Recent advances on Hybrid High-Order method for nonlinear problems

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Outline

Model problem I

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a polytopal bounded connected domain
- Let $p \in (1, +\infty)$ and $f \in L^{p'}(\Omega)$ with $p' := \frac{p}{p-1}$
- We consider the Leray–Lions problem: Find $u \in W_0^{1,p}(\Omega)$ s.t.

$$\boxed{A(u,v) := \int_\Omega \mathbf{a}(\boldsymbol{x}, \nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega fv \quad \forall v \in W_0^{1,p}(\Omega)}$$

- A typical example is the p-Laplacian: For $p \in (1, +\infty)$,

$$\mathbf{a}(\boldsymbol{x}, \xi) = |\xi|^{p-2}\xi$$

- Applications to glaciology, turbulent porous media flow, airfoil design
- **Perfect playground for discrete functional analysis tools ☺**

Model problem II

Assumption (Leray–Lions operator/v1)

For a fixed index $p \in (1, +\infty)$, $f \in L^{p'}(\Omega)$ *and* $\mathbf{a}$ *satisfies*

- *Growth.* $\mathbf{a}(\cdot, \mathbf{0}) \in L^{p'}(\Omega)^d$ *and there is* $\beta_{\mathbf{a}} > 0$ *s.t.*

$$|\mathbf{a}(\boldsymbol{x}, \xi) - \mathbf{a}(\boldsymbol{x}, \mathbf{0})| \leq \beta_{\mathbf{a}} |\xi|^{p-1} \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \xi \in \mathbb{R}^d.$$

- *Monotonicity. For a.e.* $\boldsymbol{x} \in \Omega$, *for all* $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$[\mathbf{a}(\boldsymbol{x}, \xi) - \mathbf{a}(\boldsymbol{x}, \eta)] \cdot [\xi - \eta] \geq 0.$$

- *Coercivity. There is* $\lambda_{\mathbf{a}} > 0$ *s.t.*

$$\mathbf{a}(\boldsymbol{x}, \xi) \cdot \xi \geq \lambda_{\mathbf{a}} |\xi|^p \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \xi \in \mathbb{R}^d.$$

A dependence on u can also be included in the analysis

Features of HHO methods

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions $d \geq 1$
- Arbitrary approximation order (including $k = 0$)
- Applicable to a vast range of physical problems
- Reduced computational cost after hybridization

$$N_{\mathrm{dof}}^{\mathrm{hho}} \approx \frac{1}{2} k^2 \,\mathrm{card}(\mathscr{F}_h)$$

Mesh I

Definition (Mesh regularity)

We consider a sequence $(\mathscr{T}_h)_{h\in\mathscr{H}}$ of polyhedral meshes s.t., for all $h\in\mathscr{H}$, $\mathscr{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathscr{H}}$ is

- shape-regular in the usual sense of Ciarlet;
- contact-regular, i.e., every simplex $S\subset T$ is s.t. $h_S\approx h_T$.

Consequences [DP and Ern, 2012, DP and Droniou, 2016a]:

- L^p-trace and inverse inequalities
- Approximation for broken polynomial spaces

Mesh II

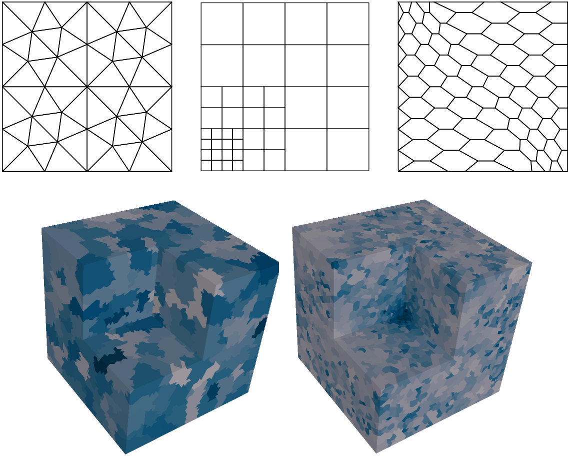


Figure: Examples of general meshes in 2d and 3d. The agglomerated 3d meshes are taken from [DP and Specogna, 2016]

Projectors on local polynomial spaces I

- The L²-orthogonal projector $\pi_T^{0,l} : L^1(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T (\pi_T^{0,l} v - v) w = 0 \text{ for all } w \in \mathbb{P}^l(T)$$

- The elliptic projector $\pi_T^{1,l} : W^{1,1}(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T \nabla(\pi_T^{1,l} v - v) \cdot \nabla w = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } \int_T (\pi_T^{1,l} v - v) = 0$$

Projectors on local polynomial spaces II

Lemma (Optimal approximation properties of local projectors)

For all $h \in \mathcal{H}$*, all* $T \in \mathcal{T}_h$*, all* $p \in [1, +\infty]$*, all* $s \in \{1, \ldots, l+1\}$*, all* $m \in \{0, \ldots, s-1\}$*, and all* $v \in W^{s,p}(T)$*, it holds with* $\star \in \{0,1\}$

$$|v - \pi_T^{\star,l} v|_{W^{m,p}(T)} + h_T^{\frac{1}{p}} |v - \pi_T^{\star,l} v|_{W^{m,p}(\mathcal{F}_T)} \lesssim h_T^{s-m} |v|_{W^{s,p}(T)}.$$

Proof.

Apply a general result from [DP and Droniou, 2016b]: every W-bounded projector has optimal approximation properties. □

Key ideas

- DOFs: polynomials of degree $k \geq 0$ at elements and faces
- Differential operators reconstructions tailored to the problem:

$$\boxed{A_{|T}(u,v) \approx \int_T \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \underline{u}_T(\boldsymbol{x})) \cdot \boldsymbol{G}_T^k \underline{v}_T(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \text{stab.}}$$

  with
  - gradient reconstruction $\boldsymbol{G}_T^k$ from local solves
  - high-order stabilisation using face-based penalty
- General meshes in any $d \geq 1$ and arbitrary polynomial degrees

DOFs and interpolation

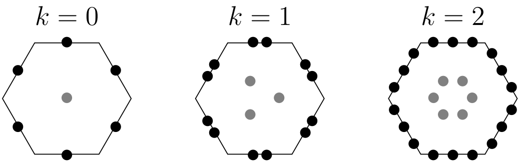


Figure: $\underline{U}_T^k$ for $k \in \{0,1,2\}$

- For $k \geq 0$ and $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

- The local interpolator $\underline{I}_T^k : W^{1,1}(T) \to \underline{U}_T^k$ is s.t.

$$\underline{I}_T^k v = (\pi_T^{0,k} v, (\pi_F^{0,k} v)_{F \in \mathcal{F}_T})$$

Operator reconstructions I

- We define the gradient reconstruction $\boldsymbol{G}_T^k : \underline{U}_T^k \mapsto \mathbb{P}^k(T)^d$ s.t.

$$\boxed{(\boldsymbol{G}_T^k \underline{v}_T, \boldsymbol{\phi})_T = -(v_T, \operatorname{div} \boldsymbol{\phi})_T + \sum_{F \in \mathscr{F}_T} (v_F, \boldsymbol{\phi} \cdot \boldsymbol{n}_{TF})_F \quad \forall \boldsymbol{\phi} \in \mathbb{P}^k(T)^d}$$

- Recalling the definition of $\underline{I}_T^k$, it holds for all $v \in W^{1,1}(T)$,

$$(\boldsymbol{G}_T^k \underline{I}_T^k v, \boldsymbol{\phi})_T = -(\cancel{\pi_T^{0,k}} v, \operatorname{div} \boldsymbol{\phi})_T + \sum_{F \in \mathscr{F}_T} (\cancel{\pi_F^{0,k}} v, \boldsymbol{\phi} \cdot \boldsymbol{n}_{TF})_F = (\nabla v, \boldsymbol{\phi})_T,$$

i.e., by definition of $\boldsymbol{\pi}_T^{0,k}$,

$$\boxed{\boldsymbol{G}_T^k \underline{I}_T^k v = \boldsymbol{\pi}_T^{0,k}(\nabla v)}$$

- As a result, $(\boldsymbol{G}_T^k \circ \underline{I}_T^k)$ has optimal $W^{s,p}$-approximation properties

Operator reconstructions II

- We also need the potential reconstruction $r_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$ s.t.

$$\boxed{(\nabla r_T^{k+1}\underline{v}_T - \boldsymbol{G}_T^k\underline{v}_T, \nabla w)_T = 0 \quad \forall w \in \mathbb{P}^{k+1}(T), \quad \int_T (r_T^{k+1}\underline{v}_T - v) = 0}$$

- Recalling the definition of $\boldsymbol{G}_T^k$ and $\underline{I}_T^k$, it holds for all $v \in W^{1,1}(T)$,

$$(\nabla r_T^{k+1}\underline{I}_T^k v, \nabla w)_T = -(\cancel{\pi_T^{0,k}} v, \triangle w)_T + \sum_{F\in\mathscr{F}_T} (\cancel{\pi_F^{0,k}} v, \nabla w\cdot\boldsymbol{n}_{TF})_F,$$

i.e. $(\nabla(r_T^{k+1}\underline{I}_T^k v - v), \nabla w)_T = 0$ and, by definition of $\pi_T^{1,k+1}$,

$$\boxed{r_T^{k+1}\underline{I}_T^k v = \pi_T^{1,k+1} v}$$

- As a result, $(r_T^{k+1} \circ \underline{I}_T^k)$ has optimal $W^{s,p}$-approximation properties

Global problem I

- For all $T \in \mathscr{T}_h$, we define the local function $\mathrm{A}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ s.t.

$$\boxed{\mathrm{A}_T(\underline{u}_T, \underline{v}_T) := \int_T \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \underline{u}_T(\boldsymbol{x})) \cdot \boldsymbol{G}_T^k \underline{v}_T(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + s_T(\underline{u}_T, \underline{v}_T)}$$

- The stabilisation term $s_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ is s.t.

$$s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathscr{F}_T} h_F^{1-p} \int_F |\delta_{TF}^k \underline{u}_T|^{p-2} \delta_{TF}^k \underline{u}_T \ \delta_{TF}^k \underline{v}_T,$$

with face-based residual operator $\delta_{TF}^k : \underline{U}_T^k \to \mathbb{P}^k(F)$ s.t.

$$\boxed{\delta_{TF}^k \underline{v}_T := \pi_F^{0,k} \left(v_F - r_T^{k+1} \underline{v}_T - \pi_T^{0,k}(v_T - r_T^{k+1} \underline{v}_T) \right)}$$

- Polynomial consistency: $\delta_{TF}^k \underline{I}_T^k w = 0$ for all $w \in \mathbb{P}^{k+1}(T)$

Global problem II

- Define the following global space with single-valued interface DOFs:

$$\underline{U}_h^k := \left(\bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F\in\mathcal{F}_h} \mathbb{P}^k(F) \right)$$

- A global function $\mathrm{A}_h : \underline{U}_h^k \times \underline{U}_h^k \to \mathbb{R}$ is assembled element-wise:

$$\mathrm{A}_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{A}_T(\underline{u}_T, \underline{v}_T)$$

- We seek $\underline{u}_h \in \underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \mid v_F = 0 \, \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$ s.t.

$$\boxed{\mathrm{A}_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

with $v_{h|T} = v_T$ for all $T \in \mathcal{T}_h$

Global problem III

- Define on $\underline{U}_h^k$ the $W^{1,p}$-like seminorm (this is a norm on $\underline{U}_{h,0}^k$)

$$\|\underline{v}_h\|_{1,p,h} := \sum_{T\in\mathcal{T}_h}\left(\|\nabla v_T\|_{L^p(T)^d}^p + \sum_{F\in\mathcal{F}_T} h_F^{1-p}\|v_F - v_T\|_{L^p(F)}^p\right)^{1/p}$$

- We have coercivity for A_h: For all $\underline{v}_h \in \underline{U}_h^k$,

$$\boxed{\|\underline{v}_h\|_{1,p,h}^p \lesssim \mathrm{A}_h(\underline{v}_h, \underline{v}_h)}$$

- Existence for $\underline{u}_h$ follows (cf. [Deimling, 1985]) with a priori estimate

$$\boxed{\|\underline{u}_h\|_{1,p,h} \le C\|f\|_{L^{p'}(\Omega)}^{\frac{1}{p-1}}}$$

Convergence to minimal regularity solutions I

Theorem (Convergence)

Up to a subsequence as $h \to 0$, with $p^* = \frac{dp}{d-p}$ if $p < d$, $+\infty$ otherwise,

- $u_h \to u$ *and* $r_h^{k+1}\underline{u}_h \to u$ *strongly in* $L^q(\Omega)$ *for all* $q < p^*$,
- $\boldsymbol{G}_h^k \underline{u}_h \to \nabla u$ *weakly in* $L^p(\Omega)^d$.

Additionally, if $\mathbf{a}$ *is strictly monotone,*

- $\boldsymbol{G}_h^k \underline{u}_h \to \nabla u$ *strongly in* $L^p(\Omega)^d$.

In this case, both u *and* $\underline{u}_h$ *are unique and the whole sequence converges.*

Discrete functional analysis results

Lemma (Discrete Sobolev embeddings)

For all Lebesgue index q s.t. $1 \leq q \leq p^*$ if $1 \leq p < d$, $1 \leq q < +\infty$ if $p \geq d$, there exists C only depending on Ω, ρ, k, q and p s.t. $\forall \underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|v_h\|_{L^q(\Omega)} \leq C\|\underline{v}_h\|_{1,p,h}.$$

Lemma (Discrete compactness)

Let $(\underline{v}_h)_{h\in\mathcal{H}}$ be s.t. $\|\underline{v}_h\|_{1,p,h} \leq C$ for a fixed $C \in \mathbb{R}$. Then, there exists $v \in W_0^{1,p}(\Omega)$ s.t., up to a subsequence as $h \to 0$,

- *$v_h \to v$ and $r_h^{k+1}\underline{v}_h \to v$ strongly in $L^q(\Omega)$ for all $q < p^*$,*
- *$\boldsymbol{G}_h^k\underline{v}_h \to \nabla v$ weakly in $L^p(\Omega)^d$.*

Error estimates I

Assumption (Leray–Lions operator/v2)

For $p \in (1, +\infty)$, $\mathbf{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *satisfies*

- *Growth. Same as before*
- *Continuity. There is* $\gamma_{\mathbf{a}} > 0$ *s.t. for a.e.* $\boldsymbol{x} \in \Omega$, $\forall \xi, \eta \in \mathbb{R}^d$

$$|\mathbf{a}(\boldsymbol{x}, \xi) - \mathbf{a}(\boldsymbol{x}, \eta)| \leq \gamma_{\mathbf{a}} |\xi - \eta| (|\xi|^{p-2} + |\eta|^{p-2}).$$

- *Monotonicity. There is* $\zeta_{\mathbf{a}} > 0$ *s.t. for a.e.* $\boldsymbol{x} \in \Omega$, $\forall \xi, \eta \in \mathbb{R}^d$,

$$[\mathbf{a}(\boldsymbol{x}, \xi) - \mathbf{a}(\boldsymbol{x}, \eta)] \cdot [\xi - \eta] \geq \zeta_{\mathbf{a}} |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2}.$$

- *Coercivity. Same as before*

Error estimates II

Theorem (Error estimate)

Assume $u \in W^{k+2,p}(\mathcal{T}_h)$, $\mathbf{a}(\cdot,\nabla u) \in W^{k+1,p'}(\mathcal{T}_h)^d$, *and let, if* $p \geq 2$,

$$\mathrm{E}_h(u) := h^{k+1}|u|_{W^{k+2,p}(\mathcal{T}_h)} + h^{\frac{k+1}{p-1}}\left(|u|^{\frac{1}{p-1}}_{W^{k+2,p}(\mathcal{T}_h)} + |\mathbf{a}(\cdot,\nabla u)|^{\frac{1}{p-1}}_{W^{k+1,p'}(\mathcal{T}_h)}\right),$$

while, if $p < 2$,

$$\mathrm{E}_h(u) := h^{(k+1)(p-1)}|u|^{p-1}_{W^{k+2,p}(\mathcal{T}_h)} + h^{k+1}|\mathbf{a}(\cdot,\nabla u)|_{W^{k+1,p'}(\mathcal{T}_h)}.$$

Then, it holds,

$$\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h} \lesssim \mathrm{E}_h(u) = \begin{cases} \mathcal{O}(h^{\frac{k+1}{p-1}}) & \text{if } p \geq 2, \\ \mathcal{O}(h^{(k+1)(p-1)}) & \text{if } p < 2. \end{cases}$$

Results coherent with [Liu and Yan, 2001] (Crouzeix–Raviart)

Numerical examples I

Trigonometric solution, $p \geq 2$

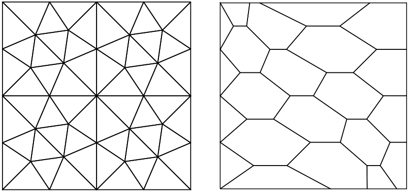


Figure: Triangular and (predominantly) hexagonal meshes

- We consider the p-Laplace problem:

$$\mathbf{a}(\boldsymbol{x}, \xi) = |\xi|^{p-2}\xi$$

- We solve the manufactured Dirichlet problem corresponding to

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$$

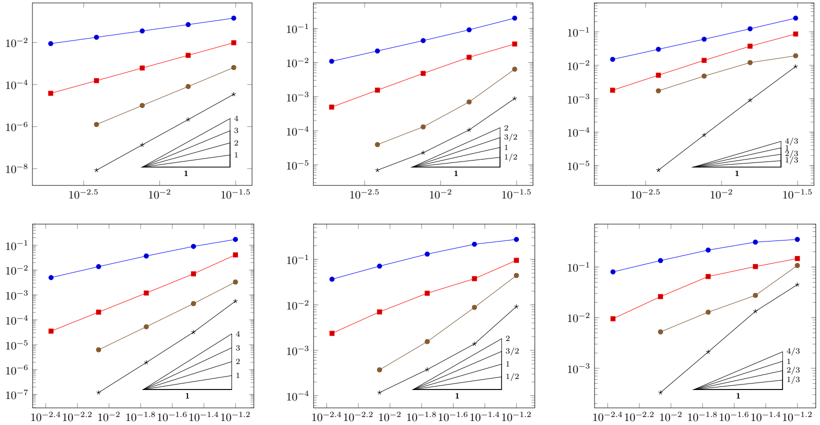
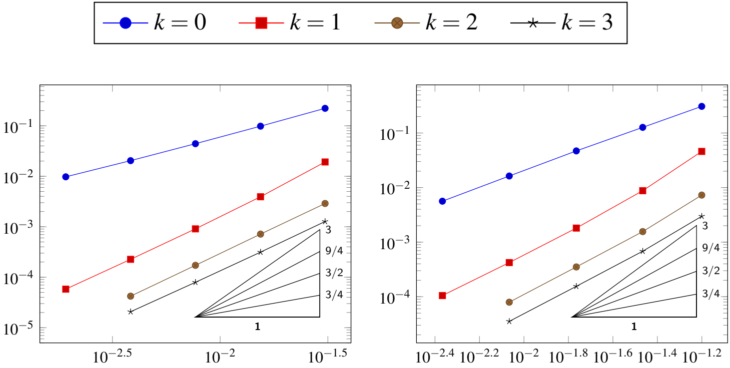


Figure: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ vs. h for $p = 2, 3, 4$ (left to right) for the triangular (above) and hexagonal (below) mesh families

- We consider the same solution but this time with $1 < p < 2$
- In this case, the derivatives of the function

$$\xi \mapsto |\xi|^{p-2}\xi$$

  are singular at $\xi = \mathbf{0}$
- This prevents the method from attaining optimal orders of convergence



(a) Triangular

(b) Hexagonal

Figure: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ versus h for the exponential solution with $p = \frac{7}{4}$.

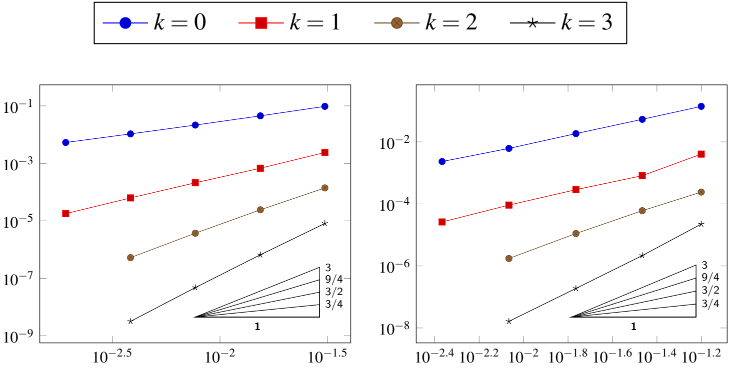
- To assess our error estimates, we therefore consider the solution

$$u(\boldsymbol{x}) = \exp(x_1 + \pi x_2)$$

- We solve again the corresponding manufactured Dirichlet problem
- This allows us to avoid dealing with singularities

Numerical examples II

Exponential solution, $1 < p < 2$



(a) Triangular (b) Hexagonal

Figure: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ versus h for the exponential solution with $p = \frac{7}{4}$.

Outline

The steady incompressible Navier–Stokes equations I

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded connected open polyhedron
- Let $\nu \in \mathbb{R}_+^*$ and $\boldsymbol{f} \in L^2(\Omega)^d$ (extension to variable ν is possible)
- The INS problem consists in finding $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\nu \triangle \boldsymbol{u} + \nabla \boldsymbol{u}\, \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, \\ \int_\Omega p &= 0 \end{aligned}$$

- We use the matrix-product notation: $\nabla \boldsymbol{v}\, \boldsymbol{w} = \left(\sum_{j=1}^d w_j \partial_j \boldsymbol{v} \right)_{1 \le i \le d}$

The steady incompressible Navier–Stokes equations II

- Let $\boldsymbol{U} := H_0^1(\Omega)^d$ and $P := L_0^2(\Omega)$
- The weak formulation reads: Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ s.t.

$$
\begin{aligned}
\nu a(\boldsymbol{u},\boldsymbol{v}) + t(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{U}, \\
-b(\boldsymbol{u},q) &= 0 && \forall q \in P,
\end{aligned}
$$

with diffusive and pressure-velocity coupling bilinear forms

$$
a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{v}, \qquad b(\boldsymbol{v},q) := -\int_\Omega (\operatorname{div}\boldsymbol{v})q,
$$

and convective trilinear form

$$
t(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \boldsymbol{v}^{\mathrm{T}} \nabla\boldsymbol{u}\,\boldsymbol{w}
$$

A key remark

- Integrating by parts and using $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$, we can prove that

$$\boxed{t(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2} \int_\Omega \boldsymbol{v}^\mathrm{T} \nabla \boldsymbol{u}\, \boldsymbol{w} - \frac{1}{2} \int_\Omega \boldsymbol{u}^\mathrm{T} \nabla \boldsymbol{v}\, \boldsymbol{w}}$$

- This shows that t is non dissipative: For all $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{U}$ it holds

$$t(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{v}) = 0$$

Discrete spaces I

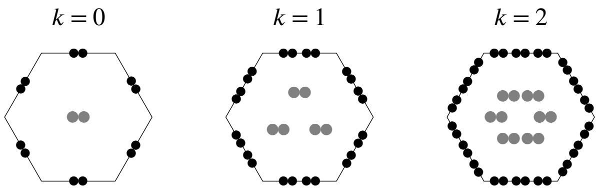


Figure: Local velocity space $\underline{\boldsymbol{U}}_T^k$ for $k \in \{0,1,2\}$

- Let a polynomial degree $k \geq 0$ be fixed and set

$$\underline{\boldsymbol{U}}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T)^d \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F)^d \right)$$

- We write the elements of $\underline{\boldsymbol{U}}_h^k$ as $\underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h} \big)$
- For the restrictions to a mesh element T we replace $h \leftarrow T$

Discrete spaces II

- The global interpolator $\underline{\boldsymbol{I}}_h^k : H^1(\Omega)^d \to \underline{\boldsymbol{U}}_h^k$ is s.t. $\forall \boldsymbol{v} \in H^1(\Omega)^d$

$$\underline{\boldsymbol{I}}_h^k \boldsymbol{v} := ((\boldsymbol{\pi}_T^{0,k} \boldsymbol{v})_{T \in \mathcal{T}_h}, (\boldsymbol{\pi}_F^{0,k} \boldsymbol{v})_{F \in \mathcal{F}_h})$$

- To account for BCs and the zero-average constraint on p, we set

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \mid \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\},$$

$$P_h^k := \left\{ q_h \in \mathbb{P}^k(\mathcal{T}_h) \,\middle|\, \int_\Omega q_h = 0 \right\}$$

- We define on $\underline{\boldsymbol{U}}_h^k$ the following seminorm (which is a norm on $\underline{\boldsymbol{U}}_{h,0}^k$):

$$\|\underline{\boldsymbol{v}}_h\|_{1,h} := \sum_{T \in \mathcal{T}_h} \left(\|\nabla \boldsymbol{v}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_F^2 \right)^{1/2}$$

Reconstructions of differential operators I

- For $l \geq 0$, the gradient reconstruction $\boldsymbol{G}_T^l : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^l(T)^{d\times d}$ is s.t.

$$\boxed{\int_T \boldsymbol{G}_T^l \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T \cdot (\operatorname{div} \boldsymbol{\tau}) + \sum_{F\in\mathscr{F}_T} \int_F \boldsymbol{v}_F \cdot (\boldsymbol{\tau} \boldsymbol{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^l(T)^{d\times d}}$$

- The choice $l = 2k$ will be used to discretize the convective term
- The global gradient reconstruction is defined setting $\forall T \in \mathscr{T}_h$

$$(\boldsymbol{G}_h^l \underline{\boldsymbol{v}}_h)_{|T} := \boldsymbol{G}_T^l \underline{\boldsymbol{v}}_T$$

Reconstructions of differential operators II

- The velocity reconstruction $\boldsymbol{r}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T)^d$ is s.t.

$$\boxed{\int_T (\nabla \boldsymbol{r}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{G}_T^k \underline{\boldsymbol{v}}_T) : \nabla \boldsymbol{w} = 0 \quad \forall \boldsymbol{w} \in \mathbb{P}^{k+1}(T)^d, \quad \int_T \boldsymbol{r}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_T = \boldsymbol{0}}$$

- Finally, the discrete divergence $D_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)$ is s.t.

$$\boxed{D_T^k = \operatorname{tr}(\boldsymbol{G}_T^k)}$$

- Global versions of the above operators are defined setting $\forall T \in \mathcal{T}_h$

$$(\boldsymbol{r}_h^{k+1} \underline{\boldsymbol{v}}_h)_{|T} := \boldsymbol{r}_T^{k+1} \underline{\boldsymbol{v}}_T, \quad (D_h^k \underline{\boldsymbol{v}}_h)_{|T} := D_T^k \underline{\boldsymbol{v}}_T$$

Viscous term

- The viscous term is discretized by means of the bilinear form a_h s.t.

$$\boxed{a_h(\underline{\boldsymbol{u}}_h,\underline{\boldsymbol{v}}_h) := \int_\Omega \boldsymbol{G}_h^k\underline{\boldsymbol{u}}_h : \boldsymbol{G}_h^k\underline{\boldsymbol{v}}_h + s_h(\underline{\boldsymbol{u}}_h,\underline{\boldsymbol{v}}_h)}$$

- Here, s_h is a stabilization bilinear form defined as follows:

$$s_h(\underline{\boldsymbol{u}}_h,\underline{\boldsymbol{v}}_h) := \sum_{T\in\mathscr{T}_h}\sum_{F\in\mathscr{F}_T} h_F^{-1}\int_F \delta_{TF}^k\underline{\boldsymbol{u}}_T \cdot \delta_{TF}^k\underline{\boldsymbol{v}}_T,$$

with face-based residual operator $\delta_{TF}^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(F)^d$ s.t.

$$\delta_{TF}^k\underline{\boldsymbol{v}}_T := \boldsymbol{\pi}_F^{0,k}\left(\boldsymbol{v}_F - \boldsymbol{r}_T^{k+1}\underline{\boldsymbol{v}}_T - \boldsymbol{\pi}_T^{0,k}(\boldsymbol{v}_T - \boldsymbol{r}_T^{k+1}\underline{\boldsymbol{v}}_T)\right)$$

- With this choice we have stability and consistency
- For variable viscosity, use the ideas of [DP and Ern, 2015]

Pressure-velocity coupling

- The pressure-velocity coupling is realized by the bilinear form

$$b_h(\underline{\boldsymbol{v}}_h, q_h) := -\int_\Omega D_h^k \underline{\boldsymbol{v}}_h q_h$$

- A crucial point is that b_h satisfies the following inf-sup condition

$$\forall q_h \in P_h^k, \quad \|q_h\|_{L^2(\Omega)} \lesssim \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \|\underline{\boldsymbol{v}}_h\|_{1,h}=1} b_h(\underline{\boldsymbol{v}}_h, q_h)$$

- **Valid on general meshes in $d \in \{2, 3\}$!**

Convective term I

- Recall the skew-symmetric expression of t written before:

$$t(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}\int_\Omega \boldsymbol{v}^{\mathrm{T}}\nabla\boldsymbol{u}\,\boldsymbol{w} - \frac{1}{2}\int_\Omega \boldsymbol{u}^{\mathrm{T}}\nabla\boldsymbol{v}\,\boldsymbol{w}$$

- Inspired by this reformulation of t, we set

$$\boxed{t_h(\underline{\boldsymbol{w}}_h,\underline{\boldsymbol{u}}_h,\underline{\boldsymbol{v}}_h) := \frac{1}{2}\int_\Omega \boldsymbol{v}_h^{\mathrm{T}}\boldsymbol{G}_h^{2k}\underline{\boldsymbol{u}}_h\,\boldsymbol{w}_h - \frac{1}{2}\int_\Omega \boldsymbol{u}_h^{\mathrm{T}}\boldsymbol{G}_h^{2k}\underline{\boldsymbol{v}}_h\,\boldsymbol{w}_h}$$

Convective term II

- In practice, one does not need to actually compute $\boldsymbol{G}_h^{2k}$ since

$$t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) = \sum_{T \in \mathscr{T}_h} t_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T),$$

where, for all $T \in \mathscr{T}_h$,

$$\boxed{\begin{aligned} t_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := &-\frac{1}{2}\int_T \boldsymbol{u}_T^{\mathrm{T}} \nabla \boldsymbol{v}_T \, \boldsymbol{w}_T + \frac{1}{2} \sum_{F \in \mathscr{F}_T} \int_F (\boldsymbol{u}_F \cdot \boldsymbol{v}_T)(\boldsymbol{w}_T \cdot \boldsymbol{n}_{TF}) \\ &+ \frac{1}{2}\int_T \boldsymbol{v}_T^{\mathrm{T}} \nabla \boldsymbol{u}_T \, \boldsymbol{w}_T - \frac{1}{2} \sum_{F \in \mathscr{F}_T} \int_F (\boldsymbol{v}_F \cdot \boldsymbol{u}_T)(\boldsymbol{w}_T \cdot \boldsymbol{n}_{TF}) \end{aligned}}$$

- By design, t_h is non dissipative: For all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h$,

$$t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) = 0$$

Discrete problem I

- The discrete problem reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ s.t.

$$\boxed{\begin{aligned} \nu a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + t_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -b_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 && \forall q_h \in P_h^k \end{aligned}}$$

- All element-based velocity DOFs and all but one pressure DOFs can be locally eliminated at each Newton iteration
- This leads to linear systems of size

$$d \,\mathrm{card}(\mathscr{F}_h^{\mathrm{i}}) \binom{k+d-1}{d-1} + \mathrm{card}(\mathscr{T}_h)$$

Discrete problem II

Theorem (Existence and a priori bounds)

There exists a solution $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ *such that*

$$\|\underline{\boldsymbol{u}}_h\|_{1,h} \lesssim \nu^{-1} \|\boldsymbol{f}\|_{L^2(\Omega)^d}, \quad \|p_h\|_{L^2(\Omega)} \lesssim \|\boldsymbol{f}\|_{L^2(\Omega)^d} + \nu^{-2} \|\boldsymbol{f}\|_{L^2(\Omega)^d}^2.$$

Theorem (Uniqueness of the discrete solution)

Assume that the right-hand side verifies

$$\|\boldsymbol{f}\|_{L^2(\Omega)^d} \leq C\nu^2$$

with $C > 0$ *small enough. Then, the solution is unique.*

Key tool: Discrete Sobolev embeddings with $p = 2$ and $p = 4$

Convergence I

Theorem (Convergence to minimal regularity solutions)

It holds up to a subsequence

- $\boldsymbol{u}_h \to \boldsymbol{u}$ *strongly in* $L^p(\Omega)^d$ *for* $p \in \begin{cases} [1,+\infty) & \text{if } d=2, \\ [1,6) & \text{if } d=3; \end{cases}$
- $\boldsymbol{G}_h^k \underline{\boldsymbol{u}}_h \to \nabla \boldsymbol{u}$ *strongly in* $L^2(\Omega)^{d\times d}$*;*
- $s_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h) \to 0$*;*
- $p_h \to p$ *strongly in* $L^2(\Omega)$*.*

Moreover, if the exact solution is unique, the whole sequence converges.

Key tool: Compactness of discrete gradients

Convergence II

Theorem (Convergence rates for small data)

Assume uniqueness for both $(\underline{\boldsymbol{u}}_h, p_h)$ *and* $(\boldsymbol{u}, p)$. *Assume, moreover, the additional regularity* $(\boldsymbol{u}, p) \in H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$, *as well as*

$$\|\boldsymbol{f}\|_{L^2(\Omega)^d} \leq C\nu^2$$

with $C > 0$ *small enough. Then, we have the following error estimate:*

$$\boxed{\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{1,h} + \nu^{-1}\|p_h - \pi_h^{0,k} p\|_{L^2(\Omega)} \lesssim h^{k+1} \mathcal{N}(\boldsymbol{u}, p)}$$

with $\mathcal{N}(\boldsymbol{u}, p) := (1 + \nu^{-1}\|\boldsymbol{u}\|_{H^2(\Omega)^d})\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d} + \nu^{-1}\|p\|_{H^{k+1}(\Omega)}$.

Key tool: Discrete Sobolev embeddings with $p = 2$ and $p = 4$

Numerical example: Kovasznay flow I

- Let ν be fixed and set

$$\mathrm{Re} := (2\nu)^{-1}, \quad \lambda := \mathrm{Re} - \left(\mathrm{Re}^2 + 4\pi^2\right)^{1/2}$$

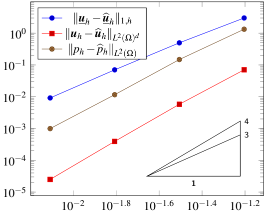
- We consider Kovasznay's exact solution:

$$u_1(\boldsymbol{x}) := 1 - \exp(\lambda x_1)\cos(2\pi x_2), \quad u_2(\boldsymbol{x}) := \frac{\lambda}{2\pi}\exp(\lambda x_1)\sin(2\pi x_2)$$

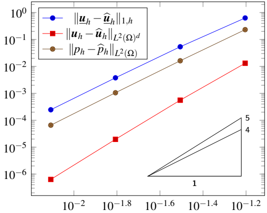
and

$$p(\boldsymbol{x}) := -\frac{1}{2}\exp(2\lambda x_1) + \frac{\lambda}{2}\left(\exp(4\lambda) - 1\right)$$

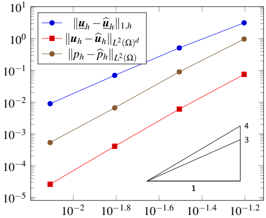
Numerical example: Kovasznay flow II



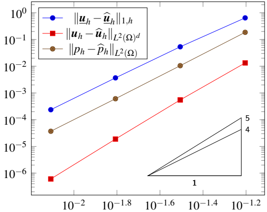
(a) t_h, $k = 2$



(b) t_h, $k = 3$



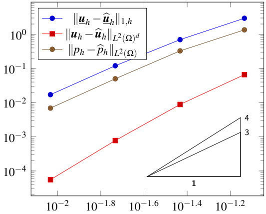
(c) t_h^{HDG}, $\eta = 0$, $k = 2$



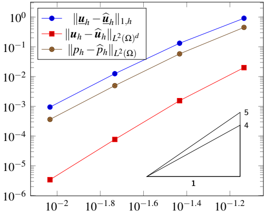
(d) t_h^{HDG}, $\eta = 0$, $k = 3$

Figure: Cartesian mesh family, errors versus h

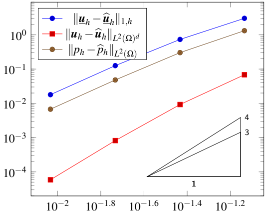
Numerical example: Kovasznay flow III



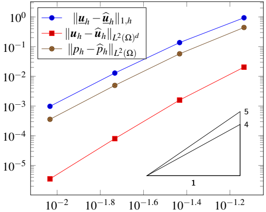
(a) t_h, $k = 2$



(b) t_h, $k = 3$



(c) t_h^{HDG}, $\eta = 0$, $k = 2$



(d) t_h^{HDG}, $\eta = 0$, $k = 3$

Figure: Hexagonal mesh family, errors versus h

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