE} \int_E (\boldsymbol{\tau} \boldsymbol{n}_E \cdot \boldsymbol{n}_E)\ \partial_{\boldsymbol{n}_E} q \\
&+ \sum_{E \in \mathcal{E}_T} \omega_{TE} \int_E \left(\partial_{\boldsymbol{t}_E} (\boldsymbol{\tau} \boldsymbol{n}_E \cdot \boldsymbol{t}_E) + \mathbf{div}\, \boldsymbol{\tau} \cdot \boldsymbol{n}_E \right) q
\end{aligned}$$

A crucial remark II

- Letting $\ell \geq 1$, taking $q \in \mathcal{P}_{\ell-1}(T)$, and inserting projectors, we have

$$\int_T \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}\ q = \int_T \boldsymbol{\pi}_{\mathcal{H},T}^{\ell-3} \boldsymbol{\tau} : \mathbf{hess}\, q - \sum_{E \in \mathcal{E}_T} \omega_{TE} \sum_{V \in \mathcal{V}_E} \omega_{EV} \left(\boldsymbol{\tau}(\boldsymbol{x}_V) \boldsymbol{n}_E \cdot \boldsymbol{t}_E \right) q(\boldsymbol{x}_V)$$
$$- \sum_{E \in \mathcal{E}_T} \omega_{TE} \int_E \pi_{\mathcal{P},E}^{\ell-2} (\boldsymbol{\tau} \boldsymbol{n}_E \cdot \boldsymbol{n}_E)\ \partial_{\boldsymbol{n}_E} q$$
$$+ \sum_{E \in \mathcal{E}_T} \omega_{TE} \int_E \pi_{\mathcal{P},E}^{\ell-1} \left(\partial_{\boldsymbol{t}_E} (\boldsymbol{\tau} \boldsymbol{n}_E \cdot \boldsymbol{t}_E) + \mathbf{div}\, \boldsymbol{\tau} \cdot \boldsymbol{n}_E \right) q$$

- The discrete $\boldsymbol{H}(\operatorname{div} \mathbf{div}, T; \mathbb{S})$ space should contain the red polynomial components to have inf-sup through Fortin's argument!

Discrete $\boldsymbol{H}(\operatorname{div}\mathbf{div}, T; \mathbb{S})$ space

- Based on the previous remark, the discrete $\boldsymbol{H}(\operatorname{div}\mathbf{div}, T; \mathbb{S})$ space is

$$\underline{\boldsymbol{\Sigma}}_T^\ell \coloneqq \Big\{ \underline{\boldsymbol{\tau}}_T = \left(\boldsymbol{\tau}_{\mathcal{H},T}, \boldsymbol{\tau}_{\mathcal{H},T}^{\rm c}, (\tau_E, D_{\boldsymbol{\tau},E})_{E\in\mathcal{E}_T}, (\boldsymbol{\tau}_V)_{V\in\mathcal{V}_T} \right) \; : \\ \boldsymbol{\tau}_{\mathcal{H},T} \in \boldsymbol{\mathcal{H}}^{\ell-3}(T) \text{ and } \boldsymbol{\tau}_{\mathcal{H},T}^{\rm c} \in \boldsymbol{\mathcal{H}}^{{\rm c},\ell}(T), \\ \tau_E \in \mathcal{P}^{\ell-2}(E) \text{ and } D_{\boldsymbol{\tau},E} \in \mathcal{P}^{\ell-1}(E) \text{ for all } E \in \mathcal{E}_T, \\ \boldsymbol{\tau}_V \in \mathbb{S} \text{ for all } V \in \mathcal{V}_T \Big\}$$

- The meaning of the components is provided by the interpolator

$$\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^\ell \boldsymbol{\tau} \coloneqq \Big(\boldsymbol{\pi}_{\mathcal{H},T}^{\ell-3}\boldsymbol{\tau}, \boldsymbol{\pi}_{\mathcal{H},T}^{{\rm c},\ell}\boldsymbol{\tau}, \\ \left(\pi_{\mathcal{P},E}^{\ell-2}(\boldsymbol{\tau}\boldsymbol{n}_E \cdot \boldsymbol{n}_E), \pi_{\mathcal{P},E}^{\ell-1}\left(\partial_{\boldsymbol{t}_E}(\boldsymbol{\tau}\boldsymbol{n}_E \cdot \boldsymbol{t}_E) + \mathbf{div}\,\boldsymbol{\tau} \cdot \boldsymbol{n}_E \right) \right)_{E\in\mathcal{E}_T}, \\ \left(\boldsymbol{\tau}(\boldsymbol{x}_V) \right)_{V\in\mathcal{V}_T} \Big)$$

Discrete div-div operator I

- Mimicking the above integration by parts formula, we let

$$\mathsf{DD}_T^{\ell-1} : \underline{\boldsymbol{\Sigma}}_T^{\ell} \to \mathcal{P}^{\ell-1}(T)$$

be s.t., for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{\ell}$ and all $q \in \mathcal{P}^{\ell-1}(T)$,

$$\int_T \mathsf{DD}_T^{\ell-1}\underline{\boldsymbol{\tau}}_T \; q = \int_T \boldsymbol{\tau}_{\mathcal{H},T} : \mathbf{hess}\, q - \sum_{E\in\mathcal{E}_T} \omega_{TE} \sum_{V\in\mathcal{V}_E} \omega_{EV} \left(\boldsymbol{\tau}_V \boldsymbol{n}_E \cdot \boldsymbol{t}_E\right) q(\boldsymbol{x}_V)$$
$$- \sum_{E\in\mathcal{E}_T} \omega_{TE} \left(\int_E \tau_E \, \partial_{\boldsymbol{n}_E} q - \int_E D_{\tau,E}\, q\right)$$

Discrete div-div operator II

- Let $\boldsymbol{\tau} \in \boldsymbol{H}^2(T;\mathbb{S})$. We have, for all $q \in \mathcal{P}^{\ell-1}(T)$,

$$
\begin{aligned}
&\int_T \mathsf{DD}_T^{\ell-1} \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{\ell} \boldsymbol{\tau}\; q \\
&= \int_T \cancel{\boldsymbol{\pi}_{\mathcal{H},T}^{\ell-3}} \boldsymbol{\tau} : \mathbf{hess}\, q - \sum_{E \in \mathcal{E}_T} \omega_{TE} \sum_{V \in \mathcal{V}_E} \omega_{EV} \left(\boldsymbol{\tau}(\boldsymbol{x}_V) \boldsymbol{n}_E \cdot \boldsymbol{t}_E \right) q(\boldsymbol{x}_V) \\
&\quad - \sum_{E \in \mathcal{E}_T} \omega_{TE} \int_E \cancel{\pi_{\mathcal{P},E}^{\ell-2}} (\boldsymbol{\tau} \boldsymbol{n}_E \cdot \boldsymbol{n}_E)\, \partial_{\boldsymbol{n}_E} q \\
&\quad + \sum_{E \in \mathcal{E}_T} \omega_{TE} \int_E \cancel{\pi_{\mathcal{P},E}^{\ell-1}} \left(\partial_{\boldsymbol{t}_E} (\boldsymbol{\tau} \boldsymbol{n}_E \cdot \boldsymbol{t}_E) + \mathbf{div}\, \boldsymbol{\tau} \cdot \boldsymbol{n}_E \right) q = \int_\Omega \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}\, q
\end{aligned}
$$

- This shows that it holds:

$$
\mathsf{DD}_T^{\ell-1} (\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{\ell} \boldsymbol{\tau}) = \pi_{\mathcal{P},T}^{\ell-1} (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \boldsymbol{H}^2(T;\mathbb{S})
$$

- The surjectivity of $\mathsf{DD}_T^{\ell-1} : \underline{\boldsymbol{\Sigma}}_T^{\ell} \to \mathcal{P}^{\ell-1}(T)$ follows

Discrete $\boldsymbol{H}^1(\Omega;\mathbb{R}^2)$ space I

$$\mathcal{RT}_1(T) \xrightarrow{\underline{\boldsymbol{I}}_{\boldsymbol{V},T}^k} \underline{\boldsymbol{V}}_T^k \xrightarrow{\underline{\boldsymbol{C}}_{\mathrm{sym},T}^{k-1}} \underline{\boldsymbol{\Sigma}}_T^{k-1} \xrightarrow{\mathsf{DD}_T^{k-2}} \mathcal{P}^{k-2}(T) \xrightarrow{0} 0.$$

- When $\boldsymbol{\tau} = \mathrm{sym}\,\mathbf{curl}\,\boldsymbol{v}$, we have

$$\begin{aligned}\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k-1}(\mathrm{sym}\,\mathbf{curl}\,\boldsymbol{v}) = \Big(&\boldsymbol{\pi}_{\boldsymbol{\mathcal{H}},T}^{k-4}(\mathrm{sym}\,\mathbf{curl}\,\boldsymbol{v}), \boldsymbol{\pi}_{\boldsymbol{\mathcal{H}},T}^{\mathrm{c},k-1}(\mathrm{sym}\,\mathbf{curl}\,\boldsymbol{v}),\\ &\big(\pi_{\mathcal{P},E}^{k-3}(\partial_{\boldsymbol{t}_E}\boldsymbol{v}\cdot\boldsymbol{n}_E), \pi_{\mathcal{P},E}^{k-2}(\partial_{\boldsymbol{t}_E}^2\boldsymbol{v}\cdot\boldsymbol{t}_E)\big)_{E\in\mathcal{E}_T},\\ &\big(\mathrm{sym}\,\mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x}_V)\big)_{V\in\mathcal{V}_T}\Big)\end{aligned}$$

- $\underline{\boldsymbol{V}}_T^k$ must allow to reconstruct all these quantities!

Discrete $\boldsymbol{H}^1(\Omega;\mathbb{R}^2)$ space II

- We consider the following space:

$$\underline{\boldsymbol{V}}_T^k := \Big\{ \underline{\boldsymbol{v}}_T = \big(\boldsymbol{v}_T, (\boldsymbol{v}_E)_{E\in\mathcal{E}_T}, (\boldsymbol{v}_V, \boldsymbol{G}_{\boldsymbol{v},V})_{V\in\mathcal{V}_T}\big) \ : \\ \boldsymbol{v}_T \in \boldsymbol{\mathcal{P}}^{k-2}(T;\mathbb{R}^2), \\ \boldsymbol{v}_E \in \boldsymbol{\mathcal{P}}^{k-4}(E;\mathbb{R}^2) \text{ for all } E\in\mathcal{E}_T, \\ \boldsymbol{v}_V \in \mathbb{R}^2 \text{ and } \boldsymbol{G}_{\boldsymbol{v},V} \in \mathbb{R}^{2\times 2} \text{ for all } V\in\mathcal{V}_T \Big\}$$

- Vertex components are readily available as $\mathbb{C}\boldsymbol{G}_{\boldsymbol{v},V}$
- Edge components come from $\boldsymbol{v}_{\mathcal{E}_T} \in \boldsymbol{\mathcal{P}}^k(\mathcal{E}_T;\mathbb{R}^2) \cap \boldsymbol{C}^0(\partial T;\mathbb{R}^2)$ s.t.

$$\forall E\in\mathcal{E}_T,\ \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k-4}(\boldsymbol{v}_{\mathcal{E}_T})_{|E} = \boldsymbol{v}_E \text{ and } \partial_{\boldsymbol{t}_E}(\boldsymbol{v}_{\mathcal{E}_T})_{|E}(\boldsymbol{x}_V) = \boldsymbol{G}_{\boldsymbol{v},V}\boldsymbol{t}_E \ \forall V\in\mathcal{V}_E, \\ \text{and } \boldsymbol{v}_{\mathcal{E}_T}(\boldsymbol{x}_V) = \boldsymbol{v}_V \ \forall V\in\mathcal{V}_T$$

Discrete $\boldsymbol{H}^1(\Omega;\mathbb{R}^2)$ space III

- Element components come from $\mathbf{C}_{\mathrm{sym},T}^{k-1} : \underline{\boldsymbol{V}}_T^k \to \boldsymbol{\mathcal{P}}^{k-1}(T;\mathbb{S})$ s.t.

$$\int_T \mathbf{C}_{\mathrm{sym},T}^{k-1}\underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T\cdot\mathbf{rot}\,\boldsymbol{\tau} + \sum_{E\in\mathcal{E}_T}\omega_{TE}\int_E \boldsymbol{v}_{\mathcal{E}_T}\cdot(\boldsymbol{\tau}\,\boldsymbol{t}_E) \quad \forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{P}}^{k-1}(T;\mathbb{S})$$

- The discrete sym curl $\underline{\boldsymbol{C}}_{\mathrm{sym},T}^{k-1} : \underline{\boldsymbol{V}}_T^k \to \underline{\boldsymbol{\Sigma}}_T^{k-1}$ is, therefore,

$$\begin{aligned}\underline{\boldsymbol{C}}_{\mathrm{sym},T}^{k-1}\underline{\boldsymbol{v}}_T \coloneqq \Big(&\boldsymbol{\pi}_{\mathcal{H},T}^{k-4}(\mathbf{C}_{\mathrm{sym},T}^{k-1}\underline{\boldsymbol{v}}_T), \boldsymbol{\pi}_{\mathcal{H},T}^{\mathrm{c},k-1}(\mathbf{C}_{\mathrm{sym},T}^{k-1}\underline{\boldsymbol{v}}_T),\\ &\big(\pi_{\mathcal{P},E}^{k-3}(\partial_{\boldsymbol{t}_E}\boldsymbol{v}_{\mathcal{E}_T}\cdot\boldsymbol{n}_E), \partial_{\boldsymbol{t}_E}^2\boldsymbol{v}_{\mathcal{E}_T}\cdot\boldsymbol{t}_E\big)_{E\in\mathcal{E}_T},\\ &\big(\mathbb{C}\boldsymbol{G}_{\boldsymbol{v},V}\big)_{V\in\mathcal{V}_T}\Big)\end{aligned}$$

Local discrete complex

Theorem (Local complex property and exactness)

The following sequence is a complex, which is exact if T is contractible:

$$\mathcal{RT}_1(T) \xrightarrow{\underline{\boldsymbol{I}}^k_{\boldsymbol{V},T}} \underline{\boldsymbol{V}}^k_T \xrightarrow{\underline{\boldsymbol{C}}^{k-1}_{\mathrm{sym},T}} \underline{\boldsymbol{\Sigma}}^{k-1}_T \xrightarrow{\mathsf{DD}^{k-2}_T} \mathcal{P}^{k-2}(T) \xrightarrow{0} 0.$$

Local tensor potential and $\mathbb{A}$-weighted product in $\underline{\boldsymbol{\Sigma}}_T^\ell$ I

- For all $E \in \mathcal{E}_T$, $P_{\boldsymbol{\Sigma},E}^\ell \underline{\boldsymbol{\tau}}_E \in \mathcal{P}^\ell(E)$ is the unique polynomial that satisfies

$$P_{\boldsymbol{\Sigma},E}^\ell \underline{\boldsymbol{\tau}}_E(\boldsymbol{x}_V) = \boldsymbol{\tau}_V \boldsymbol{n}_E \cdot \boldsymbol{n}_E \text{ for all } V \in \mathcal{V}_E \text{ and } \pi_{\mathcal{P},E}^{\ell-2}(P_{\boldsymbol{\Sigma},E}^\ell \underline{\boldsymbol{\tau}}_E) = \tau_E.$$

- We define $\boldsymbol{P}_{\boldsymbol{\Sigma},T}^\ell : \underline{\boldsymbol{\Sigma}}_T^\ell \to \mathcal{P}^\ell(T;\mathbb{S})$ s.t., $\forall (q,\boldsymbol{v}) \in \mathcal{P}^{\ell+2}(T) \times \boldsymbol{\mathcal{H}}^{\mathrm{c},\ell}(T)$,

$$\begin{aligned}
&\int_T \boldsymbol{P}_{\boldsymbol{\Sigma},T}^\ell \underline{\boldsymbol{\tau}}_T : (\mathbf{hess}\, q + \boldsymbol{v}) \\
&\quad = \int_T \mathsf{DD}_T^{\ell-1} \underline{\boldsymbol{\tau}}_T\, q + \sum_{E \in \mathcal{E}_T} \omega_{TE} \sum_{V \in \mathcal{V}_E} \omega_{EV} (\boldsymbol{\tau}_V \boldsymbol{n}_E \cdot \boldsymbol{t}_E)\, q(\boldsymbol{x}_V) \\
&\qquad + \sum_{E \in \mathcal{E}_T} \omega_{TE} \left(\int_E P_{\boldsymbol{\Sigma},E}^\ell \underline{\boldsymbol{\tau}}_E\, \partial_{\boldsymbol{n}_E} q - \int_E D_{\boldsymbol{\tau},E}\, q \right) + \int_T \boldsymbol{\tau}_{\mathcal{H},T}^{\mathrm{c}} : \boldsymbol{v}
\end{aligned}$$

Local tensor potential and $\mathbb{A}$-weighted product in $\underline{\boldsymbol{\Sigma}}_T^\ell$ II

The discrete $\mathbb{A}$-weighted product in $\underline{\boldsymbol{\Sigma}}_T^\ell$ is s.t.

$$a_T(\underline{\boldsymbol{\upsilon}}_T, \underline{\boldsymbol{\tau}}_T) \coloneqq \int_T \mathbb{A}^{-1} \boldsymbol{P}_{\boldsymbol{\Sigma},T}^\ell \underline{\boldsymbol{\upsilon}}_T : \boldsymbol{P}_{\boldsymbol{\Sigma},T}^\ell \underline{\boldsymbol{\tau}}_T + \frac{1}{D(1+\nu)} s_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\upsilon}}_T, \underline{\boldsymbol{\tau}}_T)$$

where the stabilization bilinear form is, e.g., s.t.

$$s_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\upsilon}}_T, \underline{\boldsymbol{\tau}}_T) \coloneqq [\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^\ell \boldsymbol{P}_{\boldsymbol{\Sigma},T}^\ell \underline{\boldsymbol{\upsilon}}_T - \underline{\boldsymbol{\upsilon}}_T, \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^\ell \boldsymbol{P}_{\boldsymbol{\Sigma},T}^\ell \underline{\boldsymbol{\tau}}_T - \underline{\boldsymbol{\tau}}_T]_{\boldsymbol{\Sigma},T}$$

with $[\cdot,\cdot]_{\boldsymbol{\Sigma},T}$ denoting the component L^2-product in $\underline{\boldsymbol{\Sigma}}_T^\ell$

A DDR scheme for Kirchhoff–Love plates

- Global spaces, operators, and inner products assembled as usual
- The DDR scheme for the Kirchhoff–Love plate problem reads:

  Find $(\underline{\boldsymbol{\sigma}}_h, u_h) \in \underline{\boldsymbol{\Sigma}}_h^\ell \times \mathcal{P}^{\ell-1}(\mathcal{T}_h)$ s.t.

$$
\begin{aligned}
a_h(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + b_h(\underline{\boldsymbol{\tau}}_h, u_h) &= 0 &\qquad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h^\ell, \\
-b_h(\underline{\boldsymbol{\sigma}}_h, v_h) &= \int_\Omega f\, v_h &\qquad \forall v_h \in \mathcal{P}^{\ell-1}(\mathcal{T}_h),
\end{aligned}
$$

  where

$$
a_h(\underline{\boldsymbol{\upsilon}}_h, \underline{\boldsymbol{\tau}}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{\boldsymbol{\upsilon}}_T, \underline{\boldsymbol{\tau}}_T), \quad b_h(\underline{\boldsymbol{\tau}}_h, v_h) := \sum_{T \in \mathcal{T}_h} \int_T \mathsf{DD}_T^{\ell-1} \underline{\boldsymbol{\tau}}_T\, v_T
$$

Error estimate

Theorem (Error estimate)

Assume $\boldsymbol{\sigma} \in \boldsymbol{H}^2(\Omega; \mathbb{S}) \cap \boldsymbol{H}^{\ell+1}(\mathcal{T}_h; \mathbb{S})$ *and* $u \in C^1(\overline{\Omega}) \cap H^{\ell+3}(\mathcal{T}_h)$. *Then, it holds*

$$\|\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},h}^{\ell}\boldsymbol{\sigma} - \underline{\boldsymbol{\sigma}}_h\|_{\boldsymbol{\Sigma},h} + \|\pi_{\mathcal{P},h}^{\ell-1}u - u_h\|_{L^2(\Omega)} \lesssim \gamma^{-1}h^{\ell+1}\left(\frac{1}{D(1-\nu)}|\boldsymbol{\sigma}|_{\boldsymbol{H}^{\ell+1}(\mathcal{T}_h)} + |u|_{H^{\ell+3}(\mathcal{T}_h)}\right),$$

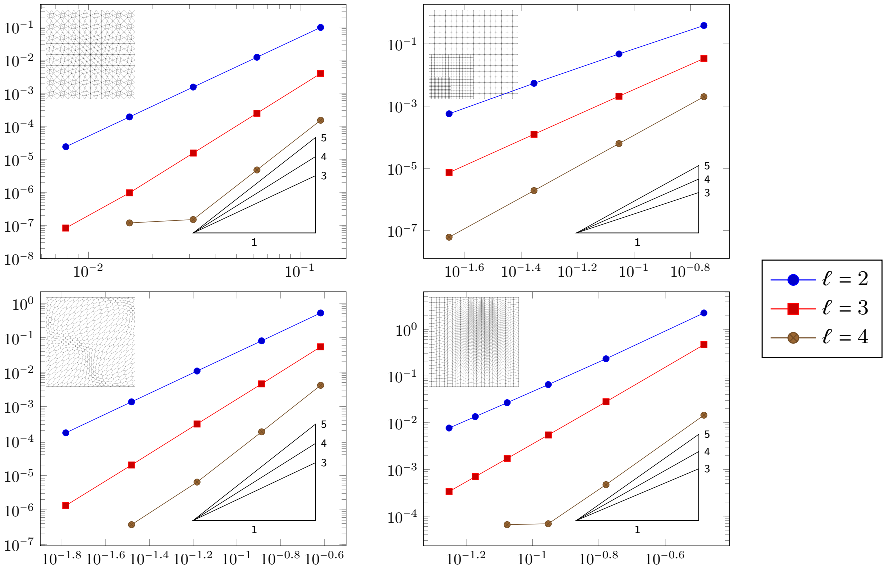
where

$$\gamma := \left[D^2\left(1 + \tfrac{1}{D^2(1-\nu)^2}\right)^2 + 1\right]^{-\frac{1}{2}}$$

and, denoting by $[\cdot,\cdot]_{\boldsymbol{\Sigma},h}$ *the global component* L^2*-product,*

$$\|\underline{\boldsymbol{\tau}}_h\|_{\boldsymbol{\Sigma},h} := [\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\tau}}_h]_{\boldsymbol{\Sigma},h}^{1/2}.$$

Convergence: Energy error vs. meshsize



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