Discrete de Rham (DDR) complexes for compatible approximations of physical problems on general meshes

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MOX, 26 May 2023

Outline

Setting I

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain with Betti numbers b_i
- We have $b_0 = 1$ (number of connected components) and $b_3 = 0$
- b_1 accounts for the number of tunnels crossing Ω

$$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$$

- b_2, on the other hand, is the number of voids encapsulated by Ω

$$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$$

Setting II

- We consider PDE models that hinge on the vector calculus operators:

$$\mathbf{grad}\, q = \begin{pmatrix} \partial_1 q \\ \partial_2 q \\ \partial_3 q \end{pmatrix}, \; \mathbf{curl}\, \boldsymbol{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}, \; \mathrm{div}\, \boldsymbol{w} = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$$

  for smooth enough functions

$$q : \Omega \to \mathbb{R}, \qquad \boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \boldsymbol{w} : \Omega \to \mathbb{R}^3$$

- The corresponding L^2-graph (domain) spaces are

$$\begin{aligned} H^1(\Omega) &:= \left\{ q \in L^2(\Omega) \;:\; \mathbf{grad}\, q \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^3 \right\}, \\ \boldsymbol{H}(\mathbf{curl}; \Omega) &:= \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \;:\; \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\}, \\ \boldsymbol{H}(\mathrm{div}; \Omega) &:= \left\{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \;:\; \mathrm{div}\, \boldsymbol{w} \in L^2(\Omega) \right\} \end{aligned}$$

- Given $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads:

  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p\cdot\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-\int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

- For $\mu > 0$ and $\boldsymbol{J} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}; \Omega)$, the magnetostatics problem reads: Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\mu \boldsymbol{H} - \mathbf{curl}\, \boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)} \\
\mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)} \\
\operatorname{div} \boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)} \\
\boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl}; \Omega) \times \boldsymbol{H}(\operatorname{div}; \Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \\
\int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \operatorname{div} \boldsymbol{A} \operatorname{div} \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}; \Omega)
\end{aligned}
$$

- Given $\kappa > 0$ and $f \in L^2(\Omega)$, the Darcy problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\kappa^{-1}\boldsymbol{u} - \mathbf{grad}\, p &= 0 && \text{in } \Omega, && \text{(Darcy's law)}\\
-\operatorname{div} \boldsymbol{u} &= f && \text{in } \Omega, && \text{(mass conservation)}\\
p &= 0 && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \kappa^{-1}\boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega p \operatorname{div} \boldsymbol{v} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega),\\
-\int_\Omega \operatorname{div} \boldsymbol{u}\, q &= \int_\Omega f\, q && \forall q \in L^2(\Omega)
\end{aligned}
$$

A unified view

- The above problems are mixed formulations involving two fields
- They can be recast into the abstract setting: Find $(\sigma, u) \in \Sigma \times U$ s.t.

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \quad \forall \tau \in \Sigma, \\ -b(\sigma,v) + c(u,v) &= g(v) \quad \forall v \in U, \end{aligned}$$

  or, equivalently, in variational formulation,

$$\mathcal{A}((\sigma,u),(\tau,v)) = f(\tau) + g(v) \qquad \forall (\tau,v) \in \Sigma \times U$$

  with

$$\mathcal{A}((\sigma,u),(\tau,v)) \coloneqq a(\sigma,\tau) + b(\tau,u) - b(\sigma,v) + c(u,v) = f(\tau) + g(v)$$

- Well-posedness holds under an inf-sup condition on $\mathcal{A}$

A unified tool for well-posedness: The de Rham complex

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Key properties depending on the topology of Ω:

$$\begin{aligned} \mathrm{Im}\,\mathbf{grad} &\subset \mathrm{Ker}\,\mathbf{curl}\,, \\ \mathrm{Im}\,\mathbf{curl} &\subset \mathrm{Ker}\,\mathrm{div}\,, \\ \boldsymbol{\Omega} \subset \mathbb{R}^3\ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} &= L^2(\Omega) \quad \text{(Darcy, magnetostatics)} \end{aligned}$$

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- Key properties depending on the topology of Ω:

$$\text{no tunnels crossing } \Omega \ (b_1 = 0) \implies \operatorname{Im}\mathbf{grad} = \operatorname{Ker}\mathbf{curl} \quad \text{(Stokes)}$$

$$\text{no voids contained in } \Omega \ (b_2 = 0) \implies \operatorname{Im}\mathbf{curl} = \operatorname{Ker}\mathrm{div} \quad \text{(magnetostatics)}$$

$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \operatorname{Im}\mathrm{div} = L^2(\Omega) \quad \text{(Darcy, magnetostatics)}$$

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$$\Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(Darcy, magnetostatics)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\mathbf{curl}\,/\mathrm{Im}\,\mathbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\mathbf{curl}$$

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$$\mathrm{Ker}\,\mathbf{curl}\,/\mathrm{Im}\,\mathbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\mathbf{curl}$$

- **Emulating these algebraic properties is key for stable discretizations**

Generalization through differential forms

- The de Rham complex generalizes to domains of $\mathbb{R}^n$ or smooth manifolds
- Denoting by d the exterior derivative and by $H\Lambda(\Omega)$ its domain,

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the vector calculus version is recovered through vector proxies

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

The (trimmed) Finite Element way

Local spaces

- Let $T \subset \mathbb{R}^3$ be a polyhedron and set, for any $k \geq -1$,

$$\mathcal{P}^k(T) \coloneqq \{\text{restrictions of 3-variate polynomials of degree} \leq k \text{ to } T\}$$

- Fix $k \geq 0$ and write, denoting by $\boldsymbol{x}_T$ a point inside T,

$$\begin{aligned}
\mathcal{P}^k(T)^3 &= \mathbf{grad}\,\mathcal{P}^{k+1}(T) \oplus (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}^{k-1}(T)^3 \eqqcolon \boldsymbol{\mathcal{G}}^k(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T) \\
&= \mathbf{curl}\,\mathcal{P}^{k+1}(T)^3 \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}^{k-1}(T) \qquad \eqqcolon \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T)
\end{aligned}$$

- Define the trimmed spaces that sit between $\mathcal{P}^k(T)^3$ and $\mathcal{P}^{k+1}(T)^3$:

$$\boldsymbol{\mathcal{N}}^{k+1}(T) \coloneqq \boldsymbol{\mathcal{G}}^k(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(T) \qquad \text{[Nédélec, 1980]}$$

$$\boldsymbol{\mathcal{RT}}^{k+1}(T) \coloneqq \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k+1}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

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$$\begin{aligned}\mathcal{P}^k(T)^3 &= \mathbf{grad}\,\mathcal{P}^{k+1}(T) \oplus (\boldsymbol{x}-\boldsymbol{x}_T)\times\mathcal{P}^{k-1}(T)^3 =: \boldsymbol{\mathcal{G}}^k(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T)\\ &= \mathbf{curl}\,\mathcal{P}^{k+1}(T)^3 \oplus (\boldsymbol{x}-\boldsymbol{x}_T)\mathcal{P}^{k-1}(T) \qquad =: \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T)\end{aligned}$$

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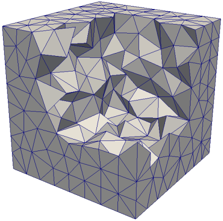
$$\boldsymbol{\mathcal{RT}}^{k+1}(T) := \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k+1}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

- The generalization $\mathcal{P}^{-,k}\Lambda^r(f)$ to r-forms on d-faces f is obtained using Koszul complements

The (trimmed) Finite Element way

Global complex

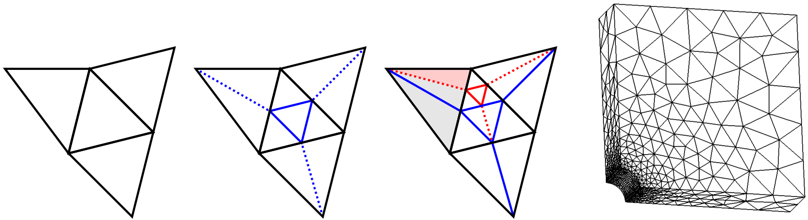


- Let $\mathcal{T}_h$ be a conforming tetrahedral mesh of Ω and let $k \geq 0$
- Local spaces can be glued together to form a global FE complex:

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \hookrightarrow & \mathcal{P}_c^{k+1}(\mathcal{T}_h) & \xrightarrow{\textbf{grad}} & \boldsymbol{N}^{k+1}(\mathcal{T}_h) & \xrightarrow{\textbf{curl}} & \boldsymbol{\mathcal{RT}}^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R} & \hookrightarrow & H^1(\Omega) & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl};\Omega) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}
$$

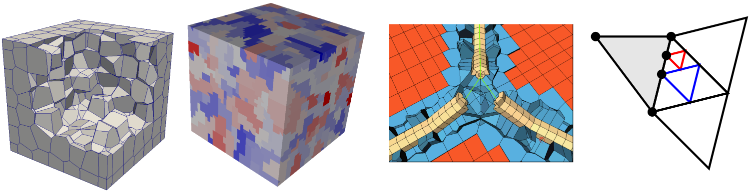
- **The gluing only works on conforming meshes (simplicial complexes)!**

The Finite Element way
Shortcomings



- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - ⟹ significant increase of DOFs on hexahedral elements

The discrete de Rham (DDR) approach I

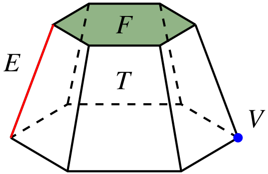


- **Key idea:** replace both spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{G}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of polyhedral meshes (CW complexes) and high-order
- Several strategies to reduce the number of unknowns on general shapes
- Natural generalization to the de Rham complex of differential forms
- On the relevance of general meshes and high-order: [Antonietti et al., 2013]

The discrete de Rham (DDR) approach II



- DDR spaces are spanned by vectors of polynomials
- Polynomial components enable consistent reconstructions of
  - vector calculus operators
  - the corresponding scalar or vector potentials
- These reconstructions emulate integration by parts (Stokes) formulas

References for this presentation

- FEEC [Arnold, Falk, Winther, 2006, Arnold, 2018]
- Introduction of DDR [DP, Droniou, Rapetti, 2020]
- DDR with Koszul complements [DP and Droniou, 2023a]
- Algebraic properties (general topologies) [DP, Droniou, Pitassi, 2023]
- Bridges with VEM [Beirão da Veiga, Dassi, DP, Droniou, 2022]
- Polytopal Exterior Calculus [Bonaldi, DP, Droniou, Hu, 2023]
- C++ open-source implementation available in HArDCore3D

Outline

The two-dimensional case

Continuous exact complex

- With F mesh face let, for $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$ smooth enough,

$$\mathbf{rot}_F\, q \coloneq (\mathbf{grad}_F\, q)^\perp \qquad \mathrm{rot}_F\, \boldsymbol{v} \coloneq \mathrm{div}_F(\boldsymbol{v}^\perp)$$

- We derive a discrete counterpart of the 2D de Rham complex:

$$\mathbb{R} \hookrightarrow H^1(F) \xrightarrow{\mathbf{grad}_F} \boldsymbol{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \xrightarrow{0} \{0\}$$

- We will need the following decomposition of $\mathcal{P}^k(F)^2$:

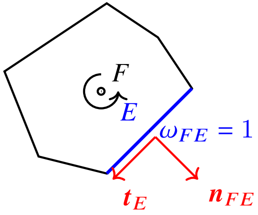
$$\mathcal{P}^k(F)^2 = \mathbf{rot}_F\, \mathcal{P}^{k+1}(F) \oplus (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F) =: \boldsymbol{\mathcal{R}}^k(F) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),$$

  and recall the 2D Raviart–Thomas space

$$\boldsymbol{\mathcal{RT}}^{k+1}(F) \coloneq \boldsymbol{\mathcal{R}}^k(F) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k+1}(F)$$

The two-dimensional case

A key remark



- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q\, \mathrm{div}_F\, \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

The two-dimensional case

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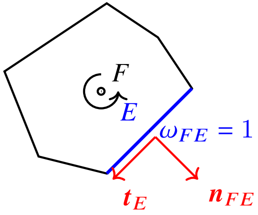

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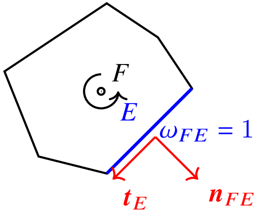


- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F \pi_{\mathcal{P},F}^{k-1} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

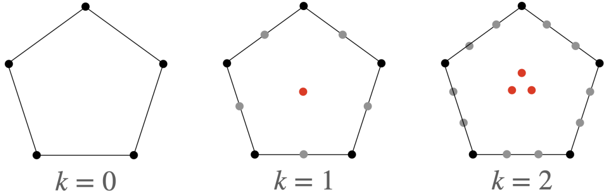
- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q_{|\partial F}$

- Based on this remark, we take as discrete counterpart of $H^1(F)$

$$\boxed{\underline{X}^k_{\mathbf{grad},F} \coloneqq \left\{\underline{q}_F = (q_F, q_{\partial F}) \,:\, q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\rm c}(\mathcal{E}_F)\right\}}$$

- Let $\underline{I}^k_{\mathbf{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathbf{grad},F}$ be s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathbf{grad},F} q \coloneqq (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$
$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \; \forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \; \forall V \in \mathcal{V}_F$$



The two-dimensional case

Reconstructions in $\underline{X}^k_{\mathbf{grad},F}$

- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F := (q_{\partial F})'_{|E}}$$

- The full face gradient $\mathbf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- The scalar trace $\gamma^{k+1}_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^{k+1}(F)$ is s.t., for all $\boldsymbol{v} \in \mathcal{R}^{\mathrm{c},k+2}(F)$,

$$\int_F \gamma^{k+1}_F \underline{q}_F \operatorname{div}_F \boldsymbol{v} = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F q_{\mathcal{E}_F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

- By construction, we have polynomial consistency:

$$\mathbf{G}^k_F (\underline{I}^k_{\mathbf{grad},F} q) = \mathbf{grad}_F\, q \text{ and } \gamma^{k+1}_F (\underline{I}^k_{\mathbf{grad},F} q) = q \text{ for all } q \in \mathcal{P}^{k+1}(F)$$

The two-dimensional case

Discrete $\boldsymbol{H}(\mathrm{rot}; F)$ space

- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) := \boldsymbol{\mathcal{R}\mathcal{T}}^{k+1}(F)^{\perp}$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\ q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) q_{|E}$$

- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) := \boldsymbol{\mathcal{RT}}^{k+1}(F)^\perp$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\ q = \int_F \boldsymbol{\pi}_{\mathcal{R},T}^{k-1} \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\boldsymbol{v} \cdot \boldsymbol{t}_E)}_{\in \mathcal{P}^k(E)} q_{|E}$$

The two-dimensional case

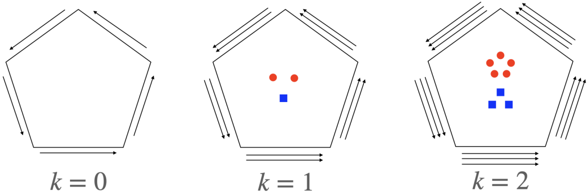
Discrete $\boldsymbol{H}(\mathrm{rot};F)$ space

- We start from: $\forall \boldsymbol{v} \in \boldsymbol{N}^{k+1}(F) := \mathcal{RT}^{k+1}(F)^{\perp}$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\; q = \int_F \boldsymbol{\pi}_{\mathcal{R},T}^{k-1}\boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \mathcal{R}^{k-1}(F)} - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\boldsymbol{v}\cdot\boldsymbol{t}_E)}_{\in \mathcal{P}^k(E)} q_{|E}$$

- This leads to the following discrete counterpart of $\boldsymbol{H}(\mathrm{rot};F)$:

$$\underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k} := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}_{\mathcal{R},F}^{\mathrm{c}}, (v_E)_{E\in\mathcal{E}_F}\big) : \boldsymbol{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F),\ \boldsymbol{v}_{\mathcal{R},F}^{\mathrm{c}} \in \mathcal{R}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \Big\}$$



- The face curl operator $C_F^k : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k \to \mathcal{P}^k(F)$ is s.t.,

$$\boxed{\int_F C_F^k \underline{\boldsymbol{v}}_F \; q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \; q \quad \forall q \in \mathcal{P}^k(F)}$$

- Let $\underline{\boldsymbol{I}}_{\mathrm{rot},F}^k : H^1(F)^2 \to \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$ collect component-wise L^2-projections
- C_F^k is polynomially consistent by construction:

$$C_F^k\big(\underline{\boldsymbol{I}}_{\mathrm{rot},F}^k \boldsymbol{v}\big) = \mathrm{rot}_F \, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F)$$

- The face curl operator $C_F^k : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)$ is s.t.,

$$\boxed{\int_F C_F^k \underline{\boldsymbol{v}}_F\ q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F\, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\ q \quad \forall q \in \mathcal{P}^k(F)}$$

- Let $\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} : H^1(F)^2 \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ collect component-wise L^2-projections
- C_F^k is polynomially consistent by construction:

$$C_F^k(\underline{\boldsymbol{I}}^k_{\mathrm{rot},F}\boldsymbol{v}) = \mathrm{rot}_F\, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F)$$

- Similarly, we can construct a tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)^2$ s.t.

$$\boldsymbol{\gamma}^k_{\mathrm{t},F}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},F}\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$$

Two-dimensional DDR complex

Space	V (vertex)	E (edge)	F (face)
$\underline{X}^k_{\mathbf{grad},F}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

- Define the discrete gradient

$$\underline{\boldsymbol{G}}^k_F \underline{q}_F := \left(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F} \mathbf{G}^k_F \underline{q}_F, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F} \mathbf{G}^k_F \underline{q}_F, (G^k_E \underline{q}_E)_{E\in\mathcal{E}_F}\right)$$

- The two-dimensional DDR complex reads

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

- If F is simply connected, this complex is exact

A glance at the general case I

- For a general domain $\Omega \subset \mathbb{R}^n$ and a form degree r, the DDR space is

$$\underline{X}_h^{k,r} := \bigtimes_{d=r}^{n} \bigtimes_{f \in \Delta_d(\mathcal{T}_h)} \mathcal{P}^{-,k}\Lambda^{d-r}(f) \text{ with } \Delta_d(\mathcal{T}_h) := \{d\text{-faces of } \mathcal{T}_h\}$$

- We recursively define, for $f \in \Delta_d(\mathcal{T}_h)$, $d = r, \ldots, n$,
  - If $r = d$,

$$P_f^{k,d}\underline{\omega}_f := \star^{-1}\omega_f \in \mathcal{P}^k\Lambda^d(f)$$

  - If $r + 1 \le d \le n$, we first let, for all $\underline{\omega}_f \in \underline{X}_f^{k,r}$ and all $\mu \in \mathcal{P}^k\Lambda^{d-r-1}(f)$,

$$\int_f \mathrm{d}_f^{k,r}\underline{\omega}_f \wedge \mu = (-1)^{r+1}\int_f \star^{-1}\omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P_{\partial f}^{r,k}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

then, for all $(\mu, \nu) \in \kappa\mathcal{P}^{k,d-r}(f) \times \kappa\mathcal{P}^{k-1,d-r+1}(f)$,

$$(-1)^{k+1}\int_f P_f^{k,r}\underline{\omega}_f \wedge (\mathrm{d}\mu + \nu) = \int_f \mathrm{d}_f^{k,f}\underline{\omega}_f \wedge \mu - \int_{\partial f} P_{\partial f}^{r,k}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu + (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \nu$$

A glance at the general case II

- The following polynomial consistency properties hold:

$$P_f^{k,r}\underline{I}_f^{k,r}\omega = \omega \qquad \forall \omega \in \mathcal{P}^k\Lambda^r(f),$$
$$\mathrm{d}_f^{k,r}\underline{I}_f^{k,r}\omega = \mathrm{d}\omega \quad \forall \omega \in \mathcal{P}^{-,k+1}\Lambda^r(f)$$

- Setting

$$\underline{\mathrm{d}}_h^{k,r}\underline{\omega}_h \coloneqq \left(\pi_f^{-,k,d-r-1}(\star\mathrm{d}_f^{k,r}\underline{\omega}_f)\right)_{f\in\Delta_d(\mathcal{T}_h),\, d\in[k+1,n]},$$

the global DDR complex of differential forms reads

$$\underline{X}_h^{k,0} \xrightarrow{\underline{\mathrm{d}}_h^{k,0}} \underline{X}_h^{k,1} \longrightarrow \cdots \longrightarrow \underline{X}_h^{k,n-1} \xrightarrow{\underline{\mathrm{d}}_h^{k,n-1}} \underline{X}_h^{k,n} \longrightarrow \{0\}$$

A glance at the general case III

For $n = 3$, we recover the DDR complex of [DP and Droniou, 2023a]:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathbf{grad},T} \xrightarrow{\underline{G}^k_T} \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \xrightarrow{\underline{C}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}$$

Space	V	E	F	T (element)
$\underline{X}^{k,0}_T \cong \underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}^{k,1}_T \cong \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$	$\mathcal{RT}^k(T)$
$\underline{X}^{k,2}_T \cong \underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\boldsymbol{\mathcal{N}}^k(T)$
$\underline{X}^{k,3}_T \cong \mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Commutation with the interpolators

Lemma (Local commutation properties)

The following diagrams commute:

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \hookrightarrow & C^\infty(\overline{T}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathrm{div}} & C^\infty(\overline{T}) & \xrightarrow{0} & \{0\} \\
 & & \downarrow{\scriptstyle \underline{I}^k_{\mathrm{grad},T}} & & \downarrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{curl},T}} & & \downarrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathrm{div},T}} & & \downarrow{\scriptstyle i_T} & & \\
\mathbb{R} & \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} & \underline{X}^k_{\mathrm{grad},T} & \xrightarrow{\underline{\boldsymbol{G}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} & \xrightarrow{\underline{\boldsymbol{C}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathrm{div},T} & \xrightarrow{D^k_T} & \mathcal{P}^k(T) & \xrightarrow{0} & \{0\}
\end{array}
$$

- Crucial for both algebraic and analytical properties
- Compatibility of projections with Helmholtz–Hodge decompositions

  $\Longrightarrow$ robustness of DDR schemes with respect to the physics:
  - Stokes [Beirão da Veiga, Dassi, DP, Droniou, 2022]
  - Reissner–Mindlin [DP and Droniou, 2023b]
  - ...

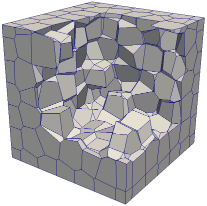
Local discrete L^2-products

- Based on the element potentials, we construct local discrete L^2-products

$$(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$$

- The L^2-products are built to be polynomially consistent

Global DDR complex



$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Global DDR spaces on a mesh $\mathcal{T}_h$ are defined gluing boundary components
- Global operators are obtained collecting local components
- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise

Cohomology of the global three-dimensional DDR complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Theorem (Cohomology of the 3D DDR complex [DP, Droniou, Pitassi, 2023])

For any $k \geq 0$, the DDR sequence forms a complex whose cohomology spaces are isomorphic to those of the continuous de Rham complex. In particular, if Ω has a trivial topology (i.e., $b_1 = b_2 = 0$), the DDR complex is exact, i.e.,

$$\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h = \mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h, \quad \mathrm{Im}\,\underline{\boldsymbol{C}}^k_h = \mathrm{Ker}\,D^k_h, \quad \mathrm{Im}\,D^k_h = \mathcal{P}^k(\mathcal{T}_h).$$

Remark (Extension to differential forms [Bonaldi, DP, Droniou, Hu, 2023])

The above result extends to the de Rham complex of differential forms.

Outline

Uniform discrete Poincaré inequality for the curl

- We assume, from this point on, that Ω has a trivial topology
- Let $(\operatorname{Ker}\underline{\boldsymbol{C}}_h^k)^\perp$ be the orthogonal of $\operatorname{Ker}\underline{\boldsymbol{C}}_h^k$ in $\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$ for $(\cdot,\cdot)_{\mathbf{curl},h}$. Then,

$$\underline{\boldsymbol{C}}_h^k : (\operatorname{Ker}\underline{\boldsymbol{C}}_h^k)^\perp \to \operatorname{Ker} D_h^k \text{ is an isomorphism}$$

- Moreover, denoting by $\|\cdot\|_{\bullet,h}$ the norm induced by $(\cdot,\cdot)_{\bullet,h}$ on $\underline{X}_{\bullet,h}^k$,

$$\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} \lesssim \|\underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} \quad \forall \underline{\boldsymbol{v}}_h \in (\operatorname{Ker}\underline{\boldsymbol{C}}_h^k)^\perp$$

Adjoint consistency of the discrete curl

Adjoint consistency measures the failure to satisfy a global IBP. For the curl,

$$\int_\Omega \boldsymbol{w}\cdot\mathbf{curl}\,\boldsymbol{v} - \int_\Omega \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{v} = 0 \text{ if } \boldsymbol{w}\times\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega$$

Theorem (Adjoint consistency for the curl)

Let $\mathcal{E}_{\mathbf{curl},h} : \left(C^0(\overline{\Omega})^3 \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)\right) \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$ *be s.t.*

$$\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h) \coloneqq (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w}, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h)_{\mathrm{div},h} - \int_\Omega \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h.$$

Then, for all $\boldsymbol{w}\in C^0(\overline{\Omega})^3 \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ *s.t.* $\boldsymbol{w}\in H^{k+2}(\mathcal{T}_h)^3$*:* $\forall\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,

$$|\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h)| \lesssim h^{k+1}\left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right).$$

Discrete problem

- We seek $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned}\int_\Omega \mu \boldsymbol{H}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 &\quad \forall\boldsymbol{\tau}\in\boldsymbol{H}(\mathbf{curl};\Omega),\\ \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{A}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v} &\quad \forall\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div};\Omega)\end{aligned}$$

- The DDR scheme is obtained with obvious substitutions:
  Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned}(\mu\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 &\quad \forall\underline{\boldsymbol{\tau}}_h\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\ (\underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D_h^k\underline{\boldsymbol{A}}_h\, D_h^k\underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) &\quad \forall\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\mathrm{div},h}\end{aligned}$$

- For general domains, we need to add orthogonality to harmonic forms

Analysis

- Define the bilinear form $\mathcal{A}_h : [\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}]^2 \to \mathbb{R}$ s.t.

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) \coloneqq (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h \, D^k_h \underline{\boldsymbol{v}}_h.$$

- Then, it holds: $\forall (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$,

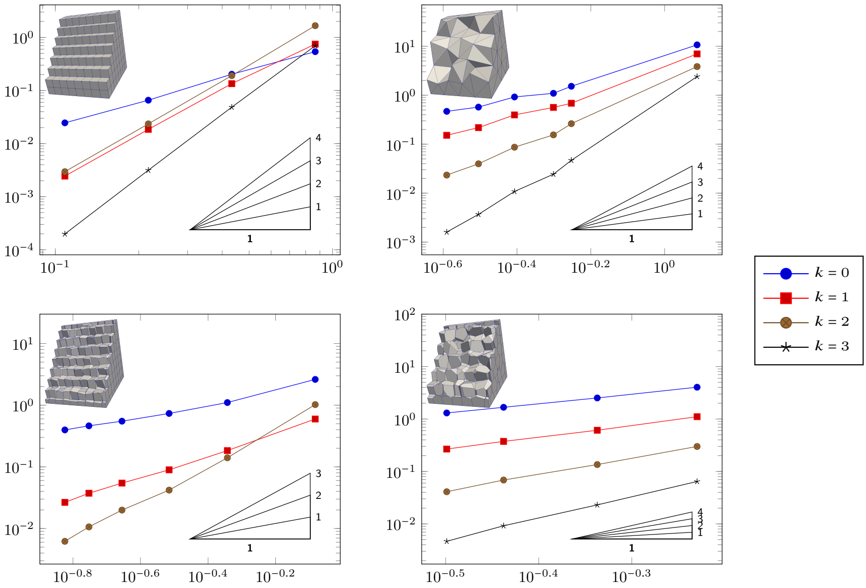
$$|||(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)|||_h \lesssim \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \setminus \{(\underline{\boldsymbol{0}}, \underline{\boldsymbol{0}})\}} \frac{\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h}$$

with $|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h^2 \coloneqq \|\underline{\boldsymbol{\tau}}_h\|^2_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h\|^2_{\mathrm{div},h} + \|\underline{\boldsymbol{v}}_h\|^2_{\mathrm{div},h} + \|D^k_h \underline{\boldsymbol{v}}_h\|^2_{L^2(\Omega)}$

- Assume $\boldsymbol{H} \in C^0(\overline{\Omega})^3 \cap H^{k+2}(\mathcal{T}_h)^3$ and $\boldsymbol{A} \in C^0(\overline{\Omega})^3 \times H^{k+2}(\mathcal{T}_h)^3$. Then,

$$|||(\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{H}, \underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h} \boldsymbol{A})|||_h \lesssim h^{k+1}$$

Numerical examples (energy error vs. meshsize)



Conclusions and perspectives

- Fully discrete approach for PDEs relating to the de Rham complex
- Key features: support of general polyhedral meshes and high-order
- Novel computational strategies made possible
- Natural extensions to differential forms

- Unified proof of analytical properties using differential forms
- Development of novel complexes (e.g., elasticity, Hessian,. . .)
- Applications (possibly beyond continuum mechanics)

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