An introduction to Discrete de Rham methods

A polytopal exterior calculus framework

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Outline

References for this presentation

- Finite Element Exterior Calculus [Arnold, Falk, Winther, 2006, Arnold, 2018]
- Finite Element Systems [Christiansen and Gillette, 2016]
- Introduction of DDR [DP, Droniou, Rapetti, 2020]
- DDR sequence based on Koszul complements [DP and Droniou, 2023]
- Application to magnetostatics [DP and Droniou, 2021]
- Polytopal Exterior Calculus [Bonaldi, DP, Droniou, Hu, 2023]
- C++ open-source implementation available in HArDCore3D

Outline

1. Motivation
2. The Discrete de Rham construction
3. Application to magnetostatics

Two model problems: Stokes

- With $\Omega \subset \mathbb{R}^3$ connected, $\nu > 0$, and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads: Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(local equilibrium)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

Two model problems: Magnetostatics

- For $\mu > 0$ and $\boldsymbol{J} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl};\Omega)$, the magnetostatics problem reads: Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\mu \boldsymbol{H} - \mathbf{curl}\, \boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)}\\
\mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)}\\
\operatorname{div} \boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)}\\
\boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\operatorname{div};\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
\int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \operatorname{div} \boldsymbol{A} \operatorname{div} \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\Omega)
\end{aligned}
$$

A unified view

- The above problems are mixed formulations involving two fields
- They can be recast into the abstract setting: Find $(u, p) \in V \times Q$ s.t.

$$\begin{aligned} Au + B^\top p &= f \quad \text{in } V', \\ -Bu + Cp &= g \quad \text{in } Q' \end{aligned}$$

- Well-posedness for this problem holds under [Brezzi and Fortin, 1991]:
  - The coercivity of A in $\operatorname{Ker} B$ and of C in $\operatorname{Ker} B^\top$
  - An inf-sup condition for B
- **Similar properties underlie the stability of numerical approximations**

A unified tool for well-posedness: The de Rham complex

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- We have key properties depending on the topology of Ω:

$$\mathrm{Im}\,\mathbf{grad} \subset \mathrm{Ker}\,\mathbf{curl},$$
$$\mathrm{Im}\,\mathbf{curl} \subset \mathrm{Ker}\,\mathrm{div}$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(magnetostatics)}$$

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- We have key properties depending on the topology of Ω:

$$\begin{aligned}
\text{no "tunnels" crossing } \Omega \ (b_1 = 0) &\implies \mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl} \quad \text{(Stokes)}\\
\text{no "voids" contained in } \Omega \ (b_2 = 0) &\implies \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div} \quad \text{(magnetostatics)}\\
\Omega \subset \mathbb{R}^3 \ (b_3 = 0) &\implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(magnetostatics)}
\end{aligned}$$

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- We have key properties depending on the topology of Ω:

$$\text{no "tunnels" crossing } \Omega \ (b_1 = 0) \implies \mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl} \quad \text{(Stokes)}$$
$$\text{no "voids" contained in } \Omega \ (b_2 = 0) \implies \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div} \quad \text{(magnetostatics)}$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(magnetostatics)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\mathbf{curl}/\mathrm{Im}\,\mathbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}/\mathrm{Im}\,\mathbf{curl}$$

- **Emulating these properties is key for stable discretizations**

Generalization through differential forms

- Denote by Ω a connected domain of $\mathbb{R}^n$, $n \geq 1$
- The de Rham complex can be generalized using differential forms:

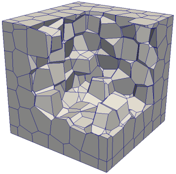
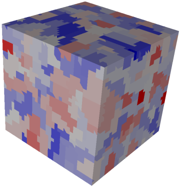
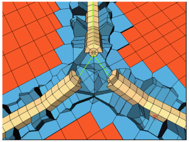
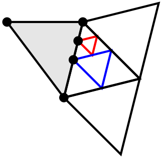
$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the following links are established through vector proxies:

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

Outline

Domain and polytopal mesh

- Assume $\Omega \subset \mathbb{R}^n$ polytopal (polygon if $n = 2$, polyhedron if $n = 3, \ldots$)
- We consider a polytopal mesh $\mathcal{M}_h$ containing all (flat) d-cells, $0 \le d \le n$
- d-cells in $\mathcal{M}_h$ are collected in $\Delta_d(\mathcal{M}_h)$, so that, when $n = 3$,
  - $\Delta_0(\mathcal{M}_h) = \mathcal{V}_h$ is the set of vertices
  - $\Delta_1(\mathcal{M}_h) = \mathcal{E}_h$ is the set of edges
  - $\Delta_2(\mathcal{M}_h) = \mathcal{F}_h$ is the set of faces
  - $\Delta_3(\mathcal{M}_h) = \mathcal{T}_h$ is the set of elements

General ideas

- Discrete spaces with polynomial components attached to mesh entities
- We recursively construct on d-cells, $d = k, \ldots, n$:
  - A discrete potential (playing the role of a k-form inside f)

$$P^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r \Lambda^k(f)$$

  - If $d \geq k+1$, a discrete exterior derivative

$$\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r \Lambda^{k+1}(f)$$

- Reconstructions mimic the Stokes formula: $\forall (\omega, \mu) \in \Lambda^\ell(f) \times \Lambda^{n-\ell-1}(f)$,

$$\int_f \mathrm{d}^\ell \omega \wedge \mu = (-1)^{\ell+1} \int_f \omega \wedge \mathrm{d}^{n-\ell-1} \mu + \int_{\partial f} \mathrm{tr}_{\partial f}\, \omega \wedge \mathrm{tr}_{\partial f}\, \mu$$

Trimmed polynomial spaces

- Let $f \in \Delta_d(\mathcal{M}_h)$, $d \in [1, n]$, fix $\boldsymbol{x}_f \in f$, and define the Koszul complement

$$\mathcal{K}_r^\ell(f) := \kappa \mathcal{P}_{r-1}\Lambda^{\ell+1}(f) \quad \text{with} \quad (\kappa\omega)_{\boldsymbol{x}}(\cdot, \ldots) := \omega_{\boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_f, \ldots)$$

- For $\ell \geq 1$ we define the trimmed polynomial spaces

$$\begin{aligned} \mathcal{P}_r^- \Lambda^0(f) &:= \mathcal{P}_r \Lambda^0(f), \\ \mathcal{P}_r^- \Lambda^\ell(f) &:= \mathrm{d}\mathcal{P}_r \Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \quad \text{if } \ell \geq 1 \end{aligned}$$

- In terms of vector proxies,

$$\forall f \equiv F \in \mathcal{F}_h, \qquad \mathcal{P}_r^- \Lambda^1(f) \cong \boldsymbol{\mathcal{N}}_r(F) = \boldsymbol{\mathcal{RT}}_r(F)^\perp,$$

$$\forall f \equiv T \in \mathcal{T}_h, \quad \begin{cases} \mathcal{P}_r^- \Lambda^1(f) \cong \boldsymbol{\mathcal{N}}_r(T) := \mathbf{grad}\, \mathcal{P}_r(T) + (\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{\mathcal{P}}_{r-1}(T), \\ \mathcal{P}_r^- \Lambda^2(f) \cong \boldsymbol{\mathcal{RT}}_r(T) := \mathbf{curl}\, \boldsymbol{\mathcal{P}}_r(T) + (\boldsymbol{x} - \boldsymbol{x}_T) \mathcal{P}_{r-1}(T) \end{cases}$$

Discrete $H\Lambda^k(\Omega)$ spaces and interpolator

$$\underline{X}^k_{r,h} \coloneqq \bigtimes_{d=k}^{n} \bigtimes_{f\in\Delta_d(\mathcal{M}_h)} \mathcal{P}^-_r\Lambda^{d-k}(f)$$

$$\underline{I}^k_{r,f} : \Lambda^k(\Omega) \ni \omega \mapsto \left(\pi^{-,d-k}_{r,f}(\star\,\mathrm{tr}_f\,\omega)\right)_{f\in\Delta_d(f),\,d\in[k,n]} \in \underline{X}^k_{r,h}$$

Space	$f_0 \equiv V$	$f_1 \equiv E$	$f_2 \equiv F$	$f_3 \equiv T$
$\underline{X}^0_{r,h}$	$\mathbb{R} = \mathcal{P}_r\Lambda^0(f_0)$	$\mathcal{P}_{r-1}\Lambda^1(f_1)$	$\mathcal{P}_{r-1}\Lambda^2(f_2)$	$\mathcal{P}_{r-1}\Lambda^3(f_3)$
$\underline{X}^1_{r,h}$		$\mathcal{P}_r\Lambda^0(f_1)$	$\mathcal{P}^-_r\Lambda^1(f_2)$	$\mathcal{P}^-_r\Lambda^2(f_3)$
$\underline{X}^2_{r,h}$			$\mathcal{P}_r\Lambda^0(f_2)$	$\mathcal{P}^-_r\Lambda^1(f_3)$
$\underline{X}^3_{r,h}$				$\mathcal{P}_r\Lambda^0(f_3)$
$\underline{X}^r_{\mathbf{grad},h}$	$\mathbb{R} = \mathcal{P}_r(V)$	$\mathcal{P}_{r-1}(E)$	$\mathcal{P}_{r-1}(F)$	$\mathcal{P}_{r-1}(T)$
$\underline{\boldsymbol{X}}^r_{\mathbf{curl},h}$		$\mathcal{P}_r(E)$	$\mathcal{RT}_r(F)$	$\mathcal{RT}_r(T)$
$\underline{\boldsymbol{X}}^r_{\mathrm{div},h}$			$\mathcal{P}_r(F)$	$\mathcal{N}_r(T)$
$\mathcal{P}_r(\mathcal{T}_h)$				$\mathcal{P}_r(T)$

Discrete potential and exterior derivative

For $d = k, \dots, n$, all $f \in \Delta_d(\mathcal{M}_h)$, and all $\underline{\omega}_f \in \underline{X}^k_{r,f}$:

- If $d = k$, we let

$$P^k_{r,f}\underline{\omega}_f \coloneqq \star^{-1}\omega_f \in \mathcal{P}_r\Lambda^d(f)$$

- If $d \geq k+1$, we first let, for all $\mu \in \mathcal{P}_r\Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu = (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

  then we enforce $\pi^{\mathcal{K},d-k}_{r,f} P^k_{r,f}\underline{\omega}_f = \star^{-1}\omega_f$ and, for all $\mu \in \mathcal{K}^{d-k-1}_{r+1}(f)$,

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge \mathrm{d}\mu = \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

The case $n = 3$ and $k = 1$ I

- For all $f \equiv T \in \mathcal{T}_h$,

$$\underline{X}^1_{r,f} \cong \underline{\boldsymbol{X}}^r_{\mathbf{curl},T} := \bigtimes_{E \in \mathcal{E}_T} \mathcal{P}_r(E) \times \bigtimes_{F \in \mathcal{F}_T} \mathcal{RT}_r(F) \times \mathcal{RT}_r(T)$$

- Let

$$\underline{\boldsymbol{v}}_T = \left((v_E)_{E \in \mathcal{E}_T}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}, \boldsymbol{v}_T\right) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},T}$$

  and denote by $\underline{\boldsymbol{v}}_Y$ its restriction to $Y \in \mathcal{E}_T \cup \mathcal{F}_T$

- For all $E \in \mathcal{E}_T$ $(d = k = 1)$, the edge tangential trace is simply

$$\gamma^r_{\mathrm{t},E} \underline{\boldsymbol{v}}_E := v_E \quad \forall E \in \mathcal{E}_T$$

The case $n = 3$ and $k = 1$ II

- For all $F \in \mathcal{F}_T$ $(d = 2)$, the face curl is given by: For all $q \in \mathcal{P}_r(F)$,

$$\int_F C_F^r \underline{v}_F \, q = \int_F \boldsymbol{v}_F \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \varepsilon_{FE} \int_E \gamma_{\mathrm{t},E}^r \underline{v}_E \, q$$

- The face tangential trace is such that, for all $(q, \boldsymbol{w}) \in \mathcal{P}_{r+1}^{\mathrm{b}}(F) \times \mathcal{R}_r^{\mathrm{c}}(F)$,

$$\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^r \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot}_F \, q + \boldsymbol{w}) = \int_F C_F^r \underline{v}_F \, q - \sum_{E \in \mathcal{E}_F} \varepsilon_{FE} \int_E \gamma_{\mathrm{t},E}^r \underline{v}_E \, q + \int_F \boldsymbol{v}_F \cdot \boldsymbol{w}$$

- For all $T \in \mathcal{T}_h$ $(d = 3)$, the element curl satisfies, for all $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_r(T)$,

$$\int_T \boldsymbol{C}_T^r \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w} = \int_T \boldsymbol{v}_T \cdot \mathbf{curl}\, \boldsymbol{w} + \sum_{F \in \mathcal{F}_T} \varepsilon_{TF} \int_F \boldsymbol{\gamma}_{\mathrm{t},F}^r \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w} \times \boldsymbol{n}_F)$$

- Finally, by similar principles, we can construct $\boldsymbol{P}_{\mathbf{curl},T}^r : \underline{\boldsymbol{X}}_{\mathbf{curl},T}^r \to \boldsymbol{\mathcal{P}}_r(T)$

Polynomial consistency

Theorem (Polynomial consistency)

For all integers $0 \le k \le d \le n$ *and all* $f \in \Delta_d(\mathcal{M}_h)$*, it holds*

$$P^k_{r,f} \underline{I}^k_{r,f} \omega = \omega \qquad \forall \omega \in \mathcal{P}_r \Lambda^k(f),$$

and, if $d \ge k+1$*,*

$$\mathrm{d}^k_{r,f} \underline{I}^k_{r,f} \omega = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}^-_{r+1} \Lambda^k(f).$$

Example (The case $n = 3$ and $k = 1$)

For $n = 3$ and $k = 1$, the above properties translate as follows:

$$\begin{aligned} \boldsymbol{P}^r_{\mathbf{curl},T} \underline{\boldsymbol{I}}^r_{\mathbf{curl},T} \boldsymbol{v} &= \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_r(T), \\ \boldsymbol{C}^r_T \underline{\boldsymbol{I}}^r_{\mathbf{curl},T} \boldsymbol{v} &= \mathbf{curl}\, \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}_{r+1}(T). \end{aligned}$$

Global discrete exterior derivative and DDR complex

- Our next goal is to connect the spaces $\underline{X}_{r,h}^{k}$ to form a well-defined sequence
- We define the global discrete exterior derivative $\underline{\mathrm{d}}_{r,h}^{k} : \underline{X}_{r,h}^{k} \to \underline{X}_{r,h}^{k+1}$ s.t.

$$\underline{\mathrm{d}}_{r,h}^{k}\underline{\omega}_{h} := \left(\pi_{r,f}^{-,d-k-1}(\star \mathrm{d}_{r,f}^{k}\underline{\omega}_{f})\right)_{f\in\Delta_{d}(\mathcal{M}_{h}),\, d\in[k+1,n]}$$

- The DDR sequence then reads

$$\underline{X}_{r,h}^{0} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{0}} \underline{X}_{r,h}^{1} \longrightarrow \cdots \longrightarrow \underline{X}_{r,h}^{n-1} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{n-1}} \underline{X}_{r,h}^{n} \longrightarrow \{0\}$$

- Specifically, for $n = 3$, we recover the one in [DP and Droniou, 2023]:

$$\underline{X}_{\mathbf{grad},h}^{r} \xrightarrow{\underline{\boldsymbol{G}}_{h}^{k}} \underline{\boldsymbol{X}}_{\mathbf{curl},h}^{r} \xrightarrow{\underline{\boldsymbol{C}}_{h}^{r}} \underline{\boldsymbol{X}}_{\mathrm{div},h}^{r} \xrightarrow{D_{h}^{r}} \mathcal{P}_{r}(\mathcal{T}_{h}) \longrightarrow \{0\}$$

Cohomology

Theorem (Cohomology of the Discrete de Rham complex)

The DDR sequence is a complex and its cohomology is isomorphic to the cohomology of the continuous de Rham complex, i.e., for all k,

$$\operatorname{Ker} \underline{\mathrm{d}}_{r,h}^{k} / \operatorname{Im} \underline{\mathrm{d}}_{r,h}^{k-1} \cong \operatorname{Ker} \mathrm{d}^{k} / \operatorname{Im} \mathrm{d}^{k-1}.$$

Example (The case $n = 3$)

For $n = 3$, in terms of vector proxies, this implies, in particular:

$$\text{no "tunnels" crossing } \Omega \ (b_1 = 0) \implies \operatorname{Im} \underline{\boldsymbol{G}}_h^k = \operatorname{Ker} \underline{\boldsymbol{C}}_h^r$$
$$\text{no "voids" contained in } \Omega \ (b_2 = 0) \implies \operatorname{Im} \underline{\boldsymbol{C}}_h^r = \operatorname{Ker} D_h^r$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \operatorname{Im} D_h^r = \mathcal{P}_r(\mathcal{T}_h)$$

Discrete L^2-products

- We can define on $\underline{X}^k_{r,h}$ a discrete L^2-product $(\cdot,\cdot)_{k,h} : \underline{X}^k_{r,h} \times \underline{X}^k_{r,h} \to \mathbb{R}$:

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} := \sum_{f \in \Delta_n(\mathcal{M}_h)} \left(\int_f P^k_{r,f}\underline{\omega}_f \wedge \star P^k_{r,f}\underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) \right)$$

- Above, $s_{k,f} : \underline{X}^k_{r,f} \times \underline{X}^k_{r,f} \to \mathbb{R}$ is a stabilisation that satisfies

$$s_{k,f}(\underline{I}^k_{r,f}\omega, \underline{\mu}_f) = 0 \qquad \forall \omega \in \mathcal{P}_r\Lambda^k(f)$$

- Numerical schemes are obtained replacing spaces, differential operators, and L^2-products with their discrete counterparts

Outline

Discrete problem I

- With $\mu = 1$, we seek $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned}\int_\Omega \boldsymbol{H}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 &\quad \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\ \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{A}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v} &\quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)\end{aligned}$$

- The DDR scheme is obtained substituting

$$\boldsymbol{H}(\mathbf{curl};\Omega) \leftarrow \underline{\boldsymbol{X}}^r_{\mathbf{curl},h}, \qquad \boldsymbol{H}(\mathrm{div};\Omega) \leftarrow \underline{\boldsymbol{X}}^r_{\mathrm{div},h}$$

and

$$\begin{aligned}\int_\Omega \boldsymbol{H}\cdot\boldsymbol{\tau} &\leftarrow (\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h}, &\qquad \int_\Omega \mathbf{curl}\,\boldsymbol{\tau}\cdot\boldsymbol{v} &\leftarrow (\underline{\boldsymbol{C}}^r_h\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h},\\ \int_\Omega \mathrm{div}\,\boldsymbol{w}\ \ \mathrm{div}\,\boldsymbol{v} &\leftarrow \int_\Omega D^r_h\underline{\boldsymbol{w}}_h\ D^r_h\underline{\boldsymbol{v}}_h, &\qquad \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v} &\leftarrow \int_\Omega \boldsymbol{J}\cdot\boldsymbol{P}^r_{\mathrm{div},h}\underline{\boldsymbol{v}}_h\end{aligned}$$

Discrete problem II

- The discrete problem reads: Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}$ s.t.

$$\begin{aligned}
(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 \qquad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h}, \\
(\underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^r_h \underline{\boldsymbol{A}}_h \, D^r_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^r_{\mathrm{div},h}
\end{aligned}$$

- Provided $b_2 = 0$, stability results from the exactness of the portion

$$\underline{X}^r_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^r_h} \underline{\boldsymbol{X}}^r_{\mathrm{div},h} \xrightarrow{D^r_h} \mathcal{P}_r(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- For smooth enough solution, the energy error is $\mathcal{O}(h^{r+1})$

Numerical examples

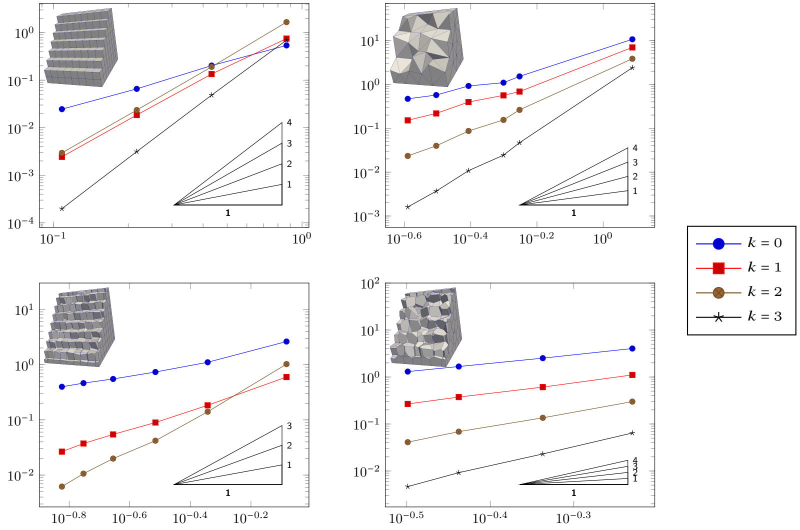


Table: Energy error vs. meshsize

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