Recent advances on Hybrid High-Order methods for linear and nonlinear problems

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POEMS 2017

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Polytopal meshes I

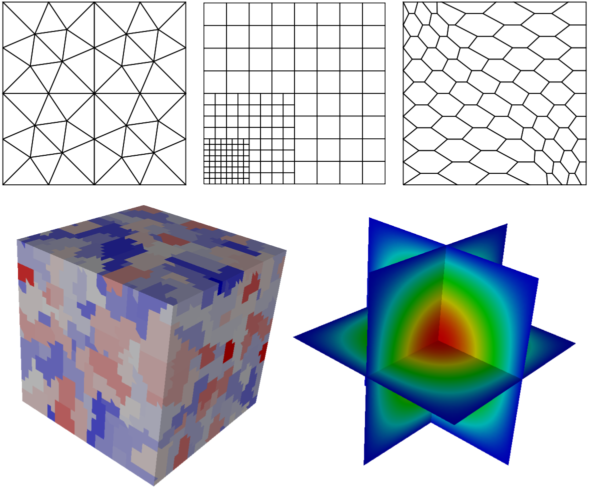

Figure: Admissible meshes in 2d and 3d, and HHO solution on the agglomerated mesh (example taken from [DP and Specogna, 2016])

Polytopal meshes II

Definition (Mesh regularity)

We consider a refined sequence $(\mathscr{T}_h)_{h\in\mathscr{H}}$ of polytopal meshes s.t., for all $h \in \mathscr{H}$, $\mathscr{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathscr{H}}$ is

- shape-regular in the sense of Ciarlet;
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences [DP and Ern, 2012]:

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces

See also [DP and Droniou, 2017a, DP and Droniou, 2017b]

Outline

Outline

1. Analysis tools for polytopal discretisations of nonlinear problems
2. Application: The incompressible Navier–Stokes equations
3. A stable gradient reconstruction

Key properties for convergence

- For linear problems, we follow the Lax–Richtmyer's principle:

$$\text{consistency} \implies (\text{stability} \iff \text{convergence})$$

- As in the FE analysis, we need some key properties:
  - Approximability
  - Asymptotic consistency
  - Stability
- For non linear problems, compactness is also required

A paradigmatic example: The p-Laplace problem

- In what follows, we focus on problems set in $W_0^{1,p}(\Omega)$, $p \in (1, +\infty)$
- Consider as an example the p-Laplace problem: Find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\operatorname{div}(\sigma(\nabla u)) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $f \in L^{p'}(\Omega)$, $p' := \frac{p}{p-1}$, and $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is s.t.

$$\sigma(\tau) := |\tau|^{p-2}\tau$$

- In weak formulation: Find $u \in W_0^{1,p}(\Omega)$ s.t., for all $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \sigma(\nabla u)\cdot\nabla v = \int_\Omega fv$$

- See [DP and Droniou, 2017a] for more general Leray–Lions operators

Discretisation of Leray–Lions type problems

- Conforming Finite Elements
  - p-Laplacian, a priori [Barrett and Liu, 1994]
  - A priori and a posteriori [Glowinski and Rappaz, 2003]
- Nonconforming FE for the p-Laplacian [Liu and Yan, 2001]
- Mixed Finite Volumes for Leray–Lions [Droniou, 2006]
- Discrete Duality FV, $d = 2$ [Andreianov, Boyer, Hubert, 2004–07]
- Mimetic FD [Antonietti, Bigoni, Verani, 2014]
- Hybrid High-Order (HHO) for general Leray–Lions operators
  - Convergence by compactness [DP and Droniou, 2017a]
  - Error estimates [DP and Droniou, 2017b]

Projectors on local polynomial spaces

- At the core of HHO are projectors on local polynomial spaces
- For X element or face, the L^2-projector $\pi_X^{0,l} : L^1(X) \to \mathbb{P}^l(X)$ is s.t.

$$(\pi_X^{0,l} v - v, w)_X = 0 \text{ for all } w \in \mathbb{P}^l(X)$$

- For $T \in \mathscr{T}_h$, the elliptic projector $\pi_T^{1,l} : W^{1,1}(T) \to \mathbb{P}^l(T)$ is s.t.

$$(\nabla(\pi_T^{1,l} v - v), \nabla w)_T = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } (\pi_T^{1,l} v - v, 1)_T = 0$$

- Both projectors have optimal approximation properties in $\mathbb{P}^l(T)$

Computing L^2-gradient projections from L^2-projections

- Let now $T \in \mathscr{T}_h$ be fixed. For $v \in W^{1,1}(T)$ and $\phi \in C^\infty(\overline{T})^d$, we have

$$(\nabla v, \phi)_T = -(v, \operatorname{div} \phi)_T + \sum_{F \in \mathscr{F}_T} (v, \phi \cdot \boldsymbol{n}_{TF})_F$$

- Specializing this formula to $\phi \in \mathbb{P}^k(T)^d$, we can write

$$(\pi_T^{0,k} \nabla v, \phi)_T = -(\pi_T^{0,k} v, \operatorname{div} \phi)_T + \sum_{F \in \mathscr{F}_T} (\pi_F^{0,k} v, \phi \cdot \boldsymbol{n}_{TF})_F,$$

since $\operatorname{div} \phi \in \mathbb{P}^{k-1}(T) \subset \mathbb{P}^k(T)$ and $\phi_{|F} \cdot \boldsymbol{n}_{TF} \in \mathbb{P}^k(F)$ for all $F \in \mathscr{F}_T$

- **Hence, $\pi_T^{0,k} \nabla v$ can be computed from $\pi_T^{0,k} v$ and $\pi_F^{0,k} v$, $F \in \mathscr{F}_T$**

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- **Hence, $\pi_T^{0,k} \nabla v$ can be computed from $\pi_T^{0,k} v$ and $\pi_F^{0,k} v$, $F \in \mathcal{F}_T$**
- **$\pi_T^{1,k+1} v$ can be computed specializing to $\phi = \nabla w \in \nabla \mathbb{P}^{k+1}(T)$**

DOFs and interpolation

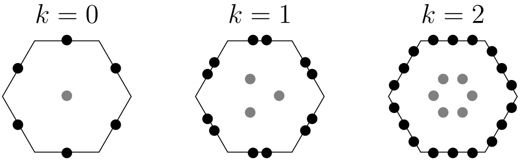

Figure: $\underline{U}_T^k$ for $k \in \{0,1,2\}$

- For $k \geq 0$ and $T \in \mathscr{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathscr{F}_T} \mathbb{P}^k(F) \right)$$

- The local interpolator $\underline{I}_T^k : W^{1,1}(T) \to \underline{U}_T^k$ is s.t.

$$\underline{I}_T^k v = (\pi_T^{0,k} v, (\pi_F^{0,k} v_{|F})_{F \in \mathscr{F}_T})$$

- (Degree k inside T: local conservation, L^2-convergence for $k = 1$)

Local reconstructions and approximability

- We define the gradient reconstruction $\boldsymbol{G}_T^k : \underline{U}_T^k \mapsto \mathbb{P}^k(T)^d$ s.t.

$$(\boldsymbol{G}_T^k \underline{v}_T, \boldsymbol{\phi})_T = -(v_T, \operatorname{div} \boldsymbol{\phi})_T + \sum_{F \in \mathscr{F}_T} (v_F, \boldsymbol{\phi} \cdot \boldsymbol{n}_{TF})_F \quad \forall \boldsymbol{\phi} \in \mathbb{P}^k(T)^d$$

- We also need the potential reconstruction $r_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$ s.t.

$$(\nabla r_T^{k+1} \underline{v}_T, \nabla w)_T = -(v_T, \triangle w)_T + \sum_{F \in \mathscr{F}_T} (v_F, \nabla w \cdot \boldsymbol{n}_{TF})_F \quad \forall w \in \mathbb{P}^{k+1}(T)^d$$

- Prescribing that $(r_T^{k+1} \underline{v}_T - v_T, 1)_T = 0$, we have for all $v \in W^{1,1}(T)$,

$$\boxed{\boldsymbol{G}_T^k \underline{I}_T^k v = \boldsymbol{\pi}_T^{0,k} \nabla v, \qquad r_T^{k+1} \underline{I}_T^k v = \pi_T^{1,k+1} v}$$

- **Approximability** of smooth functions through $\boldsymbol{G}_T^k$ and r_T^{k+1} follows
- Similar ideas are ubiquitous in POEMS (HDG, (nc)VEM,...)

Asymptotic consistency I

- Define the following global space with single-valued interface DOFs:

$$\underline{U}_h^k := \left(\bigtimes_{T \in \mathscr{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathscr{F}_h} \mathbb{P}^k(F) \right)$$

- Boundary conditions are strongly enforced considering the subspace

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \,:\, v_F = 0 \quad \forall F \in \mathscr{F}_h^{\mathrm{b}} \right\}$$

- We also define the $W_0^{1,p}$-like norm $\|\underline{v}_h\|_{1,p,h}^p := \sum_{T \in \mathscr{T}_h} \|\underline{v}_T\|_{1,p,T}^p$ where

$$\|\underline{v}_T\|_{1,p,T}^p := \|\nabla v_T\|_{L^p(T)^d}^p + \sum_{F \in \mathscr{F}_T} h_F^{1-p} \|v_F - v_T\|_{L^p(F)}^p \quad \forall T \in \mathscr{T}_h$$

Asymptotic consistency II

- A global gradient reconstruction is obtained setting, for all $\underline{v}_h \in \underline{U}_h^k$,

$$(\boldsymbol{G}_h^k \underline{v}_h)_T := \boldsymbol{G}_T^k \underline{v}_T, \qquad \forall T \in \mathscr{T}_h$$

- Define $\mathscr{E}_h : \boldsymbol{W}^{p'}(\mathrm{div};\Omega) \to [0,+\infty)$ s.t., with $(v_h)_{|T} := v_T\ \forall T \in \mathscr{T}_h$,

$$\mathscr{E}_h(\psi) := \sup_{\underline{v}_h \in \underline{U}_{h,0}^k, \|\underline{v}_h\|_{1,p,h}=1} \left| \int_\Omega \left(\boldsymbol{G}_h^k \underline{v}_h \cdot \psi + v_h \,\mathrm{div}\, \psi \right) \right|$$

- **Asymptotic consistency** holds in the form of a discrete global IBP:

$$\boxed{\lim_{h\to 0} \mathscr{E}_h(\psi) = 0 \qquad \forall \psi \in \boldsymbol{W}^{p'}(\mathrm{div};\Omega)}$$

- Moreover, one can prove that

$$\mathscr{E}_h(\psi) \lesssim h^{k+1} \|\psi\|_{W^{k+1,p'}(\mathscr{T}_h)^d} \quad \forall \psi \in \boldsymbol{W}^{p'}(\mathrm{div};\Omega) \cap W^{k+1,p'}(\mathscr{T}_h)^d$$

Stability through a boundary difference seminorm I

- We seek **stability** in the form of the uniform norm equivalence

$$\|\underline{v}_h\|_{1,p,h}^p \simeq \|\boldsymbol{G}_h^k \underline{v}_h\|_{L^p(\Omega)^d}^p + |\underline{v}_h|_{1,p,h}^p, \quad |\underline{v}_h|_{1,p,h}^p := \sum_{T\in\mathscr{T}_h} |\underline{v}_T|_{1,p,T}^p$$

- To inspire stabilisation terms, the seminorm should scale like $\mathscr{E}_h$:

$$|\underline{I}_h^k v|_{1,p,h} \lesssim h^{k+1} \|v\|_{W^{k+2,p}(\mathscr{T}_h)} \quad \forall v \in W_0^{1,p}(\Omega) \cap W^{k+2,p}(\mathscr{T}_h)$$

- A paradigmatic choice is (cf. A. Ern's talk)

$$\boxed{|\underline{v}_T|_{1,p,T}^p := \sum_{F\in\mathscr{F}_T} h_F^{1-p} \|(\delta_{TF}^k - \delta_T^k)\underline{v}_T\|_{L^p(F)}^p}$$

with high-order difference operators

$$\delta_T^k \underline{v}_T := \pi_T^{0,k}(r_T^{k+1}\underline{v}_T - v_T), \qquad \delta_{TF}^k \underline{v}_T := \pi_F^{0,k}(r_T^{k+1}\underline{v}_T - v_F) \quad \forall F \in \mathscr{F}_T$$

Stability through a boundary difference seminorm II

Crucially, we have the discrete Sobolev embeddings

Lemma (Discrete Sobolev embeddings)

For any Lebesgue exponent q s.t.

$$\begin{cases} 1 \leq q \leq p^* := \frac{dp}{d-p} & \text{if } 1 \leq p < d, \\ 1 \leq q < +\infty & \text{if } p \geq d, \end{cases}$$

we have for all $\underline{v}_h \in \underline{U}_{h,0}^k$

$$\|v_h\|_{L^q(\Omega)} \lesssim C \|\underline{v}_h\|_{1,p,h}.$$

where $a \lesssim b$ means $a \leq Cb$ with C only depending on Ω, ρ, k, q and p.

Compactness

Lemma (Discrete compactness)

Let $(\underline{v}_h)_{h\in\mathscr{H}}$ be s.t., for all $h \in \mathscr{H}$, $\|\underline{v}_h\|_{1,p,h} \le C$ for a fixed $C \in \mathbb{R}$. Then, there exists $v \in W_0^{1,p}(\Omega)$ s.t., up to a subsequence as $h \to 0$,

- $v_h \to v$ *strongly in $L^q(\Omega)$ for all* $q \in \begin{cases} [1,p^*) & \text{if } 1 \le p < d, \\ [1,+\infty) & \text{if } p \ge d; \end{cases}$
- $\boldsymbol{G}_h^k \underline{v}_h \to \nabla v$ *weakly in $L^p(\Omega)^d$.*

Remark (Alternative compact gradients)

This result extends to any gradient $\mathscr{G}_T : \underline{U}_T^k \to \mathbb{G}_T$ s.t. $\mathbb{P}^0(T)^d \subset \mathbb{G}_T$ and, for all $\underline{v}_T \in \underline{U}_T^k$ and all $\phi \in \mathbb{G}_T$,

$$(\mathscr{G}_T \underline{v}_T, \phi)_T = -(v_T, \operatorname{div} \phi)_T + \sum_{F \in \mathscr{F}_T} (v_F, \phi \cdot \boldsymbol{n}_{TF})_F.$$

This is true, in particular, for $\mathbb{G}_T = \nabla \mathbb{P}^{k+1}(T)$ and $\mathbb{G}_T = \mathbb{P}^l(T)^d$, $l \ge 0$.

An HHO scheme with external stabilisation

- Define, for all $T \in \mathscr{T}_h$, the function $A_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ s.t.

$$A_T(\underline{u}_T, \underline{v}_T) := \int_T \boldsymbol{\sigma}(\boldsymbol{G}_T^k \underline{u}_T) \cdot \boldsymbol{G}_T^k \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

  with stabilisation contribution inspired by $|\cdot|_{1,p,T}$ s.t.

$$s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathscr{F}_T} h_F^{1-p} \int_F |(\delta_{TF}^k - \delta_T^k)\underline{u}_T|^{p-2} (\delta_{TF}^k - \delta_T^k)\underline{u}_T \; (\delta_{TF}^k - \delta_T^k)\underline{v}_T$$

- The HHO scheme for the p-Laplacian reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{A_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathscr{T}_h} A_T(\underline{u}_T, \underline{v}_T) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

Well-posedness and convergence

Theorem (Well-posedness and convergence)

There exists a unique solution to the HHO scheme with a priori estimate

$$\|\underline{\boldsymbol{u}}_h\|_{1,p,h} \lesssim \|f\|_{L^{p'}(\Omega)}^{\frac{1}{p-1}}.$$

Moreover, denoting by $(\underline{u}_h)_{h\in\mathcal{H}} \in (\underline{U}_{h,0}^k)_{h\in\mathcal{H}}$ the sequence of discrete solutions on $(\mathcal{T}_h)_{h\in\mathcal{H}}$ it holds, as $h \to 0$,

- $u_h \to u$ *strongly in $L^q(\Omega)$ for all* $q \in \begin{cases} [1,p^*) & \text{if } 1 \le p < d, \\ [1,+\infty) & \text{if } p \ge d; \end{cases}$
- $\boldsymbol{G}_h^k \underline{u}_h \to \nabla u$ *strongly in $L^p(\Omega)^d$.*

No regularity on the exact solution beyond $W_0^{1,p}(\Omega)$ required!

Convergence rates

Theorem (Convergence rates)

Further assuming $u \in W^{k+2,p}(\Omega)$ *and* $\sigma(\nabla u) \in W^{k+1,p'}(\Omega)^d$, *it holds:*

$$\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h} \lesssim \begin{cases} h^{k+1}|u|_{W^{k+2,p}(\Omega)} + h^{\frac{k+1}{p-1}}\left(|u|_{W^{k+2,p}(\Omega)}^{\frac{1}{p-1}} + |\sigma(\nabla u)|_{W^{k+1,p'}(\Omega)^d}^{\frac{1}{p-1}}\right) & \text{if } p \geq 2, \\ h^{(k+1)(p-1)}|u|_{W^{k+2,p}(\Omega)}^{p-1} + h^{k+1}|\sigma(\nabla u)|_{W^{k+1,p'}(\Omega)^d} & \text{if } p < 2. \end{cases}$$

Numerical examples

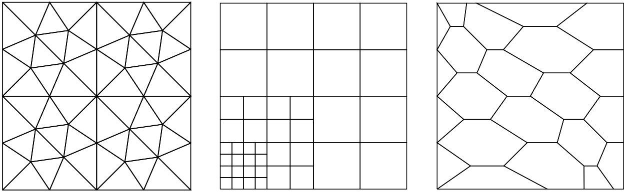

Figure: Triangular, locally refined, and predominantly hexagonal meshes

- Trigonometric solution ($p \geq 2$)

$$u(\boldsymbol{x}) = \sin(2\pi x_1)\sin(2\pi x_2)$$

- Exponential solution ($p < 2$)

$$u(\boldsymbol{x}) = \exp(x_1 + \pi x_2)$$

Numerical examples

Trigonometric solution, $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ v. h, $p \in \{2,3,4\}$

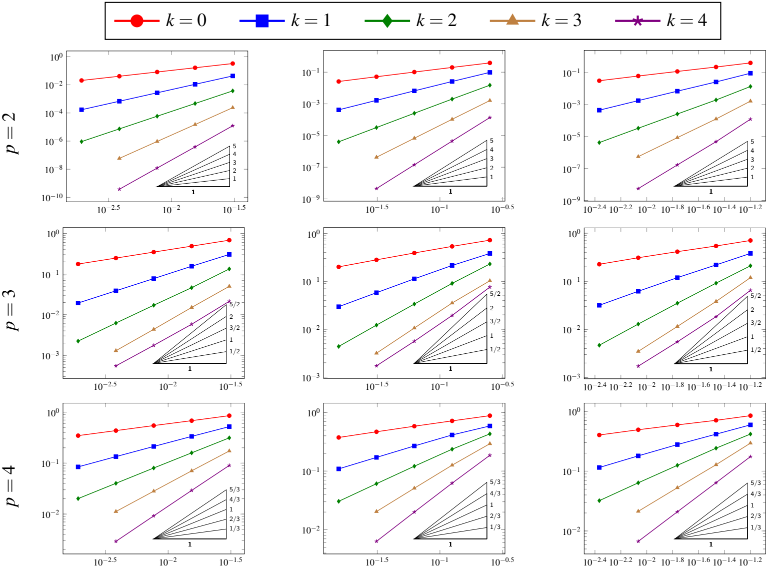

Figure: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ versus h.

Numerical examples

Exponential solution, $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ v. h, $p = 3/4$

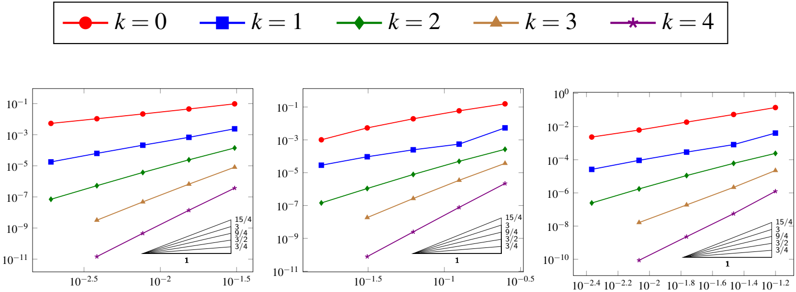

Figure: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ versus h.

Outline

The steady incompressible Navier–Stokes equations

- Letting $\nu \in \mathbb{R}_+^*$ (extension to variable ν is possible), $\boldsymbol{f} \in L^2(\Omega)^d$, and

$$\boldsymbol{U} := H_0^1(\Omega)^d, \qquad P := L_0^2(\Omega),$$

the INS problem in $d \in \{2,3\}$ reads: Find $(\boldsymbol{u},p) \in \boldsymbol{U} \times P$ s.t.

$$\begin{aligned} \nu a(\boldsymbol{u},\boldsymbol{v}) + t(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{U}, \\ -b(\boldsymbol{u},q) &= 0 && \forall q \in P, \end{aligned}$$

with bilinear forms a and b and trilinear form t s.t.

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{v}, \quad b(\boldsymbol{v},q) := -\int_\Omega (\operatorname{div}\boldsymbol{v})q, \quad t(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \boldsymbol{v}^{\mathrm{T}}\nabla\boldsymbol{u}\,\boldsymbol{w}$$

- We use the matrix-product notation: $\nabla\boldsymbol{v}\,\boldsymbol{w} = \left(\sum_{j=1}^d w_j \partial_j v_i\right)_{1\le i\le d}$

Some related works (among many on the subject)

- DG, artificial compressibility flux [Bassi et al., 2006]
- DG, agglomerated meshes [Bassi et al., 2012]
- DG, analysis by compactness [DP and Ern, 2010]
- HDG, error estimates [Nguyen, Peraire, Cockburn, 2011, Çeşmelioğlu, Cockburn, Qiu, 2016]
- VEM, $\boldsymbol{H}(\mathrm{div})$-conforming [Beirão da Veiga, Lovadina, Vacca 2016–2017]
- HHO, Stokes [Aghili, Boyaval, DP, 2015, DP, Ern, Linke, Schieweck, 2016]
- HHO, Navier–Stokes [DP and Krell, 2016]

Discrete spaces

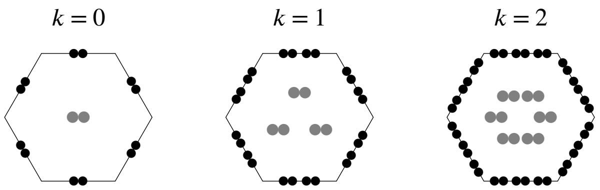

Figure: Local velocity space $\underline{\boldsymbol{U}}_T^k$ for $k \in \{0, 1, 2\}$

- We consider the vector version of the HHO discrete space
- Let a polynomial degree $k \geq 0$ be fixed and set

$$\underline{\boldsymbol{U}}_h^k := \left(\bigtimes_{T \in \mathscr{T}_h} \mathbb{P}^k(T)^d \right) \times \left(\bigtimes_{F \in \mathscr{F}_h} \mathbb{P}^k(F)^d \right)$$

- We account for BCs on $\boldsymbol{u}$ and the zero-average constraint on p in

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathscr{F}_h^{\mathrm{b}} \right\}, \qquad P_h^k := \mathbb{P}^k(\mathscr{T}_h) \cap L_0^2(\Omega)$$

Gradient and divergence reconstructions

- Let a mesh element $T \in \mathscr{T}_h$ be fixed
- For $l \geq 0$, the gradient reconstruction $\boldsymbol{G}_T^l : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^l(T)^{d\times d}$ is s.t.

$$\boxed{(\boldsymbol{G}_T^l \underline{\boldsymbol{v}}_T, \boldsymbol{\tau})_T = -(\boldsymbol{v}_T, \operatorname{div} \boldsymbol{\tau})_T + \sum_{F\in\mathscr{F}_T} (\boldsymbol{v}_F, \boldsymbol{\tau}\boldsymbol{n}_{TF})_F \quad \forall \boldsymbol{\tau} \in \mathbb{P}^l(T)^{d\times d}}$$

- **This time, we also allow $l \neq k$** ($l = 2k$ used in the convective term)
- The divergence reconstruction $D_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)$ is s.t.

$$\boxed{D_T^k = \operatorname{tr}(\boldsymbol{G}_T^k)}$$

- Global versions are defined setting

$$(\boldsymbol{G}_h^l \underline{\boldsymbol{v}}_h)_{|T} := \boldsymbol{G}_T^l \underline{\boldsymbol{v}}_T, \quad (D_h^k \underline{\boldsymbol{v}}_h)_{|T} := D_T^k \underline{\boldsymbol{v}}_T \qquad \forall T \in \mathscr{T}_h$$

Viscous term

- The viscous term is discretized as before by means of

$$\boxed{a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \int_\Omega \boldsymbol{G}_h^k \underline{\boldsymbol{u}}_h : \boldsymbol{G}_h^k \underline{\boldsymbol{v}}_h + s_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h),}$$

- Variable viscosity can be treated following [DP and Ern, 2015]
- Tools for non-Newtonian fluids are available in [Botti et al., 2016]

Pressure-velocity coupling

- The pressure-velocity coupling is realized through the bilinear form

$$\boxed{b_h(\underline{\boldsymbol{v}}_h, q_h) := -\int_\Omega D_h^k \underline{\boldsymbol{v}}_h q_h}$$

- Crucially, b_h satisfies the following (uniform) inf-sup condition

$$\boxed{\forall q_h \in P_h^k, \quad \|q_h\|_{L^2(\Omega)} \lesssim \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \|\underline{\boldsymbol{v}}_h\|_{1,h}=1} b_h(\underline{\boldsymbol{v}}_h, q_h)}$$

- **Valid on general meshes for $d \in \{2,3\}$!**

Convective term I

- For all $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ with $\operatorname{div} \boldsymbol{w} = 0$, we have

$$t(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{v}^{\mathrm{T}} \nabla \boldsymbol{u}\, \boldsymbol{w} = \frac{1}{2} \int_\Omega \boldsymbol{v}^{\mathrm{T}} \nabla \boldsymbol{u}\, \boldsymbol{w} - \frac{1}{2} \int_\Omega \boldsymbol{u}^{\mathrm{T}} \nabla \boldsymbol{v}\, \boldsymbol{w}$$

- This skew-symmetric version emphasizes that t is non-dissipative:

$$t(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{v}) = 0$$

- Inspired by this remark, we set

$$\boxed{t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \frac{1}{2} \int_\Omega \boldsymbol{v}_h^{\mathrm{T}} \boldsymbol{G}_h^{2k} \underline{\boldsymbol{u}}_h\, \boldsymbol{w}_h - \frac{1}{2} \int_\Omega \boldsymbol{u}_h^{\mathrm{T}} \boldsymbol{G}_h^{2k} \underline{\boldsymbol{v}}_h\, \boldsymbol{w}_h,}$$

- By design, t_h is also non-dissipative: For all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h$,

$$t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) = 0$$

Convective term II

Remark (Implementation)

In practice, one does not need to actually compute $\boldsymbol{G}_h^{2k}$. Simply write

$$t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) = \sum_{T \in \mathscr{T}_h} t_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T),$$

where, for all $T \in \mathscr{T}_h$,

$$\begin{aligned} t_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := & -\frac{1}{2}\int_T \boldsymbol{u}_T^{\mathrm{T}} \nabla \boldsymbol{v}_T \, \boldsymbol{w}_T + \frac{1}{2}\sum_{F \in \mathscr{F}_T} \int_F (\boldsymbol{u}_F \cdot \boldsymbol{v}_T)(\boldsymbol{w}_T \cdot \boldsymbol{n}_{TF}) \\ & + \frac{1}{2}\int_T \boldsymbol{v}_T^{\mathrm{T}} \nabla \boldsymbol{u}_T \, \boldsymbol{w}_T - \frac{1}{2}\sum_{F \in \mathscr{F}_T} \int_F (\boldsymbol{v}_F \cdot \boldsymbol{u}_T)(\boldsymbol{w}_T \cdot \boldsymbol{n}_{TF}). \end{aligned}$$

Discrete problem I

- The discrete problem reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ s.t.

$$\boxed{\begin{aligned} \nu a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + t_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -b_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 && \forall q_h \in P_h^k \end{aligned}}$$

- When using iterative solvers, static condensation can significantly reduce the number of unknowns at each iteration

Discrete problem II

Theorem (Existence and a priori bounds)

There exists a solution $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ *such that*

$$\|\underline{\boldsymbol{u}}_h\|_{1,h} \lesssim \nu^{-1} \|\boldsymbol{f}\|_{L^2(\Omega)^d}, \quad \|p_h\|_{L^2(\Omega)} \lesssim \|\boldsymbol{f}\|_{L^2(\Omega)^d} + \nu^{-2} \|\boldsymbol{f}\|_{L^2(\Omega)^d}^2.$$

Theorem (Uniqueness of the discrete solution)

Assume that the right-hand side verifies

$$\|\boldsymbol{f}\|_{L^2(\Omega)^d} \leq C\nu^2$$

with $C > 0$ *small enough. Then, the solution is unique.*

Key tool: Discrete Sobolev embeddings with $p = 2$ and $p = 4$

Convergence I

Theorem (Convergence to minimal regularity solutions)

Denote by $((\underline{\boldsymbol{u}}_h, p_h))_{h\in\mathcal{H}} \in (\underline{\boldsymbol{U}}_{h,0}^k \times P_h^k)_{h\in\mathcal{H}}$ *the sequence of discrete solutions on* $(\mathcal{T}_h)_{h\in\mathcal{H}}$*. It holds, up to a subsequence, as* $h \to 0$,

- $\boldsymbol{u}_h \to \boldsymbol{u}$ *strongly in* $L^p(\Omega)^d$ *for* $p \in \begin{cases} [1,+\infty) & \text{if } d=2, \\ [1,6) & \text{if } d=3; \end{cases}$
- $\boldsymbol{G}_h^k \underline{\boldsymbol{u}}_h \to \nabla \boldsymbol{u}$ *strongly in* $L^2(\Omega)^{d\times d}$;
- $s_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h) \to 0$;
- $p_h \to p$ *strongly in* $L^2(\Omega)$.

Moreover, if the exact solution is unique, the whole sequence converges.

Key tool: Compactness of discrete gradients

Convergence II

Theorem (Convergence rates for small data)

Assume uniqueness for both $(\underline{\boldsymbol{u}}_h, p_h)$ *and* $(\boldsymbol{u}, p)$. *Assume, moreover, the additional regularity* $(\boldsymbol{u}, p) \in H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$, *as well as*

$$\|\boldsymbol{f}\|_{L^2(\Omega)^d} \leq C\nu^2$$

with $C > 0$ *small enough. Then, we have the following error estimate:*

$$\boxed{\|\underline{\boldsymbol{u}}_h - \underline{I}_h^k \boldsymbol{u}\|_{1,h} + \nu^{-1}\|p_h - \pi_h^{0,k} p\|_{L^2(\Omega)} \lesssim h^{k+1} \mathcal{N}(\boldsymbol{u}, p)}$$

with $\mathcal{N}(\boldsymbol{u}, p) := \left(1 + \nu^{-1}\|\boldsymbol{u}\|_{H^2(\Omega)^d}\right)\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d} + \nu^{-1}\|p\|_{H^{k+1}(\Omega)}$.

Key tools: Non-dissipativity, discrete Sobolev embeddings

Numerical example: Kovasznay flow

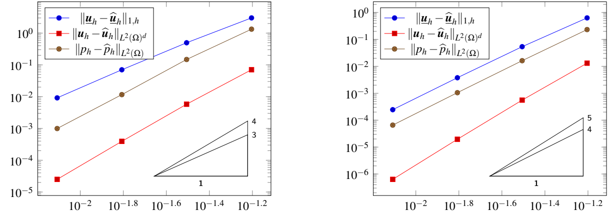

Figure: Cartesian mesh family, errors versus h, $k \in \{2,3\}$

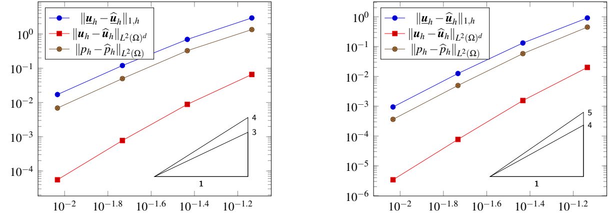

Figure: Hexagonal mesh family, errors versus h, $k \in \{2,3\}$

Numerical example: FVCA 8 steady 2d test I

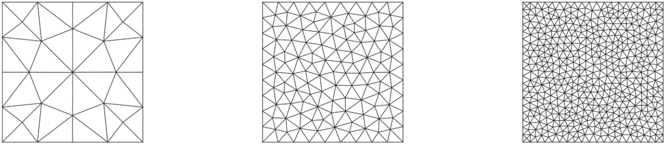

Figure: Triangular mesh family

mesh #	$\|\underline{\boldsymbol{u}}_h - \underline{I}_h^k \boldsymbol{u}\|_{1,h}$	EOC	$\|\boldsymbol{u}_h - \boldsymbol{u}\|$	EOC	$\|p - p_h\|$	EOC
1	15.67	0	0.41	0	1.5	0
2	1.65	2.67	$1.46 \cdot 10^{-2}$	3.96	$2.07 \cdot 10^{-2}$	4.98
3	$8.8 \cdot 10^{-2}$	4.14	$6.85 \cdot 10^{-4}$	4.33	$1.45 \cdot 10^{-3}$	3.72
4	$9.69 \cdot 10^{-3}$	2.3	$3.64 \cdot 10^{-5}$	3.06	$9.67 \cdot 10^{-5}$	2.81
5	$2.31 \cdot 10^{-3}$	2.06	$4.5 \cdot 10^{-6}$	3.01	$1.24 \cdot 10^{-5}$	2.94

Table: Triangular mesh family, $\nu = 10^{-3}$, $k = 1$

Numerical example: FVCA 8 steady 2d test II

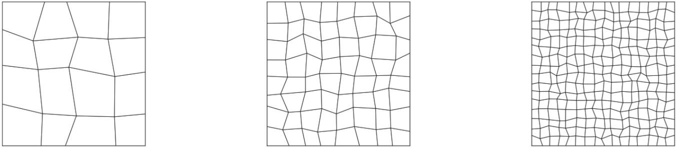

Figure: Deformed quadrangular mesh family

mesh #	$\|\underline{\boldsymbol{u}}_h - \underline{I}_h^k \boldsymbol{u}\|_{1,h}$	EOC	$\|\boldsymbol{u}_h - \boldsymbol{u}\|$	EOC	$\|p - p_h\|$	EOC
1	3.69	0	$9.65 \cdot 10^{-2}$	0	0.18	0
2	3.55	$6 \cdot 10^{-2}$	$4.7 \cdot 10^{-2}$	1.09	0.11	0.72
3	0.23	4.02	$2.53 \cdot 10^{-3}$	4.32	$4.94 \cdot 10^{-3}$	4.44
4	$4.17 \cdot 10^{-2}$	2.52	$2.58 \cdot 10^{-4}$	3.34	$5.46 \cdot 10^{-4}$	3.18
5	$8.33 \cdot 10^{-3}$	2.34	$2.47 \cdot 10^{-5}$	3.41	$5.84 \cdot 10^{-5}$	3.22
6	$1.97 \cdot 10^{-3}$	2.09	$2.85 \cdot 10^{-6}$	3.12	$6.65 \cdot 10^{-6}$	3.14

Table: Deformed quadrangular mesh family, $\nu = 10^{-3}$, $k = 1$

Outline

Internal stabilisation

- Let us go back to the p-Laplace model problem
- **Can stability be embedded into the gradient reconstruction?**
- We would like a stable gradient reconstruction $\mathscr{G}_h$ s.t., replacing

$$W_0^{1,p}(\Omega) \leftarrow \underline{U}_{h,0}^k, \quad u \leftarrow \underline{u}_h, \quad v \leftarrow \underline{v}_h, \quad \nabla \leftarrow \mathscr{G}_h$$

in the weak formulation: Find $u \in W_0^{1,p}(\Omega)$ s.t.,

$$\int_\Omega \sigma(\nabla u) \cdot \nabla v = \int_\Omega f v \quad \forall v \in W_0^{1,p}(\Omega),$$

we obtain the convergent scheme: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{\int_\Omega \sigma(\mathscr{G}_h \underline{u}_h) \cdot \mathscr{G}_h \underline{v}_h = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

- Inspired by Gradient Discretisations [Droniou et al., 2017]

Key properties

We seek $\mathscr{G}_h$ s.t., for all $T \in \mathscr{T}_h$, $\boxed{\mathscr{G}_T \underline{v}_T = \boldsymbol{G}_T^k \underline{v}_T + \boldsymbol{S}_T \underline{v}_T}$ and

(S1) L^2-stability and boundedness. For all $\underline{v}_T \in \underline{U}_T^k$ it holds that

$$\|\boldsymbol{S}_T \underline{v}_T\|_{L^2(T)^d} \simeq |\underline{v}_T|_{1,2,T} := \left(\sum_{F \in \mathscr{F}_T} h_F^{-1} \|(\delta_{TF}^k - \delta_T^k)\underline{v}_T\|_{L^2(F)}^2 \right)^{1/2}$$

(S2) Orthogonality. For all $\underline{v}_T \in \underline{U}_T^k$ and all $\phi \in \mathbb{P}^k(T)^d$,

$$(\boldsymbol{S}_T \underline{v}_T, \phi)_T = 0$$

(S3) Image. If $p \neq 2$, $\boldsymbol{S}_T$ is piecewise polynomial on a partition $\mathscr{P}_T$ of T

Lemma (Properties of $\mathscr{G}_h$-based schemes)

Under **(S1)–(S3)**, *approximability*, *asymptotic consistency*, *stability*, *and compactness are verified. Moreover, the triplet* $(\underline{U}_{h,0}^k, \underline{v}_h \mapsto v_h, \mathscr{G}_h)$ *is a convergent Gradient Scheme.*

Stable gradient reconstructions: An inspiring remark

- Setting $\delta^k_{\nabla,T} := \nabla r_T^{k+1} - \boldsymbol{G}_T^k$, we have for all $\phi \in \mathbb{P}^k(T)^d$

$$\boxed{0 = -((\delta^k_{\nabla,T} - \nabla\delta^k_T)\underline{v}_T, \phi)_T + \sum_{F\in\mathscr{F}_T}((\delta^k_{TF} - \delta^k_T)\underline{v}_T, \phi\cdot\boldsymbol{n}_{TF})_F}$$

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- Let now $\mathbb{S}_T \supset \mathbb{P}^k(T)^d$ and define the residual $\mathscr{R}_T(\underline{v}_T;\cdot) : \mathbb{S}_T \to \mathbb{R}$ s.t

$$\boxed{\mathscr{R}_T(\underline{v}_T;\eta) := -((\delta^k_{\nabla,T} - \nabla\delta^k_T)\underline{v}_T, \eta)_T + \sum_{F\in\mathscr{F}_T}((\delta^k_{TF} - \delta^k_T)\underline{v}_T, \eta\cdot\boldsymbol{n}_{TF})_F}$$

- **For $\mathbb{S}_T$ large enough, the Riesz representation of $\mathscr{R}_T(\underline{v}_T;\cdot)$ can control $|\underline{v}_T|_{1,2,T}$, and is therefore a good candidate for S_T**

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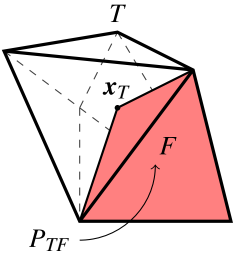
- **For $\mathbb{S}_T$ large enough, the Riesz representation of $\mathscr{R}_T(\underline{v}_T;\cdot)$ can control $|\underline{v}_T|_{1,2,T}$, and is therefore a good candidate for S_T**
- This can be interpreted as a lifting of the boundary differences on $\mathbb{S}_T$

Lifting on a Raviart–Thomas–Nédélec subspace I



$$\mathscr{P}_T := \{P_{TF} : F \in \mathscr{F}_T\}$$

- Assume T star-shaped w.r.t. to $\boldsymbol{x}_T \in T$ with $(d-1)$-simplicial faces
- These assumptions can be relaxed at the price of a heavier notation
- We consider the following choice:

$$\boxed{\mathbb{S}_T = \mathbb{RT}^{\mathrm{d},k+1}(\mathscr{P}_T) := \left\{\boldsymbol{\eta} \in L^2(T)^d : \boldsymbol{\eta}_{|P_{TF}} \in \mathbb{RT}^{k+1}(P_{TF})\ \forall F \in \mathscr{F}_T\right\}}$$

Lifting on a Raviart–Thomas–Nédélec subspace II

- The Riesz representation $\boldsymbol{S}_T$ of $\mathscr{R}(\underline{v}_T;\cdot)$ can be computed face-wise:

$$\boxed{\boldsymbol{S}_T\underline{v}_T = \sum_{F\in\mathscr{F}_T} \boldsymbol{S}_{TF}\underline{v}_T}$$

where, for all $F\in\mathscr{F}_T$, $\boldsymbol{S}_{TF}\underline{v}_T$ is s.t., for all $\eta\in\mathbb{RT}^{k+1}(P_{TF})$,

$$\boxed{(\boldsymbol{S}_{TF}\underline{v}_T,\eta)_{P_{TF}} = -((\delta^k_{\nabla,T}-\nabla\delta^k_T)\underline{v}_T,\eta)_T + ((\delta^k_{TF}-\delta^k_T)\underline{v}_T,\eta\cdot\boldsymbol{n}_{TF})_F}$$

- **The properties (S1)–(S3) are verified by construction**

Numerical examples

Trigonometric solution, $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ v. h, $p \in \{2,3,4\}$

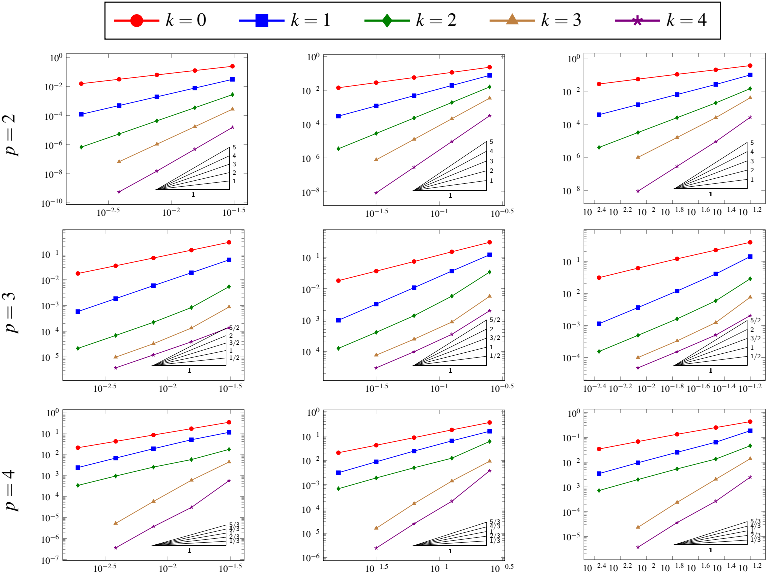
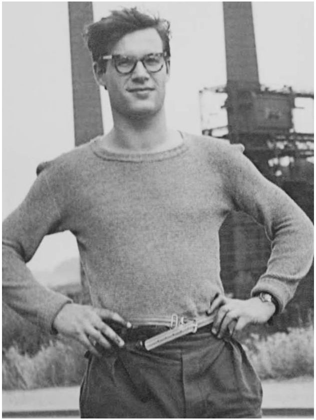


Figure: Trigonometric solution, $\|\underline{I}_h^k u - \underline{u}_h\|_{1,p,h}$ versus h.

Thank you!



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