A Hybrid High-Order method for Leray–Lions equations

D. A. Di Pietro

joint work with J. Droniou (Monash University)

Institut Montpelliérain Alexander Grothendieck

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Model problem I

- Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, be a polytopal bounded connected domain
- Let $p \in (1, +\infty)$ and $f \in L^{p'}(\Omega)$ with $p' := \frac{p}{p-1}$
- We consider the Leray–Lions problem: Find $u \in W_0^{1,p}(\Omega)$ s.t.

$$\boxed{A(u,v) := \int_\Omega \mathbf{a}(\boldsymbol{x}, \nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega f v \quad \forall v \in W_0^{1,p}(\Omega)}$$

- A typical example is the p-Laplacian: For $p \in (1, +\infty)$,

$$\mathbf{a}(\boldsymbol{x}, \nabla u) = |\nabla u|^{p-2} \nabla u$$

- Applications to glaciology, turbulent porous media flow, airfoil design
- **Perfect playground for discrete functional analysis tools ☺**

Model problem II

Assumption (Leray–Lions operator/v1)

For a fixed index $p \in (1, +\infty)$, $f \in L^{p'}(\Omega)$ *and* $\mathbf{a}$ *satisfies*

- *Growth.* $\mathbf{a}(\cdot, \mathbf{0}) \in L^{p'}(\Omega)$ *and there is* $\beta_{\mathbf{a}} > 0$ *s.t.*

$$|\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \mathbf{0})| \leqslant \beta_{\mathbf{a}} |\boldsymbol{\xi}|^{p-1} \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d.$$

- *Monotonicity.* *For a.e.* $\boldsymbol{x} \in \Omega$, *for all* $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$[\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})] \cdot [\boldsymbol{\xi} - \boldsymbol{\eta}] \geqslant 0.$$

- *Coercivity.* *There is* $\lambda_{\mathbf{a}} > 0$ *s.t.*

$$\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geqslant \lambda_{\mathbf{a}} |\boldsymbol{\xi}|^p \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d.$$

A dependence on u can also be included in the analysis

Discretization of Leray–Lions type problems

- Conforming Finite Elements
  - p-Laplacian, a priori [Barrett and Liu, 1994]
  - A priori and a posteriori [Glowinski and Rappaz, 2003]
- Nonconforming FE for the p-Laplacian [Liu and Yan, 2001]
- Mixed Finite Volumes for Leray–Lions [Droniou, 2006]
- Discrete Duality FV, $d = 2$ [Andreianov, Boyer, Hubert, 2004–07]
- Mimetic FD, quasi linear [Antonietti, Bigoni, Verani, 2014]
- Hybrid High-Order (HHO) for Leray–Lions, $p \in (1, +\infty)$
  - Convergence by compactness [DP & Droniou, Math. Comp., 2016]
  - Error estimates [DP & Droniou, submitted, 2016]
- **Ideas and tools applicable also to other POEMS (VEM, DG, HDG, WG,. . .)**

Mesh I

Definition (Mesh regularity)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the usual sense of Ciarlet;
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences:

- L^p-trace and inverse inequalities
- Approximation for broken polynomial spaces
- See [Cangiani, Georgoulis, Houston, 2014] for degenerate faces

Mesh II

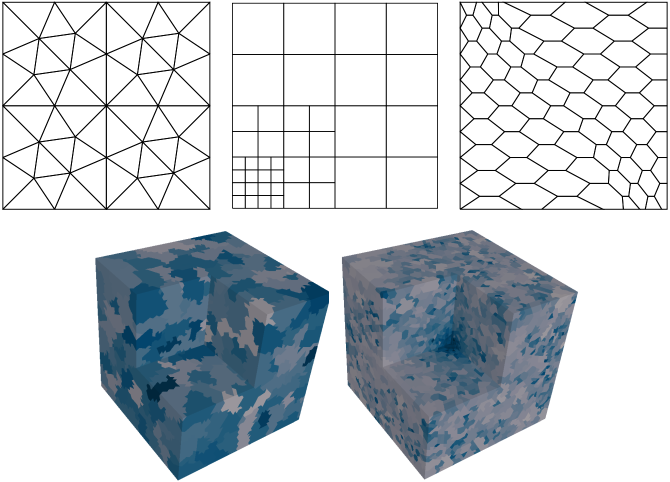

Figure: Examples of meshes in 2d and 3d: [Herbin and Hubert, 2008] and [DP and Lemaire, 2015] (above) and [DP and Specogna, 2016] (below)

Projectors on local polynomial spaces I

- The L^2-orthogonal projector $\pi_T^{0,l} : L^1(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T (\pi_T^{0,l} v - v) w = 0 \text{ for all } w \in \mathbb{P}^l(T)$$

- The elliptic projector $\pi_T^{1,l} : W^{1,1}(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T \nabla(\pi_T^{1,l} v - v)\cdot\nabla w = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } \int_T (\pi_T^{1,l} v - v) = 0$$

- The elliptic projector is at the core of other POEMS, e.g.,
  - VEM [Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, 2013]
  - HOM/nc-VEM [Lipnikov and Manzini, 2014]

Projectors on local polynomial spaces II

Lemma (Optimal approximation)

For all $p \in [1, +\infty]$, *all* $s \in \{1, \ldots, l+1\}$, *all* $m \in \{0, \ldots, s-1\}$, *and all* $v \in W^{s,p}(T)$, *it holds with* $\star \in \{0, 1\}$

$$|v - \pi_T^{\star,l} v|_{W^{m,p}(T)} + h_T^{\frac{1}{p}} |v - \pi_T^{\star,l} v|_{W^{m,p}(\mathcal{F}_T)} \lesssim h_T^{s-m} |v|_{W^{s,p}(T)}.$$

Proof.

Apply a general result from [DP and Droniou, 2016b]: every W-bounded projector has optimal approximation properties. $\square$

Key ideas

- DOFs: polynomials of degree $k \geqslant 0$ at elements and faces
- Differential operators reconstructions taylored to the problem:

$$\boxed{A_{|T}(u,v) \approx \int_T \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \underline{u}_T(\boldsymbol{x})) \cdot \boldsymbol{G}_T^k \underline{v}_T(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \text{stab.}}$$

  with
  - gradient reconstruction $\boldsymbol{G}_T^k$ from local solves
  - stabilisation using face-based penalty and high-order potential reconstruction
- General meshes in any $d \geqslant 1$ and arbitrary polynomial degrees

DOFs and interpolation

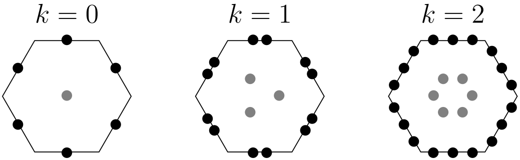

Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- For $k \geqslant 0$ and $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

- The local interpolator $\underline{I}_T^k : W^{1,1}(T) \to \underline{U}_T^k$ is s.t.

$$\underline{I}_T^k v = (\pi_T^{0,k} v, (\pi_F^{0,k} v)_{F \in \mathcal{F}_T})$$

Operator reconstructions I

- We define the gradient reconstruction $\boldsymbol{G}_T^k : \underline{U}_T^k \mapsto \mathbb{P}^k(T)^d$ s.t.

$$(\boldsymbol{G}_T^k \underline{v}_T, \boldsymbol{\phi})_T = -(v_T, \operatorname{div} \boldsymbol{\phi})_T + \sum_{F \in \mathcal{F}_T} (v_F, \boldsymbol{\phi} \cdot \boldsymbol{n}_{TF})_F \quad \forall \boldsymbol{\phi} \in \mathbb{P}^k(T)^d$$

- Recalling the definition of $\underline{I}_T^k$, it holds for all $v \in W^{1,1}(T)$,

$$(\boldsymbol{G}_T^k \underline{I}_T^k v, \boldsymbol{\phi})_T = -(\cancel{\pi_T^{0,k}} v, \operatorname{div} \boldsymbol{\phi})_T + \sum_{F \in \mathcal{F}_T} (\cancel{\pi_F^{0,k}} v, \boldsymbol{\phi} \cdot \boldsymbol{n}_{TF})_F = (\nabla v, \boldsymbol{\phi})_T,$$

i.e., by definition of $\pi_T^{0,k}$,

$$\boxed{\boldsymbol{G}_T^k \underline{I}_T^k v = \pi_T^{0,k}(\nabla v)}$$

- As a result, $(\boldsymbol{G}_T^k \circ \underline{I}_T^k)$ has optimal $W^{s,p}$-approximation properties

Operator reconstructions II

- We define the potential reconstruction $p_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$ s.t.

$$(\nabla p_T^{k+1}\underline{v}_T - \boldsymbol{G}_T^k\underline{v}_T, \nabla w)_T = 0 \quad \forall w \in \mathbb{P}^{k+1}(T)$$

  and $(p_T^{k+1}\underline{v}_T - v, 1)_T = 0$

- Recalling the definition of $\boldsymbol{G}_T^k$ and $\underline{I}_T^k$, it holds for all $v \in W^{1,1}(T)$,

$$(\nabla p_T^{k+1}\underline{I}_T^k v, \nabla w)_T = -(\cancel{\pi_T^{0,k}} v, \triangle w)_T + \sum_{F\in\mathcal{F}_T} (\cancel{\pi_F^{0,k}} v, \nabla w\cdot\boldsymbol{n}_{TF})_F = (\nabla v, \nabla w)_T,$$

  i.e., by definition of $\pi_T^{1,k+1}$,

$$\boxed{p_T^{k+1}\underline{I}_T^k v = \pi_T^{1,k+1} v}$$

- As a result, $(p_T^{k+1} \circ \underline{I}_T^k)$ has optimal $W^{s,p}$-approximation properties

Global problem I

- For all $T \in \mathcal{T}_h$, we define the local function $\mathrm{A}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ s.t.

$$\boxed{\mathrm{A}_T(\underline{u}_T, \underline{v}_T) := \int_T \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \underline{u}_T(\boldsymbol{x})) \cdot \boldsymbol{G}_T^k \underline{v}_T(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + s_T(\underline{u}_T, \underline{v}_T)}$$

- The stabilisation term $s_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ is s.t.

$$s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{1-p} \int_F \left|\delta_{TF}^k \underline{u}_T\right|^{p-2} \delta_{TF}^k \underline{u}_T \; \delta_{TF}^k \underline{v}_T,$$

with face-based residual operator $\delta_{TF}^k : \underline{U}_T^k \to \mathbb{P}^k(F)$ s.t.

$$\delta_{TF}^k \underline{v}_T := \pi_F^{0,k} \left(v_F - p_T^{k+1} \underline{v}_T - \pi_T^{0,k}(v_T - p_T^{k+1} \underline{v}_T) \right)$$

- Polynomial consistency: $\delta_{TF}^k \underline{I}_T^k v = 0$ for all $v \in \mathbb{P}^{k+1}(T)$

Global problem II

- Define the following global space with single-valued interface DOFs:

$$\underline{U}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right)$$

- A global function $\mathrm{A}_h : \underline{U}_h^k \times \underline{U}_h^k \to \mathbb{R}$ is assembled element-wise:

$$\mathrm{A}_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{A}_T(\underline{u}_T, \underline{v}_T)$$

- We seek $\underline{u}_h \in \underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \mid v_F = 0 \ \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$ s.t.

$$\boxed{\mathrm{A}_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

with $v_{h|T} = v_T$ for all $T \in \mathcal{T}_h$

Global problem III

- Define on $\underline{U}_h^k$ the $W^{1,p}$-like seminorm (this is a norm on $\underline{U}_{h,0}^k$)

$$\|\underline{v}_h\|_{1,p,h}^p := \sum_{T\in\mathcal{T}_h}\left(\|\nabla v_T\|_{L^p(T)^d}^p + \sum_{F\in\mathcal{F}_T} h_F^{1-p}\|v_F - v_T\|_{L^p(F)}^p\right)$$

- We have coercivity for A_h: For all $\underline{v}_h \in \underline{U}_h^k$,

$$\boxed{\|\underline{v}_h\|_{1,p,h}^p \lesssim \mathrm{A}_h(\underline{v}_h, \underline{v}_h)}$$

- Existence for $\underline{u}_h$ follows (cf. [Deimling, 1985]) with a priori estimate

$$\|\underline{u}_h\|_{1,p,h} \leqslant C\|f\|_{L^{p'}(\Omega)}^{\frac{1}{p-1}}$$

Convergence to minimal regularity solutions I

Theorem (Convergence)

Up to a subsequence as $h \to 0$, with $p^* = \frac{dp}{d-p}$ if $p < d$, $+\infty$ otherwise,

- $u_h \to u$ *and* $p_h^{k+1}\underline{u}_h \to u$ *strongly in $L^q(\Omega)$ for all $q < p^*$,*
- $\boldsymbol{G}_h^k \underline{u}_h \to \nabla u$ *weakly in $L^p(\Omega)^d$.*

Additionally, if $\mathbf{a}$ is strictly monotone,

- $\boldsymbol{G}_h^k \underline{u}_h \to \nabla u$ *strongly in $L^p(\Omega)^d$.*

In this case, both u and $\underline{u}_h$ are unique and the whole sequence converges.

Convergence to minimal regularity solutions II

Key discrete functional analysis results on hybrid polynomial spaces:

Lemma (Discrete Sobolev embeddings)

Let $1 \leqslant q \leqslant p^*$ if $1 \leqslant p < d$ and $1 \leqslant q < +\infty$ if $p \geqslant d$. Then, there exists C only depending on Ω, ϱ, k, q and p s.t. for all $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|v_h\|_{L^q(\Omega)} \leqslant C \|\underline{v}_h\|_{1,p,h}.$$

Lemma (Discrete compactness)

Let $(\underline{v}_h)_{h \in \mathcal{H}}$ be s.t. $\|\underline{v}_h\|_{1,p,h} \leqslant C$ for a fixed $C \in \mathbb{R}$. Then, there exists $v \in W_0^{1,p}(\Omega)$ s.t., up to a subsequence as $h \to 0$,

- $v_h \to v$ *and* $p_h^{k+1} \underline{v}_h \to v$ *strongly in* $L^q(\Omega)$ *for all* $q < p^*$,
- $\boldsymbol{G}_h^k \underline{v}_h \to \nabla v$ *weakly in* $L^p(\Omega)^d$.

Error estimates I

Assumption (Leray–Lions operator/v2)

For $p \in (1, +\infty)$, $\mathbf{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *satisfies*

- *Growth. Same as before*
- *Continuity. There is* $\gamma_{\mathbf{a}} > 0$ *s.t. for a.e.* $\boldsymbol{x} \in \Omega$, $\forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$

$$|\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| \leqslant \gamma_{\mathbf{a}} |\boldsymbol{\xi} - \boldsymbol{\eta}| (|\boldsymbol{\xi}|^{p-2} + |\boldsymbol{\eta}|^{p-2}).$$

- *Monotonicity. There is* $\zeta_{\mathbf{a}} > 0$ *s.t. for a.e.* $\boldsymbol{x} \in \Omega$, $\forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$,

$$[\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})] \cdot [\boldsymbol{\xi} - \boldsymbol{\eta}] \geqslant \zeta_{\mathbf{a}} |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 (|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p-2}.$$

- *Coercivity. Same as before*

Error estimates II

Theorem (Error estimate)

Assume $u \in W^{k+2,p}(\mathcal{T}_h)$, $\mathbf{a}(\cdot, \nabla u) \in W^{k+1,p'}(\mathcal{T}_h)^d$, *and let, if* $p \geqslant 2$,

$$\mathrm{E}_h(u) := h^{k+1}|u|_{W^{k+2,p}(\mathcal{T}_h)} + h^{\frac{k+1}{p-1}}\left(|u|^{\frac{1}{p-1}}_{W^{k+2,p}(\mathcal{T}_h)} + |\mathbf{a}(\cdot, \nabla u)|^{\frac{1}{p-1}}_{W^{k+1,p'}(\mathcal{T}_h)}\right),$$

while, if $p < 2$,

$$\mathrm{E}_h(u) := h^{(k+1)(p-1)}|u|^{p-1}_{W^{k+2,p}(\mathcal{T}_h)} + h^{k+1}|\mathbf{a}(\cdot, \nabla u)|_{W^{k+1,p'}(\mathcal{T}_h)}.$$

Then, it holds,

$$\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h} \lesssim \mathrm{E}_h(u) = \begin{cases} \mathcal{O}(h^{\frac{k+1}{p-1}}) & \text{if } p \geqslant 2, \\ \mathcal{O}(h^{(k+1)(p-1)} & \text{if } p < 2. \end{cases}$$

Results coherent with [Liu and Yan, 2001] (Crouzeix–Raviart)

Numerical example I

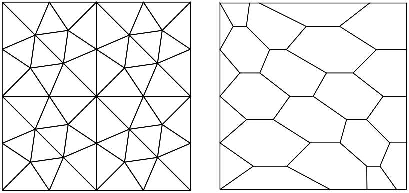

Figure: Triangular and (predominantly) hexagonal meshes

- We consider the following exact solution

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$$

- We solve the corresponding Dirichlet problem for $p \in \{2, 3, 4\}$

Numerical example II

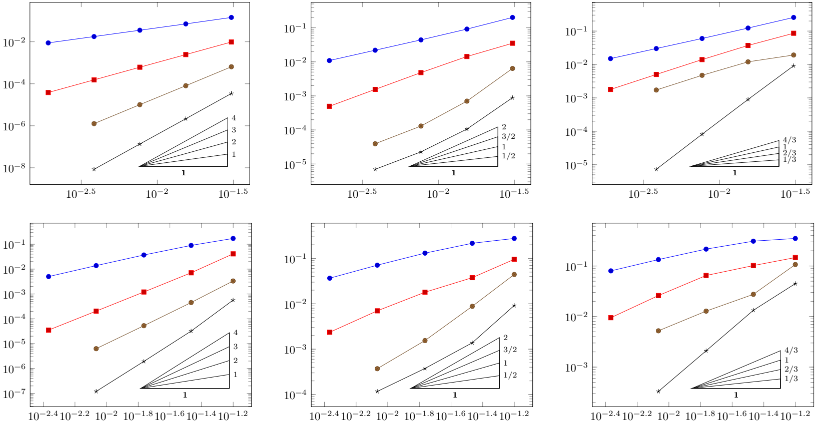

Figure: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ vs. h for $p = 2, 3, 4$ (left to right) for the triangular (above) and hexagonal (below) mesh families

Variations I

- Following [Cockburn, DP, Ern, 2016], one could replace $\underline{U}_T^k$ with

$$\underline{U}_T^{l,k} := \mathbb{P}^l(T) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right), \quad l \in \{k-1, k, k+1\}$$

- $\boldsymbol{G}_T^k$ and p_T^{k+1} remain formally the same (only their domain changes)
- The boundary residual operator, on the other hand, becomes

$$\delta_{TF}^{l,k} \underline{v}_T := \pi_F^{0,k} \left(v_F - p_T^{k+1} \underline{v}_T - \pi_T^{0,l}(v_T - p_T^{k+1} \underline{v}_T) \right)$$

Variations II

- Convergence and error estimates as for the original HHO method
- $l = k-1$ yields a HOM/nc-VEM-type scheme
  - Linear diffusion [Lipnikov and Manzini, 2014]
  - Analysis [Ayuso de Dios, Lipnikov, Manzini, 2016]
- $l = k$ corresponds to the original HHO method
- $l = k+1$ yields a Lehrenfeld–Schöberl-type HDG method
  - Linear diffusion [Lehrenfeld, 2010]
- $k = 0$ and $l = k - 1$ on simplices yields the Crouzeix–Raviart element
- **The globally-coupled unknowns coincide in all the cases!**

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