Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin Methods

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Minimal bibliography: High-order polyhedral methods

- Discontinuous Galerkin
  - Unified analysis [Arnold, Brezzi, Cockburn and Marini, 2002]
  - General meshes [DP and Ern, 2012, Cangiani, Georgoulis et al. 2014]
  - Adaptive coarsening [Bassi et al., 2012, Antonietti et al., 2013]
- Hybridizable Discontinuous Galerkin
  - LDG framework [Castillo, Cockburn, Perugia, Schötzau, 2009]
  - HDG for pure diffusion [Cockburn, Gopalakrishnan, Lazarov, 2009]
- Weak Galerkin [Wang and Ye, 2013]
- Virtual elements
  - Pure diffusion [Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, 2013]
  - Nonconforming VEM [Lipnikov and Manzini, 2014]
- Hybrid High-Order (HHO)
  - Originally introduced for linear elasticity [DP and Ern, 2015]
  - Pure diffusion [DP et al., 2014]
  - Link with HDG [Cockburn, DP, Ern, M2AN, 2016]

Outline

Outline

Model problem

- Let $\boldsymbol{\kappa} : \Omega \to \mathbb{R}^{d\times d}$ be a SPD tensor-valued field s.t.

$$0 < \underline{\kappa} \leqslant \lambda(\boldsymbol{\kappa}) \leqslant \overline{\kappa}$$

- We assume $\boldsymbol{\kappa}$ piecewise constant on a polyhedral partition P_Ω of Ω
- We consider the Darcy problem

$$\begin{aligned} -\operatorname{div}(\boldsymbol{\kappa}\nabla u) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- Weak formulation: Find $u \in U := H_0^1(\Omega)$ s.t.

$$a(u,v) := (\boldsymbol{\kappa}\nabla u, \nabla v) = (f,v) \qquad \forall v \in U$$

Mesh I

Definition (Mesh regularity and compliance)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the usual sense of Ciarlet;
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Additionally, we assume, for all $h \in \mathcal{H}$, $\mathcal{T}_h$ compliant with P_Ω, so that

$$\boldsymbol{\kappa}_T := \boldsymbol{\kappa}_{|T} \in \mathbb{P}^0(T)^{d\times d} \quad \forall T \in \mathcal{T}_h.$$

Main consequences of mesh regularity:

- L^p-trace and inverse inequalities
- Optimal $W^{s,p}$-approximation for the L^2-orthogonal projector

See [DP and Ern, 2012] ($p = 2$) and [DP and Droniou, 2016a] ($p \in [1, +\infty]$)

Mesh II

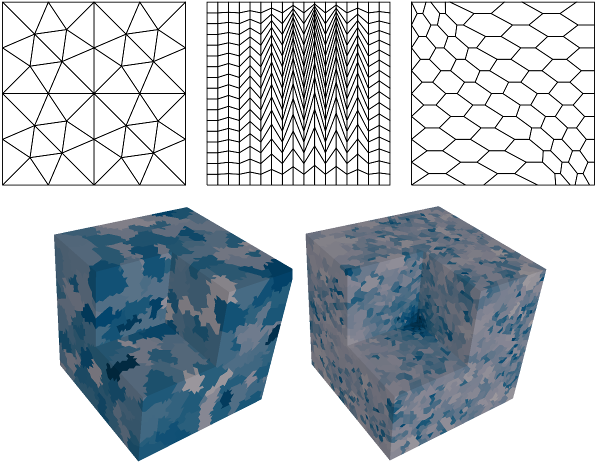


Figure: Examples of meshes in 2d and 3d: [Herbin and Hubert, 2008] and [DP and Lemaire, 2015] (above) and [DP and Specogna, 2016] (below)

Local DOF space

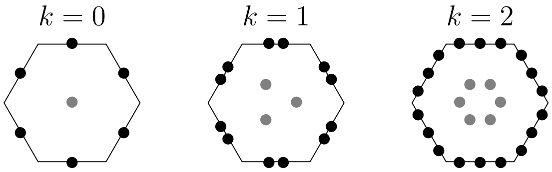


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- For all $k \geqslant 0$ and all $T \in \mathcal{T}_h$, we define the local space

$$\underline{U}_T^k := U_T^k \times U_{\partial T}^k, \qquad U_T^k := \mathbb{P}^k(T), \qquad U_{\partial T}^k := \mathbb{P}^k(\mathcal{F}_T)$$

- For a generic element of $\underline{U}_T^k$, we use the notation

$$\underline{v}_T = (v_T, v_{\partial T})$$

- Shaded DOFs can be eliminated by static condensation

Potential reconstruction I

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$p_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$$

is s.t. $(p_T^{k+1}\underline{v}_T - v_T, 1)_T = 0$ and, for all $w \in \mathbb{P}^{k+1}(T)$,

$$\boxed{(\boldsymbol{\kappa}_T \nabla p_T^{k+1}\underline{v}_T, \nabla w)_T := -(v_T, \operatorname{div}(\boldsymbol{\kappa}_T \nabla w))_T + (v_{\partial T}, \boldsymbol{\kappa}_T \nabla w \cdot \boldsymbol{n}_T)_{\partial T}}$$

- To compute p_T^{k+1}, we invert a small SPD matrix of size

$$N_{k,d} := \binom{k+1+d}{k+1} - 1$$

- **Trivially parallel task, potentially suited to GPUs!**

Potential reconstruction II

Lemma (Approximation properties of $p_T^{k+1}\underline{I}_T^k$)

Define the reduction map $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ such that

$$\underline{I}_T^k v = (\pi_T^k v, \pi_{\partial T}^k v).$$

We have, for all $v \in H^1(T)$ and all $w \in \mathbb{P}^{k+1}(T)$,

$$(\boldsymbol{\kappa}\nabla(p_T^{k+1}\underline{I}_T^k v - v), \nabla w)_T = 0.$$

Consequently, for all $v \in H^{k+2}(T)$, it holds

$$\|v - p_T^{k+1}\underline{I}_T^k v\|_T + h_T\|\nabla(v - p_T^{k+1}\underline{I}_T^k v)\|_T \leqslant C_{\boldsymbol{\kappa}} h_T^{k+2}\|v\|_{k+2,T},$$

i.e., $p_T^{k+1}\underline{I}_T^k$ has optimal approximation properties in $\mathbb{P}^{k+1}(T)$.

- For the dependence of $C_{\boldsymbol{\kappa}}$ on $\boldsymbol{\kappa}_T$ see [DP et al., 2016]
- $W^{s,p}$-approximation properties proved in [DP and Droniou, 2016b]

Stabilization I

- The following local discrete bilinear form is in general not stable

$$a_T(\underline{u}_T, \underline{v}_T) = (\boldsymbol{\kappa}_T \nabla p_T^{k+1} \underline{u}_T, \nabla p_T^{k+1} \underline{v}_T)_T$$

- As a remedy, we add a local stabilization term:

$$\boxed{a_T(\underline{u}_T, \underline{v}_T) := (\boldsymbol{\kappa}_T \nabla p_T^{k+1} \underline{u}_T, \nabla p_T^{k+1} \underline{v}_T)_T + s_T(\underline{u}_T, \underline{v}_T)}$$

- We aim at expressing coercivity w.r.t. to the local (semi-)norm

$$\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_T^2 + h_T^{-1} \|v_{\partial T} - v_T\|_{\partial T}^2$$

- We also want to preserve the approximation properties of p_T^{k+1}

Stabilization II

- A HDG-inspired choice for the stabilization would be

$$s_T^{\mathrm{hdg}}(\underline{u}_T, \underline{v}_T) = (\tau_{\partial T}(u_T - u_{\partial T}), v_T - v_{\partial T})_{\partial T}, \quad \tau_{\partial T|F} := \frac{\boldsymbol{\kappa}_T \boldsymbol{n}_{TF} \cdot \boldsymbol{n}_{TF}}{h_F}$$

- This choice is, however, suboptimal since, for all $v \in H^{k+2}(T)$,

$$\|\nabla(p_T^{k+1} \underline{I}_T^k v - v)\|_T \lesssim h^{k+1} \|v\|_{H^{k+2}(T)},$$

while we only have

$$s_T^{\mathrm{hdg}}(\underline{I}_T^k v, \underline{I}_T^k v)^{1/2} \lesssim h^k \|v\|_{H^{k+1}(T)}$$

- **We need to penalize higher-order differences!**

Stabilization III

- Define the high-order correction of cell DOFs $P_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$

$$P_T^{k+1}\underline{v}_T := v_T + \left(p_T^{k+1}\underline{v}_T - \pi_T^k p_T^{k+1}\underline{v}_T\right)$$

- We consider the stabilization bilinear form s_T s.t.

$$s_T(\underline{u}_T, \underline{v}_T) := (\tau_{\partial T}\pi_{\partial T}^k(P_T^{k+1}\underline{u}_T - u_{\partial T}), \pi_{\partial T}^k(P_T^{k+1}\underline{v}_T - v_{\partial T}))_{\partial T}$$

- With this choice we have stability: For all $\underline{v}_T \in \underline{U}_T^k$

$$\|\underline{v}_T\|_{1,T}^2 \lesssim a_T(\underline{v}_T, \underline{v}_T) \lesssim \|\underline{v}_T\|_{1,T}^2$$

- Additionally, for all $v \in H^{k+2}(T)$,

$$s_T(\underline{I}_T^k v, \underline{I}_T^k v)^{1/2} \lesssim h^{k+1}\|v\|_{H^{k+2}(T)}$$

Discrete problem

- We define the global space with single-valued interface DOFs

$$\underline{U}_h^k := U_{\mathcal{T}_h}^k \times U_{\mathcal{F}_h}^k, \qquad U_{\mathcal{T}_h}^k := \mathbb{P}^k(\mathcal{T}_h), \qquad U_{\mathcal{F}_h}^k := \mathbb{P}^k(\partial\mathcal{T}_h),$$

- We also need the subspace with strongly enforced BCs

$$\underline{U}_{h,0}^k := U_{\mathcal{T}_h}^k \times U_{\mathcal{F}_h,0}^k, \quad U_{\mathcal{F}_h,0}^k := \left\{\underline{v}_h \in \underline{U}_h^k \mid v_F \equiv 0 \ \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ such that

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = (f, v_{\mathcal{T}_h}) \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

where $v_{\mathcal{T}_h|T} := v_T$ for all $T \in \mathcal{T}_h$

Convergence I

Theorem (Energy-norm error estimate)

Assume $u \in H^{k+2}(\Omega)$ *and define the global reduction map*

$$\underline{I}_h^k u := \left((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h}\right) \in \underline{U}_{h,0}^k.$$

Then, we have the following energy error estimate:

$$\boxed{\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)},}$$

where $\|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$.

For the original LDG-H method, we have $h^{k+1/2}$

Convergence II

Theorem (L^2-norm error estimate)

Further assuming elliptic regularity and $f \in H^1(\Omega)$ if $k = 0$,

$$\boxed{\|u_{\mathcal{T}_h} - \pi_h^k u\| \lesssim h^{k+2} B(u,k),}$$

with $B(u,0) := \|f\|_{H^1(\Omega)}$, $B(u,k) := \|u\|_{H^{k+2}(\Omega)}$ if $k \geqslant 1$ and

$$u_{\mathcal{T}_h|T} = u_T \qquad \forall T \in \mathcal{T}_h.$$

For the original LDG-H method, we have h^{k+1}

Corollary (L^2-norm estimate for $p_T^{k+1}\underline{u}_T$)

Letting $\breve{u}_h \in \mathbb{P}^{k+1}(\mathcal{T}_h)$ be s.t. $\breve{u}_{h|T} = p_T^{k+1}\underline{u}_T$ for all $T \in \mathcal{T}_h$, it holds

$$\|\breve{u}_h - u\| \lesssim h^{k+2} B(u,k).$$

Numerical example I

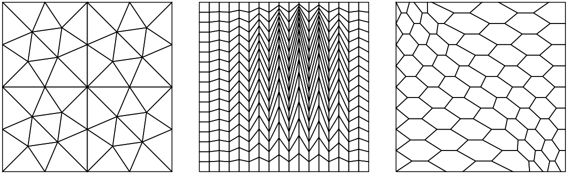


Figure: Triangular, Kershaw and hexagonal mesh families

- We consider Le Potier's exact solution on $\Omega = (0,1)^2$

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$$

- The diffusion field has rotating principal axes

$$\boldsymbol{\kappa}(\boldsymbol{x}) = \begin{pmatrix} (x_2-\overline{x}_2)^2 + \epsilon(x_1-\overline{x}_1)^2 & -(1-\epsilon)(x_1-\overline{x}_1)(x_2-\overline{x}_2) \\ -(1-\epsilon)(x_1-\overline{x}_1)(x_2-\overline{x}_2) & (x_1-\overline{x}_1)^2 + \epsilon(x_2-\overline{x}_2)^2 \end{pmatrix},$$

with anisotropy ratio $\epsilon = 1 \cdot 10^{-2}$ and center $(\overline{x}_1, \overline{x}_2) = -(0.1, 0.1)$

Numerical example II

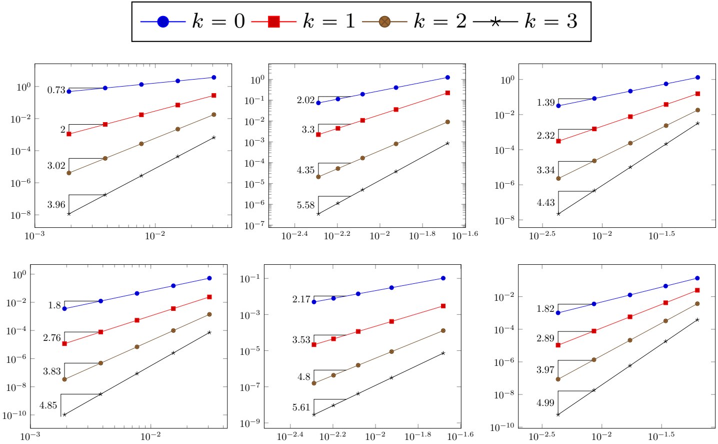
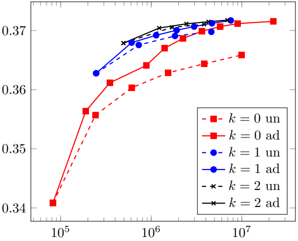


Figure: Energy- (above) and L^2-errors (below) for the three mesh families



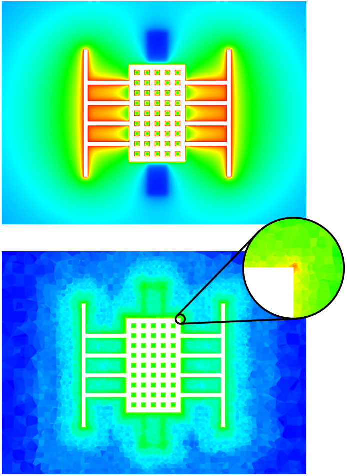
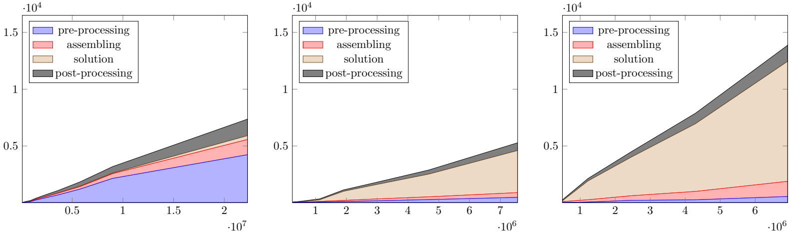


Figure: Adaptive algorithm for 3d electrostatics [DP and Specogna, 2016]

Teaser: Industrial application II



(a) $k = 0$ (b) $k = 1$ (c) $k = 2$

Figure: Computing wall time vs N_{dof}

Outline

Numerical trace formulation I

- Separating element- and face-based test functions, we have

$$\begin{aligned} \forall T \in \mathcal{T}_h, \quad a_T(\underline{u}_T, (v_T, 0)) &= (f, v_T)_T \quad && \forall v_T \in U_T^k \\ a_h(\underline{u}_h, (0, v_{\mathcal{F}_h})) &= 0 && \forall v_{\mathcal{F}_h} \in U_{\mathcal{F}_h,0}^k \end{aligned}$$

- The first set of equations define local equilibria
- The second set is a global transmission condition

Numerical trace formulation II

- For $T \in \mathcal{T}_h$, define the boundary residual $r_{\partial T}^k : \mathbb{P}^k(\mathcal{F}_T) \to \mathbb{P}^k(\mathcal{F}_T)$ s.t.

$$\forall \lambda \in \mathbb{P}^k(\mathcal{F}_T), \quad r_{\partial T}^k(\lambda) := \pi_{\partial T}^k \left(\lambda - p_T^{k+1}(0, \lambda) + \pi_T^k p_T^{k+1}(0, \lambda) \right)$$

- The penalized difference rewrites: For all $\underline{v}_T \in \underline{U}_T^k$,

$$\pi_{\partial T}^k (P_T^{k+1} \underline{v}_T - v_{\partial T}) = r_{\partial T}^k (v_T - v_{\partial T})$$

Numerical trace formulation III

- As a result, for all $T \in \mathcal{T}_h$ we have

$$s_T(\underline{u}_T, (0, v_{\partial T})) = -(\tau_{\partial T} r_{\partial T}^k(u_T - u_{\partial T}), r_{\partial T}^k(v_{\partial T}))_{\partial T}$$

  or, introducing the adjoint $r_{\partial T}^{k,*}$ of $r_{\partial T}^k$,

$$s_T(\underline{u}_T, (0, v_{\mathcal{F}_h})) = (r_{\partial T}^{k,*}(\tau_{\partial T} r_{\partial T}^k(u_T - u_{\partial T})), v_{\partial T})_{\partial T}$$

- Plugging this expression into that of a_T, we finally arrive at

$$a_T(\underline{u}_T, (0, v_{\partial T})) = -\,(\underbrace{\boldsymbol{\kappa}_T \nabla p_T^{k+1} \underline{u}_T \cdot \boldsymbol{n}_T}_{\text{consistency}} \underbrace{- r_{\partial T}^{k,*}(\tau_{\partial T} r_{\partial T}^k(u_T - u_{\partial T}))}_{\text{penalty}}, v_{\partial T})_{\partial T}$$

- **The term in red is the conservative normal flux trace**

Numerical trace formulation IV

Lemma (Numerical trace reformulation of HHO)

The discrete problem: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ *s.t.*

$$a_h(\underline{u}_h, \underline{v}_h) = (f, v_{\mathcal{T}_h}) \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^k$$

can be equivalently reformulated as follows: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ *s.t.*

$$\begin{aligned} \forall T \in \mathcal{T}_h, \quad a_T(\underline{u}_T, (v_T, 0)) &= (f, v_T)_T && \forall v_T \in U_T^k, \\ \sum_{T \in \mathcal{T}_h} (\widehat{q}_{\boldsymbol{n},T}(\underline{u}_T), v_{\mathcal{F}_h})_{\partial T} &= 0 && \forall v_{\mathcal{F}_h} \in U_{\mathcal{F}_h,0}^k, \end{aligned}$$

with conservative normal flux trace s.t., for all $T \in \mathcal{T}_h$*,*

$$\widehat{q}_{\boldsymbol{n},T}(\underline{u}_T) := \boldsymbol{\kappa}_T \nabla p_T^{k+1} \underline{u}_T \cdot \boldsymbol{n}_T - r_{\partial T}^{k,*}(\tau_{\partial T} r_{\partial T}^k (u_T - u_{\partial T})).$$

Link with HDG methods I

- Let us interpret HHO as a HDG method
- HDG methods hinge on three set of spaces:
  - $\{\boldsymbol{V}(T)\}_{T\in\mathcal{T}_h}$, for the approximation of the flux
  - $\{W(T)\}_{T\in\mathcal{T}_h}$, for the approximation of the potential
  - $\{M(F)\}_{F\in\mathcal{F}_h}$, for the approximation of the potential traces
- The definition is completed with a recipe for the normal flux trace

Link with HDG methods II

- We define the following global spaces:

$$\mathbf{V}_h := \bigtimes_{T\in\mathcal{T}_h} \mathbf{V}(T), \quad W_h \times M_h := \left(\bigtimes_{T\in\mathcal{T}_h} W(T)\right) \times \left(\bigtimes_{F\in\mathcal{F}_h} M(F)\right)$$

- We also need the subspace with strongly enforced BCs,

$$M_{h,0} := \{\widehat{w} \in M_h : \ \widehat{w} = 0 \text{ on } \partial\Omega\}$$

- The HDG method reads: Find $(\mathbf{q}_h, u_h, \widehat{u}_h) \in \mathbf{V}_h \times W_h \times M_{h,0}$ s.t.

$$\boxed{\begin{aligned} (\boldsymbol{\kappa}_T^{-1}\mathbf{q}_h, \boldsymbol{v})_T - (u_h, \operatorname{div} \boldsymbol{v})_T + (\widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}_T)_{\partial T} &= 0 & \forall \boldsymbol{v} \in \mathbf{V}(T) \\ -(\mathbf{q}_h, \nabla w)_T + (\widehat{q}_{\boldsymbol{n},T}, w)_{\partial T} &= (f, w)_T & \forall w \in W(T) \\ \sum_{T\in\mathcal{T}_h} (\widehat{q}_{\boldsymbol{n},T}, \widehat{w})_{\partial T} &= 0 & \forall \widehat{w} \in M_{h,0} \end{aligned}}$$

Link with HDG methods III

- HHO corresponds to new choices for the spaces

$$\boxed{\boldsymbol{V}(T) = \boldsymbol{\kappa}_T \nabla \mathbb{P}^{k+1}(T), \quad W(T) = \mathbb{P}^k(T), \quad M(F) = \mathbb{P}^k(F)}$$

- Notice that the flux is now reconstructed in a smaller space

$$\dim(\boldsymbol{\kappa}_T \nabla \mathbb{P}^{k+1}(T)) \leqslant \dim(\mathbb{P}^k(T)^d)$$

- Another crucial novelty is the high-order normal flux trace

$$\boxed{\widehat{q}_{\boldsymbol{n},T} = \boldsymbol{\kappa}_T \nabla p_T^{k+1} \underline{u}_T \cdot \boldsymbol{n}_T - r_{\partial T}^{k,*}(\tau_{\partial T} r_{\partial T}^k (u_T - u_{\partial T}))}$$

- **HHO can be easily adapted into existing HDG codes!**

Outline

Variations: The HHO(l) family I

- Variations are possible changing the degree of element-based DOFs
- Let $l \in \{k-1, k, k+1\}$ and consider the local space

$$\underline{U}_T^{k,l} := \mathbb{P}^l(T) \times \mathbb{P}^k(\mathcal{F}_T)$$

- The first potential reconstruction p_T^{k+1} remains formally unchanged
- The second potential reconstruction used for stabilization becomes

$$P_T^{k+1}\underline{v}_T := v_T + \left(p_T^{k+1}\underline{v}_T - \pi_T^l p_T^{k+1}\underline{v}_T\right)$$

Variations: The HHO(l) family II

- Convergence rates as for the original HHO method
- For $l = k - 1$ we recover a High-Order Mimetic scheme[1]
- For $l = k$ we find the original HHO method
- For $l = k + 1$ we have a new Lehrenfeld–Schöberl-type HDG method
- $k = 0$ and $l = k - 1$ on simplices yields the Crouzeix–Raviart element
- **The globally-coupled unknowns coincide in all the cases!**

[1] Up to equivalent stabilization

A nonconforming finite element interpretation I

- We interpret the HHO(l) methods as nonconforming FE methods
- The construction extends the ideas of [Ayuso de Dios et al., 2016]
- For a fixed element $T \in \mathcal{T}_h$, we define the local space

$$\boxed{V_T^{k,l} := \left\{\varphi \in H^1(T) \mid \nabla\varphi_{|\partial T}\cdot\boldsymbol{n}_T \in \mathbb{P}^k(\mathcal{F}_T) \text{ and } \triangle\varphi \in \mathbb{P}^l(T)\right\}}$$

- We next study the relation between $V_T^{k,l}$ and $\underline{U}_T^{k,l}$

A nonconforming finite element interpretation II

- Let $\Phi_T : \underline{U}_T^{k,l} \to V_T^{k,l}$ be s.t. $\Phi_T(\underline{v}_T)$ solves the Neumann problem

$$\boxed{\triangle \Phi_T(\underline{v}_T) = v_T - |T|_d^{-1} \left[(v_T, 1)_T - (v_{\partial T}, 1)_{\partial T} \right]}$$

and

$$\nabla \Phi_T(\underline{v}_T)_{|\partial T} \cdot \boldsymbol{n}_T = v_{\partial T}, \qquad (\Phi_T(\underline{v}_T), 1)_T = 0$$

- Both Φ_T and $\underline{I}_T^{k,l} : V_T^{k,l} \to \underline{U}_T^{k,l}$ can be proved to be injective
- Therefore, $\underline{I}_T^{k,l} : V_T^{k,l} \to \underline{U}_T^{k,l}$ is an isomorphism and we can identify

$$\boxed{V_T^{k,l} \sim \underline{U}_T^{k,l},}$$

which means that $\underline{U}_T^k$ contains the DOFs for $V_T^{k,l}$ as defined by $\underline{I}_T^k$

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