Hybrid High-Order methods for nonlinear problems

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Two crucial problems for humanity

Hybrid High-Order (HHO) methods

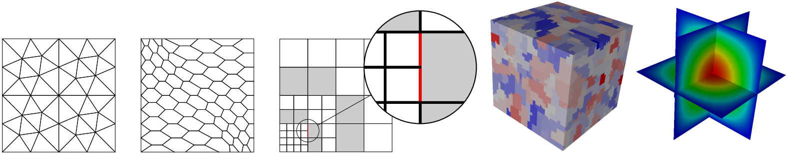


Figure: Examples of supported meshes $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ in 2d and 3d

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Natural extension to nonlinear problems
- Reduced computational cost after static condensation
- **Key idea:** replace spaces and operators with discrete counterparts

References for this presentation

- HHO for Leray–Lions problems
  - Analysis tools and convergence [DP and Droniou, 2017a]
  - Basic error estimates [DP and Droniou, 2017b]
  - Stabilization-free [DP, Droniou, Manzini, 2018]
  - Improved estimates (general meshes) [DP, Droniou, Harnist, 2021]
  - Improved estimates (standard meshes) [Carstensen and Tran, 2020]
- Applications
  - Nonlinear elasticity [Botti, DP, Sochala, 2017]
  - Nonlinear poroelasticity [Botti, DP, Sochala, 2018]
  - Non-Newtonian fluids [Botti, Castanon Quiroz, DP, Harnist, 2020]
- General introduction to HHO methods:

Di Pietro, D. A. and Droniou, J. (2020).
The Hybrid High-Order method for polytopal meshes. Design, analysis, and applications, volume 19 of *Modeling, Simulation and Application*.
Springer International Publishing.

Outline

Model problem

- Let $\Omega \subset \mathbb{R}^d$ denote a bounded connected polyhedral domain
- Let $r \in (0, +\infty)$ and $r' := \frac{r}{r-1}$
- Consider the problem: Given $f \in L^{r'}(\Omega)$, find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\nabla} u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- In weak formulation: Find $u \in W_0^{1,r}(\Omega)$ s.t.

$$\int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla} u) \cdot \boldsymbol{\nabla} v = \int_\Omega f v \qquad \forall v \in W_0^{1,r}(\Omega).$$

- The key differential operator is the gradient

Flux function

Assumption (Flux function I)

The Carathéodory function[1] $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is s.t., for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^d$,

- Growth. There exists a real number $\overline{\sigma} > 0$ s.t.

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \mathbf{0})| \leq \overline{\sigma} |\boldsymbol{\eta}|^{r-1}.$$

- Coercivity. There is a real number $\underline{\sigma} > 0$ s.t.,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq \underline{\sigma} |\boldsymbol{\eta}|^r.$$

- Monotonicity. It holds

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\xi})) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) \geq 0.$$

[1] $\boldsymbol{\sigma}(\boldsymbol{x}, \cdot)$ continuous, $\boldsymbol{\sigma}(\cdot, \boldsymbol{\eta})$ measurable

L^2-orthogonal projectors on local polynomial spaces

- Let a polynomial degree $k \geq 0$ and a mesh element or face X be fixed
- Define the polynomial space

$$\mathbb{P}^k(X) := \{\text{restriction to } X \text{ of } d\text{-variate polynomials of total degree} \leq k\}$$

- The L^2-orthogonal projector $\pi_X^k : L^2(X) \to \mathbb{P}^k(X)$ is s.t.

$$\int_X (\pi_X^k v - v) w = 0 \text{ for all } w \in \mathbb{P}^k(X)$$

- Optimal approximation properties hold [DP and Droniou, 2020]

A key remark

- Let a polytopal mesh element $T \in \mathcal{T}_h$ be fixed
- Recall the following IBP formula, valid for all $(v, \boldsymbol{\tau}) \in W^{1,1}(T) \times C^\infty(\overline{T})^d$:

$$\int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\tau} = -\int_T v \ (\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F v \ (\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF})$$

- Given an integer $k \geq 0$, taking $\boldsymbol{\tau} \in \mathbb{P}^k(T)^d$ we can write

$$\int_T \boldsymbol{\pi}_T^k(\boldsymbol{\nabla} v) \cdot \boldsymbol{\tau} = -\int_T \pi_T^k v \ (\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^k v_{|F} \ (\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF})$$

- **Hence, $\boldsymbol{\pi}_T^k(\boldsymbol{\nabla} v)$ can be computed from $\pi_T^k v$ and $(\pi_F^k v_{|F})_{F \in \mathcal{F}_T}$!**

Local HHO space and interpolator

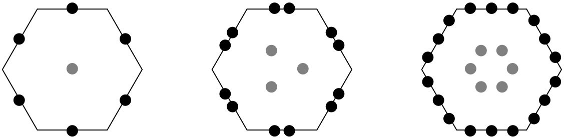


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$ and $d = 2$

- For $k \geq 0$ and $T \in \mathcal{T}_h$, define the local HHO space

$$\underline{U}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \,:\, v_T \in \mathbb{P}^k(T) \text{ and } v_F \in \mathbb{P}^k(F) \text{ for all } F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{I}_T^k : W^{1,1}(T) \to \underline{U}_T^k$ is s.t., for all $v \in W^{1,1}(T)$,

$$\underline{I}_T^k v := \left(\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T} \right)$$

Gradient reconstruction

- Let $T \in \mathcal{T}_h$. We define the local gradient reconstruction

$$\boldsymbol{G}_T^k : \underline{U}_T^k \to \mathbb{P}^k(T)^d$$

s.t., for all $\underline{v}_T \in \underline{U}_T^k$,

$$\int_T \boldsymbol{G}_T^k \underline{v}_T \cdot \boldsymbol{\tau} = -\int_T v_T \ (\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F v_F \ (\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T)^d$$

- By construction, we have,

$$\boldsymbol{G}_T^k(\underline{I}_T^k v) = \boldsymbol{\pi}_T^k(\boldsymbol{\nabla} v) \quad \forall v \in W^{1,1}(T)$$

- $(\boldsymbol{G}_T^k \circ \underline{I}_T^k)$ therefore has optimal approximation properties in $\mathbb{P}^k(T)^d$

Global HHO space and gradient reconstruction

- The global HHO space is obtained patching interface unknowns:

$$\underline{U}_h^k \coloneqq \Big\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) \;:\; v_T \in \mathbb{P}^k(T) \text{ for all } T\in\mathcal{T}_h \text{ and } v_F \in \mathbb{P}^k(F) \text{ for all } F\in\mathcal{F}_h \Big\}$$

- The global gradient $\boldsymbol{G}_h^k : \underline{U}_h^k \to \mathbb{P}^k(\mathcal{T}_h)^d$ is s.t.

$$\forall \underline{v}_h \in \underline{U}_h^k, \quad (\boldsymbol{G}_h^k \underline{v}_h)_{|T} \coloneqq \boldsymbol{G}_T^k \underline{v}_T \quad \forall T \in \mathcal{T}_h$$

- Accounting for boundary conditions, we set

$$\underline{U}_{h,0}^k \coloneqq \left\{ \underline{v}_h \in \underline{U}_h^k \;:\; v_F = 0 \text{ for all } F \in \mathcal{F}_h \text{ s.t. } F \subset \partial\Omega \right\}$$

Discrete Sobolev norms

- We need to endow $\underline{U}_h^k$ with a Sobolev structure
- We define the discrete Sobolev seminorm s.t., for all $\underline{v}_h \in \underline{U}_h^k$,

$$\|\underline{v}_h\|_{1,r,h} := \left(\sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,r,T}^r \right)^{\frac{1}{r}}$$

  where, for all $T \in \mathcal{T}_h$,

$$\|\underline{v}_T\|_{1,r,T} := \left(\|\nabla v_T\|_{L^r(T)^d}^r + \sum_{F \in \mathcal{F}_T} h_F^{1-r} \|v_F - v_T\|_{L^r(F)}^r \right)^{\frac{1}{r}}$$

- The factor h_F^{1-r} in the boundary term ensures the appropriate scaling

Discrete functional analysis results I

Theorem (Discrete Sobolev–Poincaré inequalities)

Let

$$1 \le q \le \frac{dr}{d-r} \text{ if } 1 \le r < d \text{ and } 1 \le q < +\infty \text{ if } r \ge d.$$

Then, for all $\underline{v}_h \in \underline{U}_{h,0}^k$*, letting* $v_h \in \mathbb{P}^k(\mathcal{T}_h)$ *be s.t.*

$$(v_h)_{|T} := v_T \qquad \forall T \in \mathcal{T}_h,$$

it holds, with $C > 0$ *depending only on* Ω, k, r, q, *and mesh regularity,*

$$\|v_h\|_{L^q(\Omega)} \le C \|\underline{v}_h\|_{1,r,h}.$$

Corollary (Discrete Sobolev norms)

The mapping $\|\cdot\|_{1,r,h}$ *is a norm on* $\underline{U}_{h,0}^k$.

Discrete functional analysis results II

Theorem (Discrete compactness)

Let $(\mathcal{M}_h)_{h>0}$ *be a regular mesh sequence and* $(\underline{v}_h)_{h>0} \in (\underline{U}_{h,0}^k)_{h>0}$ *s.t.*

$$\|\underline{v}_h\|_{1,r,h} \le C \text{ for all } h > 0.$$

Then, there exists $v \in W_0^{1,r}(\Omega)$ *s.t., up to a subsequence as* $h \to 0$*,*

- $v_h \to v$ *strongly in* $L^q(\Omega)$ *for all* $1 \le q < \begin{cases} \frac{dr}{d-r} & \text{if } r < d, \\ +\infty & \text{otherwise;} \end{cases}$
- $\boldsymbol{G}_h^k \underline{v}_h \rightharpoonup \boldsymbol{\nabla} v$ *weakly in* $L^r(\Omega)^d$.

Proposition (Strong convergence of the gradient for smooth functions)

With $(\mathcal{M}_h)_{h>0}$ *as before it holds, for all* $\varphi \in W^{1,r}(\Omega)$,

$$\boldsymbol{G}_h^k(\underline{I}_h^k \varphi) \to \boldsymbol{\nabla}\varphi \text{ strongly in } L^r(\Omega)^d \text{ as } h \to 0.$$

Discrete problem I

- Define the function $\mathrm{a}_h : \underline{U}_h^k \times \underline{U}_h^k \to \mathbb{R}$ s.t.

$$\mathrm{a}_h(\underline{w}_h, \underline{v}_h) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{G}_h^k \underline{w}_h) \cdot \boldsymbol{G}_h^k \underline{v}_h + \sum_{T \in \mathcal{T}_h} \mathrm{s}_T(\underline{w}_T, \underline{v}_T)$$

- Above, s_T is a stabilization obtained penalizing face residuals s.t.
  - $\|\boldsymbol{G}_T^k \underline{v}_T\|_{L^r(T)^d}^r + \mathrm{s}_T(\underline{v}_T, \underline{v}_T) \simeq \|\underline{v}_T\|_{1,r,T}^r$ uniformly in h
  - $\mathrm{s}_T(\underline{I}_T^k w, \underline{v}_T) = 0$ for all $(w, \underline{v}_T) \in \mathbb{P}^{k+1}(T) \times \underline{U}_T^k$
  - Hölder continuity and strong monotonicity hold

Discrete problem II

The discrete Leray–Lions problem reads:

$$\text{Find } \underline{u}_h \in \underline{U}_{h,0}^k \text{ s.t. } \mathrm{a}_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k$$

Lemma (Existence and a priori bound)

There is at least one solution $\underline{u}_h \in \underline{U}_{h,0}^k$, and any solution satisfies

$$\|\underline{u}_h\|_{1,r,h} \le C \|f\|_{L^{r'}(\Omega)}^{\frac{1}{r-1}},$$

with real number $C > 0$ independent of h.

Remark (Uniqueness)

Uniqueness holds replacing monotonicity with strict monotonicity.

Convergence I

Theorem (Convergence)

Let $(\mathcal{M}_h)_{h>0}$ be a regular mesh sequence and let $(\underline{u}_h)_{h>0}$ be the corresponding sequence of discrete solutions. Then, as $h \to 0$, up to a subsequence,

- $u_h \to u$ *strongly in* $L^q(\Omega)$ *with* $1 \le q < \begin{cases} \frac{dr}{d-r} & \text{if } r < d, \\ +\infty & \text{otherwise,} \end{cases}$
- $\boldsymbol{G}_h^k \underline{u}_h \rightharpoonup \boldsymbol{\nabla} u$ *weakly in* $L^r(\Omega)^d$,

with $u \in W_0^{1,r}(\Omega)$ *solution to the continuous problem. If, additionally,* $\boldsymbol{\sigma}$ *is strictly monotone, then* u *is unique and* $\boldsymbol{G}_h^k \underline{u}_h$ *converges strongly.*

Convergence II

Proof.

- Combining the a priori bound with discrete compactness, we infer the existence of $u \in W_0^{1,r}(\Omega)$ s.t. the above convergences hold
- Taking $\underline{v}_h = \underline{I}_h^k \varphi$ as test function with $\varphi \in C_c^\infty(\Omega)$ and using Minty's trick, we infer that u solves the continuous problem
- Using Vitali's theorem, we prove strong convergence of $\boldsymbol{G}_h^k \underline{u}_h$ under strict monotonicity of $\boldsymbol{\sigma}$ □

Error estimates I

Assumption (Flux function II)

In addition to Assumption I, it holds, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^d$,

- Hölder continuity. There exists a real number $\sigma^* > 0$ s.t.

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\xi})| \leq \sigma^* |\boldsymbol{\eta} - \boldsymbol{\xi}| \left(|\boldsymbol{\eta}|^{r-2} + |\boldsymbol{\xi}|^{r-2}\right).$$

- Strong monotonicity. There exists a real number $\sigma_* > 0$ s.t.

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\xi})) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) \geq \sigma_* |\boldsymbol{\eta} - \boldsymbol{\xi}|^2 \left(|\boldsymbol{\eta}| + |\boldsymbol{\xi}|\right)^{r-2}.$$

Remark (r-Laplacian)

The above assumptions are verified by the r-Laplace flux function

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta}) = |\boldsymbol{\eta}|^{r-2} \boldsymbol{\eta}.$$

Error estimates II

Theorem (Basic error estimate)

Assume $u \in W^{k+2,r}(\mathcal{T}_h)$ *and* $\boldsymbol{\sigma}(\cdot, \nabla u) \in W^{k+1,r'}(\mathcal{T}_h)^d$ *and let*

- *if* $r \geq 2$,

$$\mathcal{E}_h(u) := h^{k+1}|u|_{W^{k+2,r}(\mathcal{T}_h)} + h^{\frac{k+1}{r-1}}\left(|u|_{W^{k+2,r}(\mathcal{T}_h)}^{\frac{1}{r-1}} + |\boldsymbol{\sigma}(\cdot, \nabla u)|_{W^{k+1,r'}(\mathcal{T}_h)^d}^{\frac{1}{r-1}}\right);$$

- *if* $r < 2$,

$$\mathcal{E}_h(u) := h^{(k+1)(r-1)}|u|_{W^{k+2,r}(\mathcal{T}_h)}^{r-1} + h^{k+1}|\boldsymbol{\sigma}(\cdot, \nabla u)|_{W^{k+1,r'}(\mathcal{T}_h)^d}.$$

Then, it holds

$$\|\underline{I}_h^k u - \underline{u}_h\|_{1,r,h} \leq C\mathcal{E}_h(u),$$

with $C > 0$ *depending only on* Ω, k, r, $\underline{\sigma}$, $\overline{\sigma}$, σ_*, σ^*, *and mesh regularity.*

Improved error estimates

- The above estimate gives the following asymptotic convergence rates:

$$\begin{cases} h^{\frac{k+1}{r-1}} & \text{if } r \geq 2, \\ h^{(k+1)(r-1)} & \text{if } 1 < r < 2 \end{cases}$$

- Successively [DP, Droniou, Harnist, 2021] proved

$$h^{k+1} \text{ in the non-degenerate case for } 1 < r \leq 2,$$

  with intermediate rates depending on a degeneracy parameter
- Very recently, [Carstensen and Tran, 2020] proved convergence in

$$h^{\frac{k+1}{3-r}} \text{ for } 1 < r \leq 2$$

  for a variation of the HHO method on conforming simplicial meshes based on a stable gradient inspired by [DP, Droniou, Manzini, 2018]

Numerical example

Convergence for $r = 3$

h	$\lVert \underline{I}_h^k u - \underline{u}_h \rVert_{1,r,h}$	EOC
$k = 1$ (1)		
$3.07 \cdot 10^{-2}$	$1.71 \cdot 10^{-2}$	—
$1.54 \cdot 10^{-2}$	$4.72 \cdot 10^{-3}$	1.87
$7.68 \cdot 10^{-3}$	$1.16 \cdot 10^{-3}$	2.02
$3.84 \cdot 10^{-3}$	$2.96 \cdot 10^{-4}$	1.97
$1.92 \cdot 10^{-3}$	$7.77 \cdot 10^{-5}$	**1.93**
$k = 2$ ($\frac{3}{2}$)		
$3.07 \cdot 10^{-2}$	$2.72 \cdot 10^{-3}$	—
$1.54 \cdot 10^{-2}$	$2.32 \cdot 10^{-4}$	3.57
$7.68 \cdot 10^{-3}$	$3.32 \cdot 10^{-5}$	2.79
$3.84 \cdot 10^{-3}$	$7.25 \cdot 10^{-6}$	2.2
$1.92 \cdot 10^{-3}$	$1.81 \cdot 10^{-6}$	**2.00**
$k = 3$ (2)		
$3.07 \cdot 10^{-2}$	$3.1 \cdot 10^{-4}$	—
$1.54 \cdot 10^{-2}$	$2.97 \cdot 10^{-5}$	3.4
$7.68 \cdot 10^{-3}$	$4.4 \cdot 10^{-6}$	2.74
$3.84 \cdot 10^{-3}$	$9.76 \cdot 10^{-7}$	2.17
$1.92 \cdot 10^{-3}$	$2.41 \cdot 10^{-7}$	**2.02**

Table: Triangular mesh family

h	$\lVert \underline{I}_h^k u - \underline{u}_h \rVert_{1,r,h}$	EOC
$k = 1$ (1)		
$6.5 \cdot 10^{-2}$	$3.06 \cdot 10^{-2}$	—
$3.15 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	1.41
$1.61 \cdot 10^{-2}$	$3.35 \cdot 10^{-3}$	1.77
$9.09 \cdot 10^{-3}$	$1.25 \cdot 10^{-3}$	1.72
$4.26 \cdot 10^{-3}$	$3.58 \cdot 10^{-4}$	**1.65**
$k = 2$ ($\frac{3}{2}$)		
$6.5 \cdot 10^{-2}$	$1.18 \cdot 10^{-2}$	—
$3.15 \cdot 10^{-2}$	$2.33 \cdot 10^{-3}$	2.24
$1.61 \cdot 10^{-2}$	$4.4 \cdot 10^{-4}$	2.48
$9.09 \cdot 10^{-3}$	$1.02 \cdot 10^{-4}$	2.56
$4.26 \cdot 10^{-3}$	$1.42 \cdot 10^{-5}$	**2.60**
$k = 3$ (2)		
$6.5 \cdot 10^{-2}$	$2.75 \cdot 10^{-3}$	—
$3.15 \cdot 10^{-2}$	$2.69 \cdot 10^{-4}$	3.21
$1.61 \cdot 10^{-2}$	$4.01 \cdot 10^{-5}$	2.84
$9.09 \cdot 10^{-3}$	$1.31 \cdot 10^{-5}$	1.96
$4.26 \cdot 10^{-3}$	$2.21 \cdot 10^{-6}$	**2.35**

Table: Voronoi mesh family

Outline

Model problem I

- Let $d \in \{2, 3\}$. Given $\boldsymbol{f} : \Omega \to \mathbb{R}^d$, the nonlinear Stokes problem reads: Find $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + \boldsymbol{\nabla} p &= \boldsymbol{f} && \text{in } \Omega,\\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega,\\
\int_\Omega p &= 0,
\end{aligned}
$$

- We focus, for the sake of simplicity, on power-law fluids, for which

$$
\boldsymbol{\sigma}(\boldsymbol{\tau}) = |\boldsymbol{\tau}|^{r-2}\boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}
$$

- For $r \in (1, 2]$ the fluid is shear-thinning, for $r \geq 2$, shear-thickening
- More general strain rate-shear stress laws can be considered

Model problem II

- Define the following spaces:

$$\boldsymbol{U} := W_0^{1,r}(\Omega)^d, \quad P := \left\{ q \in L^{r'}(\Omega) \ : \ \int_\Omega q = 0 \right\}$$

- Taking $\boldsymbol{f} \in L^{r'}(\Omega)^d$, the weak formulation is: Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ s.t.

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad && \forall \boldsymbol{v} \in \boldsymbol{U}, \\ -b(\boldsymbol{u}, q) &= 0 && \forall p \in P \end{aligned}$$

  where $a : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}$ and $b : \boldsymbol{U} \times P \to \mathbb{R}$ are s.t.

$$a(\boldsymbol{w}, \boldsymbol{v}) := \int_\Omega \boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w}) : \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}, \quad b(\boldsymbol{v}, q) := -\int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \ q$$

- The extension of stability results is non-trivial

Local HHO space and seminorm

- Given $T \in \mathcal{T}_h$, the vector version of the local HHO space is

$$\underline{\boldsymbol{U}}_T^k := \left\{ \underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}) \ : \ \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \text{ for all } F \in \mathcal{F}_T \right\}$$

- We furnish $\underline{\boldsymbol{U}}_T^k$ with the strain rate $W^{1,r}$-like seminorm

$$\|\underline{\boldsymbol{v}}_T\|_{\boldsymbol{\varepsilon},r,T} := \left(\|\boldsymbol{\nabla}_{\rm s} \boldsymbol{v}_T\|_{L^r(T)^{d\times d}}^r + \sum_{F \in \mathcal{F}_T} h_F^{r-1} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^r(F)^d}^r \right)^{\frac{1}{r}}$$

- Notice that the symmetric gradient replaces the gradient!

Symmetric gradient and divergence reconstructions

- The local symmetric gradient reconstruction is s.t.

$$\boldsymbol{G}_{\mathrm{s},T}^{k} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$ and all $\boldsymbol{\tau} \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$,

$$\int_T \boldsymbol{G}_{\mathrm{s},T}^{k}\underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T \cdot (\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{v}_F \cdot (\boldsymbol{\tau}\boldsymbol{n}_{TF})$$

- A divergence reconstruction $D_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)$ is obtained setting

$$D_T^k := \mathrm{tr}(\boldsymbol{G}_{\mathrm{s},T}^{k})$$

- With $\underline{\boldsymbol{I}}_T^k$ interpolator on $\underline{\boldsymbol{U}}_T^k$ we have, for all $\boldsymbol{v} \in W^{1,1}(T)^d$,

$$\boldsymbol{G}_{\mathrm{s},T}^{k}(\underline{\boldsymbol{I}}_T^k\boldsymbol{v}) = \boldsymbol{\pi}_T^k(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}), \qquad D_T^k(\underline{\boldsymbol{I}}_T^k\boldsymbol{v}) = \pi_T^k(\boldsymbol{\nabla}\cdot\boldsymbol{v})$$

Global HHO space and strain reconstruction

- At the global level, we define the velocity space

$$\underline{\boldsymbol{U}}_h^k := \big\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}) : \\ \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \text{ for all } T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \text{ for all } F \in \mathcal{F}_h \big\}$$

along with its subspace with strongly enforced BC

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k : \boldsymbol{v}_F = \boldsymbol{0} \text{ for all } F \in \mathcal{F}_h \text{ s.t. } F \subset \partial\Omega \right\}$$

- We furnish $\underline{\boldsymbol{U}}_{h,0}^k$ with the global $W^{1,r}$-seminorm $\|\cdot\|_{\boldsymbol{\varepsilon},r,h}$
- The global strain reconstruction $\boldsymbol{G}_{\mathrm{s},h}^k : \underline{\boldsymbol{U}}_h^k \to \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is s.t.

$$\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k, \quad (\boldsymbol{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h)_{|T} := \boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T \quad \forall T \in \mathcal{T}_h$$

Viscous function I

- The viscous function $\mathrm{a}_h : \underline{\boldsymbol{U}}_h^k \times \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ is s.t.

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) \coloneqq \int_\Omega \boldsymbol{\sigma}(\boldsymbol{G}_{\mathrm{s},h}^k \underline{\boldsymbol{u}}_h) : \boldsymbol{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h + \sum_{T \in \mathcal{T}_h} \mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

- To formulate assumptions on s_T, we introduce the singular exponent

$$\tilde{r} \coloneqq \min(r, 2)$$

Viscous function II

Assumption

The stabilization function s_T is linear in its second argument and it satisfies:

- Stability. For all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$, $\|\boldsymbol{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T\|_{L^r(T)^{d\times d}}^2 + \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \simeq \|\underline{\boldsymbol{v}}_T\|_{\boldsymbol{\varepsilon},r,T}^2$
- Polynomial consistency. For all $(\boldsymbol{w}, \underline{\boldsymbol{v}}_T) \in \mathbb{P}^{k+1}(T)^d \times \underline{\boldsymbol{U}}_T^k$, $\mathrm{s}_T(\underline{\boldsymbol{I}}_T^k \boldsymbol{w}, \underline{\boldsymbol{v}}_T) = 0$
- Hölder continuity. For all $\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T, \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{U}}_T^k$, setting $\underline{\boldsymbol{e}}_T \coloneqq \underline{\boldsymbol{u}}_T - \underline{\boldsymbol{w}}_T$,

$$|\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) - \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T)| \lesssim \left(\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{u}}_T) + \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{w}}_T)\right)^{\frac{r-\tilde{r}}{r}} \mathrm{s}_T(\underline{\boldsymbol{e}}_T, \underline{\boldsymbol{e}}_T)^{\frac{\tilde{r}-1}{r}} \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T)^{\frac{1}{r}}$$

- Strong monotonicity. For all $\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{U}}_T^k$, setting $\underline{\boldsymbol{e}}_T \coloneqq \underline{\boldsymbol{u}}_T - \underline{\boldsymbol{w}}_T$,

$$\left(\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{e}}_T) - \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{e}}_T)\right)\left(\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{u}}_T) + \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{w}}_T)\right)^{\frac{2-\tilde{r}}{r}} \gtrsim \mathrm{s}_T(\underline{\boldsymbol{e}}_T, \underline{\boldsymbol{e}}_T)^{\frac{r+2-\tilde{r}}{r}}$$

Stability and polynomial consistency are incompatible for $k = 0$!

Discrete Korn inequality

Discrete stability hinges on the following result:

Theorem (Discrete Korn inequality)

Assume $k \geq 1$*. Then, for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$*, letting* $\boldsymbol{v}_h \in \mathbb{P}^k(\mathcal{T}_h)^d$ *be s.t.* $(\boldsymbol{v}_h)_{|T} := \boldsymbol{v}_T$ *for all* $T \in \mathcal{T}_h$*,*

$$\|\boldsymbol{v}_h\|_{L^r(\Omega)^d} + |\boldsymbol{v}_h|_{W^{1,r}(\mathcal{T}_h)^d} \lesssim \|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},r,h},$$

with $|\cdot|_{W^{1,r}(\mathcal{T}_h)^d}$ *broken* $W^{1,r}$*-seminorm.*

Pressure–velocity coupling

The pressure-velocity coupling bilinear form $\mathrm{b}_h : \underline{\boldsymbol{U}}_h^k \times \mathbb{P}^k(\mathcal{T}_h)$ is s.t.

$$\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) \coloneqq -\sum_{T\in\mathcal{T}_h} \int_T D_T^k \underline{\boldsymbol{v}}_T \; q_T$$

Lemma (Inf-sup stability)

Define the pressure space

$$P_h^k \coloneqq \left\{ q_h \in \mathbb{P}^k(\mathcal{T}_h) \; : \; \int_\Omega q_h = 0 \right\}.$$

Then it holds, for all $q_h \in P_h^k$,

$$\|q_h\|_{L^{r'}(\Omega)} \lesssim \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k,\, \|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},r,h}=1} \mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h).$$

Discrete problem

The discrete problem reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ s.t.

$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h & \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 & \forall q_h \in P_h^k \end{aligned}$$

Theorem (Well-posedness)

There exists a unique discrete solution $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$*, and the following a priori bounds hold:*

$$\begin{aligned} \|\underline{\boldsymbol{u}}_h\|_{\boldsymbol{\varepsilon},r,h} &\lesssim \|\boldsymbol{f}\|_{L^{r'}(\Omega)^d}^{\frac{1}{r-1}} + \|\boldsymbol{f}\|_{L^{r'}(\Omega)^d}^{\frac{1}{r+1-\tilde{r}}}, \\ \|p_h\|_{L^{r'}(\Omega)} &\lesssim \|\boldsymbol{f}\|_{L^{r'}(\Omega)^d} + \|\boldsymbol{f}\|_{L^{r'}(\Omega)^d}^{\frac{\tilde{r}-1}{r+1-\tilde{r}}}, \end{aligned}$$

with hidden multiplicative constants possibly depending on Ω, d, k, *and the mesh regularity parameter.*

Error estimate

Theorem (Error estimate)

Assume the regularity

$$\boldsymbol{u} \in W^{1,r}(\Omega)^d \cap W^{k+2,r}(\mathcal{T}_h)^d, \quad p \in W^{1,r'}(\Omega) \cap W^{(k+1)(\tilde{r}-1)}(\mathcal{T}_h).$$
$$\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}) \in W^{1,r'}(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \cap W^{(k+1)(\tilde{r}-1),r'}(\mathcal{T}_h;\mathbb{R}^{d\times d}_{\rm sym}).$$

Then,

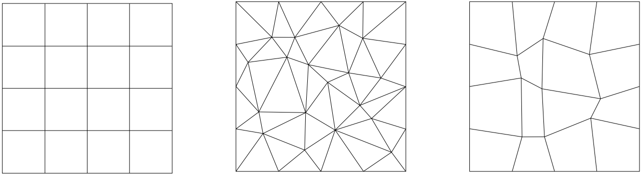
$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k\boldsymbol{u}\|_{\boldsymbol{\varepsilon},r,h} \le A h^{\frac{(k+1)(\tilde{r}-1)}{r+1-\tilde{r}}},$$
$$\|p_h - \pi_h^k p\|_{L^{r'}(\Omega)} \le B h^{(k+1)(\tilde{r}-1)} + C h^{\frac{(k+1)(\tilde{r}-1)^2}{r+1-\tilde{r}}},$$

with A, B, and C possibly depending on Ω, d, k, the mesh regularity parameter, and on bounded norms of $\boldsymbol{u}$, p, and $\boldsymbol{f}$.

Remark (Orders of convergence)

The order for the velocity is the same as for Leray–Lions problems. The asymptotic order for the pressure is $h^{(k+1)(r-1)^2}$ if $r < 2$, $\frac{k+1}{r-1}$ otherwise.

- We assess the orders of convergence using a manufactured solution
- We take $k = 1$ and let r vary in $\{1.5, 1.75, \ldots, 2.75\}$
- The regularity assumptions are mostly verified (except for $r = 1.5$, for which $\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}) \notin W^{1,r'}(\Omega, \mathbb{R}^{d\times d}_{\rm sym})$)
- We consider three families of meshes



Cartesian — Distorted triangular — Distorted quadrangular

Numerical examples II

Convergence

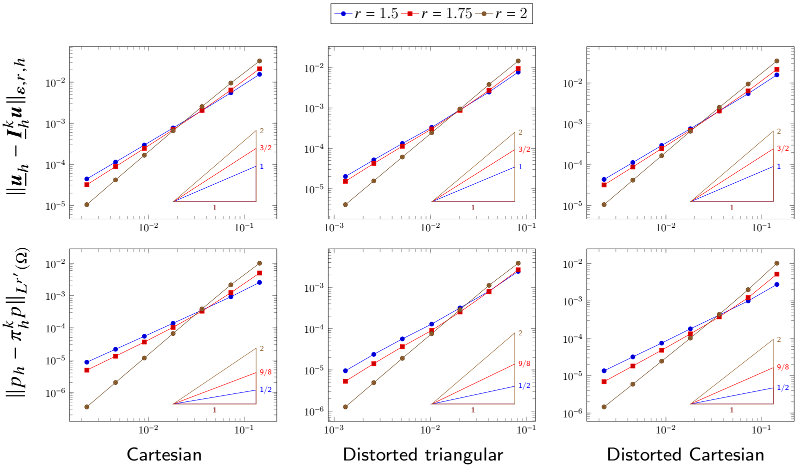


Figure: Convergence for shear-thinning fluids. The slopes indicate the expected order of convergence, i.e., $\mathcal{O}^1_{\text{vel}} = 2(r-1)$ and $\mathcal{O}^1_{\text{pre}} = 2(r-1)^2$ for $r \in \{1.5, 1.75, 2\}$.

Numerical examples III

Convergence

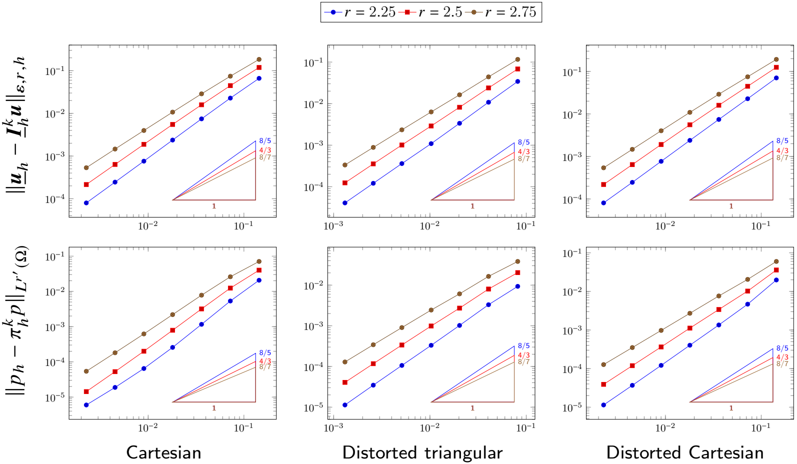
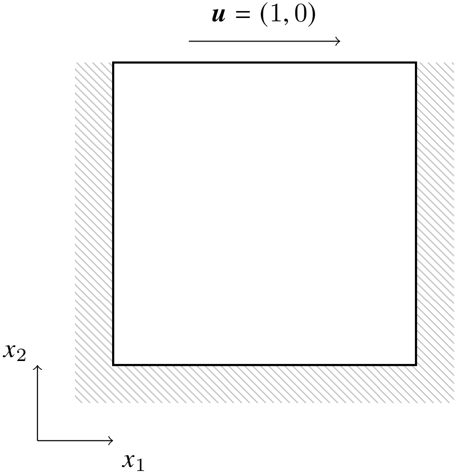


Figure: Convergence for shear-thickening fluids. The slopes indicate the expected order of convergence, i.e. $\mathcal{O}^1_{\text{vel}} = \mathcal{O}^1_{\text{pre}} = \frac{2}{r-1}$ for $r \in \{2.25, 2.5, 2.75\}$.

Lid-driven cavity I



Lid-driven cavity II

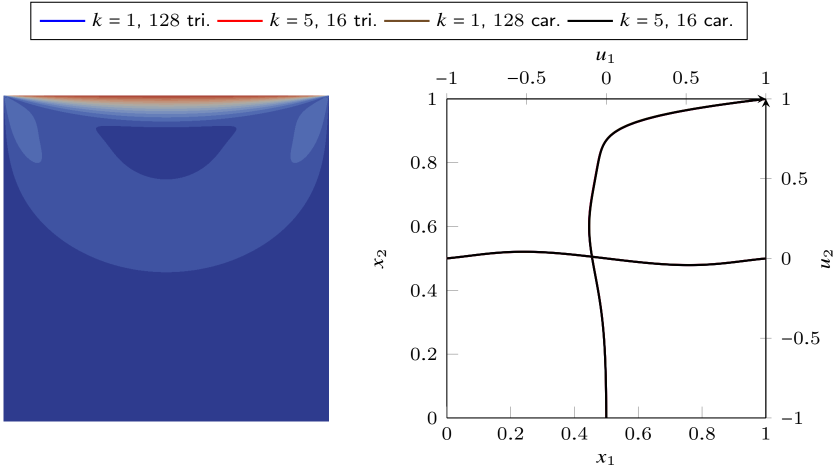


Figure: $r = 1.25$ (shear-thinning fluid)

Lid-driven cavity III

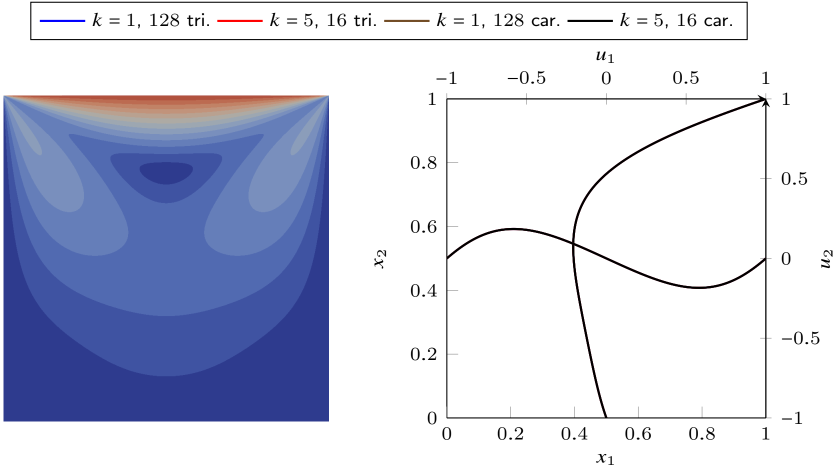


Figure: $r = 2$ (Newtonian fluid)

Lid-driven cavity IV

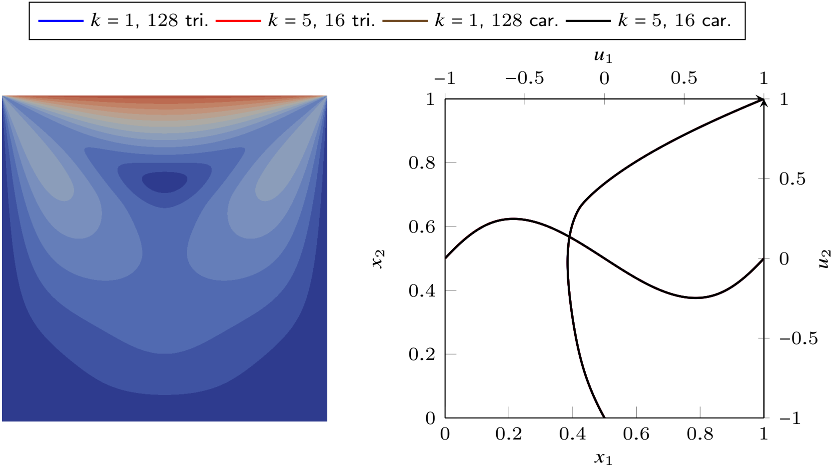


Figure: $r = 2.75$ (shear-thickening fluid)

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