Hybrid High-Order methods on general meshes

Daniele A. Di Pietro

Institut Montpelliérain Alexander Grothendieck, University of Montpellier

ZHACM Colloquium, 20 April 2016

Outline

Polyhedral methods for Advection-Diffusion-Reaction

- Discontinuous Galerkin (DG)
  - PDEs with nonnegative char. form [Houston, Schwab, Süli, 2002]
  - Locally degenerate ADR [DP, Ern, Guermond 2008]
- Hybridizable Discontinuous Galerkin (HDG)
  - Pure diffusion [Cockburn et al., 2009]
  - Diffusion-dominated ADR [Chen and Cockburn, 2014]
- Virtual elements (VEM)
  - Pure diffusion [Beirão da Veiga et al., 2013]
  - Diffusion-dominated ADR [Beirão da Veiga et al., 2016]
- Hybrid High-Order (HHO)
  - Pure diffusion [DP, Ern, Lemaire, 2014]
  - Locally degenerate ADR [DP, Ern, Droniou, 2015]
  - HHO as HDG on steroids [Cockburn, DP, Ern, 2015]
- **Link with residual distribution schemes [Abgrall et al., 2014]?**

Features of HHO

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Reproduction of desirable continuum properties
  - Integration by parts formulas
  - Kernels of operators
  - Symmetries
- Reduced computational cost after hybridization

$$N_{\text{dof}}^{\text{hho}} \approx \frac{1}{2}k^2 \operatorname{card}(\mathcal{F}_h) \qquad N_{\text{dof}}^{\text{dg}} \approx \frac{1}{6}k^3 \operatorname{card}(\mathcal{T}_h)$$

Outline

Mesh regularity I

Definition (Mesh regularity)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the sense of Ciarlet;
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences:

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces
- See [Di Pietro and Droniou, 2015] for functional analytic results

Mesh regularity II

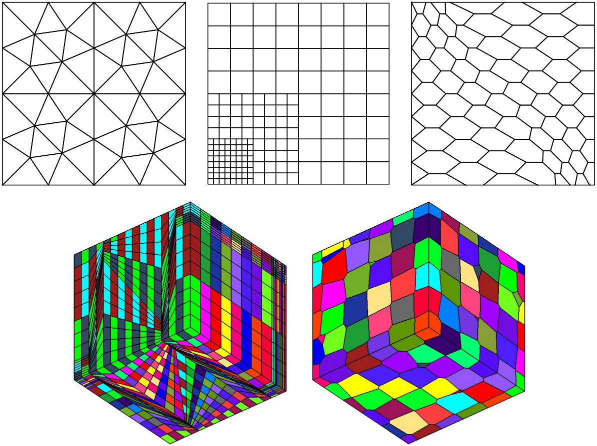


Figure: Admissible meshes in 2d and 3d: [Herbin and Hubert, 2008, FVCA5] and [Di Pietro and Lemaire, 2015] (above) and [Eymard et al., 2011, FVCA6] (below)

Model problem

- Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, denote a bounded, connected polyhedral domain
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$\begin{aligned} -\triangle u = f & \qquad \text{in } \Omega \\ u = 0 & \qquad \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

In a nutshell

- DOFs: polynomials of degree $k \geqslant 0$ at elements and faces
- Differential operators reconstructions tailored to the problem:

$$\boxed{a_{|T}(u,v) \approx (\boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{u}_T, \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{v}_T) + \text{stabilization}}$$

  with
  - high-order reconstruction p_T^{k+1} from local Neumann solves
  - stabilization via face-based penalty
- Construction yielding supercloseness on general meshes

DOFs

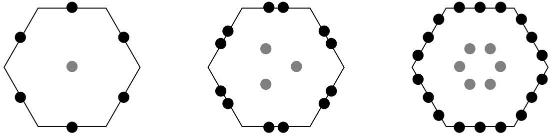


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- For $k \geqslant 0$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

- The corresponding global space has single-valued interface DOFs

$$\underline{U}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right)$$

Local potential reconstruction I

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$\mathrm{p}_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$$

is s.t. $\forall \underline{v}_T \in \underline{U}_T^k$, $(\mathrm{p}_T^{k+1}\underline{v}_T - v_T, 1)_T = 0$ and $\forall w \in \mathbb{P}^{k+1}(T)$,

$$\boxed{(\boldsymbol{\nabla}\mathrm{p}_T^{k+1}\underline{v}_T, \boldsymbol{\nabla}w)_T := -(v_T, \triangle w)_T + \sum_{F\in\mathcal{F}_T}(v_F, \boldsymbol{\nabla}w\cdot\boldsymbol{n}_{TF})_F}$$

- SPD linear system of size

$$N_{k,d} := \binom{k+1+d}{k+1} - 1$$

Local potential reconstruction II

k	$d = 1$	$d = 2$	$d = 3$
0	2	3	4
1	3	6	10
2	4	10	20
3	5	15	35

Table: Size $N_{k,d}$ of the local matrix to invert to compute $\mathrm{p}_T^{k+1}\underline{v}_T$

Local potential reconstruction III

Lemma (Approximation properties for $\mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k$)

Define the local interpolator $\underline{\mathrm{I}}_T^k : H^1(T) \to \underline{U}_T^k$ *s.t.*

$$\underline{\mathrm{I}}_T^k v = \left(\pi_T^k v, (\pi_F^k v)_{F\in\mathcal{F}_T}\right).$$

Then, $(\mathrm{p}_T^{k+1} \circ \underline{\mathrm{I}}_T^k)$ *has optimal approximation properties. In particular, for all* $T \in \mathcal{T}_h$ *and all* $v \in H^{k+2}(T)$*, it holds*

$$\|v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v\|_T + h_T\|\boldsymbol{\nabla}(v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v)\|_T \lesssim h_T^{k+2}\|v\|_{k+2,T}.$$

Local potential reconstruction IV

- Since $\triangle w \in \mathbb{P}^{k-1}(T)$ and $\nabla w_{|F}\cdot\boldsymbol{n}_{TF} \in \mathbb{P}^k(F)$,

$$\begin{aligned}(\nabla \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v, \nabla w)_T &= -(\pi_T^k v, \triangle w)_T + \sum_{F\in\mathcal{F}_T}(\pi_F^k v, \nabla w\cdot\boldsymbol{n}_{TF})_F \\ &= -(v, \triangle w)_T + \sum_{F\in\mathcal{F}_T}(v, \nabla w\cdot\boldsymbol{n}_{TF})_F \\ &= (\nabla v, \nabla w)_T\end{aligned}$$

- This shows that $(\mathrm{p}_T^{k+1}\circ\underline{\mathrm{I}}_T^k)$ is the elliptic projector on $\mathbb{P}^{k+1}(T)$:

$$(\nabla(\mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v - v), \nabla w)_T = 0 \qquad \forall w \in \mathbb{P}^{k+1}(T)$$

- The approximation properties follow using the Dupont-Scott theory

Stabilization I

- We would be tempted to approximate

$$a_{|T}(u,v) \approx (\nabla \mathrm{p}_T^{k+1}\underline{u}_T, \nabla \mathrm{p}_T^{k+1}\underline{v}_T)_T$$

- However, this choice is not stable in general
- We remedy by adding a local stabilization term

$$a_{|T}(u,v) \approx a_T(\underline{u}_T,\underline{v}_T) := (\nabla \mathrm{p}_T^{k+1}\underline{u}_T, \nabla \mathrm{p}_T^{k+1}\underline{v}_T)_T + s_T(\underline{u}_T,\underline{v}_T)$$

- Coercivity and boundedness are expressed w.r.t. to the seminorm

$$\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_T^2 + \sum_{F\in\mathcal{F}_T} \frac{1}{h_F}\|v_F - v_T\|_F^2$$

Stabilization II

- For all $T \in \mathcal{T}_h$, define the stabilization bilinear form

$$\boxed{s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\delta_{TF}^k \underline{u}_T, \delta_{TF}^k \underline{v}_T)_F}$$

with face-based residual operator $\delta_{TF}^k : \underline{U}_T^k \to \mathbb{P}^k(F)$ s.t.

$$\delta_{TF}^k \underline{v}_T := \pi_F^k (\mathrm{p}_T^{k+1} \underline{v}_T - v_F) - \pi_T^k (\mathrm{p}_T^{k+1} \underline{v}_T - v_T)$$

- With this choice, a_T satisfies for all $\underline{v}_T \in \underline{U}_T^k$,

$$\|\underline{v}_h\|_{1,T}^2 \lesssim a_T(\underline{v}_T, \underline{v}_T) \lesssim \|\underline{v}_T\|_{1,T}^2$$

Stabilization III

- Key point: s_T preserves the approximation properties of $\boldsymbol{\nabla}\mathrm{p}_T^{k+1}$
- For all $v \in H^{k+2}(T)$, letting

$$\widehat{\underline{v}}_T := \underline{\mathrm{I}}_T^k v = \left(\pi_T^k v, (\pi_F^k v)_{F\in\mathcal{F}_T}\right),$$

we have

$$\begin{aligned}\|\delta_{TF}^k \widehat{\underline{v}}_T\|_F &= \|\pi_F^k(\mathrm{p}_T^{k+1}\widehat{\underline{v}}_T - \pi_F^k v) - \pi_T^k(\mathrm{p}_T^{k+1}\widehat{\underline{v}}_T - \pi_T^k v)\|_F \\ &= \|\pi_F^k(\mathrm{p}_T^{k+1}\widehat{\underline{v}}_T - v) - \pi_T^k(\mathrm{p}_T^{k+1}\widehat{\underline{v}}_T - v)\|_F \\ &\lesssim h_T^{-1/2}\|\mathrm{p}_T^{k+1}\widehat{\underline{v}}_T - v\|_T\end{aligned}$$

- Recalling the approximation properties of p_T^{k+1}, this yields

$$\boxed{\left(\|\boldsymbol{\nabla}(\mathrm{p}_T^{k+1}\widehat{\underline{v}}_T - v)\|_T^2 + s_T(\widehat{\underline{v}}_T, \widehat{\underline{v}}_T)\right)^{1/2} \lesssim h_T^{k+1}\|v\|_{k+2,T}}$$

Discrete problem

- We enforce boundary conditions strongly considering the space

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \mid v_F \equiv 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_h} (f, v_T)_T \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

- Well-posedness follows from the coercivity of a_h

Convergence I

Theorem (Energy-norm error estimate)

Assume $u \in H^{k+2}(\mathcal{T}_h)$ *and let*

$$\underline{\widehat{u}}_h := \left((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h}\right) \in \underline{U}_{h,0}^k.$$

We have the following energy error estimate:

$$\boxed{\|\underline{u}_h - \underline{\widehat{u}}_h\|_{1,h} \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)},}$$

with H^1*-like norm on* $\underline{U}_{h,0}^k$ *given by*

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2.$$

Convergence II

Theorem (L^2-norm error estimate)

Further assuming elliptic regularity and $f \in H^1(\Omega)$ if $k = 0$,

$$\boxed{\max\left(\|\check{u}_h - u\|, \|\widehat{u}_h - u_h\|\right) \lesssim h^{k+2}\mathcal{N}_k,}$$

with $\mathcal{N}_0 := \|f\|_{H^1(\Omega)}$, $\mathcal{N}_k := \|u\|_{H^{k+2}(\mathcal{T}_h)}$ for $k \geqslant 1$, and

$$\forall T \in \mathcal{T}_h, \qquad \check{u}_{h|T} := \mathrm{p}_T^{k+1}\underline{u}_T, \quad \widehat{u}_{h|T} := \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k u, \quad u_{h|T} := u_T.$$

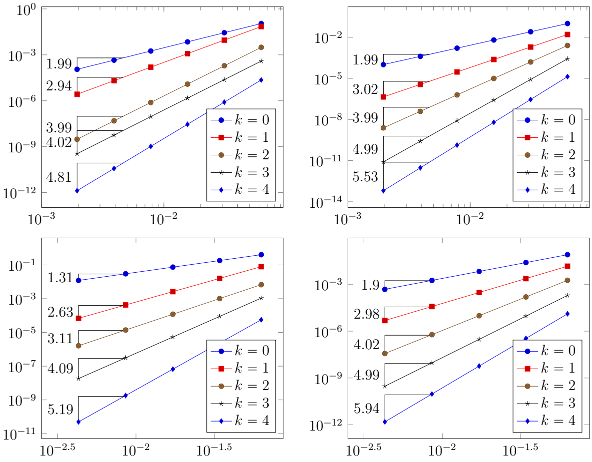


Figure: 2d test case, trigonometric solution. Energy (left) and L^2-norm (right) of the error vs. h for uniformly refined triangular (top) and hexagonal (bottom) mesh families

Numerical examples I

3d industrial test case, adaptive refinement, cost assessment

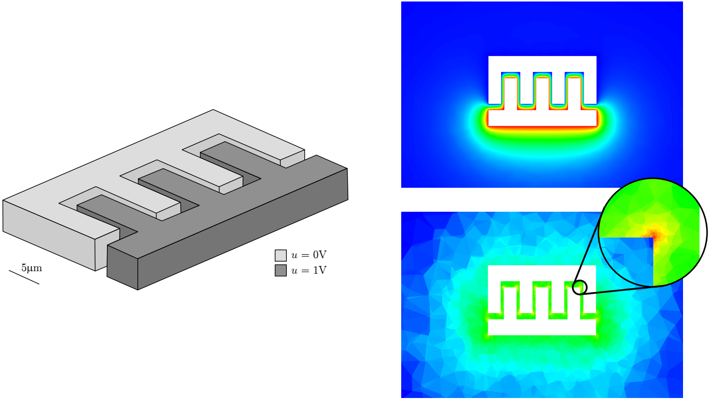
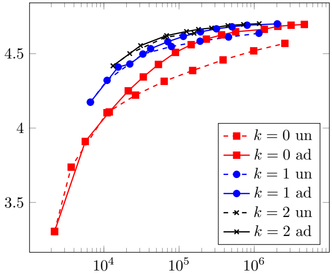


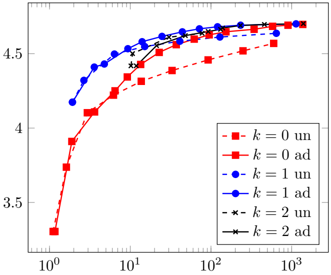
Figure: Geometry (lef), numerical solution (right, top) and final adaptive mesh (right, bottom) for the comb-drive actuator test case [Di Pietro & Specogna, 2016]

Numerical examples II

3d industrial test case, adaptive refinement, cost assessment



(a) Capacitance vs. ndofs

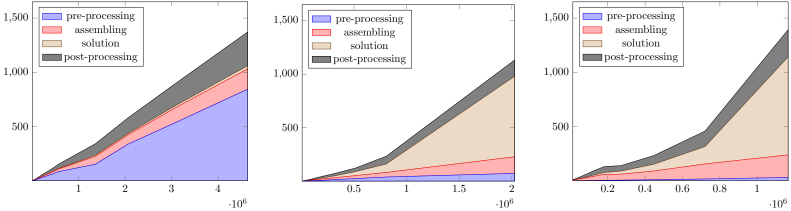


(b) Capacitance vs. computing time

Figure: Results for the comb drive benchmark.

Numerical examples III

3d industrial test case, adaptive refinement, cost assessment



(a) $k = 0$ (b) $k = 1$ (c) $k = 2$

Figure: Computing wall time (s) vs. number of DOFs for the comb drive benchmark.

Numerical examples I

3d test case, singular solution, adaptive coarsening

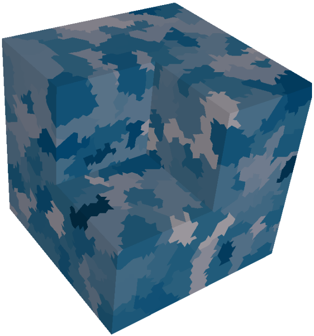
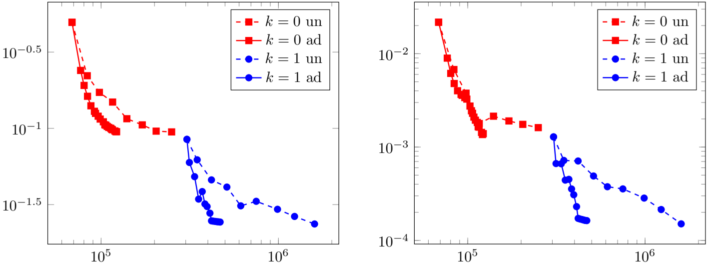


Figure: Fichera corner benchmark, adaptive mesh coarsening [Di Pietro & Specogna, 2016]

Numerical examples II

3d test case, singular solution, adaptive coarsening



(a) Energy-error vs. ndofs

(b) L^2-error vs. ndofs

Figure: Error vs. number of DOFs for the Fichera corner benchmark, adaptively coarsened meshes

Outline

Variable diffusion I

- Let $\boldsymbol{\kappa} : \Omega \to \mathbb{R}^{d\times d}$ be a polyomial SPD tensor-valued field
- We consider the Darcy problem

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\boldsymbol{\nabla}u) &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) := (\boldsymbol{\kappa}\boldsymbol{\nabla}u, \boldsymbol{\nabla}v) = (f,v) \qquad \forall v \in H_0^1(\Omega)$$

- We confer built-in $\boldsymbol{\kappa}$-dependence to p_T^{k+1}

$$\boxed{(\boldsymbol{\kappa}\boldsymbol{\nabla}\mathrm{p}_T^{k+1}\underline{v}_T, \boldsymbol{\nabla}w)_T = (\boldsymbol{\kappa}\boldsymbol{\nabla}v_T, \boldsymbol{\nabla}w)_T + \sum_{F\in\mathcal{F}_T}(v_F - v_T, \boldsymbol{\kappa}\boldsymbol{\nabla}w\cdot\boldsymbol{n}_{TF})_F}$$

Variable diffusion II

Lemma (Approximation properties of $\mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k$)

There is C independent of h_T and $\boldsymbol{\kappa}$ s.t., for all $v \in H^{k+2}(T)$, it holds with $\alpha = \frac{1}{2}$ if $\boldsymbol{\kappa}$ is piecewise constant and $\alpha = 1$ otherwise:

$$\boxed{\|v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v\|_T + h_T \|\boldsymbol{\nabla}(v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v)\|_T \leqslant C \rho_T^{\alpha} h_T^{k+2} \|v\|_{k+2,T},}$$

with heterogeneity/anisotropy ratio

$$\rho_T := \frac{\kappa_T^\sharp}{\kappa_T^\flat} \geqslant 1.$$

Discrete problem and convergence I

- We define the local bilinear form $a_{\boldsymbol{\kappa},T}$ on $\underline{U}_T^k \times \underline{U}_T^k$ as

$$a_{\boldsymbol{\kappa},T}(\underline{u}_T, \underline{v}_T) := (\boldsymbol{\kappa}\boldsymbol{\nabla}\mathrm{p}_T^{k+1}\underline{u}_T, \boldsymbol{\nabla}\mathrm{p}_T^{k+1}\underline{v}_T)_T + s_{\boldsymbol{\kappa},T}(\underline{u}_T, \underline{v}_T)$$

where, letting $\kappa_F := \|\boldsymbol{n}_{TF}\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF}\|_{L^\infty(F)}$,

$$s_{\boldsymbol{\kappa},T}(\underline{u}_T, \underline{v}_T) := \sum_{F\in\mathcal{F}_T} \frac{\kappa_F}{h_F}(\delta_{TF}^k\underline{u}_T, \delta_{TF}^k\underline{v}_T)_F$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{a_{\boldsymbol{\kappa},h}(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} a_{\boldsymbol{\kappa},T}(\underline{u}_T, \underline{v}_T) = \sum_{T\in\mathcal{T}_h} (f, v_T)_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

Discrete problem and convergence II

Theorem (Energy-error estimate)

Assume that $u \in H^{k+2}(\mathcal{T}_h)$. *Then, with*

$$\underline{\widehat{u}}_h := \left((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h}\right) \in \underline{U}_{h,0}^k,$$

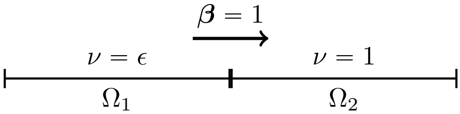
and α *as above,*

$$\|\underline{\widehat{u}}_h - \underline{u}_h\|_{\boldsymbol{\kappa},h} \lesssim \left(\sum_{T\in\mathcal{T}_h} \kappa_T^\sharp \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{k+2,T}^2\right)^{1/2}.$$

Outline

Degenerate diffusion-advection-reaction I

- Let us start with the following 1d problem:



- As $\epsilon \to 0^+$, a boundary layer develops at $x = 1/2$
- When $\epsilon = 0$, it turns into a jump discontinuity
- This was already observed in [Gastaldi and Quarteroni, 1989]

Degenerate diffusion-advection-reaction II

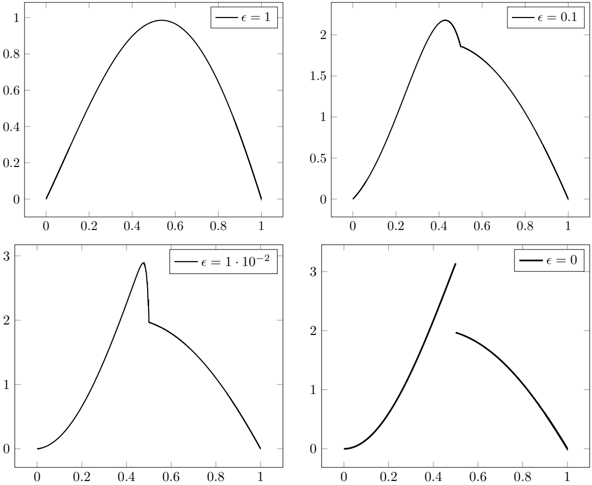


Figure: Solutions for different values of ϵ

Degenerate diffusion-advection-reaction III

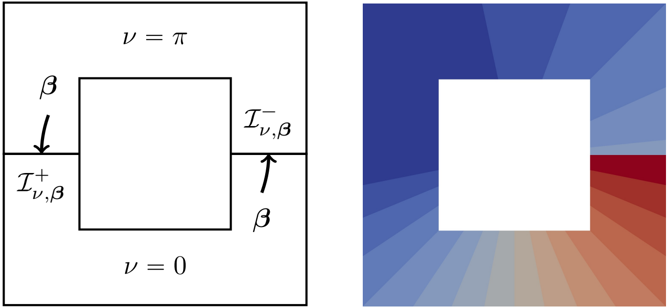


Figure: Example of degenerate diffusion-advection-reaction problem in 2d from [Di Pietro et al., 2008]. The diffusive/non-diffusive interface is $\mathcal{I}_{\nu,\boldsymbol{\beta}} := \mathcal{I}^{-}_{\nu,\boldsymbol{\beta}} \cup \mathcal{I}^{+}_{\nu,\boldsymbol{\beta}}$.

Degenerate diffusion-advection-reaction IV

- Define the diffusive/inflow portion of $\partial\Omega$

$$\Gamma_{\nu,\boldsymbol{\beta}} := \{\boldsymbol{x} \in \partial\Omega \mid \nu > 0 \text{ or } \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\}$$

- Consider the possibly degenerate problem

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}(u) + \mu u &= f && \text{in } \Omega\backslash\mathcal{I}_{\nu,\boldsymbol{\beta}}, \\ \boldsymbol{\Phi}(u) &= -\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u && \text{in } \Omega, \\ u &= g && \text{on } \Gamma_{\nu,\boldsymbol{\beta}}, \end{aligned}$$

  with $\boldsymbol{\beta} \in \mathrm{Lip}(\Omega)^d$ s.t. $\boldsymbol{\nabla}\cdot\boldsymbol{\beta} = 0$, $\mu > 0$

- On $\mathcal{I}_{\nu,\boldsymbol{\beta}}$, we enforce the interface conditions

$$[\![\boldsymbol{\Phi}(u)]\!]\cdot\boldsymbol{n}_I = 0 \text{ on } \mathcal{I}_{\nu,\boldsymbol{\beta}} \quad \text{and} \quad [\![u]\!] = 0 \text{ on } \mathcal{I}^+_{\nu,\boldsymbol{\beta}}$$

Key ideas

- Discrete advective derivative satisfying a discrete IBP formula
- Weakly enforced boundary conditions
  - Extension of Nietsche's ideas to HHO
  - Automatic detection of $\Gamma_{\nu,\boldsymbol{\beta}}$
- Upwind stabilization using cell- and face-unknowns
  - Independent control for the advective part
  - Consistency also on $\mathcal{I}^{-}_{\nu,\boldsymbol{\beta}}$, where u jumps

Features

- Polyhedral meshes and arbitrary approximation order $k \geqslant 0$
- Method valid for the full range of Peclet numbers
- Analysis capturing the variation in the order of convergence in the diffusion-dominated and advection-dominated regimes
- **No need to duplicate interface unknowns on $\mathcal{I}^{-}_{\nu,\boldsymbol{\beta}}$ (!)**

Advective derivative I

- The discrete advective derivative $\mathrm{G}^k_{\boldsymbol{\beta},T} : \underline{U}^k_T \to \mathbb{P}^k(T)$ is s.t.

$$\boxed{(\mathrm{G}^k_{\boldsymbol{\beta},T}\underline{v}_T, w)_T = -(v_T, \boldsymbol{\beta}\cdot\boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})v_F, w)_F}$$

  for all $\underline{v}_T \in \underline{U}^k_T$ and all $w \in \mathbb{P}^k(T)$
- For advective stability, we need a discrete IBP mimicking

$$(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} w, v)_\Omega + (w, \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_\Omega = ((\boldsymbol{\beta}\cdot\boldsymbol{n})w, v)_{\partial\Omega}$$

Advective derivative II

Lemma (Discrete IBP)

For all $\underline{w}_h, \underline{v}_h \in \underline{U}_h^k$ *it holds*

$$\sum_{T\in\mathcal{T}_h} \left((\mathrm{G}_{\boldsymbol{\beta},T}^k \underline{w}_T, v_T)_T + (w_T, \mathrm{G}_{\boldsymbol{\beta},T}^k \underline{v}_T)_T \right) = \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_F) w_F, v_F)_F - \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_h} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})(w_F - w_T), v_F - v_T)_F.$$

Diffusion I

- We modify the diffusion bilinear form to weakly enforce BCs
- The new bilinear form $a_{\nu,h}$ reads (after setting $\boldsymbol{\kappa} = \nu \boldsymbol{I}_d$),

$$a_{\nu,h}(\underline{w}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_{\nu,T}(\underline{w}_T, \underline{v}_T) + s_{\partial,\nu,h}(\underline{w}_h, \underline{v}_h)$$

with, for a user-defined parameter ς,

$$s_{\partial,\nu,h}(\underline{w}_h, \underline{v}_h) := \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} \left(-(\nu_F \boldsymbol{\nabla} \mathrm{p}_{T(F)}^k \underline{w}_T \cdot \boldsymbol{n}_{TF}, v_F)_F + \frac{\varsigma \nu_F}{h_F} (w_F, v_F)_F \right)$$

Diffusion II

Lemma (inf-sup stability of $a_{\nu,h}$)

Assuming that

$$\varsigma > \frac{C_{\mathrm{tr}}^2 N_\partial}{4}$$

it holds for all $\underline{v}_h \in \underline{U}_h^k$

$$a_{\nu,h}(\underline{v}_h, \underline{v}_h) =: \|\underline{v}_h\|_{\nu,h}^2 \simeq \sum_{T\in\mathcal{T}_h} \nu_T \|\underline{v}_T\|_{1,T}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \frac{\nu_F}{h_F} \|v_F\|_F^2.$$

Advection-reaction I

- For all $T \in \mathcal{T}_h$, we let

$$a_{\boldsymbol{\beta},\mu,T}(\underline{w}_T, \underline{v}_T) := -(w_T, \mathrm{G}^k_{\boldsymbol{\beta},T}\underline{v}_T)_T + \mu(w_T, v_T)_T + s^-_{\boldsymbol{\beta},T}(\underline{w}_T, \underline{v}_T)$$

  with local upwind stabilization bilinear form s.t.

$$s^-_{\boldsymbol{\beta},T}(\underline{w}_T, \underline{v}_T) := \sum_{F\in\mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^-(w_F - w_T), v_F - v_T)_F,$$

- Including weakly enforced BCs, we define

$$\boxed{a_{\boldsymbol{\beta},\mu,h}(\underline{w}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} a_{\boldsymbol{\beta},\mu,T}(\underline{w}_h, \underline{v}_h) + \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} ((\boldsymbol{\beta}\cdot\boldsymbol{n})^+ w_F, v_F)_F}$$

Advection-reaction II

Lemma (Stability of $a_{\boldsymbol{\beta},\mu,h}$)

Let $\eta := \min_{T\in\mathcal{T}_h}(1, \tau_{\mathrm{ref},T}\mu)$ *with* $\tau_{\mathrm{ref},T} := \max(\|\mu\|_{L^\infty(T)}, L_{\boldsymbol{\beta},T})^{-1}$*. Then,*

$$\forall \underline{v}_h \in \underline{U}_h^k, \qquad \eta\|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 \leqslant a_{\boldsymbol{\beta},\mu,h}(\underline{v}_h, \underline{v}_h),$$

with global advection-reaction norm

$$\|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{\boldsymbol{\beta},\mu,T}^2 + \frac{1}{2}\sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2} v_F\|_F^2,$$

and, for all $T \in \mathcal{T}_h$*,*

$$\|\underline{v}_T\|_{\boldsymbol{\beta},\mu,T}^2 := \frac{1}{2}\sum_{F\in\mathcal{F}_T} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2}(v_F - v_T)\|_F^2 + \tau_{\mathrm{ref},T}^{-1}\|v_T\|_T^2.$$

Discrete problem I

- Define the following RHS linear form accounting for BCs:

$$l_h(\underline{v}_h) := \sum_{T\in\mathcal{T}_h} (f, v_T)_T + \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \left(((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^- g, v_F)_F + \frac{\nu_F \varsigma}{h_F}(g, v_F)_F \right)$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_h^k$ s.t., $\forall \underline{v}_h \in \underline{U}_h^k$,

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := a_{\nu,h}(\underline{u}_h, \underline{v}_h) + a_{\boldsymbol{\beta},\mu,h}(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h)}$$

Discrete problem II

Lemma (Stability of a_h)

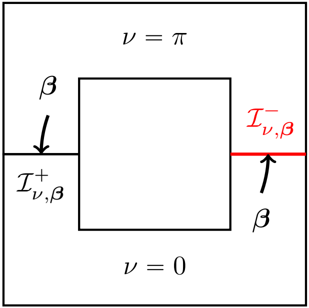
There is $\gamma_{\varrho,\varsigma} > 0$ independent of h, ν, $\boldsymbol{\beta}$ and μ s.t., for all $\underline{w}_h \in \underline{U}_h^k$,

$$\|\underline{w}_h\|_{\sharp,h} \leqslant \gamma_{\varrho,\varsigma}\zeta^{-1} \sup_{\underline{v}_h \in \underline{U}_h^k \setminus \{\underline{0}\}} \frac{a_h(\underline{w}_h, \underline{v}_h)}{\|\underline{v}_h\|_{\sharp,h}},$$

with $\zeta := \tau_{\mathrm{ref},T}\mu$ and stability norm

$$\|\underline{v}_h\|_{\sharp,h}^2 := \|\underline{v}_h\|_{\nu,h}^2 + \|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 + \sum_{T\in\mathcal{T}_h} h_T \beta_{\mathrm{ref},T}^{-1} \|\mathrm{G}_{\boldsymbol{\beta},T}^k \underline{v}_h\|_T^2.$$

A modified interpolator



- Let $F \in \mathcal{F}_h^{\mathrm{i}}$ be such that $F \subset \mathcal{I}^-_{\nu,\boldsymbol{\beta}}$
- The trace of u is two-valued on F
- We interpolate the face unknown **from the diffusive side**

Convergence I

Theorem (Error estimate)

Assume that, for all $T \in \mathcal{T}_h$, $u \in H^{k+2}(T)$ and

$$h_T L_{\boldsymbol{\beta},T} \leqslant \beta_{\mathrm{ref},T} \quad \text{and} \quad h_T \mu \leqslant \beta_{\mathrm{ref},T},$$

Then, there is $C > 0$ independent of h, ν, $\boldsymbol{\beta}$, and μ s.t.

$$\|\widehat{\underline{u}}_h - \underline{u}_h\|_{\sharp,h} \leqslant C \left(\sum_{T \in \mathcal{T}_h} \left[(\nu_T \|u\|_{k+2,T}^2 + \tau_{\mathrm{ref},T}^{-1} \|u\|_{k+1,T}^2) h_T^{2(k+1)} + \beta_{\mathrm{ref},T} \min(1, \mathrm{Pe}_T) h_T^{2(k+1/2)} \|u\|_{k+1,T}^2 \right] \right)^{1/2},$$

with local Peclet number $\mathrm{Pe}_T := \max_{F \in \mathcal{F}_T} \|\mathrm{Pe}_{TF}\|_{L^\infty(F)}$.

Convergence II

- This estimate holds across the entire range for Pe_T
- In the diffusion-dominated regime ($\mathrm{Pe}_T \leqslant h_T$), we have

$$\|\widehat{\underline{u}}_h - \underline{u}_h\|_{\sharp,h} = \mathcal{O}(h^{k+1})$$

- In the advection-dominated regime ($\mathrm{Pe}_T \geqslant 1$), we have

$$\|\widehat{\underline{u}}_h - \underline{u}_h\|_{\sharp,h} = \mathcal{O}(h^{k+1/2})$$

- In between, we have intermediate orders of convergence

Numerical example I

- Let $\Omega = (-1,1)^2 \backslash [-0.5, 0.5]^2$ and set

$$\nu(\theta, r) = \begin{cases} \pi & \text{if } 0 < \theta < \pi, \\ 0 & \text{if } \pi < \theta < 2\pi, \end{cases} \quad \boldsymbol{\beta}(\theta, r) = \frac{\boldsymbol{e}_\theta}{r}, \quad \mu = 1 \cdot 10^{-6}$$

- We consider the exact solution

$$u(\theta, r) = \begin{cases} (\theta - \pi)^2 & \text{if } 0 < \theta < \pi \\ 3\pi(\theta - \pi) & \text{if } \pi < \theta < 2\pi \end{cases}$$

Numerical example II

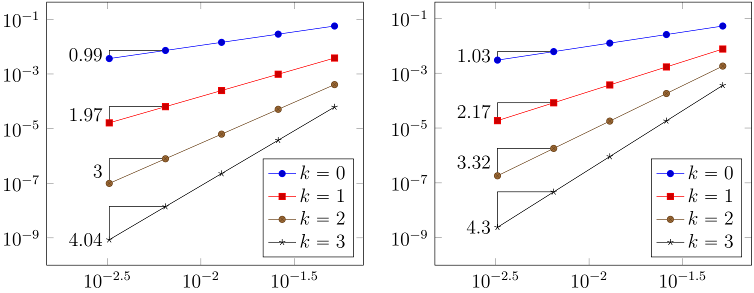


Figure: Energy (left) and L^2-norm (right) of the error vs. h

References I

Abgrall, R., Ricchiuto, M., and de Santis, D. (2014).
High-order preserving residual distribution schemes for advection-diffusion scalar problems on arbitrary grids.
SIAM J. Sci. Comput., 36(3):A955–A983.

Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L. D., and Russo, A. (2013).
Basic principles of virtual element methods.
Math. Models Methods Appl. Sci., 23:199–214.

Beirão da Veiga, L., Brezzi, F., Marini, L. D., and Russo, A. (2016).
Virtual Element Methods for general second order elliptic problems on polygonal meshes.

Chen, Y. and Cockburn, B. (2014).
Analysis of variable-degree HDG methods for convection-diffusion equations. part II: semimatching nonconforming meshes.
Math. Comp., 83(285):87–111.

Cockburn, B., Di Pietro, D. A., and Ern, A. (2015).
Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin methods.
ESAIM: Math. Model Numer. Anal. (M2AN).
Published online. DOI: 10.1051/m2an/2015051.

Cockburn, B., Gopalakrishnan, J., and Lazarov, R. (2009).
Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems.
SIAM J. Numer. Anal., 47(2):1319–1365.

Di Pietro, D. A. and Droniou, J. (2015).
A Hybrid High-Order method for Leray–Lions elliptic equations on general meshes.
Preprint arXiv:1508.01918.

Di Pietro, D. A., Droniou, J., and Ern, A. (2015).
A discontinuous-skeletal method for advection–diffusion–reaction on general meshes.
SIAM J. Numer. Anal., 53(5):2135–2157.

References II

Di Pietro, D. A. and Ern, A. (2015).
A hybrid high-order locking-free method for linear elasticity on general meshes.
Comput. Methods Appl. Mech. Engrg., 283:1–21.

Di Pietro, D. A., Ern, A., and Guermond, J.-L. (2008).
Discontinuous Galerkin methods for anisotropic semi-definite diffusion with advection.
SIAM J. Numer. Anal., 46(2):805–831.

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Methods Appl. Math., 14(4):461–472.

Di Pietro, D. A. and Lemaire, S. (2015).
An extension of the Crouzeix–Raviart space to general meshes with application to quasi-incompressible linear elasticity and Stokes flow.
Math. Comp., 84(291):1–31.

Dupont, T. and Scott, R. (1980).
Polynomial approximation of functions in Sobolev spaces.
Math. Comp., 34(150):441–463.

Eymard, R., Henry, G., Herbin, R., Hubert, F., Klöfkorn, R., and Manzini, G. (2011).
3D benchmark on discretization schemes for anisotropic diffusion problems on general grids.
In *Finite Volumes for Complex Applications VI - Problems & Perspectives*, volume 2, pages 95–130. Springer.

Gastaldi, F. and Quarteroni, A. (1989).
On the coupling of hyperbolic and parabolic systems: Analytical and numerical approach.
Appl. Numer. Math., 6:3–31.

References III

Herbin, R. and Hubert, F. (2008).
Benchmark on discretization schemes for anisotropic diffusion problems on general grids.
In Eymard, R. and Hérard, J.-M., editors, *Finite Volumes for Complex Applications V*, pages 659–692. John Wiley & Sons.

Houston, P., Schwab, C., and Süli, E. (2002).
Discontinuous hp-finite element methods for advection-diffusion-reaction problems.
SIAM J. Numer. Anal., 39(6):2133–2163.