Hybrid High-Order methods for diffusion problems on polytopes and curved elements

Daniele A. Di Pietro

from joint works with L. Botti, J. Droniou, A. Ern, S. Lemaire

Institut Montpelliérain Alexander Grothendieck, University of Montpellier

MAFELAP, 19 June 2019

Features

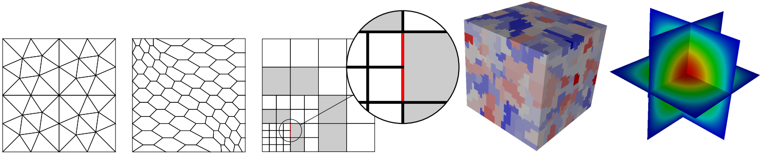


Figure: Examples of supported meshes $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ in 2d and 3d

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Physical fidelity leading to robustness in singular limits
- Natural extension to nonlinear problems
- Reduced computational cost after static condensation

References

- HHO for pure diffusion [DP, Ern, Lemaire, 2014]
- Curved faces and comparison with DG [Botti and DP, 2018]
- Optimal approximation for projectors [DP and Droniou, 2017ab]

New book!

D. A. Di Pietro and J. Droniou
The Hybrid High-Order Method for Polytopal Meshes
Design, Analysis, and Applications
516 pages, http://hal.archives-ouvertes.fr/hal-02151813

Model problem

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denote a bounded connected polyhedral domain
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in U := H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \forall v \in U$$

Projectors on local polynomial spaces

- With $X \in \mathcal{T}_h \cup \mathcal{F}_h$, the L^2-projector $\pi_X^{0,\ell} : L^2(X) \to \mathbb{P}^l(X)$ is s.t.

$$\int_X (\pi_X^{0,l} v - v) w = 0 \text{ for all } w \in \mathbb{P}^\ell(X)$$

- The elliptic projector $\pi_T^{1,\ell} : H^1(T) \to \mathbb{P}^\ell(T)$ is s.t.

$$\int_T \nabla(\pi_T^{1,\ell} v - v)\cdot\nabla w = 0 \text{ for all } w \in \mathbb{P}^\ell(T) \text{ and } \int_T (\pi_T^{1,\ell} v - v) = 0$$

- Both have optimal approximation properties in $\mathbb{P}^\ell(T)$

Computing $\pi_T^{1,k+1}$ from L^2-projections of degree k

- Recall the following IBP valid for all $v \in H^1(T)$ and all $w \in C^\infty(\overline{T})$:

$$\boxed{\int_T \nabla v \cdot \nabla w = -\int_T v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v \nabla w \cdot \boldsymbol{n}_{TF}}$$

- Specializing it to $w \in \mathbb{P}^{k+1}(T)$, we can write

$$\int_T \nabla \pi_T^{1,k+1} v \cdot \nabla w = -\int_T \pi_T^{0,k} v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^{0,k} v \nabla w \cdot \boldsymbol{n}_{TF}$$

- Moreover, it can be easily seen that

$$\int_T (\pi_T^{1,k+1} v - v) = \int_T (\pi_T^{1,k+1} v - \pi_T^{0,k} v) = 0$$

- **Hence, $\pi_T^{1,k+1} v$ can be computed from $\pi_T^{0,k} v$ and $(\pi_F^{0,k} v)_{F \in \mathcal{F}_T}$!**

Discrete unknowns

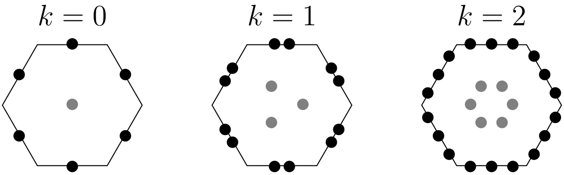


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- Let a polynomial degree $k \geq 0$ be fixed
- For all $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$\underline{U}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \ : \ v_T \in \mathbb{P}^k(T) \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ is s.t., for all $v \in H^1(T)$,

$$\underline{I}_T^k v := \left(\pi_T^{0,k} v, (\pi_F^{0,k} v)_{F \in \mathcal{F}_T} \right)$$

Local potential reconstruction

- Let $T \in \mathcal{T}_h$. We define the local potential reconstruction operator

$$r_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$$

s.t., for all $\underline{v}_T \in \underline{U}_T^k$, $\int_T (r_T^{k+1}\underline{v}_T - v_T) = 0$ and

$$\boxed{\int_T \nabla r_T^{k+1}\underline{v}_T \cdot \nabla w = -\int_T v_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F \nabla w \cdot \boldsymbol{n}_{TF} \quad \forall w \in \mathbb{P}^{k+1}(T)}$$

- By construction, we have

$$\boxed{r_T^{k+1} \circ \underline{I}_T^k = \pi_T^{1,k+1}}$$

- $(r_T^{k+1} \circ \underline{I}_T^k)$ has therefore optimal approximation properties in $\mathbb{P}^{k+1}(T)$

Stabilization I

- We would be tempted to approximate

$$a_{|T}(u,v) \approx a_{|T}(r_T^{k+1}\underline{u}_T, r_T^{k+1}\underline{v}_T)$$

- This choice, however, is not stable in general. We consider instead

$$\boxed{\mathrm{a}_T(\underline{u}_T,\underline{v}_T) \coloneqq a_{|T}(r_T^{k+1}\underline{u}_T, r_T^{k+1}\underline{v}_T) + \mathrm{s}_T(\underline{u}_T,\underline{v}_T)}$$

- The role of s_T is to ensure $\|\cdot\|_{1,T}$-coercivity with

$$\|\underline{v}_T\|_{1,T}^2 \coloneqq \|\nabla v_T\|_{L^2(T)^d}^2 + \sum_{F\in\mathcal{F}_T}\frac{1}{h_F}\|v_F - v_T\|_{L^2(F)}^2 \quad \forall \underline{v}_T \in \underline{U}_T^k$$

Stabilization II

Assumption (Stabilization bilinear form)

The bilinear form $\mathrm{s}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ satisfies the following properties:

- Symmetry and positivity. s_T is symmetric and positive semidefinite.
- Stability. It holds, with hidden constant independent of h and T,

$$\mathrm{a}_T(\underline{v}_T, \underline{v}_T)^{\frac{1}{2}} \simeq \|\underline{v}_T\|_{1,T} \quad \forall \underline{v}_T \in \underline{U}_T^k.$$

- Polynomial consistency. For all $w \in \mathbb{P}^{k+1}(T)$ and all $\underline{v}_T \in \underline{U}_T^k$,

$$\mathrm{s}_T(\underline{I}_T^k w, \underline{v}_T) = 0.$$

Stabilization III

- The following stable choice violates polynomial consistency:

$$\mathrm{s}_T^{\mathrm{hdg}}(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} \int_F (u_F - u_T)\, (v_F - v_T)$$

- To circumvent this problem, we penalize the high-order differences

$$(\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T}) := \underline{I}_T^k r_T^{k+1} \underline{v}_T - \underline{v}_T$$

- The classical HHO stabilization bilinear form reads

$$\boxed{\mathrm{s}_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} \int_F (\delta_T^k - \delta_{TF}^k)\underline{u}_T \; (\delta_T^k - \delta_{TF}^k)\underline{v}_T}$$

Discrete problem

- Define the global space with single-valued interface unknowns

$$\underline{U}_h^k := \Big\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) \;:\; v_T \in \mathbb{P}^k(T) \quad \forall T\in\mathcal{T}_h \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F\in\mathcal{F}_h \Big\}$$

  and its subspace with strongly enforced boundary conditions

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \;:\; v_F = 0 \quad \forall F\in\mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{\mathrm{a}_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{a}_T(\underline{u}_T, \underline{v}_T) = \sum_{T\in\mathcal{T}_h} \int_T f v_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

- Well-posedness follows from coercivity and discrete Poincaré

Convergence

Theorem (Energy-norm error estimate)

Assume $u \in H_0^1(\Omega) \cap H^{k+2}(\mathcal{T}_h)$. *The following energy error estimate holds:*

$$\boxed{\|\nabla_h(r_h^{k+1}\underline{u}_h - u)\| + |\underline{u}_h|_{s,h} \lesssim h^{k+1}|u|_{H^{k+2}(\mathcal{T}_h)}}$$

with $(r_h^{k+1}\underline{u}_h)_{|T} := r_T^{k+1}\underline{u}_T$ *for all* $T \in \mathcal{T}_h$ *and* $|\underline{u}_h|_{s,h}^2 := \sum_{T\in\mathcal{T}_h} \mathrm{s}_T(\underline{u}_T, \underline{u}_T)$.

Theorem (Superclose L^2-norm error estimate)

Further assuming elliptic regularity and $f \in H^1(\mathcal{T}_h)$ *if* $k = 0$,

$$\boxed{\|r_h^{k+1}\underline{u}_h - u\| \lesssim h^{k+2}\mathcal{N}_k,}$$

with $\mathcal{N}_0 := \|f\|_{H^1(\mathcal{T}_h)}$ *and* $\mathcal{N}_k := |u|_{H^{k+2}(\mathcal{T}_h)}$ *for* $k \geq 1$.

Numerical examples

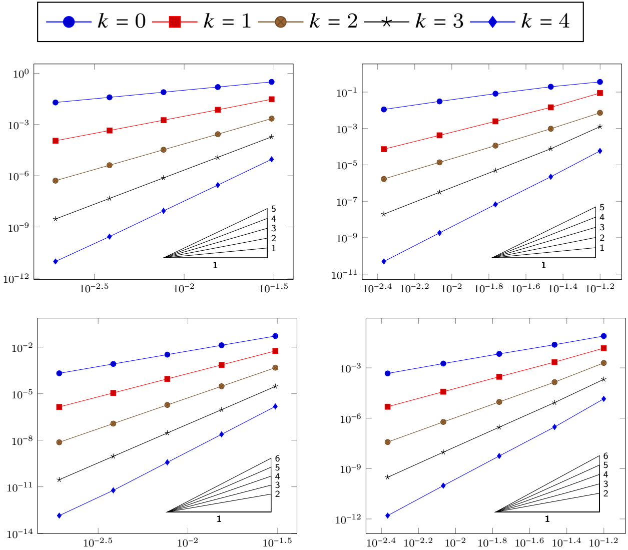


Figure: Trigonometric solution, energy norm (top) and L^2-norm vs. h (bottom) for triangular (left) and polygonal (right) mesh sequences

Extension to curved faces I

- Let $F \in \mathcal{F}_h$ denote a mesh face
- Let σ be reference face and $\boldsymbol{\Psi}_F$ an invertible mapping s.t.

$$F = \boldsymbol{\Psi}_F(\sigma)$$

- We assume that $\boldsymbol{\Psi}_F \in \mathbb{M}^m_{d-1}(\sigma)^d$ with $m \geq 1$ and

$$\mathbb{M}^m_{d-1}(\sigma) \in \left\{\mathbb{P}^m_{d-1}(\sigma), \mathbb{S}^m_{d-1}(\sigma), \mathbb{Q}^m_{d-1}(\sigma)\right\}$$

- The effective mapping order is the smallest integer $\widetilde{m}$ s.t.

$$\boldsymbol{\Psi}_F \in \mathbb{P}^{\widetilde{m}}_{d-1}(\sigma)^d$$

Extension to curved faces II

- Given an integer $l \geq k$, consider the modified HHO space:

$$\underline{U}_h^{k,l} := \Big\{ \underline{v}_T = (v_T, (v_\sigma)_{\Psi_F(\sigma)\in\mathcal{F}_T}) \ : \ v_T \in \mathbb{P}^k(T) \quad \forall T \in \mathcal{T}_h \text{ and } v_\sigma \in \mathbb{P}^l_{d-1}(\sigma) \quad \forall \Psi_F(\sigma) \in \mathcal{F}_h \Big\}$$

- We interpolate at faces mapping $v : F \to \mathbb{R}$ on $\pi_\sigma^l v \in \mathbb{P}^l_{d-1}(\sigma)$ s.t.

$$\int_\sigma (v \circ \Psi_F - \pi_\sigma^l v) z |\boldsymbol{J}_{\Psi_F}| = 0 \qquad \forall z \in \mathbb{P}^k_{d-1}(\sigma)$$

- For all $T \in \mathcal{T}_h$, $r_T^{k+1} : \underline{U}_T^{k,l} \to \mathbb{P}^{k+1}(T)$ is s.t., for all $w \in \mathbb{P}^{k+1}(T)$,

$$\int_T \boldsymbol{\nabla} r_T^{k+1} \underline{v}_T \cdot \boldsymbol{\nabla} w = -\int_T v_T \Delta w + \sum_{F=\Psi_F(\sigma)\in\mathcal{F}_T} \int_F (v_\sigma \circ \Psi_F^{-1}) \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF}$$

- **What about the commutation with the elliptic projector?**

Extension to curved faces III

Proposition (Comparison with the elliptic projector)

It holds, for all $T \in \mathcal{T}_h$:

- *If $\widetilde{m} = 1$ then, for all $v \in H^1(T)$,*

$$r_T^{k+1} \underline{I}_T^{k,l} v = \pi_T^{1,k+1} v \quad \forall l \geq k;$$

- *If $\widetilde{m} > 1$, for all $v \in H^{\widetilde{k}+1}(T)$ with $\widetilde{k} := \lfloor l/\widetilde{m} \rfloor$,*

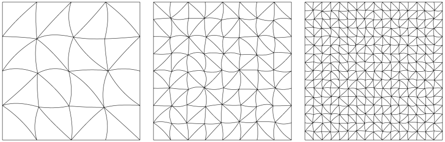
$$\|\nabla(r_T^{k+1} \underline{I}_T^{k,l} v - \pi_T^{1,k+1} v)\|_T \lesssim h_T^{\widetilde{k}} |v|_{H^{\widetilde{k}+1}(T)}.$$

Optimal error estimates are obtained with the following choice:

$$l_{\text{opt}} = \begin{cases} k & \text{if } \widetilde{m} = 1, \\ \widetilde{m}(k+1) & \text{if } \widetilde{m} > 1. \end{cases}$$

Numerical examples

$d = 2$, tri3 and tri6 meshes, quadratic solution



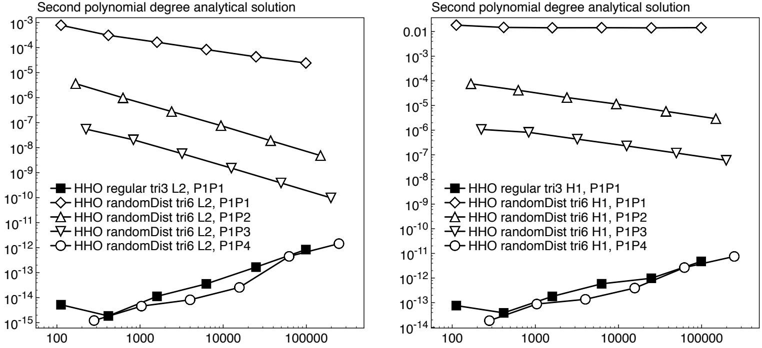


Figure: Error versus number of DOFs for HHO discretizations of the Poisson equation on regular 3-node ($\widetilde{m} = 1$) and randomly distorted 6-node triangular grids ($\widetilde{m} = 2 \implies l_{\mathrm{opt}} = 2(k+1)$).
Machine error precision expected and observed for $l = 4$.

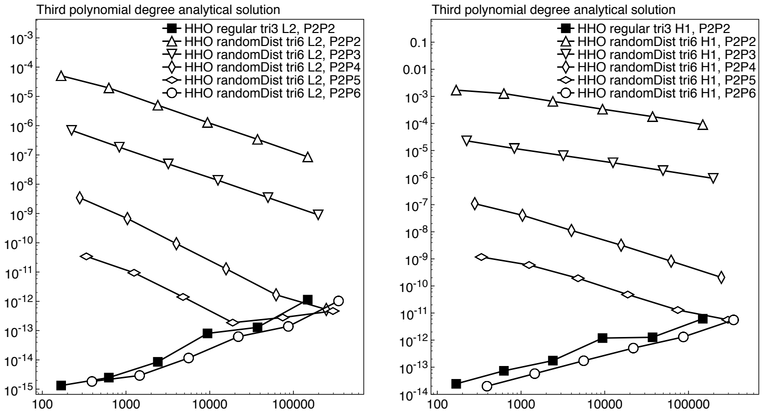
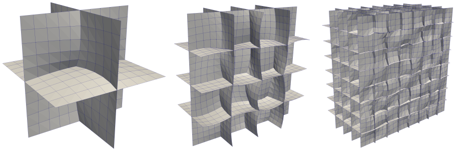
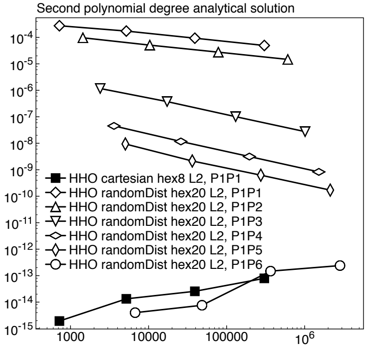


Figure: Error versus number of DOFs for HHO discretizations of the Poisson equation on regular 3-node ($\widetilde{m} = 1$) and randomly distorted 6-node triangular grids ($\widetilde{m} = 2 \implies l_{\text{opt}} = 2(k+1)$).
Machine error precision expected and observed for $l = 6$.

Numerical examples

$d = 3$, tri3 and tri6 meshes, quadratic solution





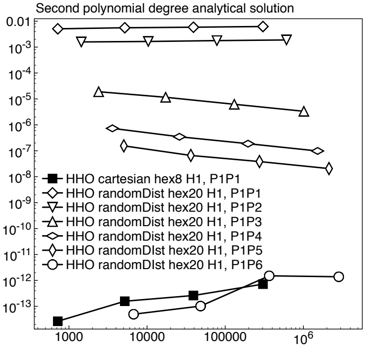


Figure: Error versus number of DOFs for HHO discretizations of the Poisson equation on regular 8-node ($\widetilde{m} = 1$) and randomly distorted 20-node hexahedral grids ($\widetilde{m} = 3 \implies l_{\text{opt}} = 3(k+1)$).
Machine error precision expected and observed for $l = 6$.

Thank you for your attention!

- **Wed1425** L. Botti, p-multilevel solution strategies for HHO
- **Wed1450** J. Droniou, HHO methods for the Brinkman model
- **Wed1515** D. Castanon-Quiroz, Pressure-robust HHO methods

References

Botti, L. and Di Pietro, D. A. (2018).
Numerical assessment of Hybrid High-Order methods on curved meshes and comparison with discontinuous Galerkin methods.
J. Comput. Phys., 370:58–84.

Di Pietro, D. A. and Droniou, J. (2017a).
A Hybrid High-Order method for Leray–Lions elliptic equations on general meshes.
Math. Comp., 86(307):2159–2191.

Di Pietro, D. A. and Droniou, J. (2017b).
$W^{s,p}$-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a Hybrid High-Order discretisation of Leray–Lions problems.
Math. Models Methods Appl. Sci., 27(5):879–908.

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Meth. Appl. Math., 14(4):461–472.