Recent advances on Hybrid High-Order methods for problems in incompressible fluid mechanics

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joint work with L. Botti and J. Droniou

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Outline

Features

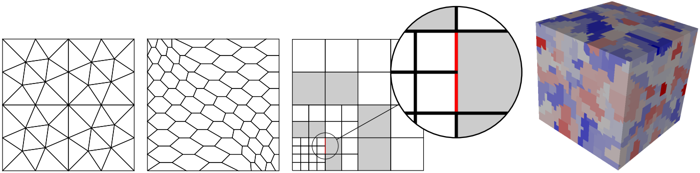

Figure: Examples of supported meshes $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ in 2d and 3d

- Capability of handling general polyhedral meshes
- Construction valid for both $d = 2$ and $d = 3$
- Arbitrary approximation order (including $k = 0$)
- Inf-sup stability on general meshes
- Robust handling of dominant advection
- Local conservation of momentum and mass
- Reduced computational cost after static condensation

HHO for incompressible flows

- HHO for Stokes [Aghili, Boyaval, DP, 2015]
- Péclet-robust HHO for Oseen [Aghili and DP, 2018]
- Darcy-robust HHO for Brinkman [Botti, DP, Droniou, 2018]
- Skew-symmetric HHO for Navier–Stokes [DP and Krell, 2018]
- **Temam's device for HHO** [Botti, DP, Droniou, 2018]
- See also D. Castanon-Quiroz's presentation

New book!

D. A. Di Pietro and J. Droniou
The Hybrid High-Order Method for Polytopal Meshes
Design, Analysis, and Applications
516 pages, http://hal.archives-ouvertes.fr/hal-02151813

Outline

1 Basics of HHO methods

2 Application to the incompressible Navier–Stokes problem

Model problem for the viscous term

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denote a bounded connected polyhedral domain
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in U := H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = \int_\Omega f v \qquad \forall v \in U$$

Projectors on local polynomial spaces

- With $X \in \mathcal{T}_h \cup \mathcal{F}_h$, the L^2-projector $\pi_X^{0,l} : L^2(X) \to \mathbb{P}^l(X)$ is s.t.

$$\pi_X^{0,l} v = \arg \min_{w \in \mathbb{P}^l(X)} \|w - v\|_X^2$$

- The elliptic projector $\pi_T^{1,l} : H^1(T) \to \mathbb{P}^l(T)$ is s.t.

$$\pi_T^{1,l} v = \arg \min_{w \in \mathbb{P}^l(T),\ \int_T (w-v)=0} \|\boldsymbol{\nabla}(w - v)\|_T^2$$

- Optimal approximation properties hold [DP and Droniou, 2017ab]

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$$\int_X (\pi_X^{0,l} v - v) w = 0 \text{ for all } w \in \mathbb{P}^l(X)$$

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- The elliptic projector $\pi_T^{1,l} : H^1(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T \nabla(\pi_T^{1,l} v - v) \cdot \nabla w = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } \int_T (\pi_T^{1,l} v - v) = 0$$

- Optimal approximation properties hold [DP and Droniou, 2017ab]

Computing $\pi_T^{1,k+1}$ from L^2-projections of degree k

- Recall the following IBP valid for all $v \in H^1(T)$ and all $w \in C^\infty(\overline{T})$:

$$\boxed{\int_T \nabla v \cdot \nabla w = -\int_T v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v \nabla w \cdot \boldsymbol{n}_{TF}}$$

- Taking $w \in \mathbb{P}^{k+1}(T)$ and using the definitions above, we can write

$$\int_T \nabla \pi_T^{1,k+1} v \cdot \nabla w = -\int_T \pi_T^{0,k} v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^{0,k} v \nabla w \cdot \boldsymbol{n}_{TF}$$

- **Hence, $\pi_T^{1,k+1} v$ can be computed from $\pi_T^{0,k} v$ and $(\pi_F^{0,k} v)_{F \in \mathcal{F}_T}$!**

Discrete unknowns

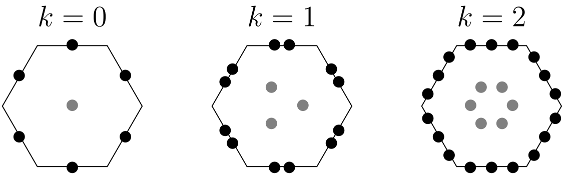

Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$ and $d = 2$

- For $k \geq 0$ and $T \in \mathcal{T}_h$, define the local space of discrete unknowns

$$\underline{U}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \;:\; v_T \in \mathbb{P}^k(T) \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ is s.t., for all $v \in H^1(T)$,

$$\underline{I}_T^k v := \left(\pi_T^{0,k} v, (\pi_F^{0,k} v)_{F \in \mathcal{F}_T} \right)$$

Local potential reconstruction

- Let $T \in \mathcal{T}_h$. We define the local potential reconstruction operator

$$r_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$$

s.t., for all $\underline{v}_T \in \underline{U}_T^k$, $\int_T (r_T^{k+1}\underline{v}_T - v_T) = 0$ and

$$\boxed{\int_T \nabla r_T^{k+1}\underline{v}_T \cdot \nabla w = -\int_T v_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F \nabla w \cdot \boldsymbol{n}_{TF} \quad \forall w \in \mathbb{P}^{k+1}(T)}$$

- By construction, we have

$$\boxed{r_T^{k+1} \circ \underline{I}_T^k = \pi_T^{1,k+1}}$$

- $(r_T^{k+1} \circ \underline{I}_T^k)$ has therefore optimal approximation properties in $\mathbb{P}^{k+1}(T)$

Local bilinear form

We approximate $a_{|T}(u, v)$ with

$$\boxed{\mathrm{a}_T(\underline{u}_T, \underline{v}_T) := a_{|T}(r_T^{k+1}\underline{u}_T, r_T^{k+1}\underline{v}_T) + \mathrm{s}_T(\underline{u}_T, \underline{v}_T)}$$

Assumption (Stabilization bilinear form)

The bilinear form $\mathrm{s}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ satisfies the following properties:

- Symmetry and positivity. s_T is symmetric and positive semidefinite.
- Stability. It holds, with hidden constant independent of h and T,

$$\mathrm{a}_T(\underline{v}_T, \underline{v}_T) \simeq \|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1}\|v_F - v_T\|_F^2 \quad \forall \underline{v}_T \in \underline{U}_T^k.$$

- Polynomial consistency. For all $w \in \mathbb{P}^{k+1}(T)$ and all $\underline{v}_T \in \underline{U}_T^k$,

$$\mathrm{s}_T(\underline{I}_T^k w, \underline{v}_T) = 0.$$

Discrete problem

- Define the global space with single-valued interface unknowns

$$\underline{U}_h^k := \Big\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) \ : \\ v_T \in \mathbb{P}^k(T) \quad \forall T \in \mathcal{T}_h \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_h \Big\}$$

  and its subspace with strongly enforced boundary conditions

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \ : \ v_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{\mathrm{a}_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{a}_T(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_h} \int_T f v_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

Convergence

Theorem (Energy-norm error estimate)

Assume $u \in H_0^1(\Omega) \cap H^{k+2}(\mathcal{T}_h)$. *The following energy error estimate holds:*

$$\boxed{\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)}}$$

where $\|\underline{v}_h\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$.

Theorem (Superconvergence in the L^2-norm)

Further assuming elliptic regularity and $f \in H^1(\mathcal{T}_h)$ *if* $k = 0$,

$$\boxed{\|u_h - \pi_h^{0,k} u\| \lesssim \begin{cases} h^{k+2} \|f\|_{H^1(\mathcal{T}_h)} & \text{if } k = 0, \\ h^{k+2} |u|_{H^{k+2}(\mathcal{T}_h)} & \text{if } k \geq 1. \end{cases}}$$

Outline

The incompressible Navier–Stokes equations

- Let $d \in \{2,3\}$, $\nu \in \mathbb{R}_+^*$, $\boldsymbol{f} \in L^2(\Omega)^d$, $\boldsymbol{U} := H_0^1(\Omega)^d$, and $P := L_0^2(\Omega)$
- The INS problem reads: Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ s.t.

$$\boxed{\begin{aligned} \nu a(\boldsymbol{u},\boldsymbol{v}) + t(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{U}, \\ -b(\boldsymbol{u},q) &= 0 && \forall q \in L^2(\Omega), \end{aligned}}$$

with viscous and pressure-velocity coupling bilinear forms

$$a(\boldsymbol{w},\boldsymbol{v}) := \int_\Omega \boldsymbol{\nabla}\boldsymbol{w} : \boldsymbol{\nabla}\boldsymbol{v}, \quad b(\boldsymbol{v},q) := -\int_\Omega q \boldsymbol{\nabla}\cdot\boldsymbol{v}$$

and convective trilinear form

$$t(\boldsymbol{w},\boldsymbol{v},\boldsymbol{z}) := \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{z} = \sum_{i=1}^{d}\sum_{j=1}^{d}\int_\Omega w_j(\partial_j v_i) z_i$$

Discrete spaces

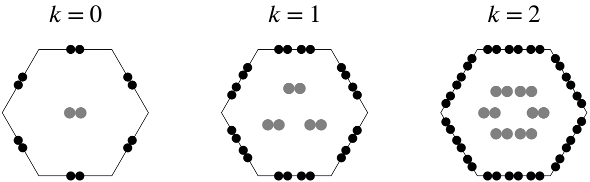

Figure: Local velocity space $\underline{\boldsymbol{U}}_T^k$ for $k \in \{0, 1, 2\}$ and $d = 2$

- For $k \geq 0$, we define the global space of discrete velocity unknowns

$$\underline{\boldsymbol{U}}_h^k := \Big\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) \ : \ \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \quad \forall T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \quad \forall F \in \mathcal{F}_h \Big\}$$

- The velocity and pressure spaces are

$$\boxed{\underline{\boldsymbol{U}}_{h,0}^k := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\} \text{ and } P_h^k := \mathbb{P}^k(\mathcal{T}_h) \cap P}$$

Viscous term

- The viscous term is discretized by means of the bilinear form a_h s.t.

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

  where, letting $\boldsymbol{r}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T)^d$ as for Poisson component-wise,

$$\boxed{\mathrm{a}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T) \coloneqq \int_T \boldsymbol{\nabla} \boldsymbol{r}_T^{k+1} \underline{\boldsymbol{w}}_T : \boldsymbol{\nabla} \boldsymbol{r}_T^{k+1} \underline{\boldsymbol{v}}_T + \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T)}$$

- Variable viscosity can be treated following [DP and Ern, 2015] for $k \geq 1$ or [Botti, DP, Guglielmana, 2019] for $k = 0$

Divergence reconstruction

- Let $\ell \geq 0$. Inspired by the IBP formula: $\forall(\boldsymbol{v}, q) \in H^1(T)^d \times C^\infty(\overline{T})$,

$$\int_T (\nabla\cdot\boldsymbol{v})\ q = -\int_T \boldsymbol{v}\cdot\nabla q + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}\cdot\boldsymbol{n}_{TF})\ q$$

  we introduce divergence reconstruction $D_T^\ell : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^\ell(T)$ s.t.

$$\boxed{\int_T D_T^\ell \underline{\boldsymbol{v}}_T\ q = -\int_T \boldsymbol{v}_T\cdot\nabla q + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}_F\cdot\boldsymbol{n}_{TF})\ q \quad \forall q \in \mathbb{P}^\ell(T)}$$

- By construction, it holds, for all $\boldsymbol{v} \in H^1(T)^d$,

$$D_T^k \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \pi_T^{0,k}(\nabla\cdot\boldsymbol{v})$$

Pressure-velocity coupling

$$\boxed{\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) := -\sum_{T\in\mathcal{T}_h}\int_T D_T^k\underline{\boldsymbol{v}}_T\ q_T}$$

Lemma (Uniform inf-sup condition)

There is $\beta > 0$ independent of h s.t.

$$\forall q_h \in P_h^k, \quad \beta\|q_h\|_{L^2(\Omega)} \le \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_{h,0}^k,\,\|\underline{\boldsymbol{v}}_h\|_{1,h}=1} \mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h).$$

Stability result valid on general meshes and for any $k \geq 0$

Convective term: A key remark

- We have the following IBP formula: For all $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{U}$,

$$\boxed{\int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{z} + \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{z}\cdot\boldsymbol{v} + \int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{z}) = 0}$$

- Using this formula with $\boldsymbol{w} = \boldsymbol{v} = \boldsymbol{z} = \boldsymbol{u}$, we get

$$t(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\cdot\boldsymbol{u} = \cancel{-\frac{1}{2}\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{u})} = 0$$

- The discrete velocity may not be divergence-free
- Following [Temam, 1979], we use instead of t

$$\boxed{t^{\mathrm{tm}}(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z}) := \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{z} + \frac{1}{2}\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{z})}$$

Directional derivative reconstruction

- Let $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{U}}_T^k$. The directional derivative reconstruction along $\underline{\boldsymbol{w}}_T$ is

$$\boldsymbol{G}_T^k(\underline{\boldsymbol{w}}_T;\cdot) : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)^d$$

s.t., for all $\boldsymbol{z} \in \mathbb{P}^k(T)^d$,

$$\boxed{\int_T \boldsymbol{G}_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{z} = \int_T (\boldsymbol{w}_T\cdot\boldsymbol{\nabla})\boldsymbol{v}_T\cdot\boldsymbol{z} + \sum_{F\in\mathcal{F}_T}\int_F (\boldsymbol{w}_F\cdot\boldsymbol{n}_{TF})(\boldsymbol{v}_F-\boldsymbol{v}_T)\cdot\boldsymbol{z}}$$

- It holds, for all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{z}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$,

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T \Big(\boldsymbol{G}_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{z}_T + \boldsymbol{v}_T\cdot\boldsymbol{G}_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{z}}_T) + D_T^{2k}\underline{\boldsymbol{w}}_T(\boldsymbol{v}_T\cdot\boldsymbol{z}_T)\Big)\\ = -\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F (\boldsymbol{w}_F\cdot\boldsymbol{n}_{TF})(\boldsymbol{v}_F-\boldsymbol{v}_T)\cdot(\boldsymbol{z}_F-\boldsymbol{z}_T).\end{aligned}$$

Convective term

$$\boxed{t^{\mathrm{tm}}(\boldsymbol{w},\boldsymbol{v},\boldsymbol{z}) \coloneqq \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{z} + \frac{1}{2}\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{z}) \quad \forall \boldsymbol{w},\boldsymbol{v},\boldsymbol{z}\in\boldsymbol{U}}$$

- Inspired by t^{tm}, we set

$$\begin{aligned}\mathrm{t}_h(\underline{\boldsymbol{w}}_h,\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{z}}_h) \coloneqq {}& \sum_{T\in\mathcal{T}_h}\int_T G_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{z}_T + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_T D_T^{2k}\underline{\boldsymbol{w}}_T(\boldsymbol{v}_T\cdot\boldsymbol{z}_T) \\ &+ \frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F (\boldsymbol{w}_F\cdot\boldsymbol{n}_{TF})(\boldsymbol{v}_F-\boldsymbol{v}_T)\cdot(\boldsymbol{z}_F-\boldsymbol{z}_T)\end{aligned}$$

- The second and third terms embody Temam's device

Discrete problem

- The discrete problem reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ s.t.

$$\boxed{\begin{aligned} \nu \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{t}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 && \forall q_h \in \mathbb{P}^k(\mathcal{T}_h) \end{aligned}}$$

- Optionally, upwind stabilisation can be added through the term

$$\mathrm{j}_h(\underline{\boldsymbol{w}}_h; \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{z}}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F \frac{\nu}{h_F} \rho(\mathrm{Pe}_{TF}(\boldsymbol{w}_F))(\boldsymbol{v}_F - \boldsymbol{v}_T) \cdot (\boldsymbol{z}_F - \boldsymbol{z}_T)$$

- Static condensation enables an efficient solution after linearisation
- Weakly enforced boundary conditions can also be considered
- Conservative fluxes can be identified

Convergence

Theorem (Convergence rates for small data)

Assume $\boldsymbol{u} \in W^{k+1,4}(\mathcal{T}_h)^d \cap H^{k+2}(\mathcal{T}_h)^d$, $p \in H^1(\Omega) \cap H^{k+1}(\Omega)$, *and*

$$\|\boldsymbol{f}\|_{L^2(\Omega)^d} \leq C\nu^2$$

with C, *independent of* h *and* ν, *small enough. Then, it holds*

$$\boxed{\begin{aligned}&\nu\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{1,h} + \|p_h - \pi_h^{0,k} p\|_{L^2(\Omega)} \\ &\qquad \lesssim h^{k+1} \left(\nu|\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^d} + \|\boldsymbol{u}\|_{W^{1,4}(\Omega)^d}|\boldsymbol{u}|_{W^{k+1,4}(\mathcal{T}_h)^d} + |p|_{H^{k+1}(\mathcal{T}_h)}\right)\end{aligned}}$$

with hidden constant independent of h *and* ν.

Lid-driven cavity I

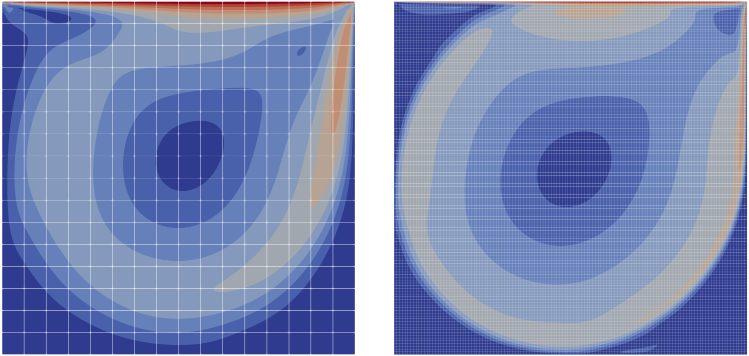

Figure: Lid-driven cavity, velocity magnitude contours (10 equispaced values in the range $[0, 1]$) for $k = 7$ computations at $\mathrm{Re} = 1{,}000$ (*left*: 16x16 grid) and $\mathrm{Re} = 20{,}000$ (*right*: 128x128 grid).

# Lid-driven cavity
Re = 1,000

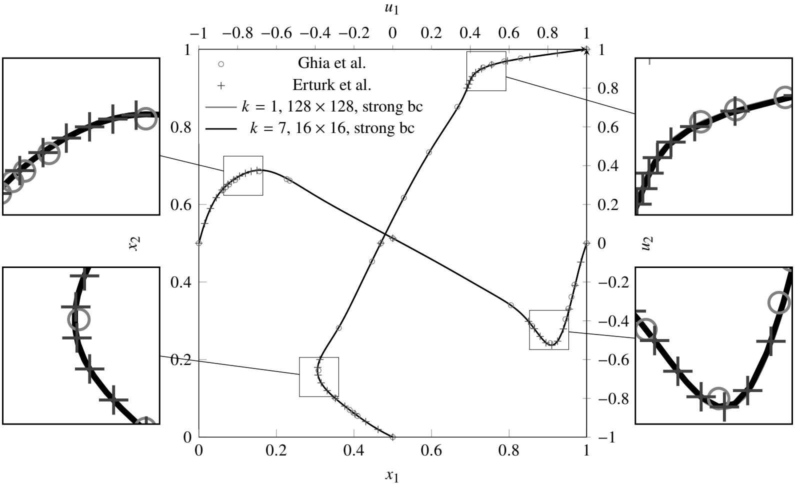

Figure: u_1 along the vertical centerline, u_2 along the horizontal centerline

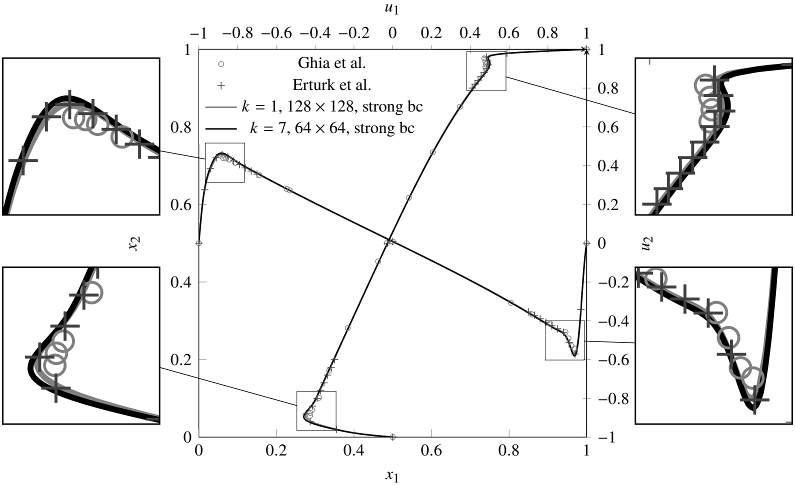

Figure: u_1 along the vertical centerline, u_2 along the horizontal centerline

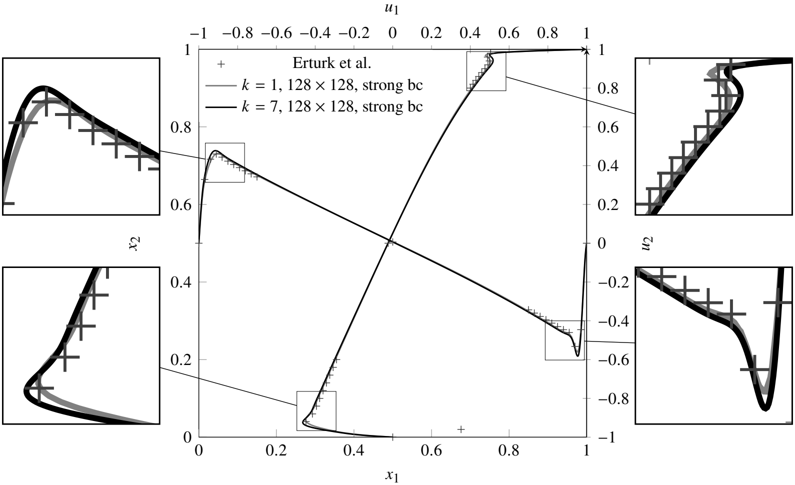

Figure: u_1 along the vertical centerline, u_2 along the horizontal centerline

Three-dimensional lid-driven cavity

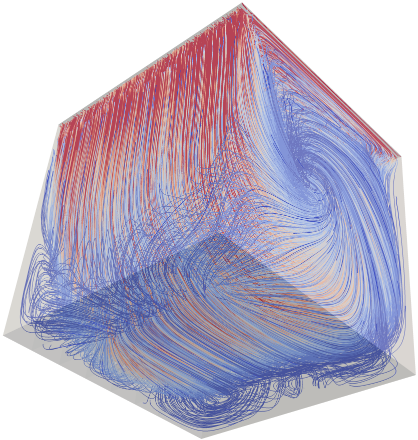

Figure: Three-dimensional lid-driven cavity, $\mathrm{Re} = 1000$, streamlines

Lid-driven cavity

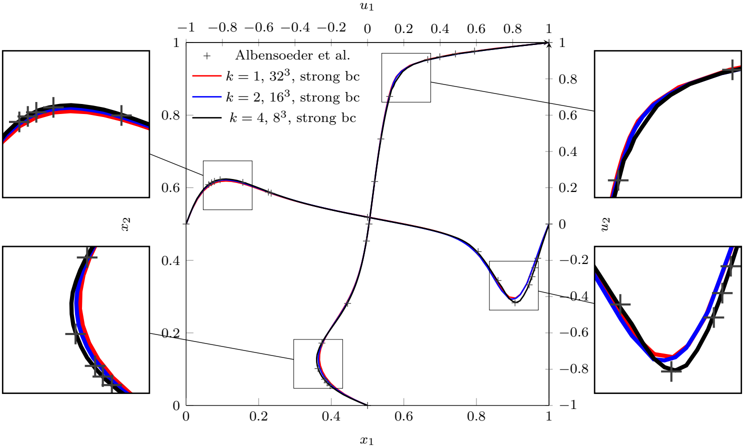

Figure: 3D Lid-driven cavity flow, horizontal component u_1 of the velocity along the vertical centerline $x_1, x_3 = \frac{1}{2}$ and the vertical component u_2 of the velocity along the horizontal centerline $x_2, x_3 = \frac{1}{2}$ for $\mathrm{Re} = 1{,}000$, $k = 1, 2, 4$

Lid-driven cavity

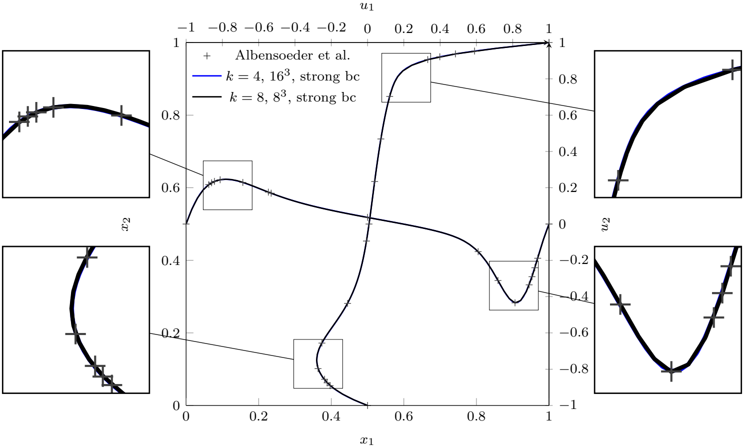

Figure: 3D Lid-driven cavity flow, horizontal component u_1 of the velocity along the vertical centerline $x_1, x_3 = \frac{1}{2}$ and the vertical component u_2 of the velocity along the horizontal centerline $x_2, x_3 = \frac{1}{2}$ for $\mathrm{Re} = 1{,}000$, $k = 4, 8$

References

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