Recent advances on nonconforming methods for diffusive problems on general meshes

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Outline

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Broken polynomial spaces on general meshes
- Admissible mesh sequences
- Sobolev embeddings

The SWIP-dG method
- Darcy flow through heterogeneous media
- Poroelasticity

Cell centered Galerkin methods
- Incomplete polynomial spaces
- Incompressible Navier–Stokes

General meshes I

- Avoid remeshing (e.g. in subsoil modeling)
- Improve domain/solution fitting
- Improve performance (fewer DOFs, reduced fill-in)
- Nonconforming/aggregative mesh adaptivity

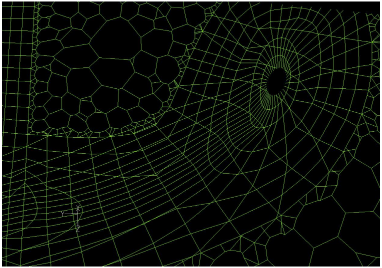


Figure: Near wellbore mesh

General meshes II

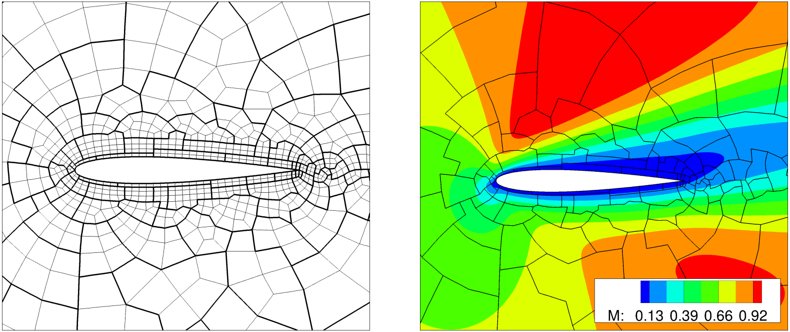


Figure: NACA0012 airfoil, computational mesh (*left*) and Mach number contours (*right*) following [Bassi et al., 2012]

- Let $\Omega \subset \mathbb{R}^d$ be an open connected bounded polyhedral domain
- Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a sequence of refined meshes of Ω
- For $k \geqslant 0$ we define the broken polynomial spaces

$$\mathbb{P}_d^k(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v_{|T} \in \mathbb{P}_d^k(T) \right\}$$

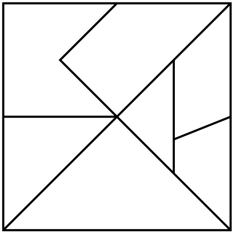


Figure: Mesh $\mathcal{T}_h$ with polygonal elements and nonmatching interfaces

Admissible mesh sequences for h-convergence II

Trace and inverse inequalities

- Every $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{S}_h$
- $(\mathfrak{S}_h)_{h\in\mathcal{H}}$ is shape-regular in the sense of Ciarlet
- Every simplex $S \subset T$ is s.t. $h_S \approx h_T$

Optimal polynomial approximation (for error estimates)

Every element T is star-shaped w.r.t. a ball of diameter $\delta_T \approx h_T$

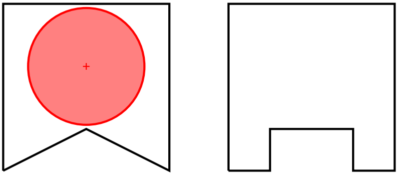


Figure: Admissible (*left*) and non-admissible (*right*) mesh elements

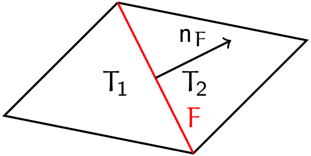


Figure: Notation for an interface $F \in \mathcal{F}_h^i$

- For $F \subset \partial T_1 \cap \partial T_2$ let

$$\{v\} := \frac{1}{2}\left(v_{|T_1} + v_{|T_2}\right), \qquad [\![v]\!] := v_{|T_1} - v_{|T_2}$$

- We introduce the discrete $W^{1,p}(\mathcal{T}_h)$-norms

$$\|v\|_{dG,p} := \left(\|\nabla_h v\|_{L^p(\Omega)^d}^p + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \|[\![v]\!]\|_{L^p(F)}^p \right)^{1/p}$$

Discrete Sobolev embeddings [DP and Ern, 2010]

Let $k \geqslant 0$. For all q such that

- $1 \leqslant q \leqslant p^* := \frac{pd}{d-p}$ if $1 \leqslant p < d$
- $1 \leqslant q < \infty$ if $d \leqslant p < \infty$

there exists $\sigma_{p,q}$ such that

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h), \qquad \|v_h\|_{L^q(\Omega)} \leqslant \sigma_{p,q} \|v_h\|_{\mathrm{dG},p}$$

Proof.

- For $p = 1$ use $\|v_h\|_{L^{1*}(\Omega)} \lesssim \|v_h\|_{\mathrm{BV}} \lesssim \|v_h\|_{\mathrm{dG},1}$
- For $1 < p < d$ use L^{1*}-estimate for $|v_h|^{\alpha}$, Hölder's and trace inequalities
- For $d \leqslant p < \infty$ use the previous point together with the comparison of broken $W^{1,p}(\mathcal{T}_h)$-norms $\square$

Sobolev embeddings in $\mathbb{P}_d^k(\mathcal{T}_h)$-spaces II

- In the Hilbertian case $p = 2$ we have the usual

$$\|v\|_{dG} := \left(\|\nabla_h v\|_{L^2(\Omega)^d}^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|_{L^2(F)}^2 \right)^{1/2}$$

- An important Sobolev embedding is the Poincaré inquality

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h) \qquad \|v_h\|_{L^2(\Omega)} \leqslant \sigma_{2,2} \|v_h\|_{dG}.$$

Outline

Motivations and goals

- Darcy flow through heterogeneous anisotropic media
  - [DP and Ern, 2011a]
- Convergence to nonsmooth solutions in faulted media
  - [DP and Ern, 2011b]
- Darcy flow through deformable porous media
  - [DP, 2011b]
- Reactive transport with singular interfaces (not detailed)
  - [Gastaldi and Quarteroni, 1989]
  - [DP et al., 2008]
- Important references for weighted averages
  - [Stenberg, 1998]
  - [Hansbo and Hansbo, 2002]
  - [Heinrich and Pietsch, 2002, Heinrich and Nicaise, 2003]
  - [Burman and Zunino, 2006]

The heterogeneous Darcy problem I

$$-\nabla\cdot(\kappa\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- There is a partition P_Ω s.t.

$$\kappa \in \mathbb{P}^0_d(P_\Omega) \text{ with } 0 < \underline{\kappa} \leqslant \kappa \leqslant \overline{\kappa}$$

- For all $h \in \mathcal{H}$, $\mathcal{T}_h$ is compatible with P_Ω
- We seek an approximate solution $u_h \in V_h$ with

$$V_h := \mathbb{P}^k_d(\mathcal{T}_h), \qquad k \geqslant 1$$

Find $u_h \in V_h$ s.t. $a_h(u_h, v_h) = \int_\Omega f v_h$ for all $v_h \in V_h$

The heterogeneous Darcy problem II

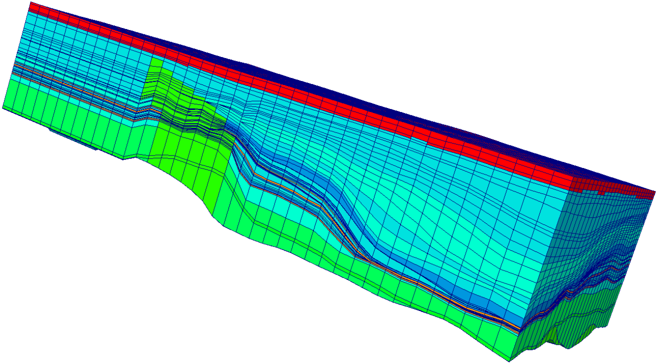


Figure: P_Ω and compatible mesh (stratigraphy of a sedimentary bassin)

The heterogeneous Darcy problem III

$$a_h^{\mathrm{sip}}(w, v_h) := \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h w\} \cdot n_F [\![v_h]\!] - \sum_{F \in \mathcal{F}_h} \int_F [\![w]\!] \{\kappa \nabla_h v_h\} \cdot n_F + \sum_{F \in \mathcal{F}_h} \int_F \frac{\eta}{h_F} [\![w]\!] [\![v_h]\!]$$

Error estimate (SIP, [Arnold, 1982])

Assume $u \in V_* := H_0^1(\Omega) \cap H^2(P_\Omega)$. Then, $\exists C \neq C(h, \kappa)$ s.t.

$$\| u - u_h \|_{dG} \leqslant C \max\left(1, \frac{\overline{\kappa}}{\underline{\kappa}}\right) \inf_{v_h \in V_h} \| u - v_h \|_{dG,*}$$

This estimate is not robust w.r.t. the heterogeneity of κ

The SWIP method I

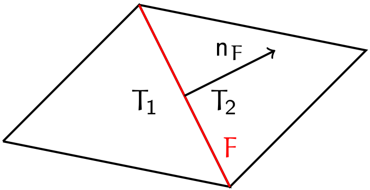


Figure: Notation for an interface $F \in \mathcal{F}_h^i$

- For $F \subset \partial T_1 \cap \partial T_2$ and $(\omega_1, \omega_2) > 0$, $\omega_1 + \omega_2 = 1$ let

$$\{v\}_\omega := \omega_1 v_{|T_1} + \omega_2 v_{|T_2}$$

- For $\omega_1 = \omega_2 = \frac{1}{2}$ we recover the standard average $\{v\}$

The SWIP method II

$$a_h^{\mathrm{swip}}(w, v_h) := \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h w\}_{\omega_\kappa} \cdot n_F [\![v_h]\!]$$

$$- \sum_{F \in \mathcal{F}_h} \int_F [\![w]\!] \{\kappa \nabla_h v_h\}_{\omega_\kappa} \cdot n_F + \sum_{F \in \mathcal{F}_h} \int_F \eta \frac{\gamma_\kappa}{h_F} [\![w]\!] [\![v_h]\!]$$

- Weighted averages + harmonic mean in penalty

$$\{\Phi\}_{\omega_\kappa} := \frac{\kappa_2}{\kappa_1 + \kappa_2} \Phi_{|T_1} + \frac{\kappa_1}{\kappa_1 + \kappa_2} \Phi_{|T_2}, \quad \gamma_\kappa := 2 \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$$

- Data-dependent energy norm on $H^1(\mathcal{T}_h)$

$$|\!|\!| v |\!|\!|_\kappa^2 := \|\kappa^{\frac{1}{2}} \nabla_h v\|_{L^2(\Omega)^d}^2 + \sum_{F \in \mathcal{F}_h} \frac{\gamma_\kappa}{h_F} \|[\![v]\!]\|_{L^2(F)}^2$$

Properties of a_h^{swip} [DP and Ern, 2011b]

Let $V_{*h} := V_h + V_*$ and assume $u \in V_*$. Then,

- Consistency. There holds

$$\forall v_h \in V_h, \qquad a_h^{\mathrm{swip}}(u, v_h) = \int_\Omega f v_h,$$

- Coercivity. There exists $C_{\mathrm{sta}} \neq C_{\mathrm{sta}}(h, \kappa)$ s.t.

$$\forall v_h \in V_h, \qquad a_h^{\mathrm{swip}}(v_h, v_h) \geqslant C_{\mathrm{sta}} |||v_h|||_\kappa^2$$

- Boundedness. There exists $C_{\mathrm{bnd}} \neq C_{\mathrm{bnd}}(h, \kappa)$ s.t.

$$\forall (w, v_h) \in V_{*h} \times V_h^{\mathrm{ccg}}, \qquad a_h^{\mathrm{swip}}(w, v_h) \leqslant C_{\mathrm{bnd}} |||w|||_{\kappa,*} |||v_h|||_\kappa.$$

Error estimate (SWIP, [DP et al., 2008])

Assume $u \in V_* = H^1_0(\Omega) \cap H^2(P_\Omega)$. Then, $\exists C \neq C(h, \kappa)$ s.t.

$$|||u - u_h|||_\kappa \leqslant C \inf_{v_h \in V_h} |||u - v_h|||_{\kappa,*}$$

Convergence rate

If, moreover $u \in H^{k+1}(P_\Omega)$, $\exists C \neq C(h, \kappa)$ s.t.

$$|||u - u_h|||_\kappa \lesssim C\overline{\kappa}^{1/2} h^k \|u\|_{H^{k+1}(P_\Omega)}.$$

- Nonconsistent for $k = 0$ except on κ-orthogonal $\mathcal{T}_h$
- Minor modifications allow to treat the case

$$u \in H^1_0(\Omega) \cap H^{3/2+\epsilon}(P_\Omega)$$

Convergence of the SWIP method to nonsmooth solutions

- However, in general we only have [Nicaise and Sändig, 1994]

$$u \in W^{2,p}(P_\Omega) \Rightarrow u \in H^{1+\alpha}(P_\Omega), \quad \alpha = 1+d\left(\frac{1}{2}-\frac{1}{p}\right) > 0$$

- Optimal convergence rates for $d = 2$ [DP and Ern, 2011a]
- Convergence by compactness for $d > 2$

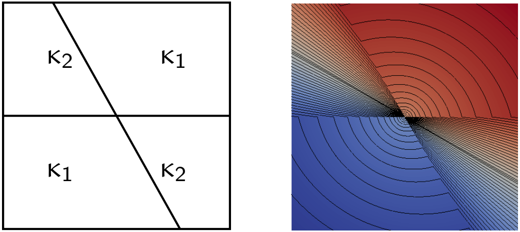


Figure: Faulted medium, $u \in H^{1.29}(P_\Omega)$, $\kappa_1/\kappa_2 = 30$

Discrete compactness I

- For $F \in \mathcal{F}_h$ and $l \geqslant 0$ the local lifting solves

$$\int_\Omega r^l_{\omega,F}([\![v]\!])\cdot\tau_h = \int_F [\![v]\!]\{\tau_h\}_\omega\cdot n_F \qquad \forall \tau_h \in \mathbb{P}^l_d(\mathcal{T}_h)^d$$

- The global lifting is defined as

$$R^l_{h,\omega}(v) := \sum_{F\in\mathcal{F}_h} r^l_{\omega,F}([\![v]\!])$$

- For all $l \geqslant 0$ we define the gradient

$$G^l_{h,\omega}(v) := \nabla_h v - R^l_{h,\omega}(v)$$

- The subscript ω is omitted if $\omega_1 = \omega_2 = 1/2$

Discrete compactness II

Compactness [DP and Ern, 2010]

Let $(v_h)_{h\in\mathcal{H}}$ be a sequence in $(\mathbb{P}_d^k(\mathcal{T}_h))_{h\in\mathcal{H}}$, $k \geqslant 0$

$$\forall h \in \mathcal{H}, \qquad \|v_h\|_{\mathrm{dG}} \leqslant C \neq C(h).$$

Then, $\exists v \in H_0^1(\Omega)$ s.t., as $h \to 0$, up to a subsequence

$$v_h \to v \qquad \text{in } L^2(\Omega),$$
$$G_h^l(v_h) \rightharpoonup \nabla v \qquad \text{for all } l \geqslant 0 \text{ weakly in } L^2(\Omega)^d.$$

Proof.

- Kolmogorov criterion to prove compactness in $L^1(\Omega)$
- Sobolev embeddings to prove compactness in $L^2(\Omega)$
- Asymptotic consistency of G_h^l yields regularity of the limit

□

Convergence [DP and Ern, 2011a]

Let $(u_h)_{h\in\mathcal{H}}$ denote the sequence of discrete solutions on the admissible mesh family $(\mathcal{T}_h)_{h\in\mathcal{H}}$. Then,

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega), \\ \nabla_h u_h &\to \nabla u && \text{strongly in } [L^2(\Omega)]^d, \\ |u_h|_J &\to 0. \end{aligned}$$

Proof.

Use the equivalent form for a_h^{swip}: For $l \in \{k-1, k\}$,

$$a_h^{\text{swip}}(u_h, v_h) = \int_\Omega \kappa G_{h,\omega_\kappa}^l(u_h) \cdot G_{h,\omega_\kappa}^l(v_h) + s_h(u_h, v_h),$$

with $s_h(\cdot,\cdot) \geqslant 0$. □

Motivations and goals

- Darcy flow through deformable porous media
  - [DP, 2011b]
- Robustness w.r.t. in the heterogeneous case
- Robustness w.r.t. incompressibility of both the medium and the fluid (not detailed here)
- Important references
  - [Wihler, 2006]
  - [Phillips and Wheeler, 2008]
  - [Ern and Meunier, 2009]
  - [Girault et al., 2011]

Biot's equations

$$\begin{aligned} -\nabla\cdot\sigma(u)+\nabla p &= f && \text{in } \Omega\times(0,t_F),\\ c_0 d_t p+\nabla\cdot(d_t u)-\nabla\cdot(\kappa\nabla p) &= 0 && \text{in } \Omega\times(0,t_F),\\ (u,p) &= 0 && \text{on } \partial\Omega\times(0,t_F),\\ (u(0),p(0)) &= (u_0,p_0) && \text{in } \Omega,\end{aligned}$$

where $\sigma(w) := 2\mu\epsilon(w)+\lambda(\nabla\cdot w)1_d$ and $\epsilon(w) := \frac{1}{2}\left(\nabla w+\nabla w^t\right)$.

- Assume $c_0 > 0$, λ, μ, and κ positive but heterogeneous
- Let $\delta t = t_F/N$ denote the time step and set $t^n := n\delta t$
- For $1 \leqslant n \leqslant N$ we seek $(u_h^n, p_h^n) \in U_h\times P_h$ with

$$U_h := \mathbb{P}_d^1(\mathcal{T}_h)^d, \qquad P_h := \mathbb{P}_d^1(\mathcal{T}_h)$$

$$
\begin{aligned}
e_h(w,v) := &\int_\Omega \sigma_h(w):\epsilon_h(v) \\
&- \sum_{F\in\mathcal{F}_h} \int_F \left(\{\sigma_h(w)\}:\langle [\![v]\!]\rangle_F \otimes n_F + [\![w]\!] \otimes n_F:\{\sigma_h(v)\}\right) \\
&+ \sum_{F\in\mathcal{F}_h} \int_F \eta \left(2\mu r_F^0([\![w]\!]):r_F^0([\![v]\!]) + \lambda l_F([\![w]\!]) l_F([\![w]\!])\right) \\
&+ \sum_{F\in\mathcal{F}_h} \int_F \frac{\eta\gamma_\mu}{h_F} [\![w]\!]\cdot[\![v]\!]
\end{aligned}
$$

$$
l_F(\varphi) := \mathrm{tr}(r_F(\varphi))
$$

A SWIP-dG method for the elasticity operator II

Properties of e_h [DP, 2011a]

Let $U_h := \mathbb{P}^1_d(\mathcal{T}_h)^d$ and $U_{*h} := U_h + [H^1_0(\Omega) \cap H^2(P_\Omega)]^d$. Then,

- Coercivity. There exists $C_{sta} \neq C_{sta}(h, \lambda, \mu)$ s.t.

$$\forall v_h \in U_h, \quad e_h(v_h, v_h) \geqslant C_{sta} |||v_h|||^2_{\mu,\lambda},$$

- Boundedness. There exists $C_{bnd} \neq C_{bnd}(h, \lambda, \mu)$ s.t.

$$\forall (w, v_h) \in U_{*h} \times U_h, \quad e_h(w, v_h) \leqslant C_{bnd} |||w|||_{\mu,\lambda,*} |||v_h|||_{\mu,\lambda}.$$

- For a discrete Korn inequality see [Brenner, 2004]
- Locking-free on conforming simplicial meshes since

$$\mathbb{CR}(\mathcal{T}_h) \subset \mathbb{P}^1_d(\mathcal{T}_h)$$

$$b_h(v_h, q_h) := -\int_\Omega \nabla_h \cdot v_h q_h + \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \cdot n_F \{q_h\}$$
$$= -\int_\Omega D_h^0(v_h) q_h, \qquad D_h^0(v_h) := \mathrm{tr}(G_h^0(v_h))$$

Discrete stability for b_h [DP, 2007]

There is $0 < \beta \neq \beta(h)$ s.t., for all $q_h \in P_h$,

$$\beta \|q_h\|_{/} \leqslant \sup_{v_h \in U_h \setminus \{0\}} \frac{b_h(v_h, q_h)}{\|v_h\|_{dG}} + |q_h|_p,$$

where $|q_h|_p^2 := \sum_{F \in \mathcal{F}_h^i} \int_F h_F [\![q_h]\!]^2$.

The discrete problem I

For $n \geqslant 1$, find $(u_h^n, p_h^n) \in U_h \times P_h$ s.t. for all $(v_h, q_h) \in U_h \times P_h$,

$$e_h(u_h^n, v_h) + b_h(v_h, p_h^n) = (f_h^n, v_h)$$

$$(c_0 \delta_t^{(1)} p_h^n, q_h) - b_h(\delta_t^{(1)} u_h^n, q_h) + a_h^{\text{swip}}(p_h^n, q_h) = 0$$

Discrete stability

Assume $f \in C^1(L^2(\Omega)^d)$. Then,

$$|||u_h^N|||_{\mu,\lambda}^2 + c_0 \|p_h^N\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N} \delta t |||p_h^n|||_{\kappa}^2 \leqslant C \exp(t_F),$$

where C depends on the mesh regularity parameters, on μ, and linearly in $|||u_0|||_{\mu,\lambda}^2$, $|||p_0|||_{\kappa}^2$, and $\|f\|_{C^1(L^2(\Omega)^d)}^2$.

Convergence

Assume $u \in C^2(U) \cap C^1(H^2(P_\Omega)^d)$ and $p \in C^0(P_*) \cap C^2(L^2(\Omega))$.
Then, there exists $C \neq C(h, \lambda, \underline{\kappa})$ s.t.

$$\| u^N - u_h^N \|_{\mu,\lambda} + \| p^N - p_h^N \|_{L^2(\Omega)} + \left(\sum_{n=0}^{N} \delta t \| p^n - p_h^n \|_\kappa^2 \right)^{\frac{1}{2}} \leqslant C(h + \delta t).$$

Second order in time can be proved using the BDF2 operator $\delta_t^{(2)}$ instead of the BE operator $\delta_t^{(1)}$ (cf. [DP and Ern, 2011b, Ch. 4])

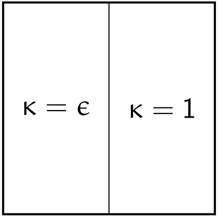
Numerical examples I

- Let $\Omega = (-1,1)^2$, $t_F = 1$, $c_0 = \lambda = \mu = 1$, and

$$\kappa = \begin{cases} 1 & \text{if } x > 0, \\ \varepsilon & \text{otherwise} \end{cases}$$

- We consider the following analytical solution in $d = 2$:

$$u_1 = e^{-t}x^2y, \qquad u_2 = -e^{-t}xy^2, \qquad p_\varepsilon = e^{-t}\cos(\kappa^{-1/2}x)$$



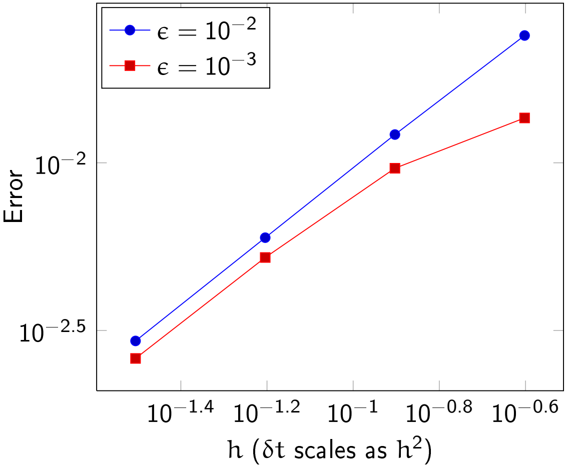


Figure: h-convergence, heterogeneous case

Outline

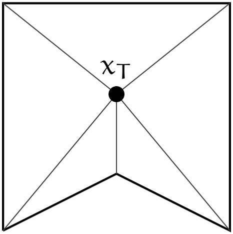
Motivations and goals

- Design consistent dG methods with 1 DOF per element
- Work on general polyhedral meshes as in dG methods
- Formulation of FV and lowest-order methods suitable for `FreeFEM`-like implementation
- See [DP, 2010, DP, 2012] and also [Botti and DP, 2011]
- Important references
  - [Aavatsmark *et al.*, 1994–11]
  - [Edwards *et al.*, 1994–11]
  - [Eymard, Gallouët, Herbin *et al.*, 2000–11]
  - [Brezzi, Lipnikov, Shashkov *et al.*, 2005–11]

Cell centers

We fix a set of points $(x_T)_{T \in \mathcal{T}_h}$ s.t.

- all $T \in \mathcal{T}_h$ is star-shaped w.r.t. x_T
- for all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$, $\mathrm{dist}(x_T, F) \approx h_T$



Discrete space

1) Fix the vector space of DOFs, e.g.,

$$\mathbb{V}_h = \mathbb{R}^{\mathcal{T}_h}, \qquad \mathbf{v}_h = (v_T)_{T \in \mathcal{T}_h} \in \mathbb{R}^{\mathcal{T}_h}$$

2) Reconstruct an asymptotically consistent gradient

$$\mathfrak{G}_h : \mathbb{V}_h \to \mathbb{P}^0_d(\mathcal{T}_h)^d$$

3) Reconstruct a broken affine function

$$\forall T \in \mathcal{T}_h, \quad \mathfrak{R}_h(\mathbf{v}_h)_{|T}(x) = v_T + \mathfrak{G}_h(\mathbf{v}_h)_{|T} \cdot (x - x_T)$$

Use as a discrete space in dG methods

$$V_h^{\mathrm{ccg}} := \mathfrak{R}_h(\mathbb{V}_h) \subset \mathbb{P}^1_d(\mathcal{T}_h)$$

Find $u_h \in V_h^{ccg}$ s.t. for all $v_h \in V_h^{ccg}$ $a_h^{swip}(u_h, v_h) = \int_\Omega f v_h$

- Consistency, coercivity, and boundedness hold *a fortiori* since

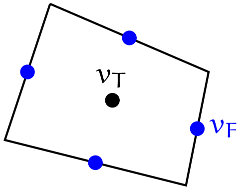
$$V_h^{ccg} \subset \mathbb{P}_d^1(\mathcal{T}_h)$$

- Fewer DOFs since

$$\dim(V_h^{ccg}) = \dim(\mathbb{P}_d^0(\mathcal{T}_h))$$

- Optimal convergence rate for $u \in H^2(P_\Omega)$
- Aubin–Nitsche trick yields optimal L^2-convergence

A gradient reconstruction based on Green's formula



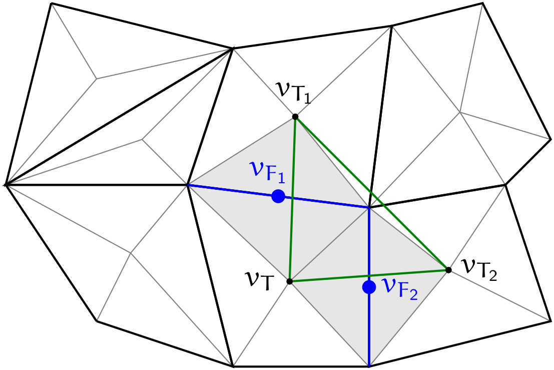
- Observe that, for all $v_h \in \mathbb{P}^0_d(\mathcal{T}_h)$ and all $T \in \mathcal{T}_h$,

$$G^0_h(v_h)_{|T} = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} \left(\{v_h\} - v_T\right) n_{T,F}$$

- Let $(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}}) \in \mathbb{R}^{\mathcal{T}_h} \times \mathbb{R}^{\mathcal{F}_h}$. For all $T \in \mathcal{T}_h$ we set

$$\mathfrak{G}_h(\mathbf{v}_h^{\mathcal{T}}, \mathbf{v}_h^{\mathcal{F}})_{|T} = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} \left(v_F - v_T\right) n_{T,F}$$

Trace interpolation



- The trace unknowns $(v_F)_{F\in\mathcal{F}_h}$ can be expressed as linear combinations of the cell unknowns $(v_T)_{T\in\mathcal{T}_h}$
- For the heterogeneous case cf. [Agélas, DP, Droniou, 2010]

Application to the incompressible Navier–Stokes equations I

$$
\begin{aligned}
-\nu\triangle u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \Omega,\\
\nabla\cdot u &= 0 && \text{in } \Omega\\
u &= 0 && \text{on } \partial\Omega,\\
\langle p\rangle_\Omega &= 0.
\end{aligned}
$$

- We consider a discretization based on the following spaces:

$$U_h := [V_h^{ccg}]^d, \quad P_h := \mathbb{P}_d^0(\mathcal{T}_h)/\mathbb{R}$$

- The discrete problem reads: For all $(v_h, q_h) \in U_h \times P_h$,

$$
\begin{aligned}
a_h^{\mathrm{swip}}(u_h, v_h) + t_h(u_h, u_h, v_h) + b_h(v_h, p_h) &= \int_\Omega f\cdot v_h\\
-b_h(u_h, q_h) + s_h(p_h, q_h) &= 0
\end{aligned}
$$

$$t_h(w,u,v) := \int_\Omega (w\cdot\nabla_h u_i)v_i - \sum_{F\in\mathcal{F}_h^i}\int_F \{w\}\cdot n_F [\![u]\!]\cdot\{v\}$$
$$+\frac{1}{2}\int_\Omega (\nabla_h\cdot w)(u\cdot v) - \frac{1}{2}\sum_{F\in\mathcal{F}_h}\int_F [\![w]\!]\cdot n_F\{u\cdot v\}$$

- Extension of Temam's device to broken spaces
- Non-dissipative since

$$t_h(v_h,v_h,v_h) = 0 \qquad \forall v_h \in U_h$$

- Asymptotically consistent for smooth and discrete functions

Lemma (Alternative expression for t_h)

For all w_h, u_h, $v_h \in U_h$ *there holds*

$$t_h(w_h, u_h, v_h) = \int_\Omega w_h \cdot \mathcal{G}_h^2(u_{h,i}) v_{h,i} + \frac{1}{2}\int_\Omega D_h^2(w_h)(u_h \cdot v_h) + \frac{1}{4}\sum_{F \in \mathcal{F}_h^i} \int_F ([\![w_h]\!] \cdot n_F)([\![u_h]\!] \cdot [\![v_h]\!]).$$

Lemma (Existence of a discrete solution)

There exists at least one discrete solution $(u_h, p_h) \in X_h$.

Convergence

Let $((u_h, p_h))_{h\in\mathcal{H}}$ be a sequence of approximate solutions on $(\mathcal{T}_h)_{h\in\mathcal{H}}$. Then, as $h \to 0$, up to a subsequence,

$$
\begin{aligned}
u_h &\to u, && \text{in } [L^2(\Omega)]^d,\\
\nabla_h u_h &\to \nabla u, && \text{in } [L^2(\Omega)]^{d,d},\\
|u_h|_J &\to 0, &&\\
p_h &\to p, && \text{in } L^2(\Omega),\\
|p_h|_p &\to 0. &&
\end{aligned}
$$

If (u, p) is unique, the whole sequence converges.

Numerical examples I

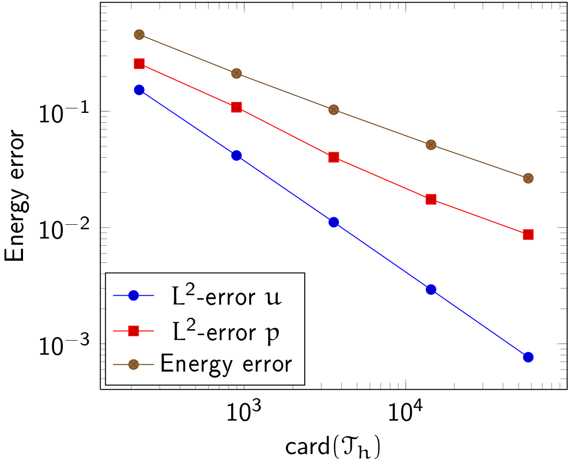


Figure: Convergence results for the Kovasznay problem

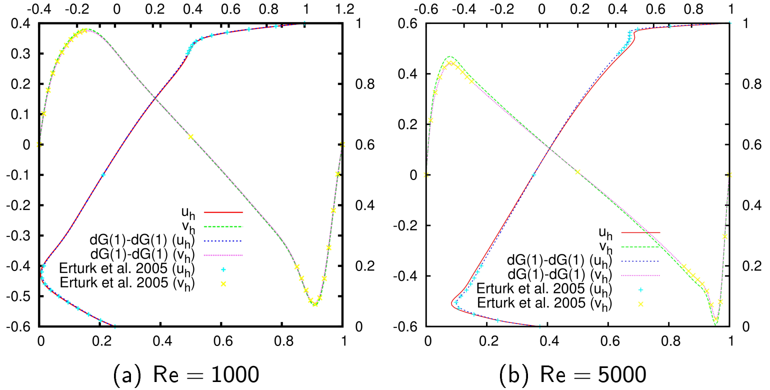
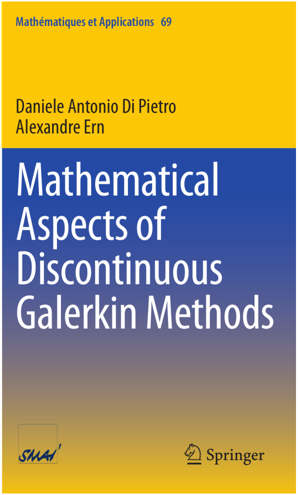


Figure: Lid-driven cavity problem in $d = 2$ (ccG vs. dG)

Thank you for your attention!



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Outline

Vanishing diffusion with advection

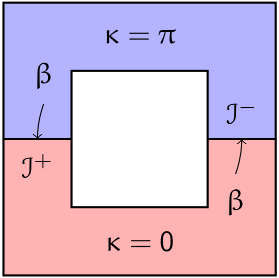
A `FreeFEM`-like library for lowest-order methods on general meshes

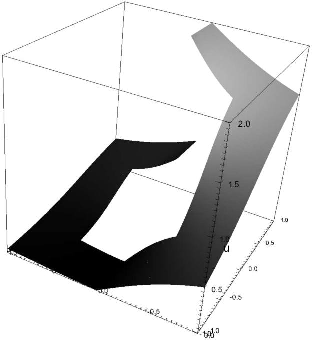
$$\nabla\cdot(-\kappa\nabla u + \beta u) + \mu u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

- Let $\beta \in [W^{1,\infty}(\Omega)]^d$, $\mu > 0$ with $0 < \mu_0 \leqslant \mu - 1/2\nabla\cdot\beta$ and

$$0 \leqslant \underline{\kappa} \leqslant \kappa \leqslant \overline{\kappa},$$

- The exact solution u may have singularities [DP et al., 2008]





Characterization of the exact solution

- Flux continuity $[\![-\kappa\nabla u + \beta u]\!]\cdot n_F = 0$ on $\mathcal{J}^{\pm}$
- Potential continuity $[\![u]\!] = 0$ on $\mathcal{J}^{+}$

Goal: Automatic detection of singular interfaces

The SWIP method for vanishing diffusion with advection III

$$a_h^{\mathrm{dar}}(w, v_h) := a_h^{\mathrm{swip}}(w, v_h) + a_h^{\mathrm{upw}}(w, v_h) + \int_\Omega \mu w v_h$$

Energy norm error estimate

Using SWIP diffusion + upwind advection, $\exists C \neq C(h, \kappa)$ s.t.

$$|||u - u_h|||_{\mathrm{dar}} \lesssim C \inf_{w_h \in V_h} |||u - w_h|||_{\mathrm{dar},*},$$

with $|||\cdot|||_{\mathrm{dar}}$ inf-sup norm and $|||\cdot|||_{\mathrm{dar},*}$ continuity norm.

- $\kappa \equiv 0 \implies$ [Johnson & Pitkäranta, 1986]
- $\beta \equiv 0,\ \kappa > 0 \implies$ [Arnold, Brezzi, Cockburn, & Marini, 2002]

Outline

Vanishing diffusion with advection

A FreeFEM-like library for lowest-order methods on general meshes

FreeFEM-like implementation in a nutshell

```
// 1) Define the discrete space
typedef FunctionSpace<span<Polynomial<d, 1> >,
                      gradient<GreenFormula<LInterpolator> >
                      >::type CCGSpace;
CCGSpace Vh(𝒯h);

// 2) Create test and trial functions
CCGSpace::TrialFunction uh(Vh, "uh");
CCGSpace::TestFunction  vh(Vh, "vh");

// 3) Define the bilinear form
Form2 ah =
  integrate(All<Cell>(𝒯h),dot(grad(uh),grad(vh)))
 -integrate(All<Face>(𝒯h),dot(N(),avg(grad(uh)))*jump(vh)
                          +dot(N(),avg(grad(vh)))*jump(uh))
 +integrate(All<Face>(𝒯h),η/H()*jump(uh)*jump(vh));

// 4) Evaluate the bilinear form
MatrixContext context(A);
evaluate(ah, context);
```

Linear combination I

- Elements of arbitrary shape may be present
- The linear operators $\mathfrak{G}_h$ and $\mathfrak{R}_h$ have unconventional stencil
  - data-dependent (cf. L-construction)
  - non-local (neighbours are involved)
- We cannot rely on reference element(s) + table of DOFs
- Instead, global DOF numbering + embedded stencil

Linear combination II

Linear operator with embedded stencil $\leftrightsquigarrow$ `LinearCombination`

- Let $\mathbb{I} \subset \mathbb{V}_h$ denote the stencil of a discrete linear operator
- A `LinearCombination` $\mathtt{lc}^r = (I, \tau_I)_{I \in \mathbb{I}}$ implements

$$\boxed{\mathtt{lc}^r(\mathbf{v}_h) = \sum_{I \in \mathbb{I}} \tau_I v_I + \tau_0 \in \mathbb{T}_r}$$

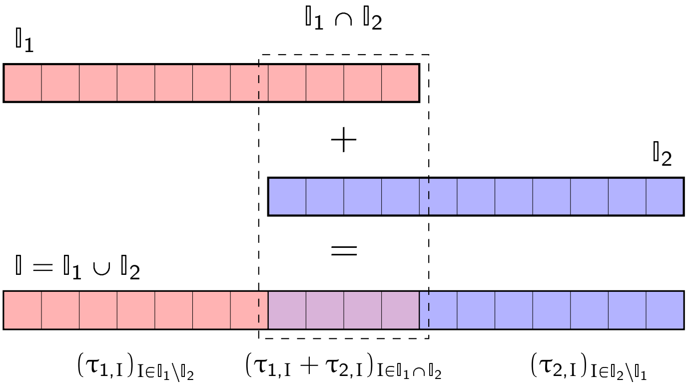
- $r \in \{0, \ldots, 2\}$ denotes the tensor rank of the result
- Algebraic composition of `LinearCombinations` is available

```
// Cell unknown v_T as a linear combination (I_T is the global DOF number)
LinearCombination<0> vT = Term(I_T,1.);

// Linear combination corresponding to 𝔊_h^grn|_T
LinearCombination<1> GT;
for(F ∈ ℱ_T)
{
  // Face unknown v_F (possibly resulting from interpolation)
  const LinearCombination<0> & vF = T_h.eval(F);
  GT += |F|_{d-1}/|T|_d (vF-vT)n_{T,F};
}

// Actually perform algebraic operations on coefficients
GT.compact();
```

Figure: Implementation of the Green gradient $\mathfrak{G}_h^{grn}$

Linear combination IV

$$\mathtt{lc}^r = \mathtt{lc}_1^r + \mathtt{lc}_2^r$$



FE-like assembly

Operator stencils $\mathbb{I}$ and $\mathbb{J}$ $\leftrightsquigarrow$ table of DOFs

- Let $u_h, v_h \in V_h^{ccg}$ and observe that

$$\int_T (\kappa \nabla_h u_h)_{|T} \cdot (\nabla_h v_h)_{|T} \leftrightsquigarrow |T|_d \, \mathtt{lc}_u \cdot \mathtt{lc}_v$$

$$\leftrightsquigarrow \mathbf{A}_T := [|T|_d \, \tau_{v,I} \cdot \tau_{u,J}]_{I \in \mathbb{I}, J \in \mathbb{J}}$$

where $\mathtt{lc}_u = (J, \tau_{u,J})_{J \in \mathbb{J}}$ and $\mathtt{lc}_v = (I, \tau_{v,I})_{I \in \mathbb{I}}$

- The assembly step reads

$$\mathbf{A}(\mathbb{I}, \mathbb{J}) \leftarrow \mathbf{A}(\mathbb{I}, \mathbb{J}) + \mathbf{A}_T$$