Hybrid High-Order methods for the incompressible Navier–Stokes equations

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Outline

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Model problem

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a polytopal domain
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

  which is a simplified model of the viscous terms in Navier–Stokes
- In weak form: Find $u \in U := H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = \int_\Omega f v \qquad \forall v \in U$$

Finite Elements

- Simple idea: Replace $U \leftarrow U_h \subset U$ and solve for $u_h \in U_h$ s.t.

$$a(u_h, v_h) = \int_\Omega f v_h \qquad \forall v_h \in U_h$$

Finite Elements

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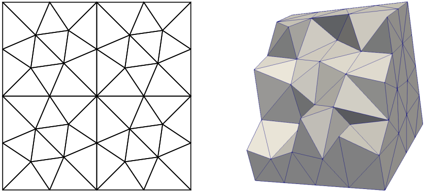


Figure: Example of Finite Element mesh in dimension $d = 2$ and $d = 3$

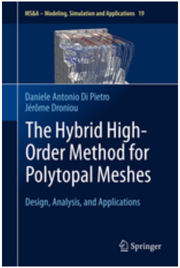
- With several limitations:
  - The construction of U_h requires a matching simplicial mesh of Ω...
  - ...making local mesh adaptation cumbersome
  - The mathematical construction lacks physical fidelity...
  - ...leading to a lack of robustness in certain regimes
  - What about non-linear problems?

Key ideas

- Define a local reconstruction r_T^{k+1} for each $T \in \mathcal{T}_h$
- Fix a space of unknowns $\underline{U}_h^k$ making the reconstructions computable
- Assemble a discrete problem as in FE from the local contributions

$$a_{|T}(u,v) \approx a_{|T}(r_T^{k+1}\underline{u}_T, r_T^{k+1}\underline{v}_T) + \text{stab.}$$

- See the monograph [DP and Droniou, 2020] for an introduction:



Features

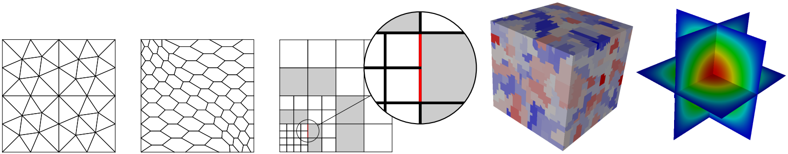


Figure: Examples of supported meshes $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ in 2d and 3d

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Physical fidelity leading to robustness in singular limits
- Natural extension to nonlinear problems
- Reduced computational cost after static condensation

Projectors on local polynomial spaces

- With $X \in \mathcal{T}_h \cup \mathcal{F}_h$, the L^2-projector $\pi_X^{0,l} : L^2(X) \to \mathbb{P}^l(X)$ is s.t.

$$\int_X (\pi_X^{0,l} v - v) w = 0 \text{ for all } w \in \mathbb{P}^l(X)$$

- The elliptic projector $\pi_T^{1,l} : H^1(T) \to \mathbb{P}^l(T)$ is s.t.

$$\int_T \boldsymbol{\nabla}(\pi_T^{1,l} v - v) \cdot \boldsymbol{\nabla} w = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } \int_T (\pi_T^{1,l} v - v) = 0$$

- Both have optimal approximation properties in $\mathbb{P}^l(T)$
- See [DP and Droniou, 2017a, DP and Droniou, 2017b]

Computing $\pi_T^{1,k+1}$ from L^2-projections of degree k

- Recall the following IBP valid for all $v \in H^1(T)$ and all $w \in C^\infty(\overline{T})$:

$$\boxed{\int_T \nabla v \cdot \nabla w = -\int_T v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v \nabla w \cdot \boldsymbol{n}_{TF}}$$

- Specializing it to $w \in \mathbb{P}^{k+1}(T)$, we can write

$$\int_T \nabla \pi_T^{1,k+1} v \cdot \nabla w = -\int_T \pi_T^{0,k} v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^{0,k} v \nabla w \cdot \boldsymbol{n}_{TF}$$

- Moreover, it can be easily seen that

$$\int_T (\pi_T^{1,k+1} v - v) = \int_T (\pi_T^{1,k+1} v - \pi_T^{0,k} v) = 0$$

- **Hence, $\pi_T^{1,k+1} v$ can be computed from $\pi_T^{0,k} v$ and $(\pi_F^{0,k} v)_{F \in \mathcal{F}_T}$!**

Discrete unknowns

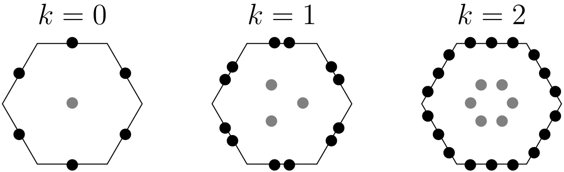


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- Let a polynomial degree $k \geq 0$ be fixed
- For all $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$\underline{U}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \ : \ v_T \in \mathbb{P}^k(T) \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ is s.t., for all $v \in H^1(T)$,

$$\underline{I}_T^k v := \left(\pi_T^{0,k} v, (\pi_F^{0,k} v)_{F \in \mathcal{F}_T} \right)$$

Local potential reconstruction

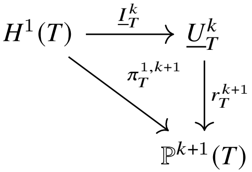
- Let $T \in \mathcal{T}_h$. We define the local potential reconstruction operator

$$r_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$$

s.t., for all $\underline{v}_T \in \underline{U}_T^k$, $\int_T (r_T^{k+1}\underline{v}_T - v_T) = 0$ and

$$\boxed{\int_T \nabla r_T^{k+1}\underline{v}_T \cdot \nabla w = -\int_T v_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F \nabla w \cdot \boldsymbol{n}_{TF} \quad \forall w \in \mathbb{P}^{k+1}(T)}$$

- It holds $r_T^{k+1} \circ \underline{I}_T^k = \pi_T^{1,k+1}$, i.e., the following diagram commutes:



Stabilization I

- We would be tempted to approximate

$$a_{|T}(u,v) \approx a_{|T}(r_T^{k+1}\underline{u}_T, r_T^{k+1}\underline{v}_T)$$

- This choice, however, is not stable in general. We consider instead

$$\boxed{\mathrm{a}_T(\underline{u}_T,\underline{v}_T) := a_{|T}(r_T^{k+1}\underline{u}_T, r_T^{k+1}\underline{v}_T) + \mathrm{s}_T(\underline{u}_T,\underline{v}_T)}$$

- The role of s_T is to ensure $\|\cdot\|_{1,T}$-coercivity with

$$\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_T^2 + \sum_{F\in\mathcal{F}_T} \frac{1}{h_F}\|v_F - v_T\|_F^2 \quad \forall \underline{v}_T \in \underline{U}_T^k$$

Stabilization II

Assumption (Stabilization bilinear form)

The bilinear form $\mathrm{s}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ satisfies the following properties:

- Symmetry and positivity. s_T is symmetric and positive semidefinite.
- Stability. It holds, with hidden constant independent of h and T,

$$\mathrm{a}_T(\underline{v}_T, \underline{v}_T)^{\frac{1}{2}} \simeq \|\underline{v}_T\|_{1,T} \quad \forall \underline{v}_T \in \underline{U}_T^k.$$

- Polynomial consistency. For all $w \in \mathbb{P}^{k+1}(T)$ and all $\underline{v}_T \in \underline{U}_T^k$,

$$\mathrm{s}_T(\underline{I}_T^k w, \underline{v}_T) = 0.$$

Stabilization III

- For all $T \in \mathcal{T}_h$, s_T can be obtained penalizing the components of

$$\underline{I}_T^k(r_T^{k+1}\underline{v}_T) - \underline{v}_T$$

- An example is

$$s_T(\underline{w}_T, \underline{v}_T) = h_T^{-2}\int_T (\pi_T^{0,k} r_T^{k+1}\underline{w}_T - w_T)(\pi_T^{0,k} r_T^{k+1}\underline{v}_T - v_T) + h_T^{-1}\sum_{F\in\mathcal{F}_T}\int_F (\pi_F^{0,k} r_T^{k+1}\underline{w}_T - w_F)(\pi_F^{0,k} r_T^{k+1}\underline{v}_T - v_F)$$

Discrete problem

- Define the global space with single-valued interface unknowns

$$\underline{U}_h^k := \Big\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) \ : \\ v_T \in \mathbb{P}^k(T) \quad \forall T \in \mathcal{T}_h \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_h \Big\}$$

  and its subspace with strongly enforced boundary conditions

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \ : \ v_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{\mathrm{a}_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{a}_T(\underline{u}_T, \underline{v}_T) = \sum_{T\in\mathcal{T}_h} \int_T f v_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

- Well-posedness follows from coercivity and discrete Poincaré

Convergence

Theorem (Energy-norm error estimate)

If $u \in H_0^1(\Omega) \cap H^{k+2}(\mathcal{T}_h)$, the following energy error estimate holds:

$$\boxed{\|\boldsymbol{\nabla}_h(r_h^{k+1}\underline{u}_h - u)\| + |\underline{u}_h|_{s,h} \lesssim h^{k+1}|u|_{H^{k+2}(\mathcal{T}_h)}}$$

with $(r_h^{k+1}\underline{u}_h)_{|T} := r_T^{k+1}\underline{u}_T$ for all $T \in \mathcal{T}_h$ and $|\underline{u}_h|_{s,h}^2 := \sum_{T \in \mathcal{T}_h} s_T(\underline{u}_T, \underline{u}_T)$.

Theorem (Superclose L^2-norm error estimate)

Further assuming elliptic regularity and $f \in H^1(\mathcal{T}_h)$ if $k = 0$,

$$\boxed{\|r_h^{k+1}\underline{u}_h - u\| \lesssim h^{k+2}\mathcal{N}_k,}$$

with $\mathcal{N}_0 := \|f\|_{H^1(\mathcal{T}_h)}$ and $\mathcal{N}_k := |u|_{H^{k+2}(\mathcal{T}_h)}$ for $k \geq 1$.

Numerical examples

2d test case, smooth solution, uniform refinement

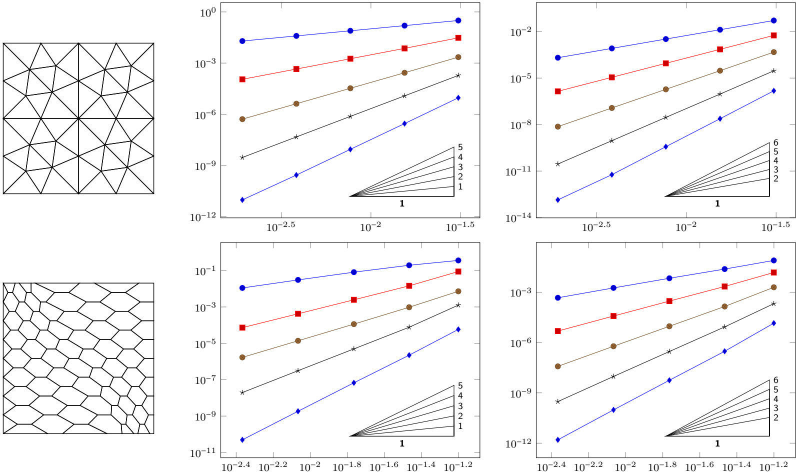


Figure: Energy (left) and L^2-errors (right) on triangular (top) and hexagonal (bottom) mesh sequences for $k = 0, \ldots, 4$

Numerical examples I

3d test case, singular solution, adaptive coarsening

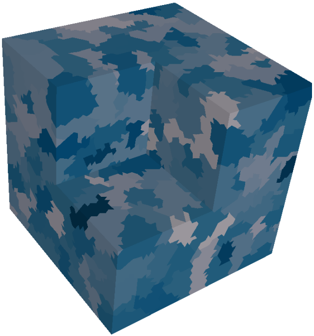
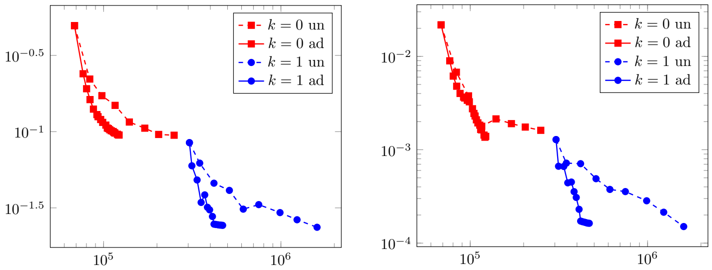


Figure: Fichera corner benchmark, adaptive mesh coarsening [DP and Specogna, 2016]

Numerical examples II

3d test case, singular solution, adaptive coarsening



(a) Energy-error vs. N_{dofs}

(b) L^2-error vs. N_{ndof}

Figure: Error vs. number of DOFs for the Fichera corner benchmark, adaptively coarsened meshes

Outline

Features

- Capability of handling general polyhedral meshes
- Construction valid for both $d = 2$ and $d = 3$
- Arbitrary approximation order (including $k = 0$)
- Inf-sup stability on general meshes
- Robust handling of dominant advection
- Local conservation of momentum and mass
- Reduced computational cost after static condensation

$$N_{\mathrm{dof},h} = d\,\mathrm{card}(\mathcal{F}_h^{\mathrm{i}})\binom{k+d-1}{d-1} + \mathrm{card}(\mathcal{T}_h)$$

HHO for incompressible flows

- MHO for Stokes [Aghili, Boyaval, DP, 2015]
- Pressure-robust HHO for Stokes [DP, Ern, Linke, Schieweck, 2016]
- Péclet-robust HHO for Oseen [Aghili and DP, 2018]
- Darcy-robust HHO for Brinkman [Botti, DP, Droniou, 2018]
- Skew-symmetric HHO for Navier–Stokes [DP and Krell, 2018]
- Temam's device for HHO [Botti, DP, Droniou, 2018]
- Curl-curl formulation [Beirão da Veiga, Dassi, DP, Droniou, 2022]

The incompressible Navier–Stokes equations

- Let $\nu > 0$, $\boldsymbol{f} \in L^2(\Omega)^d$, $\boldsymbol{U} := H_0^1(\Omega)^d$, and $P := L_0^2(\Omega)$
- The INS problem reads: Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ s.t.

$$\boxed{\begin{aligned} \nu a(\boldsymbol{u},\boldsymbol{v}) + t(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{U}, \\ -b(\boldsymbol{u},q) &= 0 && \forall q \in L^2(\Omega), \end{aligned}}$$

with viscous and pressure-velocity coupling bilinear forms

$$a(\boldsymbol{w},\boldsymbol{v}) := \int_\Omega \boldsymbol{\nabla}\boldsymbol{w} : \boldsymbol{\nabla}\boldsymbol{v}, \quad b(\boldsymbol{v},q) := -\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{v})\ q$$

and convective trilinear form

$$t(\boldsymbol{w},\boldsymbol{v},\boldsymbol{z}) := \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{z} = \sum_{i=1}^d \sum_{j=1}^d \int_\Omega w_j(\partial_j v_i) z_i$$

Discrete spaces I

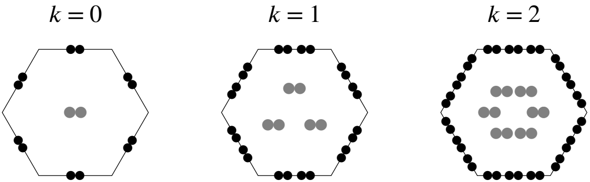


Figure: Local velocity space $\underline{\boldsymbol{U}}_T^k$ for $k \in \{0,1,2\}$

- For $k \geq 0$, we define the global space of discrete velocity unknowns

$$\underline{\boldsymbol{U}}_h^k := \Big\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) \; : \\ \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \quad \forall T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \quad \forall F \in \mathcal{F}_h \Big\}$$

- The restrictions to $T \in \mathcal{T}_h$ are $\underline{\boldsymbol{U}}_T^k$ and $\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T})$

Discrete spaces II

- The global interpolator $\underline{\boldsymbol{I}}_h^k : H^1(\Omega)^d \to \underline{\boldsymbol{U}}_h^k$ is s.t., $\forall \boldsymbol{v} \in H^1(\Omega)^d$,

$$\underline{\boldsymbol{I}}_h^k \boldsymbol{v} \coloneqq \left((\boldsymbol{\pi}_T^{0,k} \boldsymbol{v})_{T \in \mathcal{T}_h}, (\boldsymbol{\pi}_F^{0,k} \boldsymbol{v})_{F \in \mathcal{F}_h} \right)$$

- The velocity space strongly accounting for boundary conditions is

$$\underline{\boldsymbol{U}}_{h,0}^k \coloneqq \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \;:\; \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete pressure space is defined setting

$$P_h^k \coloneqq \mathbb{P}^k(\mathcal{T}_h) \cap P$$

Viscous term

- The viscous term is discretized by means of the bilinear form a_h s.t.

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

  where, letting $\boldsymbol{r}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T)^d$ as for Poisson component-wise,

$$\boxed{\mathrm{a}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T) := (\boldsymbol{\nabla}\boldsymbol{r}_T^{k+1}\underline{\boldsymbol{w}}_T, \boldsymbol{\nabla}\boldsymbol{r}_T^{k+1}\underline{\boldsymbol{v}}_T)_T + \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T)}$$

  with s_T satisfying similar properties as in the scalar case
- Variable viscosity can be treated following [DP and Ern, 2015]

Divergence reconstruction

- Let $\ell \geq 0$. Mimicking the IBP formula: $\forall(\boldsymbol{v}, q) \in H^1(T)^d \times C^\infty(\overline{T})$,

$$\int_T (\boldsymbol{\nabla}\cdot\boldsymbol{v})\ q = -\int_T \boldsymbol{v}\cdot\boldsymbol{\nabla} q + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}\cdot\boldsymbol{n}_{TF})\ q$$

we introduce divergence reconstruction $D_T^\ell : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^\ell(T)$ s.t.

$$\boxed{\int_T D_T^\ell \underline{\boldsymbol{v}}_T\ q = -\int_T \boldsymbol{v}_T\cdot\boldsymbol{\nabla} q + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}_F\cdot\boldsymbol{n}_{TF})\ q \quad \forall q \in \mathbb{P}^\ell(T)}$$

- For all $\boldsymbol{v} \in H^1(T)^d$, $D_T^k(\underline{\boldsymbol{I}}_T^k \boldsymbol{v}) = \pi_T^{0,k}(\boldsymbol{\nabla}\cdot\boldsymbol{v})$, i.e., the following diagram commutes:

$$\begin{array}{ccc} H^1(T)^d & \xrightarrow{\boldsymbol{\nabla}\cdot} & L^2(T) \\ \Big\downarrow \underline{\boldsymbol{I}}_T^k & & \Big\downarrow \pi_T^{0,k} \\ \underline{\boldsymbol{U}}_T^k & \xrightarrow{D_T^k} & \mathbb{P}^k(T) \end{array}$$

Pressure-velocity coupling I

$$\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) := -\sum_{T\in\mathcal{T}_h}\int_T D_T^k\underline{\boldsymbol{v}}_T\ q_T$$

Lemma (Uniform inf-sup condition)

There is $\beta > 0$ *independent of* h *s.t.*

$$\boxed{\forall q_h \in P_h^k,\quad \beta\|q_h\|_{L^2(\Omega)} \le \$:= \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_{h,0}^k,\ \|\underline{\boldsymbol{v}}_h\|_{1,h}=1} \mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h).}$$

Pressure-velocity coupling II

- Let $q_h \in \mathbb{P}^k(\mathcal{T}_h)$. The continuous inf-sup gives $\boldsymbol{v}_{q_h} \in H_0^1(\Omega)^d$ s.t.

$$-\boldsymbol{\nabla}\cdot\boldsymbol{v}_{q_h} = q_h \text{ and } \|\boldsymbol{v}_{q_h}\|_{H^1(\Omega)^d} \lesssim \|q_h\|_{L^2(\Omega)}$$

- We next write

$$\|q_h\|_{L^2(\Omega)}^2 = -\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{v}_{q_h})\ q_h = b(\boldsymbol{v}_{q_h}, q_h) = \mathrm{b}_h(\underline{\boldsymbol{I}}_h^k \boldsymbol{v}_{q_h}, q_h),$$

  where we have used $\pi_T^{0,k}(\boldsymbol{\nabla}\cdot\boldsymbol{v}_{q_h}) = D_T^k \underline{\boldsymbol{I}}_T^k \boldsymbol{v}_{q_h}$ for all $T \in \mathcal{T}_h$

- Using the definition of the supremum followed by

$$\|\underline{\boldsymbol{I}}_h^k \boldsymbol{v}\|_{1,h} \lesssim |\boldsymbol{v}|_{H^1(\Omega)^d} \qquad \forall \boldsymbol{v} \in H^1(T)^d,$$

  we obtain

$$\|q_h\|_{L^2(\Omega)}^2 \le \$\|\underline{\boldsymbol{I}}_h^k \boldsymbol{v}_{q_h}\|_{1,h} \lesssim \$\|\boldsymbol{v}_{q_h}\|_{H^1(\Omega)} \lesssim \$\|q_h\|_{L^2(\Omega)}$$

Stability result valid on general meshes and for any $k \ge 0$

Convective term: A key remark I

- We have the following IBP formula: For all $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z} \in H^1(\Omega)^d$,

$$\boxed{\int_\Omega (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} \cdot \boldsymbol{z} + \int_\Omega (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{z} \cdot \boldsymbol{v} + \int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{w})(\boldsymbol{v} \cdot \boldsymbol{z}) = \int_{\partial\Omega} (\boldsymbol{w} \cdot \boldsymbol{n})(\boldsymbol{v} \cdot \boldsymbol{z})}$$

- Using this formula with $\boldsymbol{w} = \boldsymbol{v} = \boldsymbol{z} = \boldsymbol{u}$, we get

$$t(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} \cdot \boldsymbol{u} = \underbrace{\cancel{-\frac{1}{2} \int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{u} \cdot \boldsymbol{u})}}_{\text{mass eq.}} + \underbrace{\cancel{\frac{1}{2} \int_{\partial\Omega} (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{u} \cdot \boldsymbol{u})}}_{\text{b.c.}} = 0$$

- **Reproducing this non-dissipation property is key!**

Convective term: A key remark II

- The discrete velocity may not be divergence-free (and zero on $\partial\Omega$)
- We can used as a starting point modified versions of t:

$$t^{\mathrm{ss}}(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z}) := \frac{1}{2}\int_\Omega (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} \cdot \boldsymbol{z} - \frac{1}{2}\int_\Omega (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{z} \cdot \boldsymbol{v}$$

or, following [Temam, 1979],

$$\boxed{t^{\mathrm{tm}}(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z}) := \int_\Omega (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} \cdot \boldsymbol{z} + \frac{1}{2}\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{w})(\boldsymbol{v} \cdot \boldsymbol{z}) - \frac{1}{2}\int_{\partial\Omega} (\boldsymbol{w} \cdot \boldsymbol{n})(\boldsymbol{v} \cdot \boldsymbol{z})}$$

- **t^{ss} and t^{tm} are non-dissipative even if $\boldsymbol{\nabla}\cdot\boldsymbol{w} \neq 0$ and $\boldsymbol{v}_{|\partial\Omega} \neq \boldsymbol{0}$**

Directional derivative reconstruction

- Let $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{U}}_T^k$ represent a velocity field on T
- We let the directional derivative reconstruction

$$G_T^k(\underline{\boldsymbol{w}}_T;\cdot) : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)^d$$

be s.t., for all $\boldsymbol{z} \in \mathbb{P}^k(T)^d$,

$$\int_T G_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{z} = \int_T (\boldsymbol{w}_T\cdot\boldsymbol{\nabla})\boldsymbol{v}_T\cdot\boldsymbol{z} + \sum_{F\in\mathcal{F}_T}\int_F (\boldsymbol{w}_F\cdot\boldsymbol{n}_{TF})(\boldsymbol{v}_F-\boldsymbol{v}_T)\cdot\boldsymbol{z}$$

Discrete global integration by parts formula

We reproduce at the discrete level the formula:

$$\int_\Omega (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{z} + \int_\Omega (\boldsymbol{w}\cdot\nabla)\boldsymbol{z}\cdot\boldsymbol{v} + \int_\Omega (\nabla\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{z}) = \int_{\partial\Omega} (\boldsymbol{w}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{z})$$

Proposition (Discrete integration by parts formula)

It holds, for all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{z}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$\sum_{T\in\mathcal{T}_h} \int_T \left(G_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{z}_T + \boldsymbol{v}_T\cdot G_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{z}}_T) + D_T^{2k}\underline{\boldsymbol{w}}_T(\boldsymbol{v}_T\cdot\boldsymbol{z}_T) \right)$$
$$= \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \int_F (\boldsymbol{w}_F\cdot\boldsymbol{n}_F)\boldsymbol{v}_F\cdot\boldsymbol{z}_F - \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{w}_F\cdot\boldsymbol{n}_{TF})(\boldsymbol{v}_F-\boldsymbol{v}_T)\cdot(\boldsymbol{z}_F-\boldsymbol{z}_T).$$

The term in red reflects the non-conformity of the method.

Convective term I

$$t^{\mathrm{tm}}(\boldsymbol{w},\boldsymbol{v},\boldsymbol{z}) \coloneqq \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{z} + \frac{1}{2}\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{z}) \quad \forall \boldsymbol{w},\boldsymbol{v},\boldsymbol{z}\in\boldsymbol{U}$$

- Inspired by t^{tm}, we set

$$\mathrm{t}_h(\underline{\boldsymbol{w}}_h,\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{z}}_h) \coloneqq \sum_{T\in\mathcal{T}_h}\int_T G_T^k(\underline{\boldsymbol{w}}_T;\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{z}_T + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_T D_T^{2k}\underline{\boldsymbol{w}}_T(\boldsymbol{v}_T\cdot\boldsymbol{z}_T)$$
$$+\frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F(\boldsymbol{w}_F\cdot\boldsymbol{n}_{TF})(\boldsymbol{v}_F-\boldsymbol{v}_T)\cdot(\boldsymbol{z}_F-\boldsymbol{z}_T)$$

- The second and third terms embody Temam's device

Discrete problem I

- The discrete problem reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ s.t.

$$\begin{aligned} \nu \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{t}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 && \forall q_h \in \mathbb{P}^k(\mathcal{T}_h) \end{aligned}$$

- Optionally, upwind stabilisation can be added through the term

$$\mathrm{j}_h(\underline{\boldsymbol{w}}_h; \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{z}}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F \frac{\nu}{h_F} \rho(\mathrm{Pe}_{TF}(\boldsymbol{w}_F))(\boldsymbol{v}_F - \boldsymbol{v}_T) \cdot (\boldsymbol{z}_F - \boldsymbol{z}_T)$$

- Weakly enforced boundary conditions can also be considered
- Conservative fluxes can be identified

Well-posedness I

Theorem (Existence and a priori bounds)

There exists a solution $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ *such that*

$$\|\underline{\boldsymbol{u}}_h\|_{1,h} \lesssim \nu^{-1}\|\boldsymbol{f}\|_{L^2(\Omega)^d}, \text{ and } \|p_h\| \lesssim \left(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \nu^{-2}\|\boldsymbol{f}\|_{L^2(\Omega)^d}^2\right),$$

with hidden constants independent of both h *and* ν*.*

Theorem (Uniqueness of the discrete solution)

Assume that it holds with C *independent of* h *and* ν *and small enough,*

$$\|\boldsymbol{f}\|_{L^2(\Omega)^d} \leq C\nu^2.$$

Then, the solution is unique.

Convergence I

Theorem (Convergence to minimal regularity solutions)

It holds up to a subsequence, as $h \to 0$,

- $\boldsymbol{u}_h \to \boldsymbol{u}$ *strongly in* $L^p(\Omega)^d$ *for* $\begin{cases} p \in [1, +\infty) & \text{if } d = 2, \\ p \in [1, 6) & \text{if } d = 3; \end{cases}$
- $\boldsymbol{\nabla}_h \boldsymbol{r}_h^{k+1} \underline{\boldsymbol{u}}_h \to \boldsymbol{\nabla} \boldsymbol{u}$ *strongly in* $L^2(\Omega)^{d \times d}$*;*
- $\mathrm{s}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h) \to 0$*;*
- $p_h \to p$ *strongly in* $L^2(\Omega)$*.*

If the exact solution is unique, then the whole sequence converges.

Key tools: Discrete Sobolev embeddings and Rellick–Kondrachov compactness results in HHO spaces from [DP and Droniou, 2017a]

Convergence II

Theorem (Convergence rates for small data)

Assume the additional regularity $\boldsymbol{u} \in W^{k+1,4}(\mathcal{T}_h)^d \cap H^{k+2}(\mathcal{T}_h)^d$ *and* $p \in H^1(\Omega) \cap H^{k+1}(\Omega)$*, as well as*

$$\|\boldsymbol{f}\|_{L^2(\Omega)^d} \leq C\nu^2$$

with C *independent of* h *and* ν *small enough. Then, it holds, with hidden constant independent of* h *and* ν*,*

$$\boxed{\begin{aligned}&\nu\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k\boldsymbol{u}\|_{1,h} + \|p_h - \pi_h^{0,k}p\|_{L^2(\Omega)} \\ &\quad\lesssim h^{k+1}\left(\nu|\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^d} + \|\boldsymbol{u}\|_{W^{1,4}(\Omega)^d}|\boldsymbol{u}|_{W^{k+1,4}(\mathcal{T}_h)^d} + |p|_{H^{k+1}(\mathcal{T}_h)}\right).\end{aligned}}$$

Convergence rate: Kovasznay flow

- Following [Kovasznay, 1948], let $\Omega \coloneqq (-0.5, 1.5) \times (0, 2)$ and set

$$\mathrm{Re} \coloneqq (2\nu)^{-1}, \qquad \lambda \coloneqq \mathrm{Re} - \left(\mathrm{Re}^2 + 4\pi^2\right)^{\frac{1}{2}}$$

- The components of the velocity are given by

$$u_1(\boldsymbol{x}) \coloneqq 1 - \exp(\lambda x_1)\cos(2\pi x_2), \qquad u_2(\boldsymbol{x}) \coloneqq \frac{\lambda}{2\pi}\exp(\lambda x_1)\sin(2\pi x_2),$$

and pressure given by

$$p(\boldsymbol{x}) \coloneqq -\frac{1}{2}\exp(2\lambda x_1) + \frac{\lambda}{2}\left(\exp(4\lambda) - 1\right)$$

- We monitor the errors

$$\boxed{\underline{\boldsymbol{e}}_h \coloneqq \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}, \qquad \epsilon_h \coloneqq p_h - \pi_h^{0,k} p}$$

Convergence rate: Kovasznay flow

Weakly enforced BC, no stabilisation, $\mathrm{Re} = 40$

N_{dof}	N_{nz}	$\|\underline{\boldsymbol{e}}_h\|_{\nu,h}$	EOC	$\|\boldsymbol{e}_h\|_{L^2(\Omega)^d}$	EOC	$\|\epsilon_h\|_{L^2(\Omega)}$	EOC	τ_{ass}	τ_{sol}
				$k = 0$					
97	1216	1.07e+00	–	3.93e-01	–	6.80e-01	–	2.68e-02	2.31e-02
353	4800	1.70e+00	-0.67	9.58e-01	-1.28	2.79e-01	1.28	3.41e-02	3.71e-02
1345	19072	1.44e+00	0.24	3.89e-01	1.30	1.32e-01	1.09	6.68e-02	8.04e-02
5249	76032	8.77e-01	0.72	1.18e-01	1.72	4.93e-02	1.42	2.15e-01	3.52e-01
20737	303616	4.78e-01	0.88	3.23e-02	1.87	1.49e-02	1.72	8.07e-01	1.95e+00
82433	1213440	2.46e-01	0.96	8.32e-03	1.96	4.08e-03	1.87	3.19e+00	1.47e+01
				$k = 1$					
177	4256	1.02e+00	–	7.27e-01	–	2.69e-01	–	1.44e-02	1.60e-02
641	16768	4.20e-01	1.28	1.66e-01	2.13	4.96e-02	2.44	3.59e-02	4.25e-02
2433	66560	1.40e-01	1.58	2.66e-02	2.64	8.60e-03	2.53	1.09e-01	1.70e-01
9473	265216	4.06e-02	1.79	3.55e-03	2.91	1.29e-03	2.74	4.62e-01	1.10e+00
37377	1058816	1.03e-02	1.97	4.37e-04	3.02	1.79e-04	2.85	1.91e+00	5.64e+00
148481	4231168	2.61e-03	1.99	5.46e-05	3.00	2.96e-05	2.60	7.07e+00	3.32e+01
				$k = 2$					
257	9152	5.50e-01	–	3.16e-01	–	1.20e-01	–	2.23e-02	2.33e-02
929	36032	7.58e-02	2.86	2.46e-02	3.68	6.03e-03	4.31	6.11e-02	7.47e-02
3521	142976	1.23e-02	2.62	1.84e-03	3.74	3.69e-04	4.03	2.41e-01	3.90e-01
13697	569600	1.70e-03	2.86	1.12e-04	4.03	3.63e-05	3.35	1.02e+00	2.21e+00
54017	2273792	2.21e-04	2.95	6.87e-06	4.03	3.84e-06	3.24	3.62e+00	1.17e+01
214529	9085952	2.80e-05	2.98	4.28e-07	4.00	3.72e-07	3.37	1.40e+01	6.76e+01
				$k = 5$					
497	34976	6.48e-03	–	1.76e-03	–	1.02e-03	–	1.23e-01	7.22e-02
1793	137600	7.07e-05	6.52	1.34e-05	7.04	4.58e-06	7.81	4.06e-01	2.95e-01
6785	545792	1.28e-06	5.79	1.10e-07	6.94	4.40e-08	6.70	1.51e+00	1.56e+00
26369	2173952	2.20e-08	5.87	8.84e-10	6.95	5.86e-10	6.23	5.67e+00	8.48e+00
103937	8677376	3.56e-10	5.95	7.20e-12	6.94	7.42e-12	6.30	2.28e+01	5.14e+01

Lid-driven cavity I

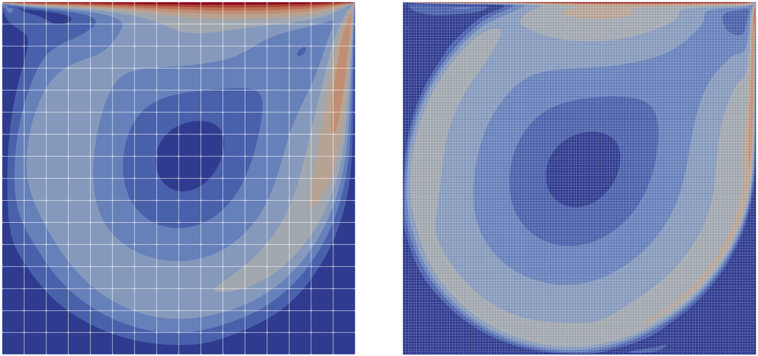


Figure: Lid-driven cavity, velocity magnitude contours (10 equispaced values in the range $[0, 1]$) for $k = 7$ computations at $\mathrm{Re} = 1{,}000$ (*left*: 16x16 grid) and $\mathrm{Re} = 20{,}000$ (*right*: 128x128 grid).

Lid-driven cavity

Re = 1,000

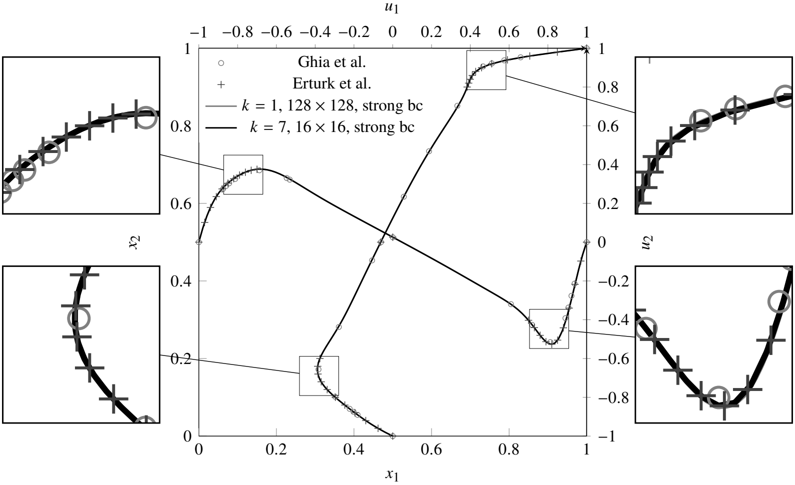


Figure: u_1 along the vertical centerline, u_2 along the horizontal centerline

Re = 10,000

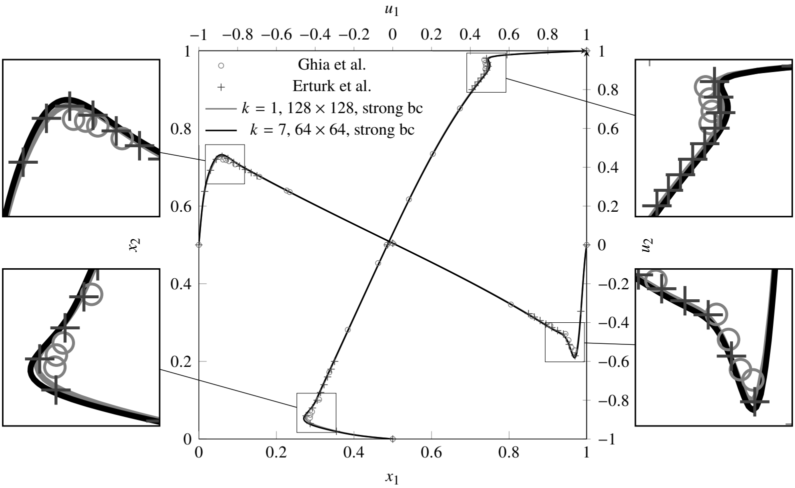


Figure: u_1 along the vertical centerline, u_2 along the horizontal centerline

Three-dimensional lid-driven cavity

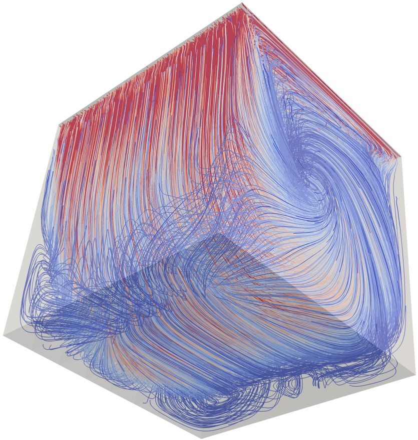


Figure: Three-dimensional lid-driven cavity, $\mathrm{Re} = 1000$, streamlines

Lid-driven cavity

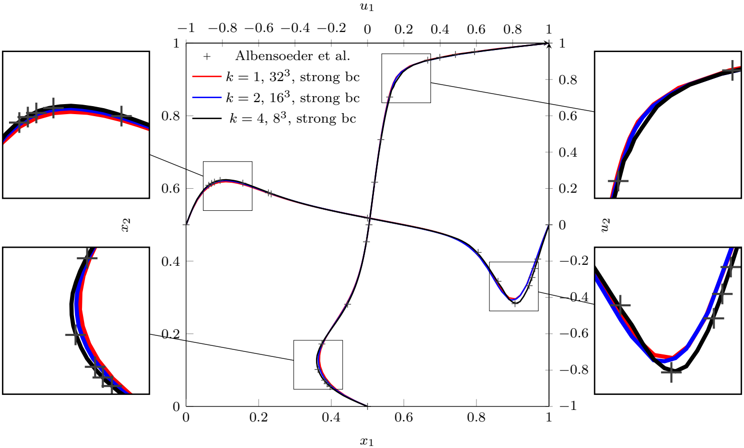


Figure: 3D Lid-driven cavity flow, horizontal component u_1 of the velocity along the vertical centerline $x_1, x_3 = \frac{1}{2}$ and the vertical component u_2 of the velocity along the horizontal centerline $x_2, x_3 = \frac{1}{2}$ for $\mathrm{Re} = 1{,}000$, $k = 1, 2, 4$

Lid-driven cavity

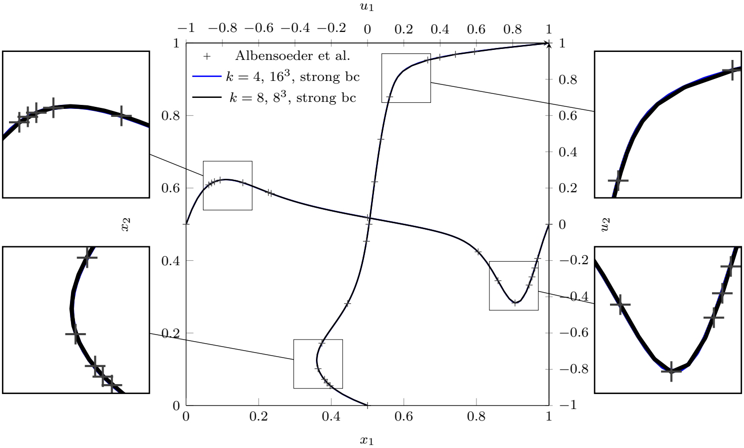


Figure: 3D Lid-driven cavity flow, horizontal component u_1 of the velocity along the vertical centerline $x_1, x_3 = \frac{1}{2}$ and the vertical component u_2 of the velocity along the horizontal centerline $x_2, x_3 = \frac{1}{2}$ for $\mathrm{Re} = 1{,}000$, $k = 4, 8$

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