An introduction to Hybrid High-Order (HHO) methods

Nonlinear elasticity and poroelasticity

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References for this presentation

Botti, M., Di Pietro, D. A., and Sochala, P. (2017).
A Hybrid High-Order method for nonlinear elasticity.
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Boffi, D., Botti, M., and Di Pietro, D. A. (2016).
A nonconforming high-order method for the Biot problem on general meshes.
SIAM J. Sci. Comput., 38(3):A1508–A1537.

Features of HHO methods

- Support of general polytopal meshes in any space dimension
- Arbitrary approximation order
- Local principle of virtual work with equilibrated tractions
- Compact stencil only invoving neighbors through faces
- Reduced cost after hybridisation for linear(ised) problems

$$N_{\mathrm{dof}}^{\mathrm{hho}} \approx \frac{1}{2}k^2\,\mathrm{card}(\mathcal{F}_h) \qquad N_{\mathrm{dof}}^{\mathrm{dg}} \approx \frac{1}{6}k^3\,\mathrm{card}(\mathcal{T}_h)$$

Polytopal meshes I

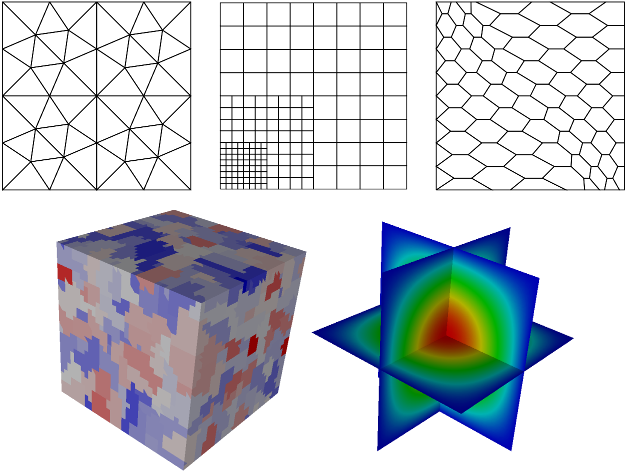


Figure: Admissible meshes. The agglomerated mesh is taken from [DP and Specogna, 2016]

Polytopal meshes II

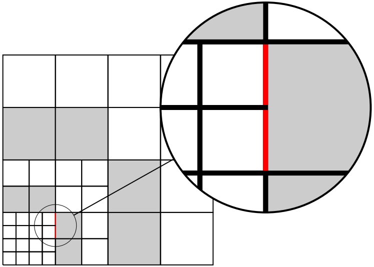


Figure: Treatment of a nonconforming junction (red) as multiple coplanar faces. Gray elements are pentagons, white elements are squares

Polytopal meshes III

Definition (Regular mesh sequence)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}} := (\mathcal{T}_h, \mathcal{F}_h)_{h\in\mathcal{H}}$ be a sequence of h-refined polytopal meshes with $\mathcal{T}_h$ set of elements and $\mathcal{F}_h$ set of faces. The sequence is regular if there exists a sequence of simplicial submeshes $(\mathfrak{T}_h)_{h\in\mathcal{H}}$

- shape-regular in the sense of Ciarlet;
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences:

- Trace and inverse inequalities
- Optimal approximation properties for broken polynomial spaces

Outline

Nonlinear elasticity I

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded connected polyhedral domain
- For $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^d)$ we seek the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ s.t.

$$
\begin{aligned}
-\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}) &= \boldsymbol{f} \qquad \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} \qquad \text{on } \partial\Omega
\end{aligned}
$$

with $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ stress-strain law
- Weak formulation: Find $\boldsymbol{u} \in H^1_0(\Omega; \mathbb{R}^d)$ such that

$$
a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}) : \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in H^1_0(\Omega; \mathbb{R}^d)
$$

with $\boldsymbol{\nabla}_{\mathrm{s}}$ denoting the symmetric (part of) the gradient

Minimal bibliography

- Error estimates under (relatively) strong assumptions on $\boldsymbol{\sigma}$ and $\boldsymbol{u}$
  - Conforming FE, standard meshes [Gatica and Stephan, 2002, Gatica et al., 2013]
  - Discontinuous Galerkin (DG), standard meshes [Ortner and Süli, 2007]
  - Virtual Elements, polyhedral meshes in 2D, low-order [Beirão da Veiga et al., 2013]
- Convergence to minimal regularity solutions
  - Gradient Discretisations [Droniou and Lamichhane, 2015]
  - DG, stronger assumptions on $\boldsymbol{\sigma}$, [Bi and Lin, 2012]
- Convergence to minimal regularity solutions and error estimates for HHO [Botti, DP, Sochala, 2017]

Stress-strain law I

Assumption (Stress-strain law I)

The Carathéodory function $\boldsymbol{\sigma}$ *is s.t.* $\boldsymbol{\sigma}(\cdot,\mathbf{0})=\mathbf{0}$. *Moreover, there exist two real numbers* $\overline{\sigma},\underline{\sigma}\in(0,+\infty)$ *s.t. for a.e.* $\boldsymbol{x}\in\Omega$ *and all* $\boldsymbol{\tau},\boldsymbol{\eta}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$,

$$
\begin{aligned}
\|\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau})\|_{d\times d} &\le \overline{\sigma}\|\boldsymbol{\tau}\|_{d\times d}, && \textit{(growth)}\\
\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}):\boldsymbol{\tau} &\ge \underline{\sigma}\|\boldsymbol{\tau}\|^2_{d\times d}, && \textit{(coercivity)}\\
(\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau})-\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})):(\boldsymbol{\tau}-\boldsymbol{\eta}) &\ge 0, && \textit{(monotonicity)}
\end{aligned}
$$

where $\|\boldsymbol{\tau}\|^2_{d\times d}\coloneqq\boldsymbol{\tau}:\boldsymbol{\tau}$ *and* $\boldsymbol{\tau}:\boldsymbol{\eta}\coloneqq\sum_{1\le i,j\le d}\tau_{ij}\eta_{ij}$.

Stress-strain law II

Example (Stress-strain laws)

- Linear elasticity. For Lamé's parameters $\mu > 0$ and $\lambda \geq 0$,

$$\boldsymbol{\sigma}(\cdot, \boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_d$$

- Hencky–Mises model. For given Lamé's functions $\widetilde{\mu}$ and $\widetilde{\lambda}$, setting $\operatorname{dev}(\boldsymbol{\tau}) := \operatorname{tr}(\boldsymbol{\tau}^2) - \frac{1}{d}\operatorname{tr}(\boldsymbol{\tau})^2$,

$$\boldsymbol{\sigma}(\cdot, \boldsymbol{\tau}) = 2\widetilde{\mu}(\operatorname{dev}(\boldsymbol{\tau}))\boldsymbol{\tau} + \widetilde{\lambda}(\operatorname{dev}(\boldsymbol{\tau}))\operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_d$$

- Isotropic damage model. For a scalar damage function $D : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ and a fourth-order tensor $\boldsymbol{C} : \Omega \to \mathbb{R}^{d^4}$,

$$\boldsymbol{\sigma}(\cdot, \boldsymbol{\tau}) = (1 - D(\boldsymbol{\tau}))\,\boldsymbol{C}(\cdot)\boldsymbol{\tau}$$

L^2-orthogonal projector I

- Let X denote an element in $\mathcal{T}_h$ or a face in $\mathcal{T}_h$ and $l \geq 0$ an integer
- The L^2-orthogonal projector $\pi_X^l : L^1(X;\mathbb{R}) \to \mathbb{P}^l(X;\mathbb{R})$ is s.t.

$$\boxed{\forall v \in L^1(\Omega;\mathbb{R}), \quad \int_X \left(\pi_X^l v - v\right) w = 0 \quad \forall w \in \mathbb{P}^l(X;\mathbb{R})}$$

- $\pi_X^l v$ is well-defined and it holds that

$$\pi_X^l v = \underset{w \in \mathbb{P}^l(X;\mathbb{R})}{\operatorname{argmin}} \|v - w\|_{L^2(X;\mathbb{R})}^2$$

- The vector- and matrix-versions $\boldsymbol{\pi}_X^l$ act component-wise

L^2-orthogonal projector II

Lemma ($W^{s,p}$-approximation properties of π_T^l)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ be a regular mesh sequence. For an integer $l \geq 0$, let an integer $s \in \{0, \ldots, l+1\}$ and a real number $p \in [1, +\infty]$ be given. Then, for all $T \in \mathcal{T}_h$, all $v \in W^{s,p}(T)$, and all $m \in \{0, \ldots, s\}$,

$$|v - \pi_T^l v|_{W^{m,p}(T)} \lesssim h_T^{s-m} |v|_{W^{s,p}(T)}$$

and, if $s \geq 1$ and $m \in \{0, \ldots, s-1\}$,

$$h_T^{\frac{1}{p}} |v - \pi_T^l v|_{W^{m,p}(\mathcal{F}_T)} \lesssim h_T^{s-m} |v|_{W^{s,p}(T)}.$$

Above, $\lesssim$ hides multiplicative constants independent of h.

See [DP and Droniou, 2017a], based on [Dupont and Scott, 1980]

Elastic projector

- Let $T \in \mathcal{T}_h$, $\mathbb{RM}_d(T)$ spanned by rigid-body motions restricted to T
- For a given integer $l \geq 1$, we define the elastic projector

$$\boldsymbol{\pi}_{\text{el},T}^{l} : W^{1,1}(T;\mathbb{R}^d) \to \mathbb{P}^l(T;\mathbb{R}^d)$$

s.t., for all $\boldsymbol{v} \in W^{1,1}(T;\mathbb{R}^d)$,

$$\int_T \boldsymbol{\nabla}_{\text{s}}(\boldsymbol{\pi}_{\text{el},T}^{l}\boldsymbol{v} - \boldsymbol{v}) : \boldsymbol{\nabla}_{\text{s}}\boldsymbol{w} = 0 \qquad \forall \boldsymbol{w} \in \mathbb{P}^l(T;\mathbb{R}^d),$$

$$\int_T \boldsymbol{\pi}_{\text{el},T}^{l}\boldsymbol{v} = \int_T \boldsymbol{v}, \quad \int_T \boldsymbol{\nabla}_{\text{ss}}\boldsymbol{\pi}_{\text{el},T}^{l}\boldsymbol{v} = \frac{1}{2}\sum_{F\in\mathcal{F}_T}\left(\boldsymbol{n}_{TF} \wedge \boldsymbol{\pi}_F^k\boldsymbol{v} - \boldsymbol{\pi}_F^k\boldsymbol{v} \wedge \boldsymbol{n}_{TF}\right)$$

- Using the abstract results of [DP and Droniou, 2017b], it can be proved that $\boldsymbol{\pi}_{\text{el},T}^{l}$ has optimal approximation properties

Computing L^2-projections of $\boldsymbol{\nabla}_{\rm s}\boldsymbol{v}$ from L^2-projections of $\boldsymbol{v}$

- For all $\boldsymbol{v} \in W^{1,1}(T;\mathbb{R}^d)$ and all $\boldsymbol{\tau} \in C^\infty(\overline{T};\mathbb{R}^{d\times d}_{\rm sym})$, it holds that

$$\boxed{\int_T \boldsymbol{\nabla}_{\rm s}\boldsymbol{v} : \boldsymbol{\tau} = -\int_T \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{v}\cdot\boldsymbol{\tau}\boldsymbol{n}_{TF}} \qquad \text{(IBP)}$$

- Specialising (IBP) to $\boldsymbol{\tau} \in \mathbb{P}^l(T;\mathbb{R}^{d\times d}_{\rm sym})$, we can write

$$\int_T \boldsymbol{\pi}_T^l\boldsymbol{\nabla}_{\rm s}\boldsymbol{v} : \boldsymbol{\tau} = -\int_T \boldsymbol{\pi}_T^{l-1}\boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{\pi}_F^l\boldsymbol{v}\cdot\boldsymbol{\tau}\boldsymbol{n}_{TF}$$

- Hence, computing $\boldsymbol{\pi}_T^l\boldsymbol{\nabla}_{\rm s}\boldsymbol{v}$ does not require a full knowledge of $\boldsymbol{v}$!
- All that is required is $\boldsymbol{\pi}_T^{l-1}\boldsymbol{v}$ and for all $F \in \mathcal{F}_T$, $\boldsymbol{\pi}_F^l\boldsymbol{v}$

Computing $\boldsymbol{\pi}_{\mathrm{el},T}^{l+1}\boldsymbol{v}$ from L^2-projections of $\boldsymbol{v}$

- Specialise now (IBP) to $\boldsymbol{\tau} = \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{w}$ with $\boldsymbol{w} \in \mathbb{P}^{l+1}(T;\mathbb{R}^d)$, to obtain

$$\int_T \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{\pi}_{\mathrm{el},T}^{l+1}\boldsymbol{v} : \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{w} = -\int_T \boldsymbol{\pi}_T^{l-1}\boldsymbol{v} \cdot (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{w}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{\pi}_F^{l}\boldsymbol{v}\cdot\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{w}\,\boldsymbol{n}_{TF}$$

- Observe, moreover, that if $l \geq 1$ then for all $\boldsymbol{w} \in \mathbb{RM}_d(T)$,

$$\int_T (\boldsymbol{\pi}_{\mathrm{el},T}^{l+1}\boldsymbol{v} - \boldsymbol{v})\cdot\boldsymbol{w} = \int_T (\boldsymbol{\pi}_{\mathrm{el},T}^{l+1}\boldsymbol{v} - \boldsymbol{\pi}_T^{l}\boldsymbol{v})\cdot\boldsymbol{w}$$

  since $\mathbb{RM}_d(T) \subset \mathbb{P}^1(T;\mathbb{R}^d) \subseteq \mathbb{P}^l(T;\mathbb{R}^d)$

- Hence, $\boldsymbol{\pi}_{\mathrm{el},T}^{l+1}\boldsymbol{v}$ is computable from $\boldsymbol{\pi}_T^{l}\boldsymbol{v}$ and for all $F \in \mathcal{F}_T$, $\boldsymbol{\pi}_F^{l}\boldsymbol{v}$

Local space of discrete unknowns and reconstructions I

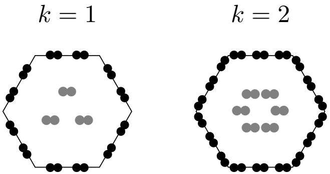


Figure: Local discrete unknowns for $k = 1, 2$. Internal unknowns can be eliminated by static condensation for linearised versions of the problem

- Let $k \geq 1$ and $T \in \mathcal{T}_h$ be fixed. The space of local unknowns is s.t.

$$\boxed{\underline{\boldsymbol{U}}_T^k := \mathbb{P}^k(T;\mathbb{R}^d) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F;\mathbb{R}^d) \right)}$$

- We denote by $\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T})$ a generic element of $\underline{\boldsymbol{U}}_T^k$
- The local interpolator $\underline{\boldsymbol{I}}_T^k : W^{1,1}(T;\mathbb{R}^d) \to \underline{\boldsymbol{U}}_T^k$ is s.t.

$$\forall \boldsymbol{v} \in W^{1,1}(T;\mathbb{R}^d), \qquad \underline{\boldsymbol{I}}_T^k \boldsymbol{v} := (\boldsymbol{\pi}_T^k \boldsymbol{v}, (\boldsymbol{\pi}_F^k \boldsymbol{v})_{F \in \mathcal{F}_T})$$

Local space of discrete unknowns and reconstructions II

- The symmetric gradient reconstruction $\boldsymbol{G}_{\mathrm{s},T}^{k} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is s.t

$$\int_T \boldsymbol{G}_{\mathrm{s},T}^{k} \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T \cdot (\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T} \int_F \boldsymbol{v}_F \cdot \boldsymbol{\tau}\boldsymbol{n}_{TF} \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

- The displacement reconstruction $\boldsymbol{r}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T;\mathbb{R}^{k+1})$ is s.t.

$$\int_T (\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{r}_T^{k+1} - \boldsymbol{G}_{\mathrm{s},T}^{k}) \underline{\boldsymbol{v}}_T : \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w} = 0 \qquad \forall \boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$$

$$\int_T (\boldsymbol{r}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_T) \cdot \boldsymbol{w} = 0 \qquad \forall \boldsymbol{w} \in \mathbb{RM}_d(T)$$

- We have the key commuting properties: For all $\boldsymbol{v} \in W^{1,1}(T;\mathbb{R}^d)$,

$$\boxed{\boldsymbol{G}_{\mathrm{s},T}^{k} \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \boldsymbol{\pi}_T^k \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}, \qquad \boldsymbol{r}_T^{k+1} \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \boldsymbol{\pi}_{\mathrm{el},T}^{k+1} \boldsymbol{v}}$$

Local contribution and stabilisation I

- Let $T \in \mathcal{T}_h$. We approximate $a_{|T}$ with $\mathrm{a}_T : \underline{\boldsymbol{U}}_T^k \times \underline{\boldsymbol{U}}_T^k \to \mathbb{R}$ s.t.

$$\boxed{\mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := \int_T \boldsymbol{\sigma}(\cdot, \boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{u}}_T) : \boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T + \mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)}$$

- Here, s_T is the stabilisation bilinear form s.t.

$$\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := \sum_{F \in \mathcal{F}_T} \frac{\gamma}{h_F} \int_F (\boldsymbol{\delta}_{TF}^k - \boldsymbol{\delta}_T^k)\underline{\boldsymbol{u}}_T \cdot (\boldsymbol{\delta}_{TF}^k - \boldsymbol{\delta}_T^k)\underline{\boldsymbol{v}}_T,$$

with γ user-defined parameter and difference operators s.t.

$$(\boldsymbol{\delta}_T^k \underline{\boldsymbol{v}}_T, (\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{v}}_T)_{F \in \mathcal{F}_T}) := \underline{\boldsymbol{I}}_T^k(\boldsymbol{r}_T^{k+1} \underline{\boldsymbol{v}}_T) - \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$$

Local contribution and stabilisation II

Proposition (Properties of s_T)

- *Stability.* *For all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$, it holds that*

$$\boxed{\|\boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T\|_{L^2(T;\mathbb{R}^{d\times d})}^2 + \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \simeq \|\underline{\boldsymbol{v}}_T\|_{\epsilon,T}^2}$$

with hidden constant independent of h and T and

$$\|\underline{\boldsymbol{v}}_T\|_{\epsilon,T}^2 \coloneqq \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}_T\|_{L^2(T;\mathbb{R}^{d\times d})}^2 + \sum_{F\in\mathcal{F}_T} \frac{1}{h_F} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^2(F;\mathbb{R}^d)}^2.$$

- *Polynomial consistency.* *For all $\boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$, it holds that*

$$\boxed{\mathrm{s}_T(\underline{\boldsymbol{I}}_T^k \boldsymbol{w}, \underline{\boldsymbol{v}}_T) = 0 \qquad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k.}$$

Local contribution and stabilisation III

Remark (Naïve stabilisation and polynomial consistency)

Stability can be achieved using the following naïve stabilisation:

$$\mathrm{s}_T^{\mathrm{hdg}}(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) = \sum_{F \in \mathcal{F}_T} \frac{\gamma}{h_F} \int_F (\boldsymbol{u}_F - \boldsymbol{u}_T) \cdot (\boldsymbol{v}_F - \boldsymbol{v}_T).$$

In this case, however, we only have polynomial consistency for $\boldsymbol{w} \in \mathbb{P}^k(T;\mathbb{R}^d)$. As a result, up to one order of convergence is lost.

Discrete problem I

- We define the global space with single-valued interface unknowns

$$\boxed{\underline{\boldsymbol{U}}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T;\mathbb{R}^d) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F;\mathbb{R}^d) \right)}$$

as well as its subspace with strongly enforced b.c.

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The global interpolator $\underline{\boldsymbol{I}}_h^k : W^{1,1}(\Omega;\mathbb{R}^d) \to \underline{\boldsymbol{U}}_h^k$ is s.t.

$$(\underline{\boldsymbol{I}}_h^k v)_{|T} := \underline{\boldsymbol{I}}_T^k v_{|T} \quad \forall T \in \mathcal{T}_h$$

Discrete problem II

- Define the function $\mathrm{a}_h : \underline{\boldsymbol{U}}_h^k \times \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ assembled element-wise:

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

- Discrete problem: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ such that

$$\boxed{\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k}$$

with $\boldsymbol{v}_h$ obtained patching element unknowns

Lemma (Existence and uniqueness)

Let $(\mathcal{M}_h)_{h \in \mathcal{H}}$ be a regular mesh sequence. Then, for all $h \in \mathcal{H}$ there exists at least one solution $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$. Additionally, if $\boldsymbol{\sigma}$ is strictly monotone, the solution is unique.

Convergence I

Theorem (Convergence)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ be a regular mesh sequence. Then, for all q s.t. $1 \le q < +\infty$ if $d = 2$, $1 \le q < 6$ if $d = 3$, as $h \to 0$, up to a subsequence,

- $\boldsymbol{u}_h \to \boldsymbol{u}$ *strongly in $L^q(\Omega;\mathbb{R}^d)$;*
- $\boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{u}}_h \to \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}$ *weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$.*

Moreover, if we assume strict monotonicity for $\boldsymbol{\sigma}$,

- $\boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{u}}_h \to \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}$ *strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$.*

If the continuous solution is unique, the whole sequence converges.

Convergence II

Assumption (Stress-strain law II)

There exist reals $\sigma^*, \sigma_* \in (0, +\infty)$ *s.t., for a.e.* $\boldsymbol{x} \in \Omega$ *and all* $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\rm sym}$,

$$\|\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})\|_{d\times d} \le \sigma^* \|\boldsymbol{\tau} - \boldsymbol{\eta}\|_{d\times d}, \qquad \text{(Lipschitz continuity)}$$

$$(\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \ge \sigma_* \|\boldsymbol{\tau} - \boldsymbol{\eta}\|^2_{d\times d}. \qquad \text{(strong monotonicity)}$$

Theorem (Error estimate)

Under the above assumption and the regularity $\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h; \mathbb{R}^d)$ *and* $\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\rm s}\boldsymbol{u}) \in H^{k+1}(\mathcal{T}_h; \mathbb{R}^{d\times d})$, *it holds that*

$$\boxed{\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u} - \boldsymbol{G}^k_{{\rm s},T}\underline{\boldsymbol{u}}_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + |\underline{\boldsymbol{u}}_h|_{{\rm s},h} \lesssim h^{k+1}\mathcal{N}_{\boldsymbol{u}},}$$

with hidden constant independent of h, $|\underline{\boldsymbol{u}}_h|^2_{{\rm s},h} := \sum_{T\in\mathcal{T}_h} {\rm s}_T(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h)$, *and* $\mathcal{N}_{\boldsymbol{u}} := \|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)} + \|\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\rm s}\boldsymbol{u})\|_{H^{k+1}(\mathcal{T}_h;\mathbb{R}^{d\times d})}$.

Convergence III

Theorem (Robust estimate for quasi-incompressible materials)

Let $\boldsymbol{\sigma}$ be such that, for all $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\rm sym}$ with $\mu > 0$ and $\lambda \geq 0$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d.$$

Then, the following locking-free error estimate holds:

$$(2\mu)^{\frac{1}{2}} \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u} - \boldsymbol{G}^k_{{\rm s},T}\underline{\boldsymbol{u}}_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \lesssim h^{k+1}\left(2\mu\|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)} + \lambda\|\boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^{k+1}(\mathcal{T}_h,\mathbb{R})}\right)$$

with hidden constant independent of h, μ, and λ.

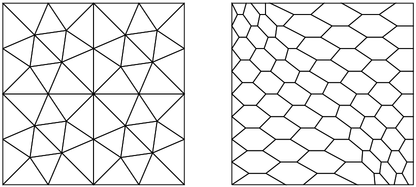
- We consider the Hencky–Mises model with $\mu = 2$ and $\lambda = 1$ and

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = ((\lambda - \mu) + \mu \exp(-\operatorname{dev}(\boldsymbol{\tau}))) \operatorname{tr}(\boldsymbol{\tau}) \boldsymbol{I}_d + \mu (2 - \exp(-\operatorname{dev}(\boldsymbol{\tau})) \boldsymbol{\tau}$$

- We solve the homogeneous Dirichlet problem with

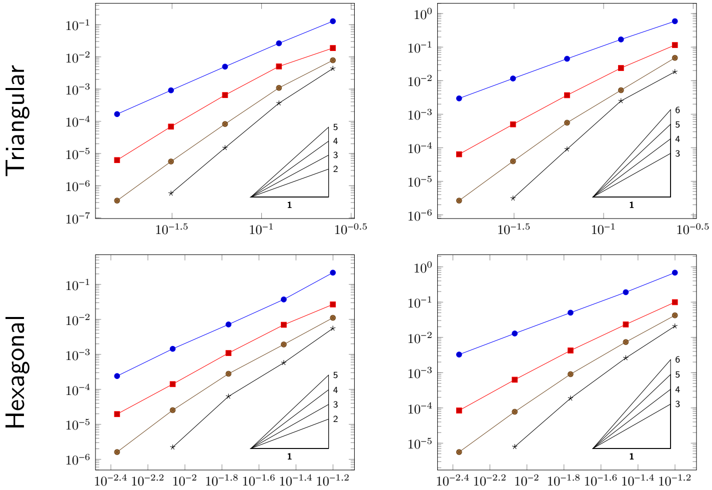
$$\boldsymbol{u}(\boldsymbol{x}) \coloneqq \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \qquad \boldsymbol{f} = -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u})$$

- Refinements of the following meshes are used:



Numerical examples II

Convergence



$\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u} - \boldsymbol{G}^k_{\mathrm{s},h}\underline{\boldsymbol{u}}_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})}$ $\|\boldsymbol{\pi}_h^k\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega;\mathbb{R}^{d})}$

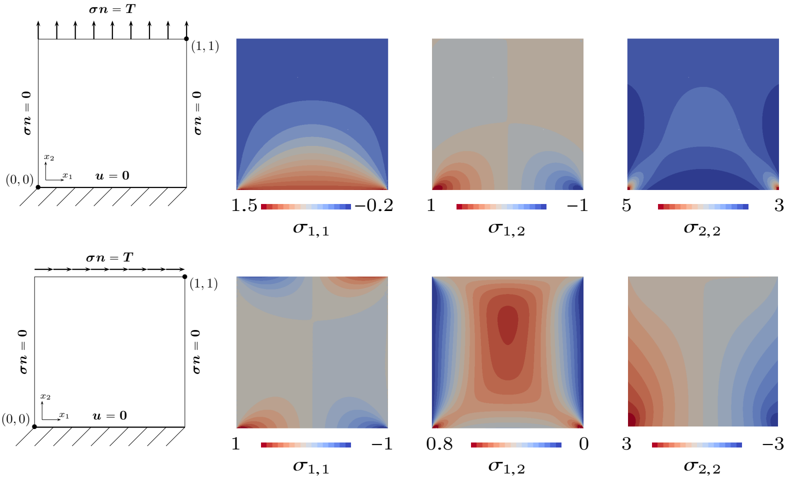
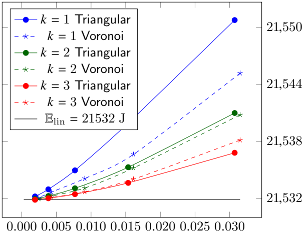


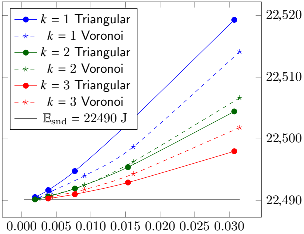
Figure: Traction and shear tests and corresponding stress components for the linear case $(10^5\mathrm{Pa})$

Numerical examples II

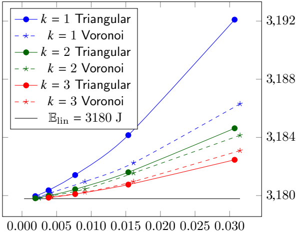
Traction and shear test cases

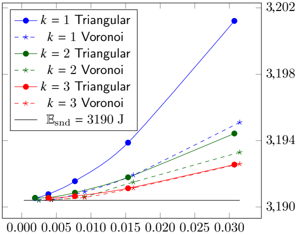
Traction





Shear





Linear Second-order[1]

[1] Obtained adding third-order terms to the energy density function

Outline

The Biot model

- Let Ω as before, $t_{\mathrm{F}} > 0$ and $\kappa : \Omega \to \mathbb{R}$ be s.t. $0 < \underline{\kappa} \le \kappa \le \overline{\kappa}$ in Ω
- Let $\boldsymbol{f}$ and g be given volumetric load and source terms
- Biot problem: Find the displacement $\boldsymbol{u}$ and the pressure p s.t.

$$\boxed{\begin{aligned} -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{\nabla}p &= \boldsymbol{f} && \text{in } \Omega\times(0,t_{\mathrm{F}}),\\ c_0 \mathrm{d}_t p + \boldsymbol{\nabla}\cdot(\mathrm{d}_t\boldsymbol{u}) - \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}p) &= g && \text{in } \Omega\times(0,t_{\mathrm{F}}), \end{aligned}}$$

  completed with initial and boundary conditions (impermeable fixed walls)
- In the incompressible case $c_0 = 0$, we further assume for any t

$$\int_\Omega p(\cdot,t) = 0 \text{ and } \int_\Omega g(\cdot,t) = 0$$

- Perspective: extension to the nonlinear, multiphase case

Minimal bibliography

- Origin of the model [Terzaghi, 1943] and [Biot, 1941, Biot, 1955]
- Finite Volumes, 3D, discontinuous coefficients [Naumovich, 2006]
- Continuous FE $\boldsymbol{u}$ + DG p [Phillips and Wheeler, 2007]
- DG $\boldsymbol{u}$ + MPFA p [Wheeler et al., 2014]
- Justification of spurious oscillations [Rodrigo et al., 2016]
- HHO $\boldsymbol{u}$ + DG p [Boffi, Botti, DP, 2016]

Features

- High-order method on general polyhedral meshes
- Inf-sup-stable hydro-mechanical coupling
- Robustness with respect to heterogeneous-anisotropic permeabilities
- Seamless treatment of the incompressible case $c_0 = 0$
- Locally equilibrated tractions and fluxes
- Numerically robust w.r.t. to spurious oscillations for small κ and τ

Discrete spaces

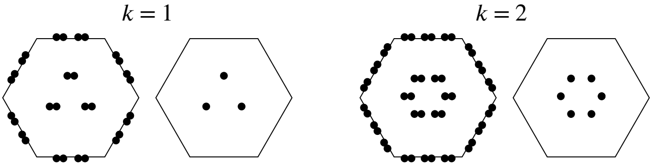


Figure: Displacement and pressure discrete unknowns for $k \in \{1, 2\}$

- Let $k \geq 1$. We approximate the displacements in the HHO space

$$\underline{\boldsymbol{U}}_{h,0}^{k} := \left\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- For the pressure, we consider the broken polynomial space

$$P_h^k := \begin{cases} \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}) & \text{if } c_0 > 0 \\ \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}) \cap L_0^2(\Omega; \mathbb{R}) & \text{if } c_0 = 0 \end{cases}$$

Discrete problem

- We consider for the sake of simplicity a uniform time mesh of size τ
- Discrete problem: For $1 \le n \le N$, $(\underline{\boldsymbol{u}}_h^n, p_h^n) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ is s.t.

$$
\boxed{
\begin{aligned}
a_h(\underline{\boldsymbol{u}}_h^n, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h^n) &= \int_\Omega \boldsymbol{f}^n \cdot \boldsymbol{v}_h \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\
(c_0 \delta_t p_h^n, q_h) - b_h(\delta_t \underline{\boldsymbol{u}}_h^n, q_h) + c_h(p_h^n, q_h) &= \int_\Omega g^n q_h \quad && \forall q_h \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})
\end{aligned}
}
$$

- For the mechanical term we use a_h defined as before

Hydro-mechanical coupling

- The hydro-mechanical coupling hinges on the bilinear form

$$b_h(\underline{\boldsymbol{v}}_h, q_h) := -\int_\Omega D_h^k \underline{\boldsymbol{v}}_h q_h, \qquad (D_h^k)_{|T} := \mathrm{tr}(\boldsymbol{G}_{\mathrm{s},T}^k) \quad \forall T \in \mathcal{T}_h$$

- $\underline{\boldsymbol{I}}_T^k$ is a Fortin interpolator: For all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$,

$$D_h^k \underline{\boldsymbol{I}}_h^k \boldsymbol{v} = \boldsymbol{\pi}_h^k(\boldsymbol{\nabla}\cdot\boldsymbol{v}), \qquad \|\underline{\boldsymbol{I}}_h \boldsymbol{v}\|_{\epsilon,h} \lesssim \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)}$$

- Hence, for all $q_h \in P_h^k$, with hidden constant independent of h,

$$\boxed{\|q_h\|_{L^2(\Omega)} \lesssim \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k,\, \|\underline{\boldsymbol{v}}_h\|_{\epsilon,h}=1} b_h(\underline{\boldsymbol{v}}_h, q_h)}$$

- This is a key point for robust L^2-norm bounds for p when $c_0 = 0$

Darcy operator I

- For the Darcy operator we use a Discontinuous Galerkin method
- For robustness in κ, we follow [DP et al., 2008]
- Key ingredients are the jump and weighted average operators

$$[\varphi]_F := \varphi_{T_1} - \varphi_{T_2}, \qquad \{\varphi\}_F := \omega_{T_1}\varphi_{T_1} + \omega_{T_2}\varphi_{T_2},$$

where $F \in \mathcal{F}_h^{\mathrm{i}}$ is s.t. $F \subset \partial T_1 \cap \partial T_2$ and

$$\omega_{T_1} := \frac{\kappa_{T_2}}{\kappa_{T_1} + \kappa_{T_2}}, \qquad \omega_{T_2} := \frac{\kappa_{T_1}}{\kappa_{T_1} + \kappa_{T_2}}$$

Darcy operator II

- The Darcy operator is discretised using the SWIP bilinear form

$$
\begin{aligned}
c_h(r_h, q_h) \coloneqq{}& \int_\Omega \kappa \boldsymbol{\nabla}_h r_h \cdot \boldsymbol{\nabla}_h q_h + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \frac{\varsigma \lambda_{\kappa,F}}{h_F} \int_F [r_h]_F [q_h]_F \\
&- \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F \left(\{\kappa \boldsymbol{\nabla}_h r_h\}_F \cdot \boldsymbol{n}_F, [q_h]_F + [r_h]_F, \{\kappa \boldsymbol{\nabla}_h q_h\}_F \cdot \boldsymbol{n}_F \right)
\end{aligned}
$$

- Here, $\varsigma > 0$ is a large enough user-defined penalty parameter and

$$
\lambda_{\kappa,F} \coloneqq \frac{2\kappa_{T_1}\kappa_{T_2}}{\kappa_{T_1} + \kappa_{T_2}}
$$

Main results I

Lemma (A priori bounds and well-posendess)

Let $\boldsymbol{\sigma}$ be such that, for all $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\rm sym}$ with $\mu > 0$ and $\lambda \geq 0$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}_d.$$

Assume $\boldsymbol{f} \in C^1([0, t_{\rm F}]; L^2(\Omega; \mathbb{R}^d))$ and $g \in C^0([0, t_{\rm F}]; L^2(\Omega; \mathbb{R}))$. Then, the discrete problem is well-posed with a priori bound

$$\|\underline{\boldsymbol{u}}_h^N\|_{{\rm a},h}^2 + \|c_0^{\frac{1}{2}} p_h^N\|_{L^2(\Omega;\mathbb{R})}^2 + \|p_h^N - \overline{p}_h^N\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{n=1}^{N} \tau \|p_h^n\|_{{\rm c},h}^2 \lesssim 1$$

where the hidden constant depends on bounded norms of p^0, $\boldsymbol{f}$, and g and we have set $\overline{p}_h^N := \int_\Omega p_h^N$.

Main results II

Theorem (Error estimate)

Let $\boldsymbol{\sigma}$ as above. Assume elliptic regularity, $p \in C^1([0,t_{\mathrm{F}}];H^{k+1}(P_\Omega;\mathbb{R}))$, $p \in C^2([0,t_{\mathrm{F}}];L^2(\Omega;\mathbb{R}))$ if $c_0 > 0$, and $\boldsymbol{u} \in C^2([0,t_{\mathrm{F}}],H^1(P_\Omega;\mathbb{R}^d)) \cap C^1([0,t_{\mathrm{F}}];H^{k+2}(P_\Omega;\mathbb{R}^d))$. Then, setting

$$\underline{\boldsymbol{e}}_h^n := \underline{\boldsymbol{u}}_h^n - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}^n, \quad \rho_h^n := p_h^n - \pi_h^k p^n, \quad \overline{\rho}_h^n := (\rho_h^n, 1),$$

it holds

$$\|\underline{\boldsymbol{e}}_h^N\|_{\mathrm{a},h}^2 + \|c_0^{\frac{1}{2}} \rho_h^N\|_{L^2(\Omega;\mathbb{R})}^2 + \|\rho_h^N - \overline{\rho}_h^N\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{n=1}^N \tau \|\rho_h^n\|_{\mathrm{c},h}^2 \lesssim \left(h^{k+1} + \tau\right)^2,$$

with hidden constant depending on bounded norms of $\boldsymbol{u}$ and p and increasing linearly with $\alpha^{\frac{1}{2}}$ where $\alpha := \overline{\kappa}/\underline{\kappa}$ is the anisotropy ratio.

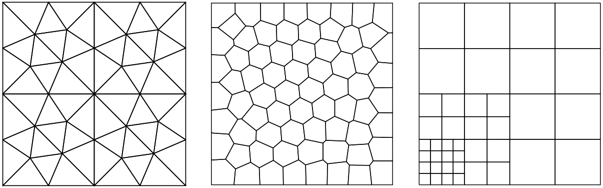


Figure: Meshes for the convergence test case

- We let $\Omega = (0,1)^2$, $c_0 = 0$, $\mu = 1$, $\lambda = 1$, and $\kappa = \boldsymbol{I}_2$ on
- The right-hand side is inferred from the (non-physical) exact solution

$$
\begin{aligned}
u_1(\boldsymbol{x}, t) &= -\sin(\pi t)\cos(\pi x_1)\cos(\pi x_2), \\
u_2(\boldsymbol{x}, t) &= \sin(\pi t)\sin(\pi x_1)\sin(\pi x_2), \\
p(\boldsymbol{x}, t) &= -\cos(\pi t)\sin(\pi x_1)\cos(\pi x_2)
\end{aligned}
$$

Numerical examples II

Convergence

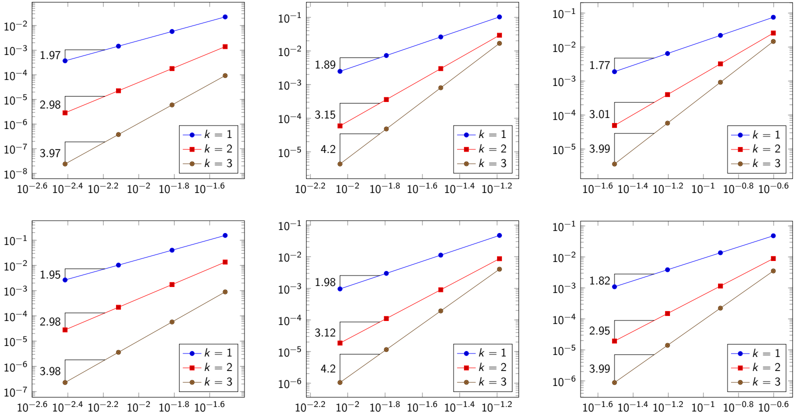


Figure: L^2-error on the pressure (top) and H^1-error on the displacement (bottom) vs. h for (from left to right) the triangular, Voronoi, and locally refined meshes

Barry and Mercer's test case

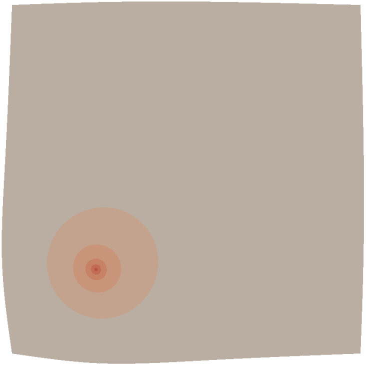


Figure: Barry and Mercer's exact solution modelling fluid injection and production from a well

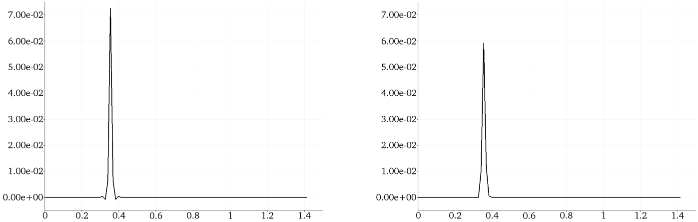
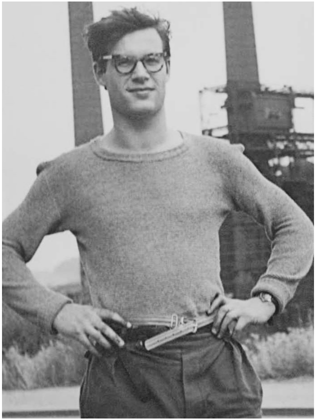


Figure: Pressure profiles along $(0,0)$–$(1,1)$ for $\boldsymbol{\kappa} = 1 \cdot 10^{-6} \boldsymbol{I}_d$ and $\boldsymbol{\tau} = 1 \cdot 10^{-4}$. Small oscillations visible on the Cartesian mesh (left, $\operatorname{card} \mathcal{T}_h = 4{,}028$), no oscillations are present on the Voronoi mesh (right, $\operatorname{card} \mathcal{T}_h = 4{,}192$)

Alexander Grothendieck's legacy



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