Hybrid High-Order methods

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Institut Henri Poincaré, 13-14 September 2016

AcknHHOwledgements

- Joubine Aghili (Université de Montpellier)
- Sébastien Boyaval (École des Ponts ParisTech)
- Daniele Boffi (Università di Pavia)
- Michele Botti (Université de Montpellier)
- Florent Chave (Université de Montpellier and Politecnico di Milano)
- Bernardo Cockburn (University of Minnesota)
- Jérôme Droniou (Monash University)
- Alexandre Ern (École des Ponts ParisTech)
- Stella Krell (Université de Nice)
- Simon Lemaire (École Polytechnique Fédérale Lausanne)
- Alexander Linke (Weierstraß Institute Berlin)
- Fabien Marche (Université de Montpellier)
- Franck Pigeonneau (Saint-Gobain Recherche)
- Berardo Ruffini (Université de Montpellier)
- Friedhelm Schieweck (Otto-von-Güricke Universität)
- Ruben Specogna (Università di Udine)

Minimal bibliography: Lowest-order polyhedral methods

- Mimetic Finite Differences
  - Application to polyhedral meshes [Kuznetsov et al., 2004]
  - Convergence analysis [Brezzi et al., 2005]
- Mixed/Hybrid Finite Volumes
  - Pure diffusion (mixed) [Droniou and Eymard, 2006]
  - Pure diffusion (primal) [Eymard et al., 2010]
  - Link with MFD [Droniou et al., 2010]
- Discrete Duality Finite Volumes [Domelevo and Omnes, 2005]
- More recently
  - Cell-centered Galerkin [DP, 2012]
  - Compatible Discrete Operators [Bonelle and Ern, 2014]
  - Generalized Crouzeix–Raviart [DP and Lemaire, 2015]

Minimal bibliography: High-order polyhedral methods

- Discontinuous Galerkin
  - Unified analysis [Arnold, Brezzi, Cockburn and Marini, 2002]
  - General meshes [DP and Ern, 2010–2012]
  - Adaptive coarsening [Bassi et al., 2012, Antonietti et al., 2013]
- Hybridizable Discontinuous Galerkin
  - Pure diffusion [Cockburn et al., 2009]
- Weak Galerkin
  - Second-order elliptic problems [Wang and Ye, 2013]
- Virtual elements
  - Pure diffusion [Beirão da Veiga et al., 2013a]
  - Nonconforming VEM [Ayuso de Dios et al., 2016]
- Hybrid High-Order (HHO)
  - Pure diffusion [DP et al., 2014]
  - Locally degenerate transport [DP et al., 2015]

Suggested readings

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Methods Appl. Math., 14(4):461–472.
DOI: 10.1515/cmam-2014-0018.

Di Pietro, D. A. and Ern, A. (2015).
A hybrid high-order locking-free method for linear elasticity on general meshes.
Comput. Methods Appl. Mech. Engrg., 283:1–21.
DOI: 10.1016/j.cma.2014.09.009.

Di Pietro, D. A., Droniou, J., and Ern, A. (2015).
A discontinuous-skeletal method for advection-diffusion-reaction on general meshes.
SIAM J. Numer. Anal., 53(5):2135–2157.
DOI: 10.1137/140993971.

Cockburn, B., Di Pietro, D. A., and Ern, A. (2016).
Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin methods.
ESAIM: Math. Model Numer. Anal. (M2AN), 50(3):635–650.
DOI: 10.1051/m2an/2015051.

Di Pietro, D. A. and Droniou, J. (2016a).
A Hybrid High-Order method for Leray–Lions elliptic equations on general meshes.
Math. Comp.
Accepted for publication. Preprint arXiv:1508.01918 [math.NA].

Di Pietro, D. A. and Droniou, J. (2016b).
$W^{s,p}$-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a Hybrid High-Order discretisation of Leray–Lions problems.
Submitted. Preprint arXiv:1606.02832 [math.NA].

Features of HHO

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Applicable to a vast range of physical problem
- Reduced computational cost after hybridization

$$N_{\mathrm{dof}}^{\mathrm{hho}} \approx \frac{1}{2} k^2 \,\mathrm{card}(\mathcal{F}_h) \qquad N_{\mathrm{dof}}^{\mathrm{dg}} \approx \frac{1}{6} k^3 \,\mathrm{card}(\mathcal{T}_h)$$

Outline

1. Basic principles of HHO
   - Polyhedral meshes
   - A HHO method for the Poisson problem
   - Variable diffusion, local conservation and variations

2. Applications
   - A vector example: linear elasticity
   - A nonlinear example: Leray–Lions problems
   - A singularly perturbed example: vanishing diffusion w/advection

Outline

Mesh regularity I

Definition (Mesh regularity)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the usual sense of Ciarlet;
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences:

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces

See [DP and Ern, 2012] and [DP and Droniou, 2016a and 2016b]

Mesh regularity II

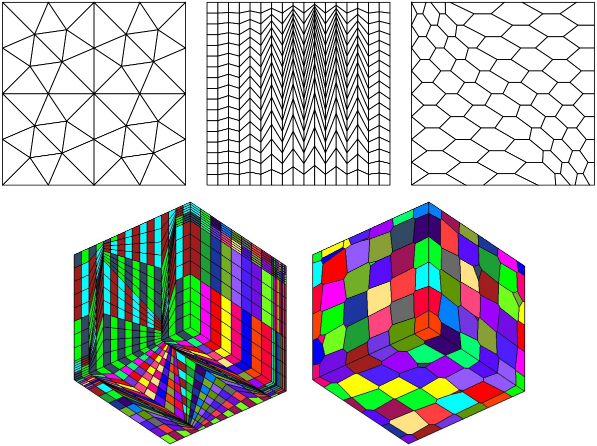


Figure: Admissible meshes in 2d and 3d: [Herbin and Hubert, 2008, FVCA5] and [DP and Lemaire, 2015] (above) and [Eymard et al., 2011, FVCA6] (below)

Projectors on local polynomial spaces I

- Key ingredients are projectors on local polynomial spaces
- The L^2-orthogonal projector $\pi_T^l : L^1(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T (\pi_T^l v - v) w = 0 \text{ for all } w \in \mathbb{P}^l(T)$$

- For all face $F \in \mathcal{F}_h$, we also need the L^2-projector π_F^l on $\mathbb{P}^l(F)$
- The elliptic projector $\varpi_T^l : W^{1,1}(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T \boldsymbol{\nabla}(\varpi_T^l v - v) \cdot \boldsymbol{\nabla} w = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } \int_T (\varpi_T^l v - v) = 0$$

Projectors on local polynomial spaces II

Lemma (Optimal $W^{s,p}$-approximation)

For all $p \in [1, +\infty]$, *all* $s \in \{1, \ldots, l+1\}$, *all* $m \in \{0, \ldots, s-1\}$, *and all* $v \in W^{s,p}(T)$, *it holds with* $\Pi_T^l = \pi_T^l$ *or* $\Pi_T^l = \varpi_T^l$

$$|v - \Pi_T^l v|_{W^{m,p}(T)} + h_T^{\frac{1}{p}} |v - \Pi_T^l v|_{W^{m,p}(\mathcal{F}_T)} \lesssim h_T^{s-m} |v|_{W^{s,p}(T)}.$$

Proof.

Apply a general result from [DP and Droniou, 2016b]: every W-bounded projector has optimal approximation properties. This result hinges on the approximation theory of [Dupont and Scott, 1980]. □

Outline

Model problem

- Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, denote a bounded, connected polyhedral domain
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$\begin{aligned} -\triangle u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Key ideas

- DOFs: polynomials of degree $k \geqslant 0$ at elements and faces
- Differential operator reconstructions tailored to the problem:

$$\boxed{a_{|T}(u,v) \approx (\nabla p_T^{k+1}\underline{u}_T, \nabla p_T^{k+1}\underline{v}_T)_T + \text{stab.}}$$

  with
  - high-order reconstruction p_T^{k+1} from local Neumann solves
  - stabilization via face-based penalty
- Construction yielding supercloseness on general meshes

DOFs

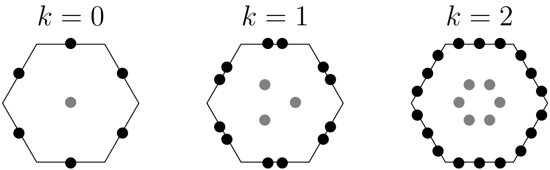


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- For $k \geqslant 0$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

- The global space has single-valued interface DOFs

$$\underline{U}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right)$$

Local potential reconstruction I

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$p_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$$

is s.t. $\forall \underline{v}_T \in \underline{U}_T^k$, $(p_T^{k+1}\underline{v}_T - v_T, 1)_T = 0$ and $\forall w \in \mathbb{P}^{k+1}(T)$,

$$\boxed{(\boldsymbol{\nabla} p_T^{k+1}\underline{v}_T, \boldsymbol{\nabla} w)_T := -(v_T, \triangle w)_T + \sum_{F\in\mathcal{F}_T} (v_F, \boldsymbol{\nabla} w\cdot\boldsymbol{n}_{TF})_F}$$

- To compute p_T^{k+1}, we invert a small SPD matrix of size

$$N_{k,d} := \binom{k+1+d}{k+1}$$

- **Trivially parallel task, potentially suited to GPUs!**

Local potential reconstruction II

Lemma (Approximation properties for $p_T^{k+1} \circ \underline{I}_T^k$)

Define the local reduction map $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ *s.t.*

$$\underline{I}_T^k : v \mapsto \left(\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T}\right).$$

Then, for all $T \in \mathcal{T}_h$ *and all* $v \in H^{k+2}(T)$,

$$\begin{aligned}\|v - p_T^{k+1}\underline{I}_T^k v\|_T + h_T \|\boldsymbol{\nabla}(v - p_T^{k+1}\underline{I}_T^k v)\|_T \\ + h_T^{1/2}\|v - p_T^{k+1}\underline{I}_T^k v\|_{\partial T} + h_T^{3/2}\|\boldsymbol{\nabla}(v - p_T^{k+1}\underline{I}_T^k v)\|_{\partial T} \\ \lesssim h_T^{k+2}\|v\|_{H^{k+2}(T)}.\end{aligned}$$

Local potential reconstruction III

- Since $\triangle w \in \mathbb{P}^{k-1}(T)$ and $\boldsymbol{\nabla} w_{|F} \cdot \boldsymbol{n}_{TF} \in \mathbb{P}^k(F)$,

$$
\begin{aligned}
(\boldsymbol{\nabla} p_T^{k+1} \underline{I}_T^k v, \boldsymbol{\nabla} w)_T &= -(\pi_T^k v, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (\pi_F^k v, \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF})_F \\
&= -(v, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (v, \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF})_F \\
&= (\boldsymbol{\nabla} v, \boldsymbol{\nabla} w)_T
\end{aligned}
$$

- As a result, recalling the definition of the elliptic projector,

$$
p_T^{k+1} \circ \underline{I}_T^k = \varpi_T^{k+1}
$$

- The approximation properties follow

Stabilization I

- The following local discrete bilinear form is in general not stable

$$a_T(\underline{u}_T, \underline{v}_T) = (\boldsymbol{\nabla} p_T^{k+1} \underline{u}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{v}_T)_T$$

- As a remedy, we add a local stabilization term:

$$\boxed{a_T(\underline{u}_T, \underline{v}_T) := (\boldsymbol{\nabla} p_T^{k+1} \underline{u}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{v}_T)_T + s_T(\underline{u}_T, \underline{v}_T)}$$

- We aim at expressing coercivity w.r.t. to the local seminorm

$$\|\underline{v}_T\|_{1,T}^2 := \|\boldsymbol{\nabla} v_T\|_T^2 + |\underline{v}_T|_{1,\partial T}^2, \quad |\underline{v}_T|_{1,\partial T}^2 := \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} \|v_F - v_T\|_F^2$$

Stabilization II

- A naive choice for the stabilization would be (cf. HDG)

$$s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} (u_F - u_T, v_F - v_T)_F$$

- This choice is, however, suboptimal since, for all $v \in H^{k+2}(T)$,

$$\|\boldsymbol{\nabla}(p_T^{k+1} \underline{I}_T^k v - v)\|_T \lesssim h^{\color{red}{k+1}} \|v\|_{H^{k+2}(T)},$$

  but we only have

$$s_T(\underline{I}_T^k v, \underline{I}_T^k v)^{1/2} \lesssim h^{\color{red}{k}} \|v\|_{H^{k+1}(T)}$$

- **We need to penalize higher-order differences!**

Stabilization III

- Let us introduce the face residual operator $\delta_{TF}^k : \underline{U}_T^k \to \mathbb{P}^k(F)$ s.t.

$$\delta_{TF}^k(\underline{v}_T) := \pi_F^k(v_F - p_T^{k+1}\underline{v}_T) - \pi_T^k(v_T - p_T^{k+1}\underline{v}_T)$$

- Consider the following least-square penalty bilinear form:

$$\boxed{s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} \frac{1}{h_F}(\delta_{TF}^k \underline{u}_T, \delta_{TF}^k \underline{v}_T)_F}$$

Stabilization IV

- Let us first investigate the consistency properties of s_T
- Using approximation for $p_T^{k+1} \circ \underline{I}_T^k$ we have, for all $v \in H^{k+2}(T)$

$$
\begin{aligned}
\|\delta_{TF}^k \underline{I}_T^k v\|_F &= \|\pi_F^k(v - p_T^{k+1}\underline{I}_T^k v) - \pi_T^k(v - p_T^{k+1}\underline{I}_T^k v)\|_F \\
&\leqslant \|\pi_F^k(v - p_T^{k+1}\underline{I}_T^k v)\|_F + \|\pi_T^k(v - p_T^{k+1}\underline{I}_T^k v)\|_F \\
&\lesssim \|v - p_T^{k+1}\underline{I}_T^k v\|_F + h_T^{-1/2}\|v - p_T^{k+1}\underline{I}_T^k v\|_T \\
&\lesssim h_T^{k+3/2}\|v\|_{H^{k+2}(T)}
\end{aligned}
$$

- Hence, this time

$$
\boxed{s_T(\underline{I}_T^k v, \underline{I}_T^k v)^{1/2} \lesssim h_T^{k+1}\|v\|_{H^{k+2}(T)}}
$$

Stabilization V

- Alternative interpretation: Define $\widehat{p}_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$ s.t.

$$\widehat{p}_T^{k+1}\underline{v}_T := v_T + (p_T^{k+1}\underline{v}_T - \pi_T^k p_T^{k+1}\underline{v}_T)$$

- $\widehat{p}_T^{k+1}\underline{v}_T$ is a high-order correction of element DOFs
- It can be proved that s_T admits the equivalent formulation

$$\boxed{s_T(\underline{u}_T, \underline{v}_T) := \sum_{F\in\mathcal{F}_T} \frac{1}{h_F}(\pi_F^k(\widehat{p}_T^{k+1}\underline{u}_T - u_F), \pi_F^k(\widehat{p}_T^{k+1}\underline{v}_T - v_F))_F}$$

Stabilization VI

Lemma (Stability and boundedness)

There is $\eta > 0$ independent of h s.t., for all $T \in \mathcal{T}_h$ and all $\underline{v}_T \in \underline{U}_T^k$,

$$\eta^{-1}\|\underline{v}_T\|_{1,T}^2 \leqslant \|\underline{v}_T\|_{a,T}^2 \leqslant \eta\|\underline{v}_T\|_{1,T}^2,$$

where

$$\|\underline{v}_T\|_{a,T}^2 := a_T(\underline{v}_T, \underline{v}_T) = \|\boldsymbol{\nabla} p_T^{k+1}\underline{v}_T\|_T^2 + s_T(\underline{v}_T, \underline{v}_T).$$

Stabilization VII

- We prove the first inequality and leave the second as an exercise
- Let $T \in \mathcal{T}_h$ and $\underline{v}_T \in \underline{U}_T^k$. By definition of $p_T^{k+1}\underline{v}_T$ with $w = v_T$,

$$\begin{aligned}\|\boldsymbol{\nabla} v_T\|_T^2 &= (\boldsymbol{\nabla} p_T^{k+1}\underline{v}_T, \boldsymbol{\nabla} v_T)_T - \sum_{F\in\mathcal{F}_T} (v_F - v_T, \boldsymbol{\nabla} v_T{\cdot}\boldsymbol{n}_{TF})_F \\ &\leqslant \|\boldsymbol{\nabla} p_T^{k+1}\underline{v}_T\|_T^2 + \frac{1}{2}\|\boldsymbol{\nabla} v_T\|_T^2 + N_\partial C_{\rm tr}^2 |\underline{v}_T|_{1,\partial T}^2\end{aligned}$$

- As a result,

$$\|\boldsymbol{\nabla} v_T\|_T^2 \lesssim \|\boldsymbol{\nabla} p_T^{k+1}\underline{v}_T\|_T^2 + |\underline{v}_T|_{1,\partial T}^2$$

Stabilization VIII

- Let now $F \in \mathcal{F}_T$. Adding and subtracting $\pi_F^k \widehat{p}_T^{k+1} \underline{v}_T$ we have

$$\|v_F - v_T\|_F \leqslant \|\pi_F^k (v_F - \widehat{p}_T^{k+1} \underline{v}_T)\|_F + \|\pi_F^k (p_T^{k+1} \underline{v}_T - \pi_T^k p_T^{k+1} \underline{v}_T)\|_F$$

and, using the discrete trace and local Poincaré inequalities

$$\begin{aligned} \|\pi_F^k (p_T^{k+1} \underline{v}_T - \pi_T^k p_T^{k+1} \underline{v}_T)\|_F &\lesssim h_F^{-1/2} \|p_T^{k+1} \underline{v}_T - \pi_T^k p_T^{k+1} \underline{v}_T\|_T \\ &\lesssim h_F^{1/2} \|\boldsymbol{\nabla} p_T^{k+1} \underline{v}_T\|_T \end{aligned}$$

- Fromt the above inequality it is readily inferred that

$$|\underline{v}_T|_{1,\partial T}^2 \lesssim s_T(\underline{v}_T, \underline{v}_T) + \|\boldsymbol{\nabla} p_T^{k+1} \underline{v}_T\|_T^2$$

- The coercivity bound follows recalling the estimate on $\|\boldsymbol{\nabla} v_T\|_T$

Discrete problem

- We enforce boundary conditions strongly considering the space

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \mid v_F \equiv 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_h} (f, v_T)_T \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

- Well-posedness follows from the $\|\cdot\|_{1,h}$-coercivity of a_h with

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$$

- In the implementation, element-based DOFs are statically condensed

Convergence I

Theorem (Energy-error estimate)

Assume $u \in H^{k+2}(\Omega)$ *and define the global reduction map*

$$\underline{I}_h^k u := \big((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h}\big) \in \underline{U}_{h,0}^k.$$

Then, we have the following energy error estimate:

$$\boxed{\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)}.}$$

Corollary (Error estimate on the flux)

It holds with $p_h^{k+1} : \underline{U}_h^k \to \mathbb{P}^{k+1}(\mathcal{T}_h)$ *s.t.* ${p_h^{k+1}}_{|T} = p_T^{k+1}$,

$$\|\boldsymbol{\nabla}_h p_h^{k+1}\underline{u}_h - \boldsymbol{\nabla} u\| + s_h(\underline{u}_h, \underline{u}_h) \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)},$$

with s_h *obtained assemblying the local contributions* s_T*.*

Convergence II

Theorem (Supercloseness)

Further assuming elliptic regularity and $f \in H^1(\Omega)$ *if* $k = 0$,

$$\boxed{\|u_h - \pi_h^k u\| \lesssim h^{k+2} B(u,k),}$$

with $B(u,0) := \|f\|_{H^1(\Omega)}$, $B(u,k) := \|u\|_{H^{k+2}(\Omega)}$ *if* $k \geqslant 1$ *and*

$$u_{h|T} = u_T \qquad \forall T \in \mathcal{T}_h.$$

Corollary (L^2-error estimate)

It holds

$$\|p_h^{k+1} \underline{u}_h - u\| \lesssim h^{k+2} B(u,k).$$

Convergence III

- Let prove the energy-error estimate. With $\widehat{\underline{u}}_h := \underline{I}_h^k u$, it holds

$$\|\widehat{\underline{u}}_h - \underline{u}_h\|_{a,h} \leqslant \eta^{1/2} \sup_{\underline{v}_h \in \underline{U}_{h,0}^k,\, \|\underline{v}_h\|_{1,h}=1} a_h(\widehat{\underline{u}}_h - \underline{u}_h, \underline{v}_h)$$

- Hence, we can estimate the error as

$$\|\widehat{\underline{u}}_h - \underline{u}_h\|_{a,h} \leqslant \eta^{1/2} \sup_{\underline{v}_h \in \underline{U}_{h,0}^k,\, \|\underline{v}_h\|_{1,h}=1} \mathcal{E}_h(\underline{v}_h),$$

with consistency error s.t.

$$\mathcal{E}_h(\underline{v}_h) := a_h(\widehat{\underline{u}}_h, \underline{v}_h) - l_h(\underline{v}_h)$$

- We next bound $\mathcal{E}_h(\underline{v}_h)$ for a generic $\underline{v}_h \in \underline{U}_{h,0}^k$ s.t. $\|\underline{v}_h\|_{1,h} = 1$

Convergence IV

- Since $f = -\triangle u$ a.e. in Ω, an element-wise partial integration yields

$$l_h(\underline{v}_h) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v_T)_T + \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} ({\color{red}v_F} - v_T, \boldsymbol{\nabla} u{\cdot}\boldsymbol{n}_{TF})_F,$$

  where we have used flux continuity and $v_F \equiv 0$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$
- Choosing $w = \check{u}_T := p_T^{k+1}\widehat{u}$ in the definition of $p_T^{k+1}\underline{v}_T$, we infer

$$a_h(\widehat{\underline{u}}_h, \underline{v}_h) = \sum_{T\in\mathcal{T}_h} \{(\boldsymbol{\nabla}\check{u}_T, \boldsymbol{\nabla} v_T)_T + s_T(\widehat{\underline{u}}_T, \underline{v}_T)\} + \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} (v_F - v_T, \boldsymbol{\nabla}\check{u}_T{\cdot}\boldsymbol{n}_{TF})_F$$

Convergence V

- Combining the previous relations, we arrive at

$$\mathcal{E}_h(\underline{v}_h) = \sum_{T\in\mathcal{T}_h} \left\{ \cancel{(\boldsymbol{\nabla}(\check{u}_T - u), \boldsymbol{\nabla} v_T)_T} + s_T(\underline{\hat{u}}_T, \underline{v}_T) \right\} + \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} (\boldsymbol{\nabla}(\check{u}_T - u)\cdot\boldsymbol{n}_{TF}, v_F - v_T)_F := \mathfrak{T}_1 + \mathfrak{T}_2$$

- Using the Cauchy–Schwarz inequality and approximation, we infer

$$|\mathfrak{T}_1| \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)} \underbrace{\|\underline{v}_h\|_{1,h}}_{=1} = h^{k+1}\|u\|_{H^{k+2}(\Omega)}$$

- Using the Cauchy–Schwarz, trace inequalities, and approximation

$$|\mathfrak{T}_2| \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)} \underbrace{\|\underline{v}_h\|_{1,h}}_{=1} = h^{k+1}\|u\|_{H^{k+2}(\Omega)}$$

Convergence VI

- Using the above bounds, we conclude that

$$\|\underline{\widehat{u}}_h - \underline{u}_h\|_{a,h} \leqslant \eta^{1/2} \sup_{\underline{v}_h \in \underline{U}_{h,0}^k,\, \|\underline{v}_h\|_{1,h}=1} \mathcal{E}_h(\underline{v}_h) \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)}$$

- Finally, using the $\|\cdot\|_{1,h}$-coercivity of a_h yields

$$\eta^{-1/2} \|\underline{\widehat{u}}_h - \underline{u}_h\|_{1,h} \leqslant \|\underline{\widehat{u}}_h - \underline{u}_h\|_{a,h} \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)}$$

Numerical examples

2d test case, smooth solution, uniform refinement

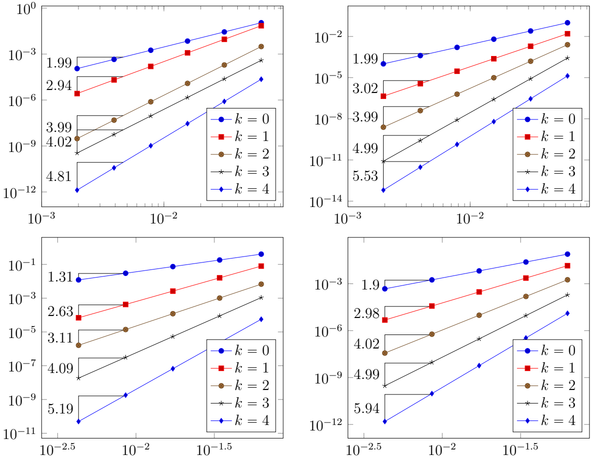


Figure: 2d test case, trigonometric solution. Energy (left) and L^2-norm (right) of the error vs. h for uniformly refined triangular (top) and hexagonal (bottom) mesh families

- Let $\Omega := (-1,1)^3 \backslash [0,1]^3$. We consider the Fichera exact solution

$$u(\boldsymbol{x}) = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{4}}$$

  corresponding to the forcing term

$$f(\boldsymbol{x}) = -\frac{3}{4}(x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{4}}$$

- We consider the adaptive procedure of [DP and Specogna, 2016]

Theorem (A posteriori error estimate)

It holds with $p_h^{k+1}\underline{u}_h \in \mathbb{P}^{k+1}(\mathcal{T}_h)$ *s.t.* $(p_h^{k+1}\underline{u}_h)_{|T} = p_T^{k+1}\underline{u}_T \ \forall T \in \mathcal{T}_h$,

$$\|\nabla_h(p_h^{k+1}\underline{u}_h - u)\|^2 \leqslant \sum_{T\in\mathcal{T}_h} \left(\eta_{\text{nc},T}^2 + (\eta_{\text{res},T} + \eta_{\text{sta},T})^2\right),$$

where, denoting by $u_h^* \in H_0^1(\Omega)$ *the nodal interpolate of* $p_h^{k+1}\underline{u}_h$,

$$\begin{aligned}
\eta_{\text{nc},T} &:= \|\nabla(p_T^{k+1}\underline{u}_T - u_h^*)\|_T,\\
\eta_{\text{res},T} &:= C_{\text{P},T} h_T \|(f + \triangle p_T^{k+1}\underline{u}_T) - \pi_T^0(f + \triangle p_T^{k+1}\underline{u}_T)\|_T,\\
\eta_{\text{sta},T} &:= C_{\text{F},T} h_T^{1/2} \|R_{\partial T}^{*,k}(\tau_{\partial T} R_{\partial T}^k(u_T - u_{\partial T}))\|_{\partial T},
\end{aligned}$$

with $R_{\partial T}^k$, $R_{\partial T}^{*,k}$ *and* $\tau_{\partial T}$ *defined as for flux the formulation (cf. below).*

Numerical examples III

3d test case, singular solution, adaptive refinement

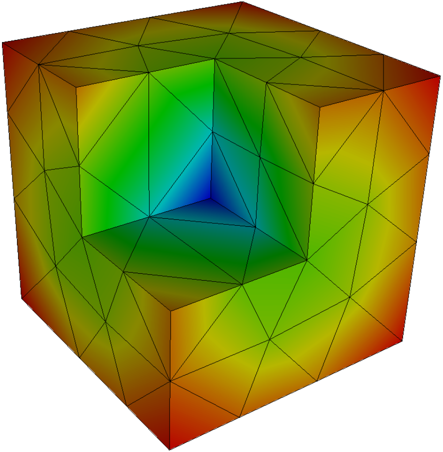
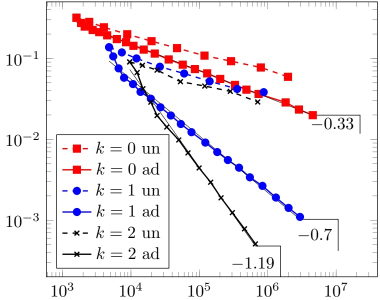
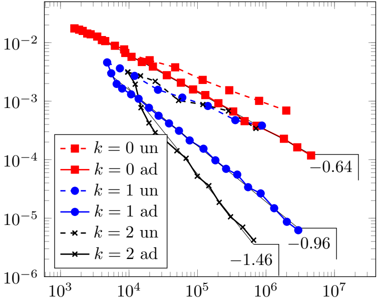


Figure: HHO solution on a sequence of adaptively refined simplicial meshes

Numerical examples IV
3d test case, singular solution, adaptive refinement



(a) Energy error vs. N_{dof}



(b) L^2-error vs. N_{dof}

Figure: Results for Fichera's test case

Numerical examples V

3d test case, singular solution, adaptive refinement

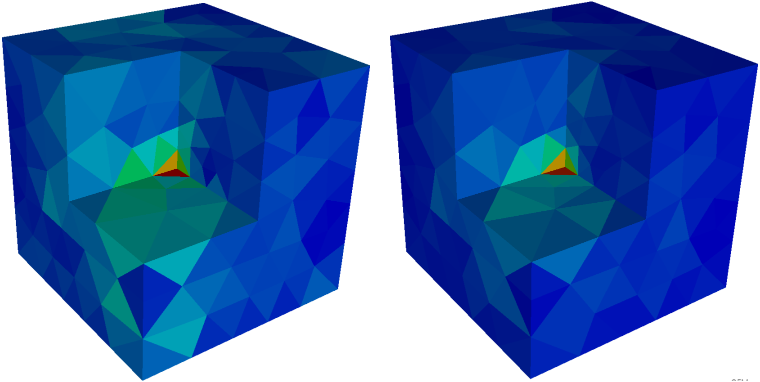


Figure: Estimated (left) and true (right) error distribution

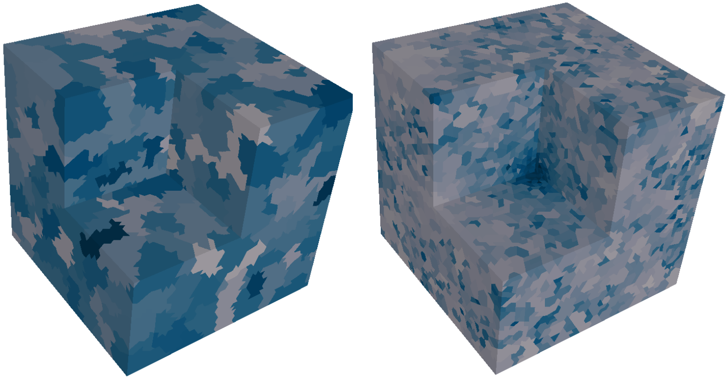
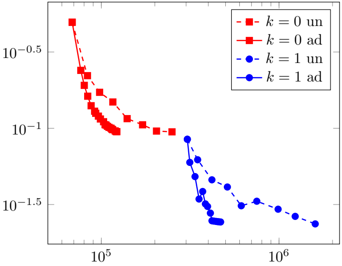


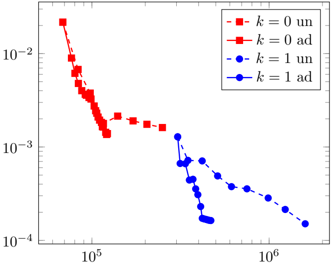
Figure: Fichera corner benchmark, adaptive mesh coarsening

Numerical examples II

Fichera's 3d test case, adaptive coarsening



(a) Energy-error vs. N_{dof}



(b) L^2-error vs. N_{dof}

Figure: Error vs. N_{dof} for the Fichera corner benchmark, adaptively coarsened meshes

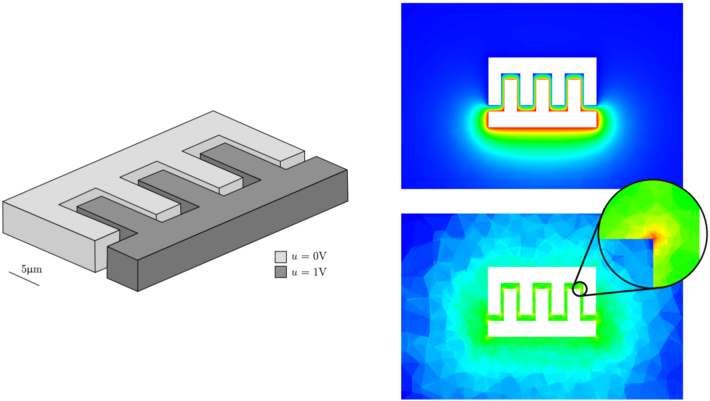
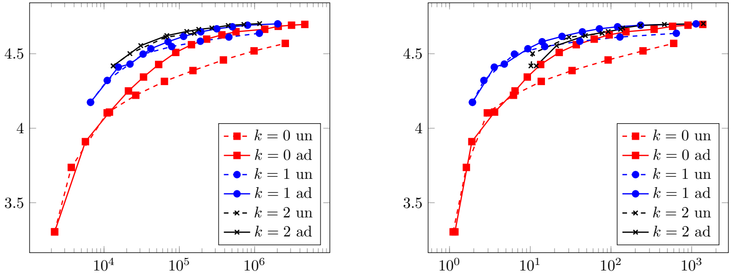
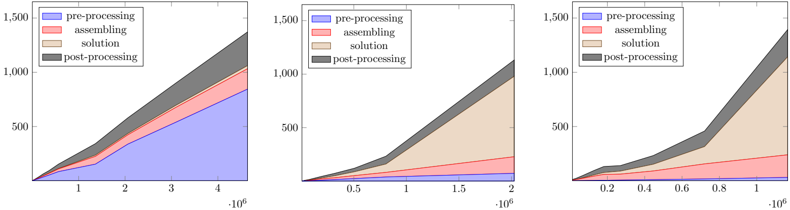


Figure: Geometry (left), numerical solution (right, top) and final adaptive mesh (right, bottom) for the comb-drive actuator test case [DP and Specogna, 2016]



(a) Capacitance vs. N_{dof} (b) Capacitance vs. computing time

Figure: Results for the comb drive benchmark.



(a) $k = 0$ (b) $k = 1$ (c) $k = 2$

Figure: Computing wall time (s) vs. N_{dof} for the comb drive benchmark

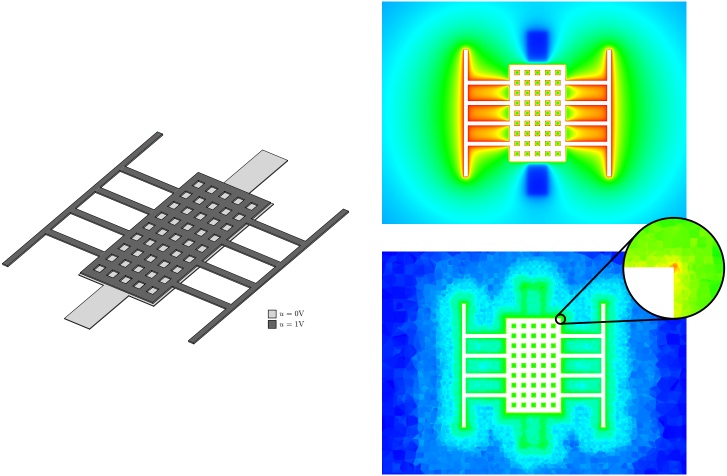
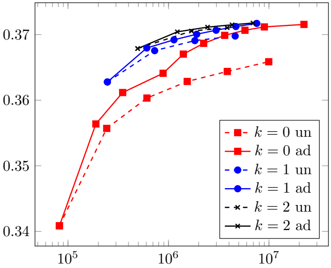
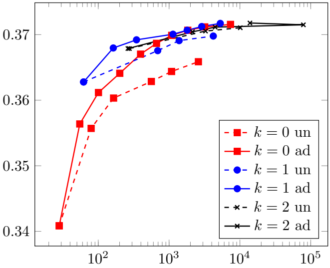


Figure: Geometry (left), numerical solution (right, top) and final adaptive mesh (right, bottom) for the MEMS test case [DP and Specogna, 2016]



(a) Capacitance vs. N_{dof}



(b) Capacitance vs. computing time

Figure: Results for the MEMS switch benchmark

Numerical examples VI
3d industrial test case, adaptive refinement, cost assessment

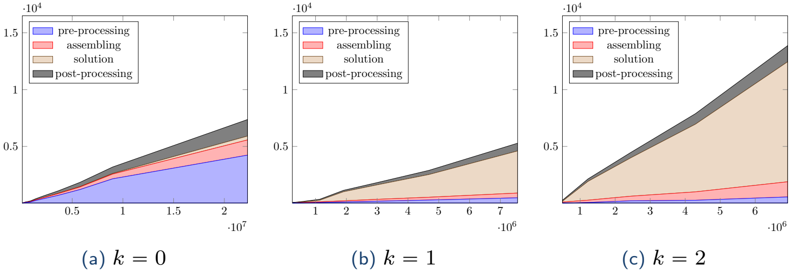
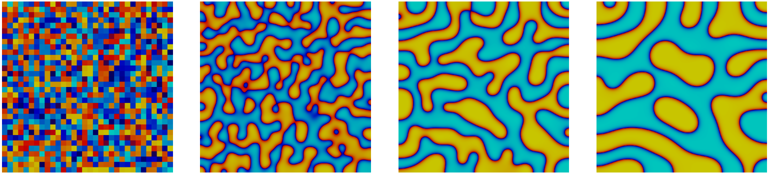
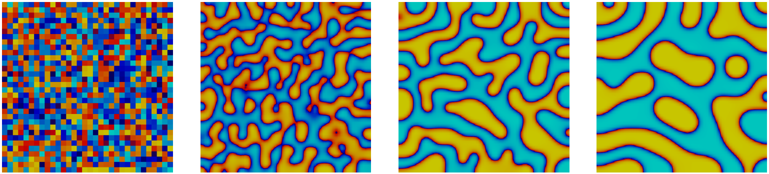


Figure: Computing wall time vs N_{dof} for the MEMS switch benchmark.



(a) 128×128 uniform Cartesian mesh, $k = 0$



(b) 64×64 uniform Cartesian mesh, $k = 1$

Figure: Spinoidal decomposition [Chave et al., 2016]

Outline

Porous media and HHO

Figure: Last week at Cargese (when the cat's away, the mice will play)

Variable diffusion I

- Let $\boldsymbol{\nu} : \Omega \to \mathbb{R}^{d\times d}$ be a SPD tensor-valued field s.t.

$$\forall T \in \mathcal{T}_h, \qquad 0 < \underline{\nu}_T \leqslant \lambda(\boldsymbol{\nu}) \leqslant \overline{\nu}_T$$

- For the sake of simplicity, we assume $\boldsymbol{\nu}$ polynomial on $\mathcal{T}_h$,

$$\exists l \in \mathbb{N}^*, \qquad \boldsymbol{\nu} \in \mathbb{P}^l(\mathcal{T}_h)^{d\times d}$$

- We consider the Darcy problem

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(\boldsymbol{\nu}\boldsymbol{\nabla}u) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Variable diffusion II

$$\boxed{(\boldsymbol{\nu}\boldsymbol{\nabla} p_T^{k+1}\underline{v}_T, \boldsymbol{\nabla} w)_T = (\boldsymbol{\nu}\boldsymbol{\nabla} v_T, \boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T}(v_F - v_T, \boldsymbol{\nu}\boldsymbol{\nabla} w\cdot\boldsymbol{n}_{TF})_F}$$

Lemma (Approximation properties of $p_T^{k+1}\underline{I}_T^k$)

For all $v \in H^{k+2}(T)$, with $\alpha = \frac{1}{2}$ if $l = 0$ and $\alpha = 1$ if $l \geqslant 1$,

$$\|v - p_T^{k+1}\underline{I}_T^k v\|_T + h_T\|\boldsymbol{\nabla}(v - p_T^{k+1}\underline{I}_T^k v)\|_T \leqslant C\rho_T^{\alpha} h_T^{k+2}\|v\|_{k+2,T},$$

with local heterogeneity/anisotropy ratio

$$\rho_T := \frac{\overline{\nu}_T}{\underline{\nu}_T} \geqslant 1.$$

Variable diffusion III

Theorem (Energy-error estimate)

Assume that $u \in H^{k+2}(\mathcal{T}_h)$ *and set*

$$a_{\boldsymbol{\nu},T}(\underline{u}_T, \underline{v}_T) := (\boldsymbol{\nu}\boldsymbol{\nabla} p_T^{k+1}\underline{u}_T, \boldsymbol{\nabla} p_T^{k+1}\underline{v}_T)_T + s_{\boldsymbol{\nu},T}(\underline{u}_T, \underline{v}_T)$$

where, letting $\nu_{TF} := \|\boldsymbol{n}_{TF}\cdot\boldsymbol{\nu}_{|T}\cdot\boldsymbol{n}_{TF}\|_{L^\infty(F)}$,

$$s_{\boldsymbol{\nu},T}(\underline{u}_T, \underline{v}_T) := \sum_{F\in\mathcal{F}_T} \frac{\nu_{TF}}{h_F}(\pi_F^k(\widehat{p}_T^{k+1}\underline{u}_T - u_F), \pi_F^k(\widehat{p}_T^{k+1}\underline{v}_T - v_F))_F.$$

Then, with α *as above and* $\|\cdot\|_{\boldsymbol{\nu},h}$ *denoting the norm defined by* $a_{\boldsymbol{\nu},h}$,

$$\boxed{\|\underline{u}_h - \underline{I}_h^k u\|_{\boldsymbol{\nu},h} \lesssim \left(\sum_{T\in\mathcal{T}_h} \overline{\nu}_T \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{k+2,T}^2\right)^{1/2}.}$$

Le Potier's test case I

- We consider Le Potier's exact solution

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2),$$

- The diffusion field has rotating principal axes

$$\boldsymbol{\nu}(\boldsymbol{x}) = \begin{pmatrix} (x_2 - \overline{x}_2)^2 + \epsilon(x_1 - \overline{x}_1)^2 & -(1-\epsilon)(x_1 - \overline{x}_1)(x_2 - \overline{x}_2) \\ -(1-\epsilon)(x_1 - \overline{x}_1)(x_2 - \overline{x}_2) & (x_1 - \overline{x}_1)^2 + \epsilon(x_2 - \overline{x}_2)^2 \end{pmatrix},$$

with anisotropy ratio and rotation center

$$\epsilon = \rho^{-1} = 1 \cdot 10^{-2}, \qquad (\overline{x}_1, \overline{x}_2) = -(0.1, 0.1)$$

Le Potier's test case II

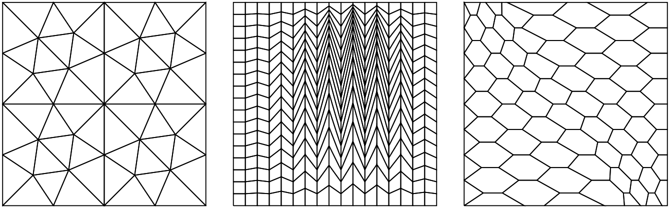


Figure: Triangular, Kershaw and hexagonal mesh families

Le Potier's test case III

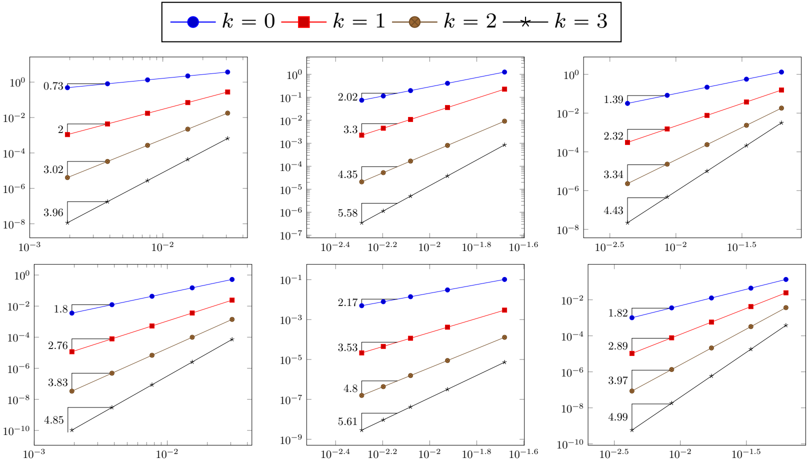


Figure: $\|\cdot\|_{1,h}$-norm (above) and L^2-norm (below) of the error vs. h for (from left to right) the triangular, Kershaw and hexagonal mesh families

Teaser: Fractured porous media

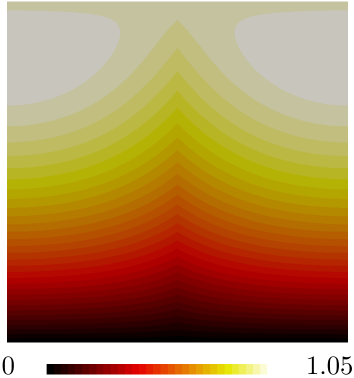


Figure: Flow in fractured porous media. Simulations by F. Chave (second H on the beach).

Local conservation and numerical fluxes I

- A highly prized property in practice is local conservation
- At the discrete level, we wish to mimick the local balance

$$(\boldsymbol{\nu}_T \boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_T - \sum_{F \in \mathcal{F}_T} (\boldsymbol{\nu}_T \boldsymbol{\nabla} u \cdot \boldsymbol{n}_{TF}, v)_F = (f, v)_T \quad \forall v \in H^1(T)$$

where, for every interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$,

$$\boldsymbol{\nu}_{T_1} \boldsymbol{\nabla} u \cdot \boldsymbol{n}_{T_1 F} + \boldsymbol{\nu}_{T_2} \boldsymbol{\nabla} u \cdot \boldsymbol{n}_{T_2 F} = 0$$

- This requires to identify numerical fluxes

Local conservation and numerical fluxes II

- Define the boundary residual operator $R^k_{\partial T} : \mathbb{P}^k(\mathcal{F}_T) \to \mathbb{P}^k(\mathcal{F}_T)$

$$R^k_{\partial T}\varphi_{|F} := \pi^k_F \left(\varphi_{|F} - p^{k+1}_T(0,\varphi) + \pi^k_T p^{k+1}_T(0,\varphi)\right) \quad \forall F \in \mathcal{F}_T$$

- Denote by $R^{*,k}_{\partial T}$ its adjoint and let $\tau_{\partial T}$ and $u_{\partial T}$ be s.t.

$$\tau_{\partial T|F} = \frac{\nu_{TF}}{h_F} \quad \text{and} \quad u_{\partial T|F} = u_F \quad \forall F \in \mathcal{F}_T$$

- Then, the penalty term can be rewritten in conservative form as

$$\boxed{s_T(\underline{u}_T, \underline{v}_T) = \sum_{F \in \mathcal{F}_T} (R^{*,k}_{\partial T}(\tau_{\partial T} R^k_{\partial T}(u_{\partial T} - u_T)), v_F - v_T))_F}$$

Local conservation and numerical fluxes III

Lemma (Flux formulation)

The HHO solution $\underline{u}_h \in \underline{U}_{h,0}^k$ *satisfies, for all* $T \in \mathcal{T}_h$ *and all* $v_T \in \mathbb{P}^k(T)$

$$(\boldsymbol{\nu}\boldsymbol{\nabla} p_T^{k+1}\underline{u}_T, \boldsymbol{\nabla} v_T)_T - \sum_{F \in \mathcal{F}_T} (\Phi_{TF}(\underline{u}_T), v_T)_F = (f, v_T)_T,$$

with numerical flux

$$\Phi_{TF}(\underline{u}_T) := \boldsymbol{\nu}_T \boldsymbol{\nabla} p_T^{k+1}\underline{u}_T \cdot \boldsymbol{n}_{TF} - R_{\partial T}^{*,k}(\tau_{\partial T} R_{\partial T}^k (u_{\partial T} - u_T)),$$

s.t., for every interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$,

$$\Phi_{T_1 F}(\underline{u}_{T_1}) + \Phi_{T_2 F}(\underline{u}_{T_2}) = 0.$$

Link with HDG

- The flux formulation shows that (cf. [Cockburn, DP and Ern, 2015])

  HHO = HDG on steroids

- Smaller local problems to eliminate flux unknowns:

$$\boldsymbol{\nabla}\mathbb{P}^{k+1}(T) \quad \text{vs.} \quad \mathbb{P}^{k}(T)^{d}$$

- Superconvergence of the potential in the L^2-norm

$$h^{k+2} \quad \text{vs.} \quad h^{k+1}$$

- **HHO can be adapted into existing HDG codes!**

The HHO(l) family

- Let $T \in \mathcal{T}_h$, $k-1 \leqslant l \leqslant k+1$, and consider the local space

$$\underline{U}_T^{k,l} := \mathbb{P}^l(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

- Convergence rates as for the original HHO method and
  - $l = k-1$: High-Order Mimetic (up to variants in stabilization)
  - $l = k$: original HHO method
  - $l = k+1$: new HDG method
- $k = 0$ and $l = k-1$ on simplices yields the Crouzeix–Raviart element
- **The globally-coupled unknowns coincide in all the cases!**

A nonconforming finite element interpretation I

- We interpret the HHO(l) methods as nonconforming FE methods
- The construction extends the ideas of [Ayuso de Dios et al., 2016]
- For a fixed element $T \in \mathcal{T}_h$, we define the local space

$$\boxed{V_T^{k,l} := \left\{\varphi \in H^1(T) \mid \; \boldsymbol{\nabla}\varphi_{|F}\cdot\boldsymbol{n}_F \in \mathbb{P}^k(F)\ \forall F \in \mathcal{F}_T \text{ and } \triangle\varphi \in \mathbb{P}^l(T) \; \right\}}$$

- We next study the relation between $V_T^{k,l}$ and $\underline{U}_T^{k,l}$

A nonconforming finite element interpretation II

- Let $\Phi_T : \underline{U}_T^{k,l} \to V_T^{k,l}$ be s.t. $\Phi_T(\underline{v}_T)$ solves the Neumann problem

$$\boxed{\triangle\Phi_T(\underline{v}_T) = v_T - \frac{1}{|T|_d}\left(\int_T v_T - \sum_{F\in\mathcal{F}_T}\int_F v_F\right)}$$

with boundary and closure conditions

$$\boldsymbol{\nabla}\Phi_T(\underline{v}_T)_{|F}\cdot\boldsymbol{n}_{TF} = v_F \quad \forall F\in\mathcal{F}_T, \qquad \int_T (\Phi_T(\underline{v}_T) - v_T) = 0$$

- Both Φ_T and $\underline{I}_T^{k,l} : V_T^{k,l} \to \underline{U}_T^{k,l}$ are injective
- Therefore, $\underline{I}_T^{k,l} : V_T^{k,l} \to \underline{U}_T^{k,l}$ is an isomorphism and we can identify

$$\boxed{V_T^{k,l} \sim \underline{U}_T^{k,l}}$$

A nonconforming finite element interpretation III

- $\underline{U}_T^k$ contains the DOFs for $V_T^{k,l}$ as defined by $\underline{I}_T^k$
- Functions in $V_T^{k,l}$ are not directly available, but DOFs in $\underline{U}_T^k$ are
- We define the computable projection $\Pi_T^{k+1} : V_T^{k,l} \to \mathbb{P}^{k+1}(T)$ s.t.

$$\boxed{\Pi_T^{k+1}\varphi := p_T^{k+1}\underline{I}_T^{k,l}\varphi}$$

- Moreover, for all $\varphi \in V_T^{k,l}$, the face residual rewrites

$$\boxed{\delta_{TF}^k \underline{I}_T^k \varphi = \pi_F^k(\Pi_T^{k+1}\varphi - \varphi) - \pi_T^k(\Pi_T^{k+1}\varphi - \varphi)}$$

The case $l = k + 1$

- Some simplifications hold for the case $k = l + 1$
- As a matter of fact, one has

$$\widehat{p}_T^{k,l}\underline{v}_T = v_T + (p_T^{k+1}\underline{v}_T - \pi_T^{k+1}p_T^{k+1}\underline{v}_T) = v_T$$

- Hence, the stabilization bilinear form s_T simply rewrites

$$s_T^{\mathrm{hdg}}(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} \frac{1}{h_F}(\pi_F^k(u_T - u_F), \pi_F^k(v_T - v_F))_F$$

- This corresponds to a new HDG-like method

Outline

Yesterday's course in a nutshell

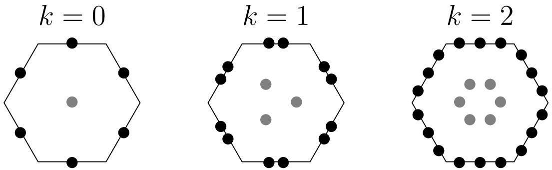


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- High-order potential reconstruction p_T^{k+1} from Neumann solves
- High-order face-based stabilisation bilinear form s_T
- Global problem from the assembly of local bilinear forms

$$\boxed{a_T(\underline{u}_T, \underline{v}_T) = (\boldsymbol{\nabla} p_T^{k+1} \underline{u}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{v}_T) + s_T(\underline{u}_T, \underline{v}_T)}$$

- Construction yielding supercloseness on general meshes

Bibliography: Linear elasticity

- On standard meshes
  - PEERS [Arnold et al., 1984]
  - Nonconforming primal* $\mathbb{P}^1$ [Brenner and Sung, 1992]
  - Nonconforming mixed [Arnold and Winther, 2003]
  - Conforming mixed polynomial [Arnold and Winther, 2002]
  - Stabilized nonconforming primal [Hansbo and Larson, 2003]
- On polyhedral meshes
  - Conforming primal VE [Beirão da Veiga, Brezzi and Marini, 2013]
  - Generalized nonconforming $\mathbb{P}^1$ [DP and Lemaire, 2015]
  - Nonconforming primal HHO [DP and Ern, 2015]

Continuous setting I

- Let $d \in \{2,3\}$. We consider the problem: Find $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ s.t.

$$\begin{aligned} -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} \qquad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \qquad \text{on } \partial\Omega \end{aligned}$$

  with real Lamé parameters $\lambda \geqslant 0$ and $\mu > 0$ and

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\nabla}_{\rm s}\boldsymbol{u} + \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{I}_d$$

- $\lambda \to +\infty$ corresponds to quasi-incompressible materials
- More general BCs can be considered with minor modifications

Continuous setting II

- Assume $\boldsymbol{f} \in L^2(\Omega)^d$ and set $\boldsymbol{U} := H_0^1(\Omega)^d$
- The weak formulation reads: Find $\boldsymbol{u} \in \boldsymbol{U}$ s.t.

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{U},$$

with bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) := 2\mu(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}) + \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u}, \boldsymbol{\nabla}\cdot\boldsymbol{v})$$

Lemma (A priori estimate)

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain. Then, there is $C_\Omega > 0$ only depending on Ω s.t.

$$\|\boldsymbol{u}\|_{H^2(\Omega)^d} + \|\lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^1(\Omega)} \leqslant C_\Omega\|\boldsymbol{f}\|_{L^2(\Omega)^d}.$$

Rigid body motions

- Applied to vector fields, the operator $\boldsymbol{\nabla}_{\mathrm{s}}$ yields strains
- Let $d = 3$. Its kernel $\mathrm{RM}(\Omega)$ contains rigid-body motions

$$\mathrm{RM}(\Omega) := \left\{ \boldsymbol{v} \in H^1(\Omega)^3 \mid \exists \boldsymbol{\alpha}, \boldsymbol{\omega} \in \mathbb{R}^3,\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\alpha} + \boldsymbol{\omega} \otimes \boldsymbol{x} \right\}$$

- We note for further use that

$$\mathbb{P}^0(\Omega)^3 \subset \mathrm{RM}(\Omega) \subset \mathbb{P}^1(\Omega)^3$$

Features

- High-order method on general polyhedral meshes
- Locking-free primal formulation
- Global SPD system
- Strongly symmetric strain and stress tensors
- Low computational cost
  - In 3d, 9 DOFs/face for the lowest-order version $k = 1$
  - Compact stencil (face neighbours)
  - Simplified data exchange w.r. to vertex DOFs

DOFs and reduction map I

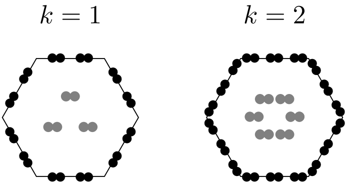


Figure: $\underline{\boldsymbol{U}}_T^k$ for $k \in \{1, 2\}$

- For $k \geqslant 1$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{\boldsymbol{U}}_T^k := \mathbb{P}^k(T)^d \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F)^d \right)$$

- The global space has single-valued interface DOFs

$$\underline{\boldsymbol{U}}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T)^d \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F)^d \right)$$

Displacement reconstruction I

- Let $T \in \mathcal{T}_h$. The local displacement reconstruction operator

$$\boldsymbol{p}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T)^d$$

is s.t., for all $\underline{\boldsymbol{v}}_T = \big(\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\boldsymbol{U}}_T^k$ and $\boldsymbol{w} \in \mathbb{P}^{k+1}(T)^d$,

$$\boxed{(\boldsymbol{\nabla}_{\rm s}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T, \boldsymbol{\nabla}_{\rm s}\boldsymbol{w})_T = -(\boldsymbol{v}_T, \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T}(\boldsymbol{v}_F, \boldsymbol{\nabla}_{\rm s}\boldsymbol{w}\boldsymbol{n}_{TF})_F}$$

- Rigid-body motions are prescribed from $\underline{\boldsymbol{v}}_T$ setting

$$\int_T \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T = \int_T \boldsymbol{v}_T, \quad \int_T \boldsymbol{\nabla}_{\rm ss}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T = \sum_{F\in\mathcal{F}_T}\int_F \frac{1}{2}(\boldsymbol{n}_{TF}\otimes\boldsymbol{v}_F - \boldsymbol{v}_F\otimes\boldsymbol{n}_{TF})$$

Displacement reconstruction II

Lemma (Approximation properties for $\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k$)

There exists $C > 0$ independent of h s.t., $\forall T \in \mathcal{T}_h$, $\forall \boldsymbol{v} \in H^{k+2}(T)^d$,

$$\begin{aligned}
&\|\boldsymbol{v} - \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v}\|_T + h_T\|\boldsymbol{\nabla}(\boldsymbol{v} - \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v})\|_T \\
&\qquad + h_T^{1/2}\|\boldsymbol{v} - \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v}\|_{\partial T} + h_T^{3/2}\|\boldsymbol{\nabla}(\boldsymbol{v} - \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v})\|_{\partial T} \\
&\qquad\qquad \leqslant Ch_T^{k+2}\|\boldsymbol{v}\|_{H^{k+2}(T)^d}.
\end{aligned}$$

Proceeding as for Poisson, one can prove the Euler equation

$$(\boldsymbol{\nabla}_{\rm s}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v} - \boldsymbol{\nabla}_{\rm s}\boldsymbol{v}, \boldsymbol{\nabla}_{\rm s}\boldsymbol{w})_T = 0 \qquad \forall \boldsymbol{w} \in \mathbb{P}^{k+1}(T)^d,$$

and the approximation properties follow since $\boldsymbol{p}_T^{k+1} \circ \underline{\boldsymbol{I}}_T^k$ is bounded.

Stabilization I

- Define, for $T \in \mathcal{T}_h$, the stabilization bilinear form s_T as

$$s_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_F^k(\widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{u}}_T - \boldsymbol{u}_F), \pi_F^k(\widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_F))_F,$$

with displacement reconstruction $\widehat{\boldsymbol{p}}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T)^d$ s.t.

$$\widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{v}}_T := \boldsymbol{v}_T + (\boldsymbol{p}_T^{k+1} \underline{\boldsymbol{v}}_T - \pi_T^k \boldsymbol{p}_T^{k+1} \underline{\boldsymbol{v}}_T)$$

- We express stability w.r. to the discrete strain norm

$$\|\underline{\boldsymbol{v}}_T\|_{\varepsilon,T}^2 := \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_F^2$$

Stabilization II

Lemma (Stability and approximation)

Let $T \in \mathcal{T}_h$ and assume $k \geqslant 1$. Then,

$$\|\boldsymbol{\nabla}_{\rm s} \boldsymbol{p}_T^{k+1} \underline{\boldsymbol{v}}_T\|_T^2 + s_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \approx \|\underline{\boldsymbol{v}}_T\|_{\varepsilon,T}^2.$$

Moreover, for all $\boldsymbol{v} \in H^{k+2}(T)^d$, we have

$$\left(\|\boldsymbol{\nabla}_{\rm s}(\boldsymbol{p}_T^{k+1} \underline{\boldsymbol{I}}_T^k \boldsymbol{v} - \boldsymbol{v})\|_T^2 + s_T(\underline{\boldsymbol{I}}_T^k \boldsymbol{v}, \underline{\boldsymbol{I}}_T^k \boldsymbol{v})\right)^{1/2} \lesssim h_T^{k+1} \|\boldsymbol{v}\|_{H^{k+2}(T)^d}.$$

Classical result for $k = 0$: Crouzeix–Raviart does not meet Korn!

Stabilization III

- For all $F \in \mathcal{F}_T$ one has, inserting $\pm\pi_F^k \widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{v}}_T$,

$$\|\boldsymbol{v}_F - \boldsymbol{v}_T\|_F \lesssim \|\pi_F^k(\boldsymbol{v}_F - \widehat{\boldsymbol{p}}_T^{k+1}\underline{\boldsymbol{v}}_T)\|_F + h_F^{-1/2}\|\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T - \pi_T^k \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T\|_T$$

- For any function $\boldsymbol{w} \in H^1(T)^d$ with rigid-body motions $\boldsymbol{w}_{\rm RM}$,

$$\|\boldsymbol{w} - \pi_T^k \boldsymbol{w}\|_T = \|(\boldsymbol{w} - \boldsymbol{w}_{\rm RM}) - \pi_T^k(\boldsymbol{w} - \boldsymbol{w}_{\rm RM})\|_T \lesssim h_T \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{w}\|_T$$

  where $\pi_T^k \boldsymbol{w}_{\rm RM} = \boldsymbol{w}_{\rm RM}$ requires $k \geqslant 1$ to have

$$\mathrm{RM}(T) \subset \mathbb{P}^k(T)^d$$

- Clearly, this reasoning breaks down for $k = 0$

Divergence reconstruction

- We define the local local discrete divergence operator

$$D_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)$$

s.t., for all $\underline{\boldsymbol{v}}_T = \big(\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\boldsymbol{U}}_T^k$ and all $q \in \mathbb{P}^k(T)$,

$$\boxed{(D_T^k \underline{\boldsymbol{v}}_T, q)_T := -(\boldsymbol{v}_T, \boldsymbol{\nabla} q)_T + \sum_{F\in\mathcal{F}_T} (\boldsymbol{v}_F \cdot \boldsymbol{n}_{TF}, q)_F}$$

- By construction, we have the following commuting diagram:

$$\begin{array}{ccc}
\boldsymbol{H}^1(T) & \xrightarrow{\boldsymbol{\nabla}\cdot} & L^2(T) \\
{\scriptstyle \underline{\boldsymbol{I}}_T^k}\big\downarrow & & \big\downarrow{\scriptstyle \pi_T^k} \\
\underline{\boldsymbol{U}}_T^k & \xrightarrow{D_T^k} & \mathbb{P}^k(T)
\end{array}$$

Discrete problem

- We define the local bilinear form a_T on $\underline{\boldsymbol{U}}_T^k \times \underline{\boldsymbol{U}}_T^k$ as

$$a_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := 2\mu(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{u}}_T, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T)_T \\ + \lambda(D_T^k\underline{\boldsymbol{u}}_T, D_T^k\underline{\boldsymbol{v}}_T) + (2\mu)s_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

- The discrete problem reads: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ s.t.

$$\boxed{a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T\in\mathcal{T}_h} a_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{f}, \boldsymbol{v}_T)_T \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k}$$

with $\underline{\boldsymbol{U}}_{h,0}^k$ incorporating boundary conditions

Convergence I

Theorem (Energy-norm error estimate)

Assume $k \geqslant 1$ *and the additional regularity* $\boldsymbol{u} \in H^{k+2}(\Omega)^d$. *Then, there exists* $C > 0$ *independent of* h, μ, *and* λ *s.t.*

$$(2\mu)^{1/2} \|\underline{\boldsymbol{u}}_h - \widehat{\underline{\boldsymbol{u}}}_h\|_{a,h} \leqslant C h^{k+1} B(\boldsymbol{u}, k),$$

with

$$B(\boldsymbol{u}, k) := (2\mu)\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d} + \|\lambda \boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^{k+1}(\Omega)}.$$

Convergence II

- Locking-free if $B(\boldsymbol{u}, k)$ is bounded uniformly in λ
- For $d = 2$ and Ω convex, one has using Cattabriga's regularity

$$B(\boldsymbol{u}, 0) = \|\boldsymbol{u}\|_{H^2(\Omega)^d} + \|\lambda \boldsymbol{\nabla}\cdot \boldsymbol{u}\|_{H^1(\Omega)} \leqslant C_\mu \|\boldsymbol{f}\|$$

- More generally, for $k \geqslant 1$, we need the regularity shift

$$\boxed{B(\boldsymbol{u}, k) \leqslant C_\mu \|\boldsymbol{f}\|_{H^k(\Omega)^d}}$$

- **Key point: commuting property for D_T^k**

Convergence III

Theorem (L^2-error estimate for the displacement)

Further assuming elliptic regularity for Ω, it holds with $C > 0$ independent of λ and h,

$$\|\boldsymbol{u}_h - \pi_h^k \boldsymbol{u}\| \leqslant C h^{k+2} B(\boldsymbol{u}, k),$$

with $\boldsymbol{u}_h$ s.t. $\boldsymbol{u}_{h|T} = \boldsymbol{u}_T$ for all $T \in \mathcal{T}_h$.

Numerical example I

- We consider the following exact solution:

$$\boldsymbol{u}(\boldsymbol{x}) = \big(\sin(\pi x_1)\sin(\pi x_2) + (2\lambda)^{-1}x_1, \cos(\pi x_1)\cos(\pi x_2) + (2\lambda)^{-1}x_2\big)$$

- The solution u has vanishing divergence in the limit $\lambda \to +\infty$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{\lambda}$$

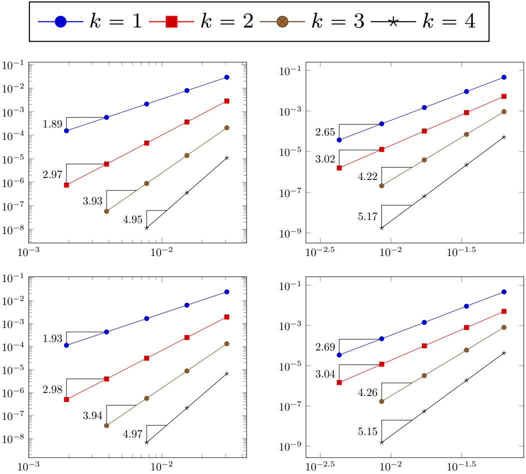


Figure: Energy error with $\lambda = 1$ (above) and $\lambda = 1000$ (below) vs. h for the triangular (left) and hexagonal (right) mesh families

Numerical example III

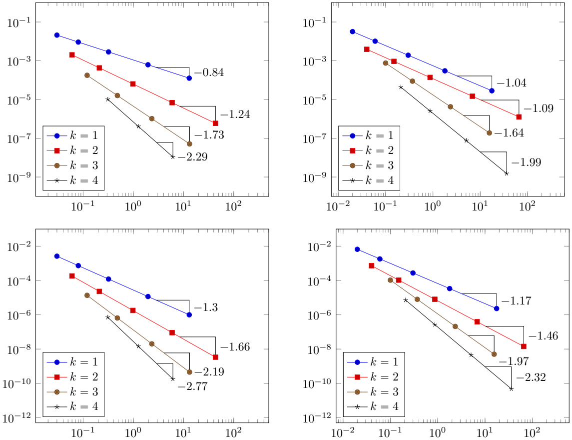


Figure: Energy (above) and displacement (below) error vs. τ_{tot} (s) for the triangular and hexagonal mesh families

Teaser: Poromechanics

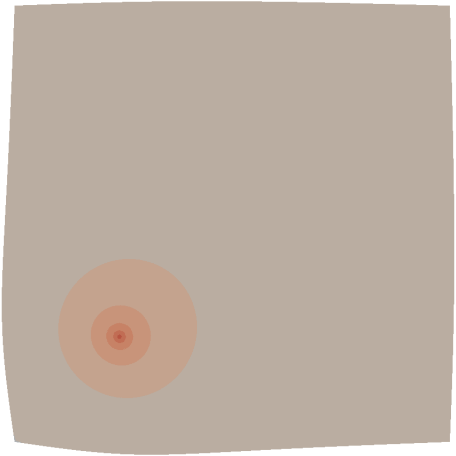


Figure: HHO + dG applied to poro-elasticity, [Boffi, Botti, DP, 2016]

Outline

Model problem I

- Let $p \in (1, +\infty)$ and $f \in L^{p'}(\Omega)$ with $p' := \frac{p}{p-1}$
- We consider the Leray–Lions problem: Find $u \in W_0^{1,p}(\Omega)$ s.t.

$$\boxed{A(u,v) := \int_\Omega \mathbf{a}(\boldsymbol{x}, \boldsymbol{\nabla} u(\boldsymbol{x})) \cdot \boldsymbol{\nabla} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega f v \quad \forall v \in W_0^{1,p}(\Omega)}$$

- A typical example is the p-Laplacian: For $p \in (1, +\infty)$,

$$\mathbf{a}(\boldsymbol{x}, \boldsymbol{\nabla} u) = |\boldsymbol{\nabla} u|^{p-2} \boldsymbol{\nabla} u$$

- Applications to glaciology, turbulent porous media flow, airfoil design
- **Perfect playground for discrete functional analysis tools**

Model problem II

Assumption (Leray–Lions operator/v1)

For a fixed index $p \in (1, +\infty)$, $f \in L^{p'}(\Omega)$ *and* $\mathbf{a}$ *satisfies*

- *Growth.* $\mathbf{a}(\cdot, \mathbf{0}) \in L^{p'}(\Omega)$ *and there is* $\beta_{\mathbf{a}} > 0$ *s.t.*

$$|\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \mathbf{0})| \leqslant \beta_{\mathbf{a}} |\boldsymbol{\xi}|^{p-1} \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d.$$

- *Monotonicity. For a.e.* $\boldsymbol{x} \in \Omega$, *for all* $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$[\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})] \cdot [\boldsymbol{\xi} - \boldsymbol{\eta}] \geqslant 0.$$

- *Coercivity. There is* $\lambda_{\mathbf{a}} > 0$ *s.t.*

$$\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geqslant \lambda_{\mathbf{a}} |\boldsymbol{\xi}|^p \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d.$$

A dependence on u can also be included in the analysis

Discretization of Leray–Lions type problems

- Conforming Finite Elements
  - p-Laplacian, a priori [Barrett and Liu, 1994]
  - A priori and a posteriori [Glowinski and Rappaz, 2003]
- Nonconforming FE for the p-Laplacian [Liu and Yan, 2001]
- Mixed Finite Volumes for Leray–Lions [Droniou, 2006]
- Discrete Duality FV, $d = 2$ [Andreianov, Boyer, Hubert, 2004–07]
- Mimetic FD, quasi linear [Antonietti, Bigoni, Verani, 2014]
- Hybrid High-Order (HHO) for Leray–Lions, $p \in (1, +\infty)$
  - Convergence by compactness [DP & Droniou, Math. Comp., 2016]
  - Error estimates [DP & Droniou, submitted, 2016]

Key ideas

- DOFs: polynomials of degree $k \geqslant 0$ at elements and faces
- Differential operators reconstructions taylored to the problem:

$$\boxed{A_{|T}(u,v) \approx \int_T \mathbf{a}(\boldsymbol{x}, G_T^k \underline{u}_T(\boldsymbol{x})) \cdot G_T^k \underline{v}_T(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \text{stab.}}$$

  with
  - gradient reconstruction G_T^k from local solves
  - stabilisation using face-based penalty and high-order potential reconstruction
- General meshes in any $d \geqslant 1$ and arbitrary polynomial degrees

Operator reconstructions I

- We define the gradient reconstruction $G_T^k : \underline{U}_T^k \mapsto \mathbb{P}^k(T)^d$ s.t.

$$(G_T^k \underline{v}_T, \boldsymbol{\phi})_T = -(v_T, \boldsymbol{\nabla}\cdot\boldsymbol{\phi})_T + \sum_{F\in\mathcal{F}_T} (v_F, \boldsymbol{\phi}\cdot\boldsymbol{n}_{TF})_F \quad \forall \boldsymbol{\phi} \in \mathbb{P}^k(T)^d$$

- Recalling the definition of $\underline{I}_T^k$, it holds for all $v \in W^{1,1}(T)$,

$$(G_T^k \underline{I}_T^k v, \boldsymbol{\phi})_T = -(\cancel{\pi_T^k} v, \boldsymbol{\nabla}\cdot\boldsymbol{\phi})_T + \sum_{F\in\mathcal{F}_T} (\cancel{\pi_F^k} v, \boldsymbol{\phi}\cdot\boldsymbol{n}_{TF})_F = (\boldsymbol{\nabla} v, \boldsymbol{\phi})_T,$$

i.e., by definition of π_T^k,

$$\boxed{G_T^k \underline{I}_T^k v = \pi_T^k(\boldsymbol{\nabla} v)}$$

- As a result, $(G_T^k \circ \underline{I}_T^k)$ has optimal $W^{s,p}$-approximation properties

Operator reconstructions II

- We define the potential reconstruction $p_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$ s.t.

$$(\nabla p_T^{k+1} \underline{v}_T - G_T^k \underline{v}_T, \nabla w)_T = 0 \quad \forall w \in \mathbb{P}^{k+1}(T)$$

  and $(p_T^{k+1} \underline{v}_T - v, 1)_T = 0$

- Recalling the definition of G_T^k and $\underline{I}_T^k$, it holds for all $v \in W^{1,1}(T)$,

$$(\nabla p_T^{k+1} \underline{I}_T^k v, \nabla w)_T = -(\cancel{\pi_T^k} v, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (\cancel{\pi_F^k} v, \nabla w \cdot \boldsymbol{n}_{TF})_F = (\nabla v, \nabla w)_T,$$

  i.e., by definition of ϖ_T^{k+1},

$$\boxed{p_T^{k+1} \underline{I}_T^k v = \varpi_T^{k+1} v}$$

- As a result, $(p_T^{k+1} \circ \underline{I}_T^k)$ has optimal $W^{s,p}$-approximation properties

Global problem I

- For all $T \in \mathcal{T}_h$, we define the local function $\mathrm{A}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ s.t.

$$\boxed{\mathrm{A}_T(\underline{u}_T, \underline{v}_T) := \int_T \mathbf{a}(\boldsymbol{x}, G_T^k \underline{u}_T(\boldsymbol{x})) \cdot G_T^k \underline{v}_T(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + s_T(\underline{u}_T, \underline{v}_T)}$$

- The stabilisation term $s_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ is s.t.

$$s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{1-p} \int_F \left| \delta_{TF}^k \underline{u}_T \right|^{p-2} \delta_{TF}^k \underline{u}_T \ \delta_{TF}^k \underline{v}_T,$$

with face-based residual operator $\delta_{TF}^k : \underline{U}_T^k \to \mathbb{P}^k(F)$ s.t.

$$\delta_{TF}^k \underline{v}_T := \pi_F^k \left(v_F - p_T^{k+1} \underline{v}_T - \pi_T^k (v_T - p_T^{k+1} \underline{v}_T) \right)$$

- Polynomial consistency: $\delta_{TF}^k \underline{I}_T^k v = 0$ for all $v \in \mathbb{P}^{k+1}(T)$

Global problem II

- Define the following global space with single-valued interface DOFs:

$$\underline{U}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right)$$

- A global function $\mathrm{A}_h : \underline{U}_h^k \times \underline{U}_h^k \to \mathbb{R}$ is assembled element-wise:

$$\mathrm{A}_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{A}_T(\underline{u}_T, \underline{v}_T)$$

- We seek $\underline{u}_h \in \underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \mid v_F = 0 \ \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$ s.t.

$$\boxed{\mathrm{A}_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

with $v_{h|T} = v_T$ for all $T \in \mathcal{T}_h$

Global problem III

- Define on $\underline{U}_h^k$ the $W^{1,p}$-like seminorm (this is a norm on $\underline{U}_{h,0}^k$)

$$\|\underline{v}_h\|_{1,p,h}^p := \sum_{T\in\mathcal{T}_h} \left(\|\nabla v_T\|_{L^p(T)^d}^p + \sum_{F\in\mathcal{F}_T} h_F^{1-p} \|v_F - v_T\|_{L^p(F)}^p \right)$$

- We have coercivity for A_h: For all $\underline{v}_h \in \underline{U}_h^k$,

$$\boxed{\|\underline{v}_h\|_{1,p,h}^p \lesssim \mathrm{A}_h(\underline{v}_h, \underline{v}_h)}$$

- Existence for $\underline{u}_h$ follows (cf. [Deimling, 1985]) with a priori estimate

$$\|\underline{u}_h\|_{1,p,h} \leqslant C \|f\|_{L^{p'}(\Omega)}^{\frac{1}{p-1}}$$

Convergence to minimal regularity solutions I

Theorem (Convergence)

Up to a subsequence as $h \to 0$, with $p^* = \frac{dp}{d-p}$ if $p < d$, $+\infty$ otherwise,

- $u_h \to u$ *and* $p_h^{k+1}\underline{u}_h \to u$ *strongly in $L^q(\Omega)$ for all $q < p^*$,*
- $G_h^k \underline{u}_h \to \boldsymbol{\nabla} u$ *weakly in $L^p(\Omega)^d$.*

Additionally, if $\mathbf{a}$ is strictly monotone,

- $G_h^k \underline{u}_h \to \boldsymbol{\nabla} u$ *strongly in $L^p(\Omega)^d$.*

In this case, both u and $\underline{u}_h$ are unique and the whole sequence converges.

Convergence to minimal regularity solutions II

Key discrete functional analysis results on hybrid polynomial spaces:

Lemma (Discrete Sobolev embeddings)

Let $1 \leqslant q \leqslant p^*$ *if* $1 \leqslant p < d$ *and* $1 \leqslant q < +\infty$ *if* $p \geqslant d$. *Then, there exists* C *only depending on* Ω, ϱ, k, q *and* p *s.t. for all* $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|v_h\|_{L^q(\Omega)} \leqslant C\|\underline{v}_h\|_{1,p,h}.$$

Lemma (Discrete compactness)

Let $(\underline{v}_h)_{h\in\mathcal{H}}$ *be s.t.* $\|\underline{v}_h\|_{1,p,h} \leqslant C$ *for a fixed* $C \in \mathbb{R}$. *Then, there exists* $v \in W_0^{1,p}(\Omega)$ *s.t., up to a subsequence as* $h \to 0$,

- $v_h \to v$ *and* $p_h^{k+1}\underline{v}_h \to v$ *strongly in* $L^q(\Omega)$ *for all* $q < p^*$,
- $G_h^k\underline{v}_h \to \boldsymbol{\nabla} v$ *weakly in* $L^p(\Omega)^d$.

Error estimates I

Assumption (Leray–Lions operator/v2)

For $p \in (1, +\infty)$, $\mathbf{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *satisfies*

- *Growth. Same as before*
- *Continuity. There is* $\gamma_{\mathbf{a}} > 0$ *s.t. for a.e.* $\boldsymbol{x} \in \Omega$, $\forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$

$$|\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| \leqslant \gamma_{\mathbf{a}} |\boldsymbol{\xi} - \boldsymbol{\eta}| (|\boldsymbol{\xi}|^{p-2} + |\boldsymbol{\eta}|^{p-2}).$$

- *Monotonicity. There is* $\zeta_{\mathbf{a}} > 0$ *s.t. for a.e.* $\boldsymbol{x} \in \Omega$, $\forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$,

$$[\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})] \cdot [\boldsymbol{\xi} - \boldsymbol{\eta}] \geqslant \zeta_{\mathbf{a}} |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 (|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p-2}.$$

- *Coercivity. Same as before*

Error estimates II

Theorem (Error estimate)

Assume $u \in W^{k+2,p}(\mathcal{T}_h)$, $\mathbf{a}(\cdot, \boldsymbol{\nabla} u) \in W^{k+1,p'}(\mathcal{T}_h)^d$, *and let, if* $p \geqslant 2$,

$$\mathrm{E}_h(u) := h^{k+1}|u|_{W^{k+2,p}(\mathcal{T}_h)} + h^{\frac{k+1}{p-1}}\left(|u|^{\frac{1}{p-1}}_{W^{k+2,p}(\mathcal{T}_h)} + |\mathbf{a}(\cdot, \boldsymbol{\nabla} u)|^{\frac{1}{p-1}}_{W^{k+1,p'}(\mathcal{T}_h)}\right),$$

while, if $p < 2$,

$$\mathrm{E}_h(u) := h^{(k+1)(p-1)}|u|^{p-1}_{W^{k+2,p}(\mathcal{T}_h)} + h^{k+1}|\mathbf{a}(\cdot, \boldsymbol{\nabla} u)|_{W^{k+1,p'}(\mathcal{T}_h)}.$$

Then, it holds,

$$\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h} \lesssim \mathrm{E}_h(u) = \begin{cases} \mathcal{O}(h^{\frac{k+1}{p-1}}) & \text{if } p \geqslant 2, \\ \mathcal{O}(h^{(k+1)(p-1)}) & \text{if } p < 2. \end{cases}$$

Results coherent with [Liu and Yan, 2001] (Crouzeix–Raviart)

Numerical example I

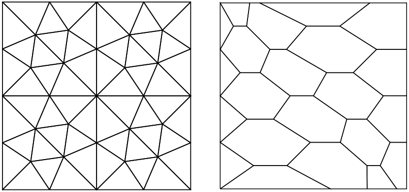


Figure: Triangular and (predominantly) hexagonal meshes

- We consider the following exact solution

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$$

- We solve the corresponding Dirichlet problem for $p \in \{2, 3, 4\}$

Numerical example II

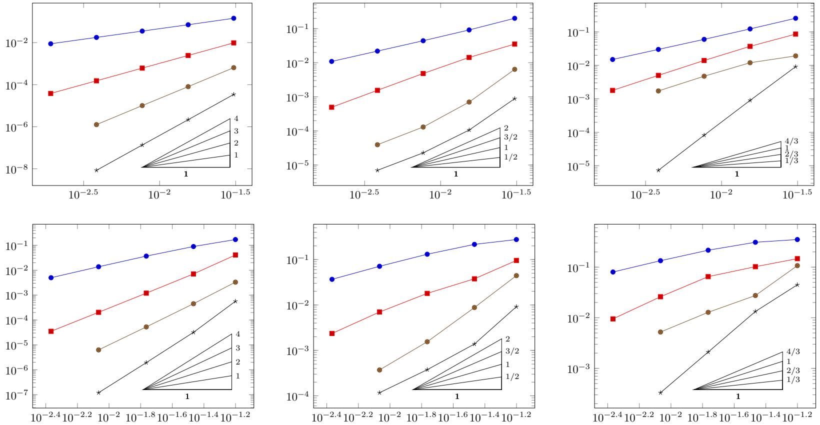


Figure: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ vs. h for $p = 2, 3, 4$ (left to right) for the triangular (above) and hexagonal (below) mesh families

Variations I

- Following [Cockburn, DP, Ern, 2016], one could replace $\underline{U}_T^k$ with

$$\underline{U}_T^{l,k} := \mathbb{P}^l(T) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right), \quad l \in \{k-1, k, k+1\}$$

- G_T^k and p_T^{k+1} remain formally the same (only their domain changes)
- The boundary residual operator, on the other hand, becomes

$$\delta_{TF}^{l,k} \underline{v}_T := \pi_F^k \left(v_F - p_T^{k+1} \underline{v}_T - \pi_T^l (v_T - p_T^{k+1} \underline{v}_T) \right)$$

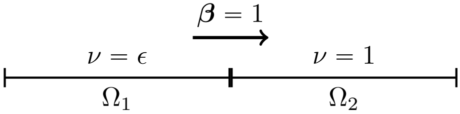
Variations II

- Convergence and error estimates as for the original HHO method
- $l = k-1$ yields a HOM/nc-VEM-type scheme
  - Linear diffusion [Ayuso de Dios, Lipnikov, Manzini, 2016]
- $l = k$ corresponds to the original HHO method
- $l = k+1$ yields a Lehrenfeld–Schöberl-type HDG method
  - Linear diffusion [Lehrenfeld, 2010]
- $k = 0$ and $l = k - 1$ on simplices yields the Crouzeix–Raviart element
- **The globally-coupled unknowns coincide in all the cases!**

Outline

Continuous setting I

- Consider the 1d problem, cf. [Gastaldi and Quarteroni, 1989]:



- As $\epsilon \to 0^+$, a boundary layer develops at $x = 1/2$
- When $\epsilon = 0$, it turns into a jump discontinuity

Continuous setting II

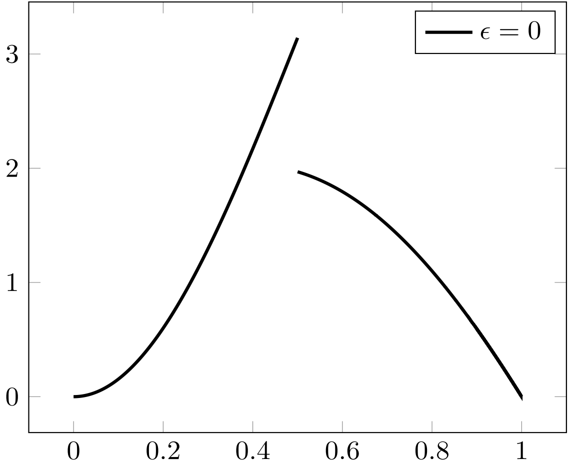


Figure: Solutions for different values of ϵ

Continuous setting III

- Let us now consider $d \geqslant 1$ with diffusion coefficient $\nu : \Omega \to \mathbb{R}^+$
- Let $P_\Omega := \{\Omega_i\}$ denote a polyhedral partition of Ω
- We assume $\nu \in \mathbb{P}^0(P_\Omega)$ and s.t.

$$\nu \geqslant \underline{\nu} \geqslant 0 \text{ a.e. in } \Omega$$

- **ν can vanish in some subdomain Ω_i!**
- Full diffusion tensors could also be considered

Continuous setting IV

- We assume that both advection and reaction are present
- The advective velocity $\boldsymbol{\beta} : \Omega \to \mathbb{R}^d$ is assumed s.t.

$$\boldsymbol{\beta} \in \mathrm{Lip}(\Omega)^d$$

- For the sake of simplicity, we also take $\boldsymbol{\beta}$ incompressible,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\beta} \equiv 0$$

- For the reaction coefficient $\mu : \Omega \to \mathbb{R}$, we assume

$$\mu \in L^\infty(\Omega) \text{ and } \mu \geqslant \mu_0 > 0 \text{ a.e. in } \Omega$$

Continuous setting V

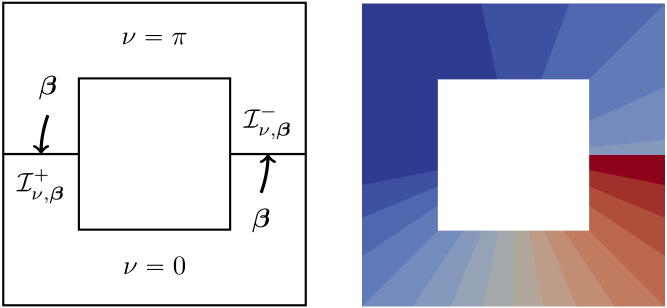


Figure: Two-dimensional example from [DP, Ern and Guermond, 2008]

Continuous setting VI

- We define $\mathcal{I}_\nu$ as the set of points in Ω in $\partial\Omega_i \cap \partial\Omega_j$ s.t.

$$\nu_{|\Omega_i} > \nu_{|\Omega_j} = 0$$

- Boundary conditions can only be enforced on

$$\Gamma_{\nu,\boldsymbol{\beta}} := \{\boldsymbol{x} \in \partial\Omega \mid \nu > 0 \text{ or } \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\}$$

- For well-posedness, transmission conditions are required on

$$\mathcal{I}^{\pm}_{\nu,\boldsymbol{\beta}} := \{\boldsymbol{x} \in \mathcal{I}_\nu \mid \pm(\boldsymbol{\beta}\cdot\boldsymbol{n}_{\Omega_i})(\boldsymbol{x}) > 0\}$$

Continuous setting VII

- Let $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_{\nu,\boldsymbol{\beta}})$. We seek $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} \boldsymbol{\nabla}\cdot(-\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u) + \mu u = f \quad & \text{in } \Omega\backslash\mathcal{I}_\nu, \\ u = g \quad & \text{on } \Gamma_{\nu,\boldsymbol{\beta}} \end{aligned}$$

- The transmission conditions that warrant well-posedness are

$$\begin{aligned} [-\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u]\cdot\boldsymbol{n}_{\Omega_i} = 0 \quad & \text{on } \mathcal{I}_\nu, \\ [u] = 0 \quad & \text{on } \mathcal{I}^+_{\nu,\boldsymbol{\beta}} \end{aligned}$$

- **The solution u can jump across $\mathcal{I}^-_{\nu,\boldsymbol{\beta}}$!**
- For a weak formulation, cf. [DP, Ern and Guermond, 2008]

Key ideas

- Discrete advective derivative satisfying a discrete IBP formula
- Upwind stabilization using element and face unknowns
  - Independent control for the advective part
  - Consistency also on $\mathcal{I}^-_{\nu,\boldsymbol{\beta}}$, where u jumps
- Weakly enforced boundary conditions
  - Extension of Nitsche's ideas to HHO
  - Automatic detection of $\Gamma_{\nu,\boldsymbol{\beta}}$

Features

- Polyhedral meshes and arbitrary approximation order $k \geqslant 0$
- Method valid for the full range of local Peclet numbers
- Analysis capturing the variation in the convergence rate
- **No need to duplicate interface unknowns on $\mathcal{I}_{\nu,\boldsymbol{\beta}}^{-}$ (!)**

Advective derivative I

- The discrete advective derivative

$$G^k_{\boldsymbol{\beta},T} : \underline{U}^k_T \to \mathbb{P}^k(T)$$

is s.t., for all $\underline{v}_T \in \underline{U}^k_T$ and all $w \in \mathbb{P}^k(T)$,

$$\boxed{(G^k_{\boldsymbol{\beta},T}\underline{v}_T, w)_T = -(v_T, \boldsymbol{\beta}\cdot\boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})v_F, w)_F}$$

- For stability, we need a discrete IBP formula mimicking

$$(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} w, v)_\Omega + (w, \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_\Omega = ((\boldsymbol{\beta}\cdot\boldsymbol{n})w, v)_{\partial\Omega}$$

Advective derivative II

Lemma (Discrete IBP formula)

For all $\underline{w}_h, \underline{v}_h \in \underline{U}_h^k$ *it holds*

$$\sum_{T\in\mathcal{T}_h} \left\{ (G^k_{\boldsymbol{\beta},T}\underline{w}_T, v_T)_T + (w_T, G^k_{\boldsymbol{\beta},T}\underline{v}_T)_T \right\} = \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_F)w_F, v_F)_F$$
$$- \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})(w_F - w_T), v_F - v_T)_F.$$

To control the term in red, we use element-face upwinding

Advection-reaction I

- For all $T \in \mathcal{T}_h$, we let

$$a_{\boldsymbol{\beta},\mu,T}(\underline{w}_T, \underline{v}_T) := -(w_T, G^k_{\boldsymbol{\beta},T}\underline{v}_T)_T + \mu(w_T, v_T)_T + s^-_{\boldsymbol{\beta},T}(\underline{w}_T, \underline{v}_T)$$

with local upwind stabilization bilinear form s.t.

$$s^-_{\boldsymbol{\beta},T}(\underline{w}_T, \underline{v}_T) := \sum_{F\in\mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^-(w_F - w_T), v_F - v_T)_F,$$

- Including weak enforcement of BCs, we let

$$\boxed{a_{\boldsymbol{\beta},\mu,h}(\underline{w}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} a_{\boldsymbol{\beta},\mu,T}(\underline{w}_h, \underline{v}_h) + \sum_{F\in\mathcal{F}_h^{\rm b}} ((\boldsymbol{\beta}\cdot\boldsymbol{n})^+ w_F, v_F)_F}$$

Advection-reaction II

Lemma (Stability of $a_{\boldsymbol{\beta},\mu,h}$)

Let $\eta := \min_{T\in\mathcal{T}_h}(1, \tau_{\mathrm{ref},T}\mu)$, $\tau_{\mathrm{ref},T} := \{\max(\|\mu\|_{L^\infty(T)}, L_{\boldsymbol{\beta},T})\}^{-1}$. *Then,*

$$\boxed{\forall \underline{v}_h \in \underline{U}_h^k, \qquad \eta\|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 \leqslant a_{\boldsymbol{\beta},\mu,h}(\underline{v}_h, \underline{v}_h),}$$

with global advection-reaction norm

$$\|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{\boldsymbol{\beta},\mu,T}^2 + \frac{1}{2}\sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2} v_F\|_F^2,$$

and, for all $T \in \mathcal{T}_h$,

$$\|\underline{v}_T\|_{\boldsymbol{\beta},\mu,T}^2 := \frac{1}{2}\sum_{F\in\mathcal{F}_T} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2}(v_F - v_T)\|_F^2 + \tau_{\mathrm{ref},T}^{-1}\|v_T\|_T^2.$$

Weakly enforced BCs for diffusion I

- We modify the diffusion bilinear form to weakly enforce BCs
- The new bilinear form $a_{\nu,h}$ reads (after setting $\boldsymbol{\nu} = \nu \boldsymbol{I}_d$),

$$\boxed{a_{\nu,h}(\underline{w}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_{\nu,T}(\underline{w}_T, \underline{v}_T) + s_{\partial,\nu,h}(\underline{w}_h, \underline{v}_h)}$$

with, for a user-defined penalty parameter $\varsigma > 0$,

$$s_{\partial,\nu,h}(\underline{w}_h, \underline{v}_h) := \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} \left\{ -(\nu_F \boldsymbol{\nabla} p_T^{k+1} \underline{w}_T \cdot \boldsymbol{n}_{TF}, v_F)_F + \frac{\varsigma \nu_F}{h_F} (w_F, v_F)_F \right\}$$

- Symmetric and skew-symmetric variations could also be devised

Weakly enforced BCs for diffusion II

Lemma (Stability of $a_{\nu,h}$)

Assuming that $\varsigma > C_{\rm tr}^2 N_\partial/4$ *it holds, for all* $\underline{v}_h \in \underline{U}_h^k$,

$$a_{\nu,h}(\underline{v}_h, \underline{v}_h) =: \|\underline{v}_h\|_{\nu,h}^2 \simeq \sum_{T\in\mathcal{T}_h} \nu_T \|\underline{v}_T\|_{1,T}^2 + \sum_{F\in\mathcal{F}_h^{\rm b}} \frac{\nu_F}{h_F} \|v_F\|_F^2.$$

Discrete problem I

- Let, accounting for boundary conditions,

$$l_h(\underline{v}_h) := \sum_{T\in\mathcal{T}_h}(f, v_T)_T + \sum_{F\in\mathcal{F}_h^{\mathrm{b}}}\left\{((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^- g, v_F)_F + \frac{\nu_F\varsigma}{h_F}(g, v_F)_F\right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_h^k$ s.t., $\forall \underline{v}_h \in \underline{U}_h^k$,

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := a_{\nu,h}(\underline{u}_h, \underline{v}_h) + a_{\boldsymbol{\beta},\mu,h}(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h)}$$

Discrete problem II

Lemma (Stability of a_h)

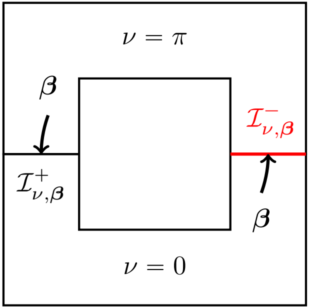
There is $\gamma_\varrho > 0$ independent of h, ν, $\boldsymbol{\beta}$ and μ s.t.

$$\forall \underline{w}_h \in \underline{U}_h^k, \qquad \|\underline{w}_h\|_{\sharp,h} \leqslant \gamma_\varrho \zeta^{-1} \sup_{\underline{v}_h \in \underline{U}_h^k \setminus \{0\}} \frac{a_h(\underline{w}_h, \underline{v}_h)}{\|\underline{v}_h\|_{\sharp,h}},$$

with $\zeta := \tau_{\mathrm{ref},T}\mu$ and stability norm

$$\|\underline{v}_h\|_{\sharp,h}^2 := \|\underline{v}_h\|_{\nu,h}^2 + \|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 + \sum_{T \in \mathcal{T}_h} h_T \beta_{\mathrm{ref},T}^{-1} \|G_{\boldsymbol{\beta},T}^k \underline{v}_h\|_T^2$$

A modified reduction map



- Let $F \in \mathcal{F}_h^{\mathrm{i}}$ be such that $F \subset \mathcal{I}^{-}_{\nu,\boldsymbol{\beta}}$
- The trace of u is two-valued on F
- We interpolate the face unknown **from the diffusive side**

Convergence I

Theorem (Error estimate)

Assume that, for all $T \in \mathcal{T}_h$, $u \in H^{k+2}(T)$ and

$$h_T L_{\boldsymbol{\beta},T} \leqslant \beta_{\mathrm{ref},T} \quad \textit{and} \quad h_T \mu \leqslant \beta_{\mathrm{ref},T},$$

Then, there is $C > 0$ independent of h, ν, $\boldsymbol{\beta}$, and μ s.t.

$$\| \underline{I}_h^k u - \underline{u}_h \|_{\sharp,h}^2 \leqslant C \sum_{T \in \mathcal{T}_h} \left\{ B_T^{\mathrm{d}}(u,k) h_T^{2(k+1)} + B_T^{\mathrm{a}}(u,k) \min(1, \mathrm{Pe}_T) h_T^{2(k+\frac{1}{2})} \right\},$$

with Pe_T denoting the local Péclet number.

Convergence II

- This estimate holds across the entire range for Pe_T
- In the diffusion-dominated regime $\mathrm{Pe}_T \leqslant h_T$, we have

$$\|\underline{I}_h^k u - \underline{u}_h\|_{\sharp,h} = \mathcal{O}(h^{k+1})$$

- In the advection-dominated regime $\mathrm{Pe}_T \geqslant 1$, we have

$$\|\underline{I}_h^k u - \underline{u}_h\|_{\sharp,h} = \mathcal{O}(h^{k+1/2})$$

- In between, we have intermediate orders of convergence

Numerical example I

- Let $\Omega = (-1,1)^2 \backslash [-0.5, 0.5]^2$ and set

$$\boldsymbol{\nu}(\theta, r) = \begin{cases} \pi & \text{if } 0 < \theta < \pi, \\ 0 & \text{if } \pi < \theta < 2\pi, \end{cases} \quad \boldsymbol{\beta}(\theta, r) = \frac{\boldsymbol{e}_\theta}{r}, \quad \mu = 1 \cdot 10^{-6}$$

- We consider the exact solution

$$u(\theta, r) = \begin{cases} (\theta - \pi)^2 & \text{if } 0 < \theta < \pi \\ 3\pi(\theta - \pi) & \text{if } \pi < \theta < 2\pi \end{cases}$$

Numerical example II

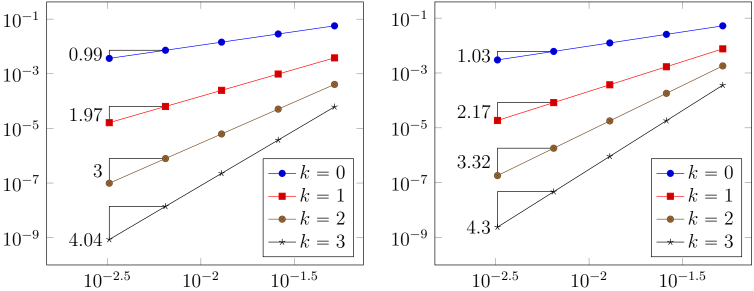


Figure: Energy (left) and L^2-norm (right) of the error vs. h

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