Hybrid High-Order (HHO) methods for quasi-incompressible linear elasticity on general meshes

Daniele A. Di Pietro   Alexandre Ern

I3M, University of Montpellier 2

WCCM XI, July 24, 2014

Introduction

- Same problem as in L. Beirão da Veiga's talk
- For $\Omega \subset \mathbb{R}^d$, ∇_s symmetric gradient, and Lamé coefficients

$$0 < \mu < +\infty, \qquad 0 \leqslant \lambda \leqslant +\infty,$$

we consider the linear elasticity problem

$$\begin{aligned} -\nabla\cdot \left(2\mu\nabla_s \underline{u} + \lambda(\nabla\cdot\underline{u})\underline{\underline{I}}_d\right) &= \underline{f} && \text{in } \Omega \\ \underline{u} &= \underline{0} && \text{on } \partial\Omega \end{aligned}$$

- The weak formulation reads: Find $\underline{u} \in \underline{U}_0 := H_0^1(\Omega)^d$ s.t.

$$\boxed{(2\mu\nabla_s\underline{u}, \nabla_s\underline{v}) + (\lambda\nabla\cdot\underline{u}, \nabla\cdot\underline{v}) = (\underline{f}, \underline{v}) \qquad \forall \underline{v} \in \underline{U}_0}$$

- More general bcs can be treated with minor modifications

Some references for linear elasticity

- Incompressible limit $\lambda \to +\infty$ requires to accurately represent nontrivial divergence-free fields
  - Classical low-order conforming FE suffer from numerical locking
  - Mixed methods [Franca & Stenberg 91; Brezzi & Fortin 91]
  - Nonconforming methods [Brenner & Sung 92]
- Low-order schemes on general meshes
  - MFD [Beirão da Veiga, Gyrya, Lipnikov & Manzini 09]
  - Generalized Crouzeix–Raviart [DP & Lemaire 14]
  - Gradient schemes [Droniou & Lamichane 14]
- Hybridizable Discontinuous Galerkin [Soon, Cockburn & Stolarski 09]
- High-order VEM on general meshes for planar elasticity with vertex, edge and cell DOFs [Beirão da Veiga, Brezzi & Marini 13]

Key ideas for HHO

- Generalized DOFs: polynomials of order $k \geqslant 1$ at elements and faces
- Reconstruction of differential operators taylored to the problem
  - Symmetric gradient obtained solving local pure-traction problems
  - Divergence satisfying a commuting diagram property
  - Face-based penalty linking cell- and face-DOFs
- Main benefits
  - Fairly general polygonal/polyhedral meshes
  - SPD global linear system
  - High-order: stress cv. rate $(k+1)$, displacement cv. rate $(k+2)$
  - Compact-stencil + static condensation = $9\ ^{\text{DOFs}}/_{\text{face}}$ $(d=3,\ k=1)$, no vertex unknowns
- References
  - Linear elasticity [DP & Ern 14, hal-00918482]
  - Poisson [DP, Ern & Lemaire 14, DOI: 10.1515/cmam-2014-0018]
  - Variable diffusion [DP & Ern 14, hal-01023302]

Mesh regularity

Definition (Mesh regularity)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of poly{gonal,hedral} meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and

- $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is shape-regular in the sense of Ciarlet;
- $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is contact regular: every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences:

- Trace and inverse inequalities
- Optimal approximation properties for broken polynomial spaces

DOFs

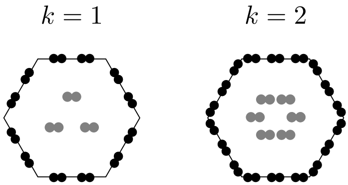


Figure : $\underline{\mathsf{U}}_T^k$ for $k \in \{1,2\}$

- For all $k \geqslant 1$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{\mathsf{U}}_T^k := \mathbb{P}_d^k(T)^d \times \left\{ \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}_{d-1}^k(F)^d \right\}$$

- The global space is obtained by patching interface DOFs

$$\underline{\mathsf{U}}_h^k := \left\{ \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}_d^k(T)^d \right\} \times \left\{ \bigtimes_{F\in\mathcal{F}_h} \mathbb{P}_{d-1}^k(F)^d \right\}$$

Displacement gradient reconstruction I

- Let $T \in \mathcal{T}_h$. The local displacement reconstruction operator

$$\underline{r}_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$$

is s.t., for all $\underline{\mathsf{v}} = \big(\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\mathsf{U}}_T^k$ and $\underline{w} \in \mathbb{P}_d^{k+1}(T)^d$,

$$\begin{aligned}(\nabla_{\rm s}\underline{r}_T^k\underline{\mathsf{v}}, \nabla_{\rm s}\underline{w})_T &:= (\nabla_{\rm s}\underline{\mathsf{v}}_T, \nabla_{\rm s}\underline{w})_T + \sum_{F\in\mathcal{F}_T}(\underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T, \nabla_{\rm s}\underline{w}\underline{n}_{TF})_F \\ &= -(\underline{\mathsf{v}}_T, \nabla\cdot\nabla_{\rm s}\underline{w})_T + \sum_{F\in\mathcal{F}_T}(\underline{\mathsf{v}}_F, \nabla_{\rm s}\underline{w}\underline{n}_{TF})_F\end{aligned}$$

with rigid-body motions prescribed from $\underline{\mathsf{v}}$

- SPD linear system of size $d\binom{k+1+d}{k+1}$ (12 for $d = 2$ and $k = 1$)

Displacement gradient reconstruction II

Lemma (Optimal approximation properties for r_T^k)

Let $T \in \mathcal{T}_h$ and define the local interpolator $I_T^k : H^1(T)^d \to \underline{\mathsf{U}}_T^k$ s.t.,

$$\forall \underline{v} \in H^1(T)^d, \qquad I_T^k \underline{v} = \left(\pi_T^k \underline{v}, (\pi_F^k \underline{v})_{F \in \mathcal{F}_T}\right) \in \underline{\mathsf{U}}_T^k$$

Then, for all $\underline{u} \in H^{k+2}(T)^d$ with $\widehat{\underline{\mathsf{u}}} := I_T^k \underline{u}$, it holds

$$\boxed{\|\underline{r}_T^k \widehat{\underline{\mathsf{u}}} - \underline{u}\|_T + h_T \|\nabla_{\mathrm{s}}(\underline{r}_T^k \widehat{\underline{\mathsf{u}}} - \underline{u})\|_T \lesssim h_T^{k+2} \|\underline{u}\|_{H^{k+2}(T)^d}.}$$

Symmetric gradient reconstruction I

- We define the symmetric gradient reconstruction operator

$$\underline{\underline{E}}_T^k : \underline{\mathsf{U}}_T^k \to \nabla_{\mathrm{s}} \mathbb{P}_d^{k+1}(T)^d$$

s.t., for all $\underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$,

$$\boxed{\underline{\underline{E}}_T^k \underline{\mathsf{v}} := \nabla_{\mathrm{s}} \underline{r}_T^k \underline{\mathsf{v}}}$$

- We wish stability of $\underline{\underline{E}}_T^k$ in the following discrete strain (semi-)norm

$$\|\underline{\mathsf{v}}\|_{\epsilon,T}^2 := \|\nabla_{\mathrm{s}} \mathsf{v}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\mathsf{v}_F - \mathsf{v}_T\|_F^2$$

- Stabilization should preserve the approximation properties of $\underline{\underline{E}}_T^k$

Symmetric gradient reconstruction II

- Define, for $T \in \mathcal{T}_h$, the stabilization bilinear form s_T as

$$s_T(\underline{\mathsf{u}},\underline{\mathsf{v}}) := \sum_{F\in\mathcal{F}_T} h_F^{-1}(\pi_F^k(\underline{R}_T^k\underline{\mathsf{u}} - \underline{\mathsf{u}}_F), \pi_F^k(\underline{R}_T^k\underline{\mathsf{v}} - \underline{\mathsf{v}}_F))_F,$$

with local displacement reconstruction $\underline{R}_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$ s.t.

$$\forall \underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k, \qquad \underline{R}_T^k\underline{\mathsf{v}} := \underline{\mathsf{v}}_T + (\underline{r}_T^k\underline{\mathsf{v}} - \pi_T^k \underline{r}_T^k\underline{\mathsf{v}})$$

where $\underline{\mathsf{v}}_T$ is perturbed using the highest-order part of $\underline{r}_T^k\underline{\mathsf{v}}$

- Then, using $k \geqslant 1$ and a local Korn's inequality, we can prove

$$\|\underline{\mathsf{v}}\|_{\varepsilon,T}^2 \lesssim \|\underline{\underline{E}}_T^k\underline{\mathsf{v}}\|_T^2 + s_T(\underline{\mathsf{v}},\underline{\mathsf{v}}) \lesssim \|\underline{\mathsf{v}}\|_{\varepsilon,T}^2$$

Symmetric gradient reconstruction III

- Key point: s_T preserves the approximation properties of $\underline{\underline{E}}_T^k$
- Let $u \in H^{k+2}(T)$ and set $\underline{\hat{u}} := \underline{I}_T^k u = \big(\pi_T^k u, (\pi_F^k u)_{F\in\mathcal{F}_T}\big)$
- Then, it holds

$$\begin{aligned}\|\pi_F^k(\underline{R}_T^k\underline{\hat{u}} - \hat{u}_F)\|_F &= \|\pi_F^k\big(\pi_T^k u + \underline{r}_T^k\underline{\hat{u}} - \pi_T^k\underline{r}_T^k\underline{\hat{u}} - \pi_F^k u\big)\|_F \\ &\leqslant \|\pi_F^k\big(\underline{r}_T^k\underline{\hat{u}} - u\big)\|_F + \|\pi_T^k\big(u - \underline{r}_T^k\underline{\hat{u}}\big)\|_F \\ &\lesssim h_T^{-1/2}\|\underline{r}_T^k\underline{\hat{u}} - u\|_T\end{aligned}$$

which, recalling the approximation properties of $\underline{\underline{E}}_T^k$ and $\underline{r}_T^k$, yields

$$\boxed{\left\{\|\underline{\underline{E}}_T^k\underline{\hat{u}} - \nabla_{\mathrm{s}} u\|_T^2 + s_T(\underline{\hat{u}},\underline{\hat{u}})\right\}^{1/2} \lesssim h_T^{k+1}\|u\|_{H^{k+2}(T)}}$$

Divergence reconstruction

- We define the local local discrete divergence operator

$$D_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^k(T)$$

s.t., for all $\underline{\mathsf{v}} = \big(\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\mathsf{U}}_T^k$ and all $q \in \mathbb{P}_d^k(T)$,

$$\boxed{(D_T^k \underline{\mathsf{v}}, q)_T := -(\mathsf{v}_T, \nabla q)_T + \sum_{F\in\mathcal{F}_T} (\mathsf{v}_F \cdot \underline{n}_{TF}, q)_F}$$

- The following diagram commutes and I_T^k is a Fortin operator:

$$\begin{array}{ccc} \underline{U}(T) & \xrightarrow{\nabla\cdot} & L^2(T) \\ {\scriptstyle I_T^k}\downarrow & & \downarrow{\scriptstyle \pi_T^k} \\ \underline{\mathsf{U}}_T^k & \xrightarrow{D_T^k} & \mathbb{P}_d^k(T) \end{array}$$

Discrete problem

- We define the local bilinear form a_T on $\underline{U}_T^k \times \underline{U}_T^k$ as

$$\boxed{a_T(\underline{u},\underline{v}) := 2\mu\left\{(\underline{\underline{E}}_T^k\underline{u},\underline{\underline{E}}_T^k\underline{v})_T + s_T(\underline{u},\underline{v})\right\} + \lambda(D_T^k\underline{u},D_T^k\underline{v}),}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{a_h(\underline{u}_h,\underline{v}_h) := \sum_{T\in\mathcal{T}_h} a_T(L_T\underline{u}_h, L_T\underline{v}_h) = \sum_{T\in\mathcal{T}_h} (\underline{f},\underline{v}_T)_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

with L_T restriction operator and bc strongly enforced considering

$$\underline{U}_{h,0}^k := \left\{\underline{v}_h = \left((\underline{v}_T)_{T\in\mathcal{T}_h},(\underline{v}_F)_{F\in\mathcal{F}_h}\right) \in \underline{U}_h^k \mid \underline{v}_F \equiv \underline{0}\ \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\}$$

- Well-posedness follows observing that, $\forall \underline{v}_h \in \underline{U}_{h,0}^k$,

$$(2\mu)\sum_{T\in\mathcal{T}_h} \|L_T\underline{v}_h\|_{\varepsilon,T}^2 \lesssim a_h(\underline{v}_h,\underline{v}_h) := \|\underline{v}_h\|_{\mathrm{en},h}^2$$

Convergence results I

Theorem (Convergence)

Let $k \geqslant 1$, *set*

$$\widehat{\underline{\mathsf{u}}}_h := \left((\pi_T^k \underline{u})_{T\in\mathcal{T}_h}, (\pi_F^k \underline{u})_{F\in\mathcal{F}_h}\right) \in \underline{\mathsf{U}}_{h,0}^k,$$

and assume $\underline{u} \in H^{k+2}(\mathcal{T}_h)^d$ *and* $\underline{\underline{\sigma}} \in H^{k+1}(\mathcal{T}_h)^{d\times d}$. *Then,*

$$\boxed{(2\mu)^{1/2}\|\underline{\mathsf{u}}_h - \widehat{\underline{\mathsf{u}}}_h\|_{\mathrm{en},h} \leqslant Ch^{k+1}\left(2\mu\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \lambda\|\nabla\cdot\underline{u}\|_{H^{k+1}(\mathcal{T}_h)}\right),}$$

with C *independent of* h, μ, *and* λ. *Hence, the method is locking-free provided the usual regularity shift holds.*

Convergence results II

Theorem (Supercloseness of the displacement)

Further assuming elliptic regularity, the following holds:

$$\left\{ \sum_{T\in\mathcal{T}_h} \|\widehat{\mathsf{u}}_T - \mathsf{u}_T\|_T^2 \right\}^{1/2} \lesssim h^{k+2}\big(2\mu\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \lambda\|\nabla\cdot\underline{u}\|_{H^{k+1}(\mathcal{T}_h)}\big).$$

Corollary (L^2-error estimate for $\underline{r}_T^k\underline{\mathsf{u}}_h$ and $\underline{R}_T^k\underline{\mathsf{u}}_h$)

Under the same assumptions, we have

$$\|\underline{u} - \check{\underline{u}}_h\| \lesssim h^{k+2}\big(2\mu\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \lambda\|\nabla\cdot\underline{u}\|_{H^{k+1}(\mathcal{T}_h)}\big),$$

where, for all $T \in \mathcal{T}_h$,

$$\check{\underline{u}}_{h|T} = \underline{r}_T^k \underline{I}_T^k \underline{u} \quad \text{or} \quad \check{\underline{u}}_{h|T} = \underline{R}_T^k \underline{I}_T^k \underline{u}.$$

Numerical validation I

- We consider the following exact solution:

$$\underline{u}\big(\sin(\pi x_1)\sin(\pi x_2) + (2\lambda)^{-1}x_1, \cos(\pi x_1)\cos(\pi x_2) + (2\lambda)^{-1}x_2\big)$$

- The solution u has vanishing divergence in the limit $\lambda \to +\infty$

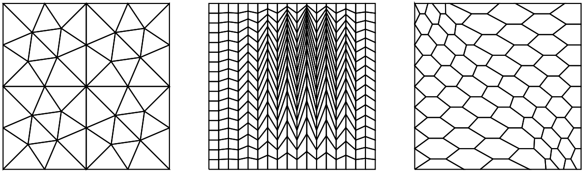


Figure : Meshes for the numerical example

Numerical validation II

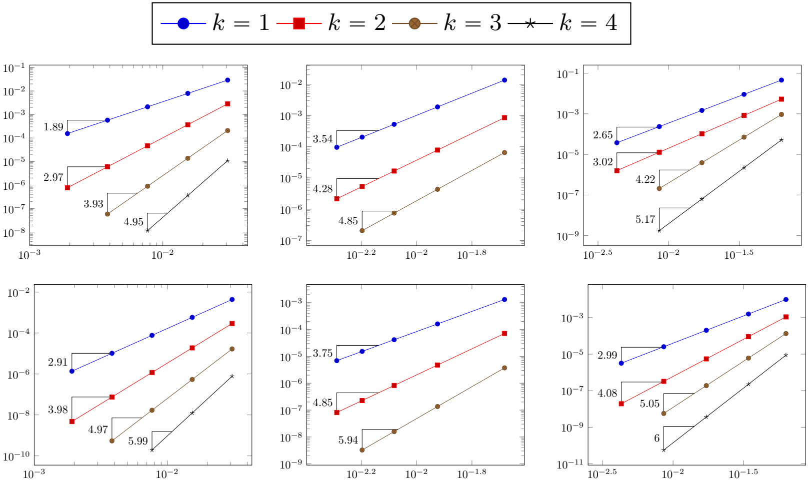


Figure : Energy (above) and displacement (below) errors vs. h for $\lambda = 1$

Numerical validation III

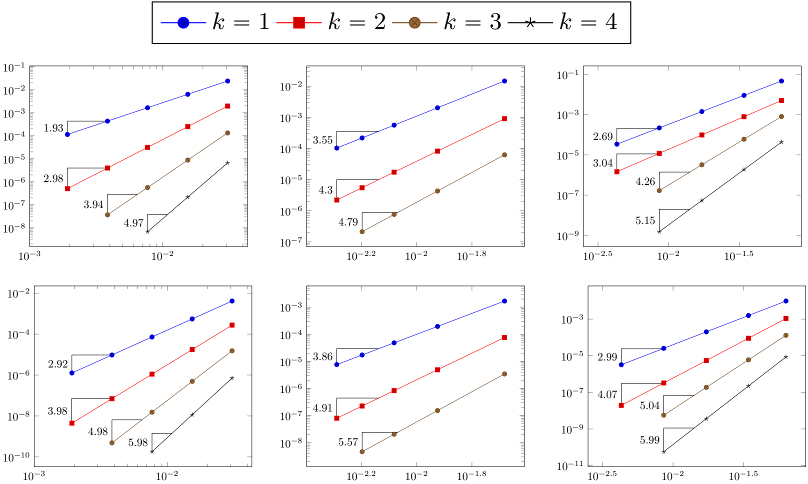


Figure : Energy (above) and displacement (below) errors vs. h for $\lambda = 1000$

Numerical validation IV

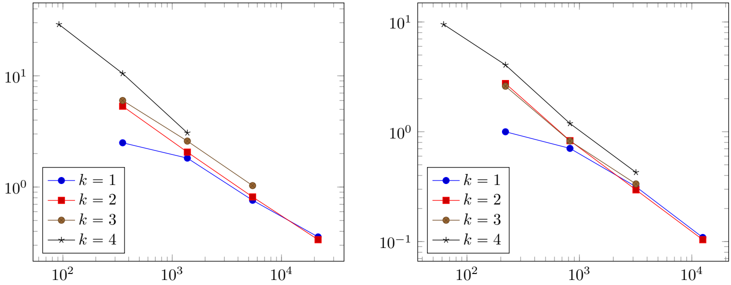


Figure : $\tau_{\text{ass}}/\tau_{\text{sol}}$ vs. $\text{card}(\mathcal{F}_h)$ for the triangular (left) and hexagonal (right) mesh families

Numerical validation V

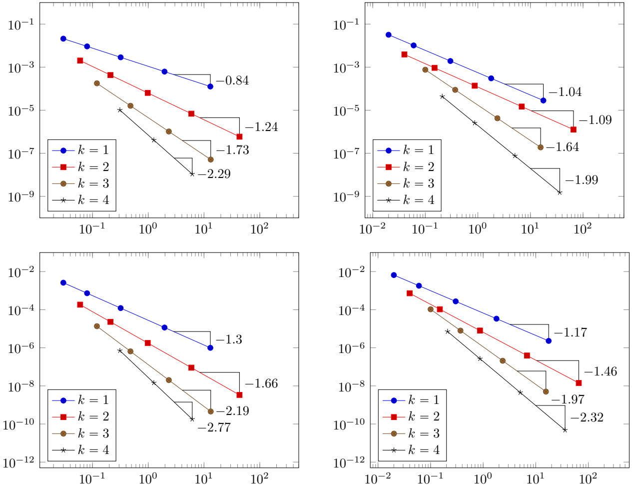


Figure : Energy (above) and displacement (below) error vs. τ_{tot} (s) for the triangular and hexagonal mesh families

Cook's membrane test case I

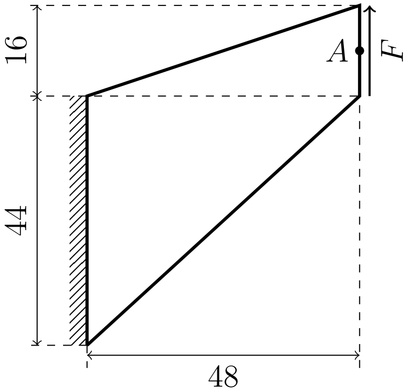
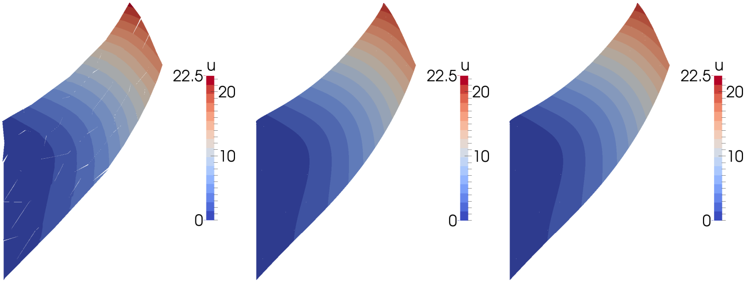


Figure : Cook's membrane test case ($\mu = 0.375$, $\lambda = 7.5 \cdot 10^6$)

Cook's membrane test case II



(a) 22 elements (b) 4,192 elements (c) 4,192 elements

Figure : Deformed configuration for the coarsest, intermediate, and finest hexagonal meshes, $k = 1$. The color represents the magnitude of the displacement field.

Cook's membrane test case III

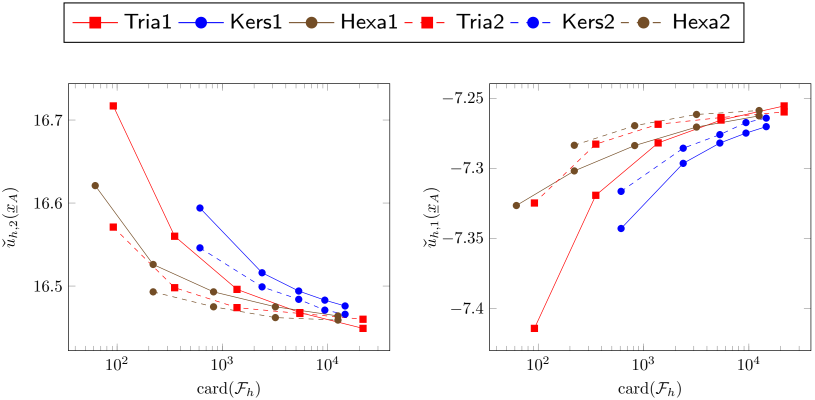


Figure : Vertical (left) and horizontal (right) displacement at A