Basic principles of polytopal approximations of partial differential equations

Daniele A. Di Pietro

Scuola Superiore Meridionale
Napoli, 02/05/2024

Outline

Outline

Setting

- Let $\Omega \subset \mathbb{R}^d$ be an open connected polytopal domain
- We focus on the Poisson problem: Given $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- A possible weak formulation reads: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega)$$

  where $H^1(\Omega) := \left\{ v \in L^2(\Omega) \,:\, \nabla v \in L^2(\Omega)^d \right\}$ and $H_0^1(\Omega)$ is its zero-trace subspace
- The well-posedness of this problem hinges on the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \le C_\Omega \|\nabla v\|_{L^2(\Omega)^d}$$

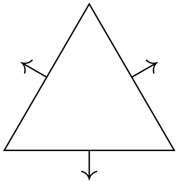
Local polynomial spaces and L^2-orthogonal projector

- Denote by $\mathbb{P}_d^\ell$ the space of d-variate polynomials of total degree $\leq \ell$
- Let $Y \subset \mathbb{R}^d$ and denote by $\mathcal{P}^\ell(Y)$ the restriction of $\mathbb{P}_d^\ell$ to Y
- Given $v \in L^2(Y)$, its L^2-orthogonal projection on $\mathcal{P}^\ell(Y)$ is s.t.

$$\int_Y (v - \pi_{\mathcal{P}^\ell(Y)} v) w = 0 \quad \forall w \in \mathcal{P}^\ell(Y)$$

- $\pi_{\mathcal{P}^\ell(Y)} v$ optimally approximates v in all Sobolev seminorms under mild assumptions on Y; see [DP and Droniou, 2020, Chapter 1] for details

Raviart–Thomas–Nédélec and Crouzeix–Raviart elements I

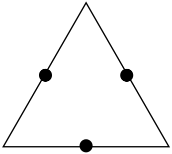


- Denote by T a d-simplex
- Let $\mathcal{RTN}^1(T) := \mathcal{P}^0(T)^d + x\mathcal{P}^0(T)$
- Define the degrees of freedom $\sigma := (\sigma_F)_{F\in\mathcal{F}_T}$ s.t.

$$\sigma_F : H^1(T)^d \ni v \mapsto \frac{1}{|F|}\int_F v\cdot n_{TF} \in \mathbb{R}$$

- $(T, \mathcal{RTN}^1(T), \sigma)$ is a FE [Raviart and Thomas, 1977, Nédélec, 1980]

Raviart–Thomas–Nédélec and Crouzeix–Raviart elements II



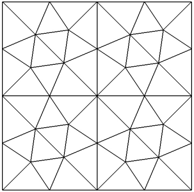
- Let now $\mathcal{P}^1(T)$ be the space of affine functions on T
- Define the degrees of freedom $\sigma \coloneqq (\sigma_F)_{F \in \mathcal{F}_T}$ s.t.

$$\sigma_F : H^1(T) \ni v \mapsto \frac{1}{|F|} \int_F v \in \mathbb{R}$$

- $(T, \mathcal{P}^1(T), \sigma)$ is a FE [Crouzeix and Raviart, 1973]

Outline

A non-conforming finite element scheme I



- Let $\mathcal{T}_h$ be a conforming simplicial mesh of Ω
- Let $\mathcal{CR}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$ be the Crouzeix–Raviart space on $\mathcal{T}_h$ and set

$$\mathcal{CR}_0(\mathcal{T}_h) := \left\{ v_h \in \mathcal{CR}(\mathcal{T}_h) \;:\; \pi_{\mathcal{P}^0(F)} v = 0 \text{ for all } F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- With ∇_h broken gradient, the scheme reads: Find $u_h \in \mathcal{CR}_0(\mathcal{T}_h)$ s.t.

$$a_h(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h \quad \forall v_h \in \mathcal{CR}_0(\mathcal{T}_h)$$

Stability analysis I

Lemma (Discrete Poincaré inequality in the Crouzeix–Raviart space)

For all $v_h \in \mathcal{CR}_0(\mathcal{T}_h)$,

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

- For each $T \in \mathcal{T}_h$, the Poincaré–Wirtinger inequality gives

$$\|v_h\|_{L^2(T)} \lesssim \|\pi_{\mathcal{P}^0(T)} v\|_{L^2(T)} + h_T \|\nabla v_T\|_{L^2(T)^d}$$

- Hence, letting $\overline{v}_h \coloneqq \pi_{\mathcal{P}^0(\mathcal{T}_h)} v_h$, it suffices to prove that

$$\|\overline{v}_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

- Since $\operatorname{div} : \mathcal{RTN}^1(\mathcal{T}_h) \to \mathcal{P}^0(\mathcal{T}_h)$ is surjective, there is $\tau_h \in \mathcal{RTN}^1(\mathcal{T}_h)$ s.t.

$$\operatorname{div} \tau_h = \overline{v}_h \text{ and } \|\tau_h\|_{H(\operatorname{div};\Omega)} \lesssim \|\overline{v}_h\|_{L^2(\Omega)}$$

Stability analysis II

- We write, letting $v_T \coloneqq (v_h)_{|T}$ for all $T \in \mathcal{T}_h$,

$$\|\overline{v}_h\|_{L^2(\Omega)}^2 = \int_\Omega v_h \operatorname{div} \tau_h = -\int_\Omega \nabla_h v_h \cdot \tau_h + \underbrace{\sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F v_T(\tau_h \cdot n_{TF})}_{=:\mathfrak{T}}$$

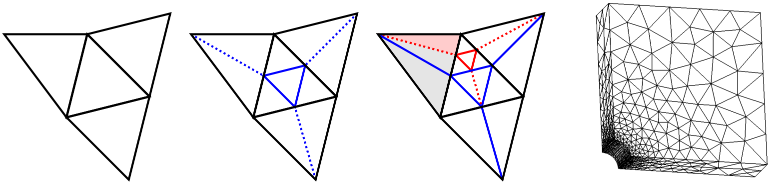
- Since $\tau_h \cdot n_{TF} \in \mathcal{P}^0(F)$, we can replace $v_T \leftarrow \pi_{\mathcal{P}^0(F)} v_T =: \color{red}{v_F}$ in the boundary term. Rearranging the sums, we have

$$\mathfrak{T} = \sum_{F\in\mathcal{F}_h^{\rm i}} \sum_{T\in\mathcal{T}_F} \int_F \color{red}{v_F}\color{black}(\tau_h \cdot n_{TF}) + \sum_{F\in\mathcal{F}_h^{\rm b}} \int_F \color{red}{v_F}\color{black}(\tau_h \cdot n_F) = 0,$$

  by single-valuedness of v_F for $F \in \mathcal{F}_h^{\rm i}$ and $v_F = 0$ for $F \in \mathcal{F}_h^{\rm b}$
- We conclude using Cauchy–Schwarz inequalities to write

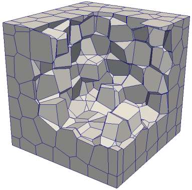
$$\|\overline{v}_h\|_{L^2(\Omega)}^2 \le \|\nabla_h v_h\|_{L^2(\Omega)^d} \|\tau_h\|_{L^2(\Omega)^d} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d} \|\overline{v}_h\|_{L^2(\Omega)}$$

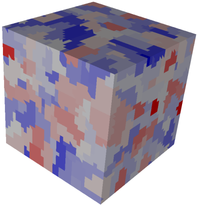
Limitations of the finite element approach

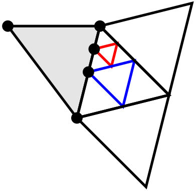


- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Treating more general meshes in the FE spirit would significantly increase the space dimension [Droniou et al., 2021]
- The extension to high-order is not straightforward

Fully discrete polytopal approach







- Key idea: replace both spaces and operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Several strategies to reduce the number of unknowns on general shapes
- Elegant analysis framework available

A few key references

- Introduction of Hybrid High-Order (HHO) methods [DP et al., 2014]
- Fully discrete analysis framework [DP and Droniou, 2018]
- A monograph on HHO methods [DP and Droniou, 2020]
- Introduction of Discrete de Rham (DDR) methods [DP et al., 2020]
- DDR for the de Rham complex of differential forms [Bonaldi et al., 2023]

Outline

A paradigm shift

- Let $v_T \in \mathcal{P}^1(T)$ and set $v_F \coloneqq \pi_{\mathcal{P}^0(F)} v_{|F}$ for all $F \in \mathcal{F}_T$
- We have, for all $\tau \in \mathcal{P}^0(T)^d$,

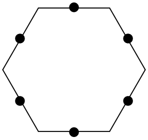
$$\int_T \nabla v_T \cdot \tau = -\int_T v_T \operatorname{div} \tau + \sum_{F \in \mathcal{F}_T} \int_F v_T (\tau \cdot n_{TF}) = \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF})$$

- Moreover, with $\overline{x}_Y$ center of mass of $Y \in \{T\} \cup \mathcal{F}_T$,

$$\pi_{\mathcal{P}^0(T)} v = v(\overline{x}_T) = \frac{1}{\mathrm{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} v(\overline{x}_F) = \frac{1}{\mathrm{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F$$

- These formulas remain valid when T is a general polytope!

Generalization to polytopes



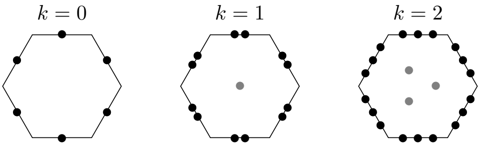
- Let $\underline{V}_T^0 \coloneqq \left\{ \underline{v}_T \coloneqq (v_F)_{F\in\mathcal{F}_T} \;:\; v_F \in \mathcal{P}^0(F) \text{ for all } F \in \mathcal{F}_T \right\}$
- Let $\underline{v}_T \in \underline{V}_T^0$. The gradient reconstruction $G_T^0 : \underline{V}_T^0 \to \mathcal{P}^0(T)^d$ is s.t.

$$\int_T G_T^0 \underline{v}_T \cdot \tau = \sum_{F\in\mathcal{F}_T} \int_F v_F(\tau\cdot n_{TF}) \quad \forall \tau \in \mathcal{P}^0(T)^d$$

- A potential reconstruction $r_T^1 : \underline{V}_T^0 \to \mathcal{P}^1(T)$ is defined enforcing

$$\nabla r_T^1 \underline{v}_T = G_T^0 \underline{v}_T \text{ and } \pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = \frac{1}{\mathrm{card}(\mathcal{F}_T)} \sum_{F\in\mathcal{F}_T} \frac{1}{|F|} \int_F v_F$$

Extension to arbitrary-order I



- Let $k \geq 0$ and define the Hybrid High-Order (HHO) space

$$\underline{V}_T^k := \big\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \; : \; v_T \in \mathcal{P}^{k-1}(T) \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_T \big\}$$

- Let $\underline{v}_T \in \underline{V}_T^k$. We define $G_T^k : \underline{V}_T^k \to \mathcal{P}^k(T)^d$ s.t.

$$\int_T G_T^k \underline{v}_T \cdot \tau = -\int_T v_T \cdot \operatorname{div} \tau + \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF}) \quad \forall \tau \in \mathcal{P}^k(T)^d$$

Extension to arbitrary-order II

- The potential reconstruction $r_T^{k+1} : \underline{V}_T^k \to \mathcal{P}^{k+1}(T)$ is s.t.

$$\int_T \nabla r_T^{k+1} \underline{v}_T \cdot \nabla w = \int_T G_T^k \underline{v}_T \cdot \nabla w \text{ for all } w \in \mathcal{P}^{k+1}(T)$$

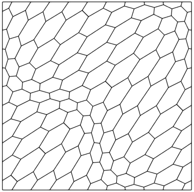
and

$$\pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = \pi_{\mathcal{P}^0(T)} v_T,$$

where

$$v_T := \frac{1}{\mathrm{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F \text{ if } k = 0$$

Discrete Poincaré inequality in HHO spaces I



- Given a polytopal mesh $\mathcal{T}_h$ of Ω, $\underline{V}_h^k$ is the global HHO space and

$$\underline{V}_{h,0}^k := \left\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_T}) \in \underline{V}_h^k \;:\; v_F = 0 \text{ for all } F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- We define on $\underline{V}_h^k$ the norm

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$$

where $\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F\in\mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2$ for all $T \in \mathcal{T}_h$

Discrete Poincaré inequality in HHO spaces II

Lemma (Poincaré inequality in HHO spaces)

For any $\underline{v}_h \in \underline{V}_{h,0}^k$*, letting* $v_h \in \mathcal{P}^{\max(k-1,0)}(\mathcal{T}_h)$ *be s.t.* $(v_h)_{|T} \coloneqq v_T$ *for all* $T \in \mathcal{T}_h$*, it holds*

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h},$$

hence $\|\cdot\|_{1,h}$ *is a norm on* $\underline{V}_{h,0}^k$*.*

- Since $\operatorname{div} : H^1(\Omega)^d \to L^2(\Omega)$ is surjective, there is $\tau \in H^1(\Omega)^d$ s.t.

$$\operatorname{div} \tau = v_h \text{ and } \|\tau\|_{H^1(\Omega)^d} \lesssim \|v_h\|_{L^2(\Omega)}$$

- Notice that here we cannot seek τ in $\mathcal{RTN}^{k+1}(\mathcal{T}_h)$ since $\mathcal{T}_h$ is not conforming simplicial!

Discrete Poincaré inequality in HHO spaces III

- We go on writing

$$
\begin{aligned}
\|v_h\|_{L^2(\Omega)}^2 &= \int_\Omega v_h \operatorname{div} \tau \\
&= \sum_{T\in\mathcal{T}_h} \left(-\int_T \nabla v_h \cdot \tau + \sum_{F\in\mathcal{F}_T} \int_F (v_T - v_F)(\tau\cdot n_{TF}) \right) \\
&\le \left[\sum_{T\in\mathcal{T}_h} \left(\|\nabla v_h\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F\in\mathcal{F}_T} \|v_T - v_F\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}} \\
&\quad \times \left[\sum_{T\in\mathcal{T}_h} \left(\|\tau\|_{L^2(T)^2}^2 + h_T \|\tau\|_{L^2(\partial T)}^2 \right) \right]^{\frac{1}{2}}
\end{aligned}
$$

- Recalling the definition of $\|\cdot\|_{1,h}$ and using trace inequalities along with $h_T \le h_\Omega \lesssim 1$, we get

$$
\|v_h\|_{L^2(\Omega)}^2 \lesssim \|\underline{v}_h\|_{1,h} \|\tau\|_{H^1(T)^d} \lesssim \|\underline{v}_h\|_{1,h} \|v_h\|_{L^2(\Omega)}
$$

An HHO scheme

We consider the following scheme: Find $\underline{u}_h \in \underline{V}^k_{h,0}$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) \coloneqq \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{V}^k_{h,0}$$

where, for all $T \in \mathcal{T}_h$,

$$a_T(\underline{u}_T, \underline{v}_T) \coloneqq \int_T G^k_T \underline{u}_T \cdot G^k_T \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

and the symmetric semi-definite bilinear form s_T satisfies

$$\|\underline{v}_T\|_{1,T} \lesssim a_T(\underline{v}_T, \underline{v}_T)^{\frac{1}{2}} \lesssim \|\underline{v}_T\|_{1,T} \quad \forall \underline{v}_T \in \underline{V}^k_T \tag{ST1}$$

Stability analysis

Lemma (Well-posedness of the HHO discrete problem)

The HHO problem admits a unique solution that satisfies

$$\|\underline{u}_h\|_{1,h} \lesssim \|f\|_{L^2(\Omega)}.$$

- Squaring and summing (ST1) over $T \in \mathcal{T}_h$, we have

$$\|\underline{v}_h\|_{1,h}^2 \lesssim a_h(\underline{v}_h, \underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k,$$

  which expresses the coercivity of a_h
- Using the Cauchy–Schwarz and discrete Poincaré inequalities,

$$\int_\Omega f v_h \le \|f\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} \|\underline{v}_h\|_{1,h}$$

- Letting $\underline{v}_h = \underline{u}_h$ above, the conclusion follows

Error analysis I

- Let $\underline{I}_h^k : H^1(\Omega) \to \underline{V}_h^k$ be s.t.

$$\underline{I}_h^k v := ((\pi_{\mathcal{P}^{k-1}(T)} v)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{P}^k(F)} v)_{F \in \mathcal{F}_h}) \quad \forall v \in H^1(\Omega)$$

- We aim at estimating the error

$$\underline{e}_h := \underline{u}_h - \underline{I}_h^k u \in \underline{V}_{h,0}^k$$

- It holds, for all $\underline{v}_h \in \underline{V}_{h,0}^k$,

$$a_h(\underline{e}_h, \underline{v}_h) = a_h(\underline{u}_h, \underline{v}_h) - a_h(\underline{I}_h^k u, \underline{v}_h) = \int_\Omega f v_h - a_h(\underline{I}_h^k u, \underline{v}_h) =: \mathcal{E}_h(\underline{v}_h)$$

- A straightforward modification of the stability proof gives

$$\|\underline{e}_h\|_{1,h} \le \sup_{\underline{v}_h \in \underline{V}_{h,0}^k \setminus \{\underline{0}\}} \frac{\mathcal{E}_h(\underline{v}_h)}{\|\underline{v}_h\|_{1,h}}$$

Error analysis II

We reformulate the components of the consistency error $\mathcal{E}_h(\underline{v}_h)$:

$$\int_\Omega f v_h = -\sum_{T\in\mathcal{T}_h}\int_T \Delta u v_h = \sum_{T\in\mathcal{T}_h}\left(\int_T \nabla u\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F (\nabla u\cdot n_{TF})(v_F - v_T)\right)$$

$$\begin{aligned}
a_h(\underline{I}_h^k u,\underline{v}_h) &= \sum_{T\in\mathcal{T}_h}\left(\int_T G_T^k(\underline{I}_T^k u)\cdot G_T^k\underline{v}_T + s_T(\underline{I}_T^k u,\underline{v}_T)\right)\\
&= \sum_{T\in\mathcal{T}_h}\left(\int_T G_T^k(\underline{I}_T^k u)\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F (G_T(\underline{I}_T^k u)\cdot n_{TF})(v_F - v_T)\right)\\
&\quad + \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u,\underline{v}_T)
\end{aligned}$$

Error analysis III

Gathering the above results, we get

$$\mathcal{E}_h(\underline{v}_h) = \sum_{T\in\mathcal{T}_h} \int_T \left[\nabla u - G_T^k(\underline{I}_T^k u)\right] \cdot \nabla v_T + \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F \left[\nabla u - (G_T(\underline{I}_T^k u) \cdot n_{TF})\right] (v_F - v_T) - \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T) =: \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3$$

Approximation properties of the discrete gradient I

- By definition, for all $T \in \mathcal{T}_h$, all $v \in H^1(T)$, and all $\tau \in \mathcal{P}^k(T)^d$,

$$\int_T G_T^k(\underline{I}_T^k v) \cdot \tau = -\int_T \pi_{\mathcal{P}^{k-1}(T)} v \operatorname{div} \tau + \sum_{F \in \mathcal{F}_T} \int_F \pi_{\mathcal{P}^k(F)} v (\tau \cdot n_{TF})$$

- Noticing that $\operatorname{div} \tau \in \mathcal{P}^{k-1}(T)$ and $\tau \cdot n_{TF} \in \mathcal{P}^k(F)$, we can remove the projectors and integrate by parts to obtain

$$\int_T G_T^k(\underline{I}_T^k v) \cdot \tau = \int_T \nabla u \cdot \tau \quad \forall \tau \in \mathcal{P}^k(T)^d$$

- This shows that $G_T^k \circ \underline{I}_T^k = \pi_{\mathcal{P}^k(T)^d} \circ \nabla$, i.e.,

$$\begin{array}{ccc} H^1(T) & \xrightarrow{\nabla} & L^2(T)^d \\ \Big\downarrow \underline{I}_T^k & & \Big\downarrow \pi_{\mathcal{P}^k(T)^d} \\ \underline{V}_T^k & \xrightarrow{G_T^k} & \mathcal{P}^k(T)^d \end{array}$$

Approximation properties of the discrete gradient II

- Noticing that $\nabla v_T \in \mathcal{P}^k(T)^d$,

$$\int_T \left[\nabla u - G_T^k(\underline{I}_T^k u)\right] \cdot \nabla v_T = \int_T \left[\nabla u - \pi_{\mathcal{P}^k(T)^d}(\nabla u)\right] \cdot \nabla v_T = 0$$

  we infer

$$\mathfrak{T}_1 = 0$$

- Using Cauchy–Schwarz inequalities and the definition of $\|\cdot\|_{1,h}$,

$$\mathfrak{T}_2 \leq \left(\sum_{T \in \mathcal{T}_h} \|\nabla u - \pi_{\mathcal{P}^k(T)^d}(\nabla u)\|_{\partial T}^2 \right)^{\frac{1}{2}} \|\underline{v}_h\|_{1,h}$$

- If, additionally, $u \in H^{r+2}(\mathcal{T}_h)$ for some $r \in \{0, \ldots, k\}$,

$$\mathfrak{T}_2 \lesssim h^{r+1} |u|_{H^{r+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Polynomial consistency of the stabilization I

- To have $\mathfrak{T}_3$ scale as $\mathfrak{T}_2$, we further assume polynomial consistency:

$$s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{V}_T^k \tag{ST2}$$

- For all $w \in H^{r+2}(T)$, setting $|\cdot|_{s,T} \coloneqq s_T(\cdot,\cdot)^{\frac{1}{2}}$, we have

$$\begin{aligned}|\underline{I}_T^k w|_{s,T} &\overset{\text{(ST2)}}{=} \min_{v \in \mathcal{P}^{k+1}(T)} |\underline{I}_T^k (w - v)|_{s,T} \\ &\overset{\text{(ST1)}}{\lesssim} \min_{v \in \mathcal{P}^{k+1}(T)} \|\underline{I}_T^k (w - v)\|_{1,T} \lesssim h_T^{r+1} |w|_{H^{r+2}(T)}\end{aligned}$$

hence, by Cauchy–Schwarz inequalities and again (ST1),

$$\mathfrak{T}_3 \lesssim h^{r+1} |u|_{H^{r+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Polynomial consistency of the stabilization II

Example

Let, for all $T \in \mathcal{T}_h$ and all $\underline{v}_T \in \underline{V}_T^k$,

$$(\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T}) \coloneqq \underline{v}_T - \underline{I}_T^k (r_T^{k+1} \underline{v}_T).$$

The stabilization bilinear form

$$s_T(\underline{w}_T, \underline{v}_T) \coloneqq h_T^{-2} \int_T \delta_T^k \underline{w}_T \, \delta_T^k \underline{v}_T + h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F \delta_{TF}^k \underline{w}_T \, \delta_{TF}^k \underline{v}_T$$

satisfies properties (ST1)–(ST2).

Error estimate for smooth solutions

Theorem (Error estimate for the HHO scheme)

Denote by $u \in H^1_0(\Omega)$ the solution to the Poisson problem and by $\underline{u}_h \in \underline{V}_h^k$ its HHO approximation. Then, under (ST1)–(ST2)*, and further assuming $u \in H^{r+2}(\mathcal{T}_h)$ for some $r \in \{0, \ldots, k\}$, it holds*

$$\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{r+1} |u|_{H^{r+2}(\mathcal{T}_h)}.$$

Numerical example

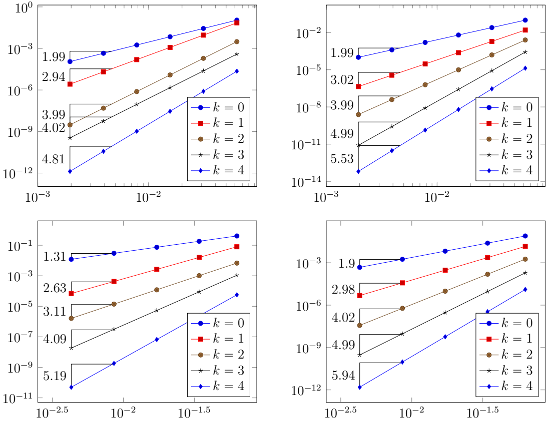


Figure: $\|\underline{e}_h\|_{1,h}$ and $\|e_h\|_{L^2(\Omega)}$ as a function of h for uniformly refined triangular (top) and hexagonal (bottom) mesh families

Funded by the European Union (ERC Synergy, NEMESIS, project number 101115663). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Thank you for your attention!

`https://erc-nemesis.eu/events/workshop-montpellier`

References I

Bonaldi, F., Di Pietro, D. A., Droniou, J., and Hu, K. (2023).
An exterior calculus framework for polytopal methods.
http://arxiv.org/abs/2303.11093.

Crouzeix, M. and Raviart, P.-A. (1973).
Conforming and nonconforming finite element methods for solving the stationary Stokes equations.
RAIRO Modél. Math. Anal. Num., 7(3):33–75.

Di Pietro, D. A. and Droniou, J. (2018).
A third Strang lemma for schemes in fully discrete formulation.
Calcolo, 55(40).

Di Pietro, D. A. and Droniou, J. (2020).
The Hybrid High-Order method for polytopal meshes.
Number 19 in Modeling, Simulation and Application. Springer International Publishing.

Di Pietro, D. A., Droniou, J., and Rapetti, F. (2020).
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.
Math. Models Methods Appl. Sci., 30(9):1809–1855.

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Meth. Appl. Math., 14(4):461–472.

Droniou, J., Eymard, R., Gallouët, T., and Herbin, R. (2021).
Polyhedral Methods in Geosciences, chapter Non-conforming finite elements on polytopal meshes, pages 1–35.
Number 27 in SEMA-SIMAI. Springer.

Nédélec, J.-C. (1980).
Mixed finite elements in $\mathbf{R}^3$.
Numer. Math., 35(3):315–341.

References II

Raviart, P. A. and Thomas, J. M. (1977).
A mixed finite element method for 2nd order elliptic problems.
In Galligani, I. and Magenes, E., editors, *Mathematical Aspects of the Finite Element Method*. Springer, New York.