A primal discontinuous Galerkin method with static condensation on very general meshes

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Finite element-type methods on general meshes

A far from exhaustive list

- Polynomial spaces on the mesh elements
  - IP – interior penalty discontinuous Galerkin (Baker'73, Wheeler'78, Arnold'82)
  - LDG – local discontinuous Galerkin (Cockburn & Shu'98)
- Polynomial spaces on the mesh skeleton (edges in 2D, faces in 3D)
  - HDG – hybridizable discontinuous Galerkin (Cockburn et al'09)
  - HHO – Hybrid High-Order (Di Pietro et al'14)
  - VEM – Virtual Elements (Beirão da Veiga et al'13)

The number of DOFs needed to achieve the accuracy $O(h^k)$ in the case of the diffusion problem:

	2D	3D
IP, LDG	$\sim \frac{1}{2}k^2 N_{elements}$	$\sim \frac{1}{6}k^3 N_{elements}$
HDG, HHO	$\sim kN_{edges}$	$\sim \frac{1}{2}k^2 N_{faces}$

(for large k)

Outline of the talk

- A reminder of the static condensation for the continuous FEM
- scSIP - a symmetric interior penalty DG method with static condensation for the diffusion problem

	2D	3D
scSIP	$(2k+1)N_{elements}$	$(k+1)^2 N_{elements}$
HDG, HHO	$\sim kN_{edges}$	$\sim \frac{1}{2}k^2 N_{faces}$

(AL, preprint arXiv 2018)

- *A priori* error estimates
- Numerical illustrations
- Admissible meshes
- An extension to Stokes

Governing equations and notations

- The diffusion equation in $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 with Dirichlet bc

$$-\partial_i(A_{ij}(x)\partial_j u) = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$

  assuming the summation over $i, j = 1, \dots, d$
- The differential operator $\mathcal{L}$ is defined by

$$\mathcal{L}u = -\partial_i(A_{ij}(x)\partial_j u)$$

- The bilinear form

$$a(u, v) := \int_\Omega A\nabla u \cdot \nabla v, \qquad A = (A_{ij})_{1\le i,j\le d}$$

- Assmptions on the coefficient matrix: $\exists 0 < \alpha \le \beta, M > 0$

$$\alpha|\xi|^2 \le \xi^T A(x)\xi \le \beta|\xi|^2, \quad \forall \xi \in \mathbb{R}^d,\ x \in \Omega$$

  and

$$|\nabla A_{ij}(x)| \le M, \quad \forall x \in \Omega,\ i, j = 1, \dots, d$$

Static condensation for continuous FEM
A reminder

- Assume (for the moment) $\Omega \subset \mathbb{R}^2$ a polygon, $\mathcal{T}_h$ a regular triangular mesh on Ω, and $g = 0$
- The usual continuous $\mathbb{P}^k$ FE on $\mathcal{T}_h$

$$V_h = \{v \in H_0^1(\Omega) : v_T \in \mathbb{P}^k(t) \quad \forall T \in \mathcal{T}_h\}$$

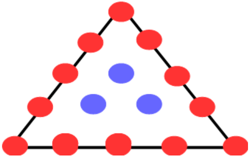
- The continuous FE solution

$$u_h \in V_h : \quad a(u_h, v_h) = \int_\Omega f v_h, \quad \forall v_h \in V_h$$

- If $k \geq 3$, the DOFs attached to interior interpolation nodes can be locally eliminated

Illustration for $k = 4$

- Global DOFs
- Local DOFs



- Decompose V_h into the local and global "features"

$$V_h = V_h^{loc} \oplus^{\perp_a} V_h'$$

$$V_h^{loc} = \{v \in V_h : v_{\partial T} = 0 \quad \forall T \in \mathcal{T}_h\}$$
$$V_h' = \{v' \in V_h : a(v, v') = 0 \quad \forall v \in V_h^{loc}\}$$

- Decomposition of the FE solution $u_h = \underbrace{u_h^{loc}}_{\in W_h^{loc}} + \underbrace{u_h'}_{\in W_h'}$

- Local and global problems

$$u_h^{loc} \in W_h^{loc} : \quad a(u_h^{loc}, v_h^{loc}) = \int_\Omega f v_h^{loc}, \quad \forall v_h^{loc} \in V_h^{loc}$$
$$u_h' \in W_h' : \quad a(u_h', v_h') = \int_\Omega f v_h', \quad \forall v_h' \in V_h' \qquad (1)$$

- The size of (1) is $\sim k$ in 2D ($\sim k^2$ in 3D) contrary to $\sim k^2$ ($\sim k^3$) for the original problem

Static condensation for continuous FEM

A reformulation in terms of orthogonal projections

- The local problems are solved separately on every triangle: $\forall T \in \mathcal{T}_h$

$$u_h^{loc,T} := u_h^{loc}|_T \in V_h^{loc,T} := \{v \in \mathbb{P}^k(T) : v|_{\partial T} = 0\}$$

  satisfies

$$\int_T A\nabla u_h^{loc,T} \cdot \nabla v_h^{loc,T} = \int_\Omega f v_h^{loc,T}, \quad \forall v_h^{loc,T} \in V_h^{loc,T}$$

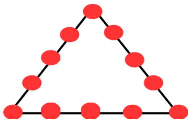
- With the orthogonal projection $\pi_T : L^2(T) \to V_h^{loc,T}$

$$\pi_T \mathcal{L}(u_h^{loc}|_T) = \pi_T f, \quad \forall T \in \mathcal{T}_h$$

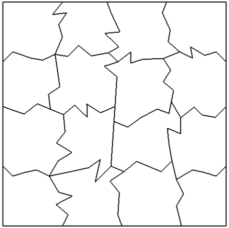
- The global subspace V_h' is populated by the solutions to

$$\pi_T \mathcal{L}(v_h'|_T) = 0, \quad \forall T \in \mathcal{T}_h$$

- The basis functions of V_h' can be associated to the nodes on the edges



- Let $\mathcal{T}_h$ be a general mesh on $\Omega \subset \mathbb{R}^d$ – a collection of non-overlapping subdomains;
  - here is an example in 2D
  - we tolerate the curved edges/faces as well



- Let $V_h = \{v \in L^2(\Omega) : v|_T \in \mathbb{P}^k(T), \forall T \in \mathcal{T}_h\}$
- The SIP method:
  find $u_h \in V_h$ such that $a_h(u_h, v_h) = L_h(v_h), \ \forall v_h \in V_h$ with

$$a_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T A\nabla u \cdot \nabla v$$

$$- \sum_{E \in \mathcal{E}_h} \int_E (\{A\nabla u \cdot n\}[v] + \{A\nabla v \cdot n\}[u]) + \sum_{E \in \mathcal{E}_h} \frac{\gamma}{h_E} \int_E [u][v]$$

- Let $\mathcal{T}_h$ be a general mesh on $\Omega \subset \mathbb{R}^d$ – a collection of non-overlapping subdomains;
  - here is an example in 2D
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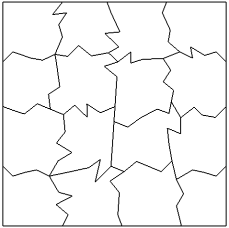
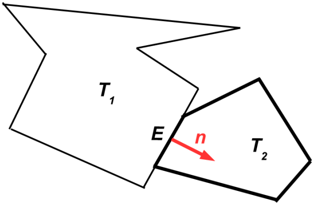


- Let $V_h = \{v \in L^2(\Omega) : v|_T \in \mathbb{P}^k(T), \forall T \in \mathcal{T}_h\}$
- The SIP method:
  find $u_h \in V_h$ such that $a_h(u_h, v_h) = L_h(v_h), \ \forall v_h \in V_h$ with

$$L_h(v) = \sum_{T \in \mathcal{T}_h} \int_T fv + \sum_{E \in \mathcal{E}_h^b} \int_E g \left(\frac{\gamma}{h_E} v - A\nabla v \cdot n \right)$$

Notations and error estimates

- $\mathcal{E}_h$ – edges/faces of the mesh $\mathcal{T}_h$



n – the unit normal on an edge E

$[v]|_E := v|_{T_1} - v|_{T_2}$

$\{v\}|_E := \frac{1}{2}\left(v|_{T_1} + v|_{T_2}\right)$

$h_E = 2\left(\frac{1}{h_{T_1}} + \frac{1}{h_{T_2}}\right)^{-1}$

- $\mathcal{E}_h^b \subset \mathcal{E}_h$ – the edges/faces on $\partial\Omega$; On any $E \in \mathcal{E}_h^b$:
  n is the outward looking, $[v] = v$, $\{v\} = v$, $h_E = h_T$

Theorem

Under mesh regularity, usual assumptions on V_h, and γ big enough

$$|u - u_h|_{H^1(\mathcal{T}_h)} \leq Ch^k |u|_{H^{k+1}(\Omega)}$$

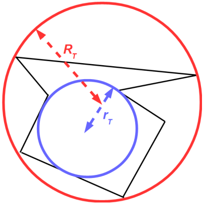
Adding the usual elliptic regularity assumption,

$$\|u - u_h\|_{L^2(\Omega)} \leq C|u|_{H^{k+1}(\Omega)} h^{k+1}$$

Mesh regularity and assumptions on V_h

cf the book by Cangiani *et al* (2017)

- We assume that $\mathcal{T}_h$ is shape regular:



For any $T \in \mathcal{T}_h$, there exist two balls $B_T^{in} \subset T \subset B_T$ with radiuses r_T and R_T such that

$$R_T \le \rho_1 r_T$$

with a regularity parameter $\rho_1 > 1$

- Optimal interpolation: there exists $I_h : H^{k+1}(\Omega) \to V_h$

$$\left(\sum_{T\in\mathcal{T}_h} \left(|v - I_h v|^2_{H^1(T)} + \frac{1}{h_T^2} \|v - I_h v\|^2_{L^2(T)} + h_T^2 |v - I_h v|^2_{H^2(T)} \right.\right.$$
$$\left.\left. + h_T \|\nabla v - \nabla I_h v\|^2_{L^2(\partial T)} + \frac{1}{h_T} \|v - I_h v\|^2_{L^2(\partial T)} \right) \right)^{\frac{1}{2}} \le C \left(\sum_{T\in\mathcal{T}_h} h_T^{2k} |v|_{H^{k+1}(T)} \right)^{\frac{1}{2}}$$

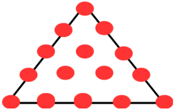
- Inverse inequalities: for any $v_h \in V_h$ and any $T \in \mathcal{T}_h$

$$\|v_h\|_{L^2(\partial T)} \le \frac{C}{\sqrt{h_T}} \|v_h\|_{L^2(T)}, \quad \|\nabla v_h\|_{L^2(\partial T)} \le \frac{C}{\sqrt{h_T}} \|\nabla v_h\|_{L^2(T)}, \quad |v_h|_{H^2(T)} \le \frac{C}{h_T} |v_h|_{H^1(T)}$$

Static condensation for discontinuous FEM ?

Not efficient/impossible if one follows the usual recipe of eliminating the local DOFs

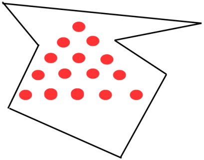
On a triangular mesh, all the standard DOFs associated to the interpolation nodes become non-local in the SIP method



- One can extract a localizable subspace of $\mathbb{P}^k(T)$, $T \in \mathcal{T}_h$ as

$$V_h^{loc,SIP,T} = \{v \in \mathbb{P}^k(T) : v|_{\partial T} = n \cdot \nabla v|_{\partial T} = 0\}$$

  but $\dim(\mathbb{P}^k(T) \setminus V_h^{loc,SIP,T})$ is rather big
- It gets worse on a general mesh (mesh elements with many edges)



Static condensation for discontinuous FEM ?

Yes, if one mimics the reformulation with orthogonal projections

- Redefine the local polynomial space

$$V_h^{loc,T} = \mathbb{P}^{k-2}(T)$$

- Re-introduce the orthogonal projection $\pi_{T,k-2} : L^2(T) \to V_h^{loc,T}$
- Redefine the local contributions to the solution as $u_h^{loc} \in V_h$

$$\pi_{T,k-2}\mathcal{L}(u_h^{loc}|_T) = \pi_{T,k-2}f, \quad \forall T \in \mathcal{T}_h$$

- The global subspace V_h' is populated by the solutions to

$$\pi_{T,k-2}\mathcal{L}(v_h'|_T) = 0, \quad \forall T \in \mathcal{T}_h$$

There is no schematic representation for the DOFs in V_h'

The basis functions of V_h' are no longer associated to some nodes. One can only say that $V_h'|_T$ is a subspace of $\mathbb{P}^k(T)$. In practice, one should precompute a basis for $V_h'|_T$.

- Compute $u_h^{loc} \in V_h$ by solving

$$\int_T \mathcal{L}(u_h^{loc}|_T) q_T = \int_T f q_T, \quad \forall q_T \in \mathbb{P}^{k-2}(T), T \in \mathcal{T}_h$$

- Define the subspace of V_h

$$V_h' = \left\{ v_h' \in V_h : \int_T \mathcal{L}(v_h'|_T) q_T = 0, \quad \forall q_T \in \mathbb{P}^{k-2}(T), T \in \mathcal{T}_h \right\}$$

- Compute $u_h' \in V_h'$ such that

$$a_h(u_h', v_h') = L_h(v_h') - a_h(u_h^{loc}, v_h'), \quad \forall v_h' \in V_h'$$

- Set

$$u_h = u_h^{loc} + u_h'$$

Lemma

Provided $h \leq h_0$, $\forall T \in \mathcal{T}_h$, $\forall q_T \in \mathbb{P}^{k-2}(T)$ $\exists u_T \in \mathbb{P}^k(T)$ such that

$$\int_T q_T(\mathcal{L}u_T) \geq \frac{1}{2}\|q_T\|^2_{L^2(T)}$$

and

$$|u_T|^2_{H^1(T)} + \frac{1}{h_T}\|u_T\|^2_{L^2(\partial T)} \leq Ch_T^2\|q_T\|^2_{L^2(T)}$$

h_0, C depend only on mesh regularity and α, β, M

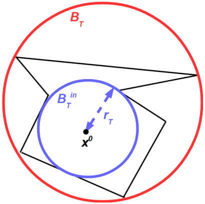
Remarks

1. One can put $h_0 = +\infty$ if A is constant on T
2. This proves that the local problem in scSIP has a solution

Proof of the lemma

- Let χ_T be the polynomial of degree 2 vanishing on ∂B_T^{in}

$$\chi_T(x) = \left(\sum_{i=1}^{d} (x_i - x_i^0)^2 - r_T^2 \right)$$



- Set $A_{ij}^0 = A_{ij}(x^0)$ and $\mathcal{L}^0 = -\partial_i A_{ij}^0 \partial_j$ and consider the linear map

$$Q : \mathbb{P}^{k-2}(T) \to \mathbb{P}^{k-2}(T)$$
$$Q(v) = \mathcal{L}^0(\chi_T v)$$

- $Ker(Q) = \{0\}$. Indeed, if $Q(v) = 0$ then $\chi_T v$ solves

$$\mathcal{L}^0(\chi_T v) = 0 \text{ in } B_T^{in}, \quad \chi_T v = 0 \text{ on } \partial B_T^{in} \qquad \Rightarrow \qquad v = 0$$

- Q is thus one-to-one: $\forall q_T \in \mathbb{P}^{k-2}(T)$, $\exists u_T := \chi_T Q^{-1}(q_T)$

$$\mathcal{L}^0 u_T = q_T$$

Proof of the lemma

- By scaling,

$$|u_T|_{W^{2,\infty}(B_T)} + \frac{1}{h_T}|u_T|_{W^{1,\infty}(B_T)} + \frac{1}{h_T^2}\|u_T\|_{L^\infty(B_T)} \leq \frac{C}{h_T^{d/2}}\|q_T\|_{L^2(B_T^{in})}$$

  Thus,

$$|u_T|_{H^1(T)} \leq |T|^{1/2}|u_T|_{W^{1,\infty}(B_T)} \leq Ch_T\|q_T\|_{L^2(T)}$$

- Similarly, $\|u_T\|_{L^2(T)} \leq Ch_T^2\|q_T\|_{L^2(T)}$ so that

$$\|u_T\|_{L^2(\partial T)} \leq Ch_T^{3/2}\|q_T\|_{L^2(T)}$$

  by the trace inverse inequality.

- In the case of variable coefficients (for h small enough)

$$\int_T q_T \mathcal{L} u_T = \int_T q_T \mathcal{L}^0 u_T + \int_T q_T \partial_i((A_{ij} - A_{ij}^0)\partial_j u_T)$$

$$\geq \|q_T\|_{L^2(T)}^2 - \|q_T\|_{L^2(T)}|T|^{1/2}h_T\|\nabla A\|_{L^\infty(T)}\frac{C}{h_T^{d/2}}\|q_T\|_{L^2(B_T^{in})}$$

$$\geq \|q_T\|_{L^2(T)}^2 - Ch_T\|q_T\|_{L^2(T)}^2 \geq \frac{1}{2}\|q_T\|_{L^2(T)}$$

Two bilinear forms

- Recall the bilinear form a_h. It is known to be coercive

$$a_h(v_h, v_h) \geq c \, ||| v_h |||^2, \quad \forall v_h \in V_h$$

$$||| v |||^2 = \sum_{T \in \mathcal{T}_h} \left(|v|^2_{H^1(T)} + \frac{1}{h_T} \| [v] \|^2_{L^2(\partial T)} \right)$$

- Introduce the bilinear form

$$b_h(\mu, v) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \mu \mathcal{L} u$$

and the space

$$M_h = \{ v \in L^2(\Omega) : v|_T \in \mathbb{P}^{k-2}(T), \forall T \in \mathcal{T}_h \}$$

The previous lemma implies the inf-sup

$$\inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b_h(\mu_h, v_h)}{\|\mu_h\|_h \, ||| v_h |||} \geq \delta$$

with $\|\mu\|_h^2 = \sum_{T \in \mathcal{T}_h} h_T^2 \|\mu\|^2_{L^2(T)}$

A saddle point reformulation

Lemma

u_h given by the scSIP method can also be recovered as a solution to the saddle point problem:
Find $u_h \in V_h$, $\lambda_h \in M_h$ such that

$$
\begin{aligned}
a_h(u_h, v_h) + b_h(\lambda_h, v_h) &= L_h(v_h), && \forall v_h \in V_h \\
b_h(\mu_h, u_h) &= \sum_{T \in \mathcal{T}_h} h_T^2 \int_T f \mu_h, && \forall \mu_h \in M_h
\end{aligned}
$$

The solution (u_h, λ_h), and thus scSIP's u_h is unique

Consistency

The exact solution u together with $\lambda = 0$ satisfy

$$
\begin{aligned}
a_h(u, v_h) + b_h(\lambda, v_h) &= L_h(v_h), && \forall v_h \in V_h \\
b_h(\mu_h, u) &= \sum_{T \in \mathcal{T}_h} h_T^2 \int_T f \mu_h, && \forall \mu_h \in M_h
\end{aligned}
$$

Proof of the saddle point equivalence lemma

- Existence and uniqueness of (u_h, λ_h) follows from the standard theory of saddle point problems (a_h is coercive, b_h satisifies inf-sup)
- Let $u_h = u_h^{loc} + u_h'$ be given by the scSIP method. Then
  1. For all $v_h' \in V_h'$

     $$a_h(u_h^{loc} + u_h', v_h') = L_h(v_h')$$

     Recalling that $V_h' = \{v_h' \in V_h : b_h(\mu_h, v_h') = 0 \ \forall \mu_h \in M_h\}$, one can add the Lagrange multiplier $\lambda_h \in M_h$ above so that it holds for all $v_h \in V_h$
  2. Recall again $b_h(\mu_h, u_h') = 0$ and observe

     $$b_h(\mu_h, u_h^{loc} + u_h') = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T f q_h, \quad \forall \mu_h \in M_h$$

  3. Since any u_h given by the scSIP method can be recovered as a solution to the saddle point problem, scSIP's u_h is unique

A priori error estimates for scSIP

Theorem

Under the same assumption as before, the scSIP method produces the unique solution $u_h \in V_h$, which satisfies

$$|u - u_h|_{H^1(\mathcal{T}_h)} \leq Ch^k |u|_{H^{k+1}(\Omega)}$$

$$\|u - u_h\|_{L^2(\Omega)} \leq C|u|_{H^{k+1}(\Omega)} h^{k+1}$$

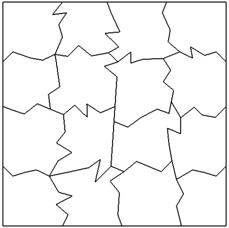
Proof by the inf-sup theory and optimal approximation

$$\begin{aligned}
&|||u_h - I_h u||| + \|\lambda_h\|_h \\
&\qquad \leq C \Bigg(|||u - I_h u\,|||^2 + \sum_{E \in \mathcal{E}_h} h_E \|\{A\nabla(u - I_h u)\cdot n\}\|^2_{L^2(E)} \\
&\qquad\qquad + \sum_{T \in \mathcal{T}_h} h_T^2 \|\mathcal{L}(u - I_h u)\|^2_{L^2(T)} \Bigg)^{\frac{1}{2}} \\
&\qquad \leq Ch^k |u|_{H^{k+1}}
\end{aligned}$$

Notes on implementation of SIP ans scSIP methods

- Introduce the bases of $\mathbb{P}^k(T_l)$ and $\mathbb{P}^{k-2}(T_l)$ on $T_l \in \mathcal{T}_h$:
$$\{\phi_i^{(l)}\}_{i=1,\ldots,N_k} \qquad \{\psi_j^{(l)}\}_{j=1,\ldots,N_{k-2}}$$
- Form the matrices $A^{(lm)} = \{a_h(\phi_i^{(l)}, \phi_j^{(l)}\}_{ij}$ for any two neighboring elements T_l and T_m
- Form the matrices $\mathbf{B}^{(l)} = \{\int_{T_l} \psi_i^{(l)} \mathcal{L} \phi_j^{(l)}\}_{ij}$
- Calculate the scSIP local contributions $\overrightarrow{u}_{loc}^{(l)}$ and the basis functions $\overrightarrow{u}^{(l,s)}$ for V_h' on every $T_l \in \mathcal{T}_h$ by
$$\begin{cases} \overrightarrow{u}_{loc}^{(l)} + (\mathbf{B}^{(l)})^T \overrightarrow{\lambda}_{loc}^{(l)} &= 0 \\ \mathbf{B}^{(l)} \overrightarrow{u}_{loc}^{(l)} &= \overrightarrow{F_\psi}^{(l)} \end{cases} \quad \text{and} \quad \begin{cases} \overrightarrow{u}^{(l,s)} + (\mathbf{B}^{(l)})^T \overrightarrow{\lambda}^{(l,s)} &= \overrightarrow{e}^s \\ \mathbf{B}^{(l)} \overrightarrow{u}^{(l,s)} &= 0 \end{cases}$$
with $\overrightarrow{F_\psi}^{(l)} = \{\int_{T_l} f \psi_i^{(l)}\}$
- Put the vectors $\overrightarrow{u}^{(l,s)}$ into the matrix $\mathbf{M}^{(l)}$ on $T_l \in \mathcal{T}_h$ (after selecting the linearly independent ones by Gramm-Schmdt)
- Form the reduced matrices $\mathbf{A}'^{(lm)} = (\mathbf{M}^{(l)})^T \mathbf{A}^{(lm)} \mathbf{M}^{(m)}$ of size $N_k' \times N_k'$
- The global problems for SIP and scSIP methods

$$\underbrace{\mathbf{A}\overrightarrow{U} = \overrightarrow{F}}_{problem\ \ of\ \ size\ \ N_k} \qquad \underbrace{\mathbf{A}'\overrightarrow{U}' = \overrightarrow{F}'}_{problem\ \ of\ \ size\ \ (N_k - N_{k-2})}$$



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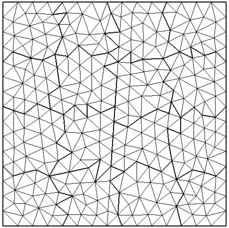
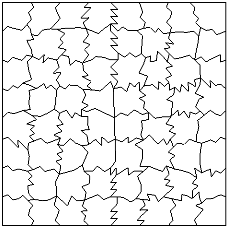


4×4 cells



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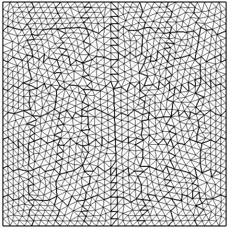
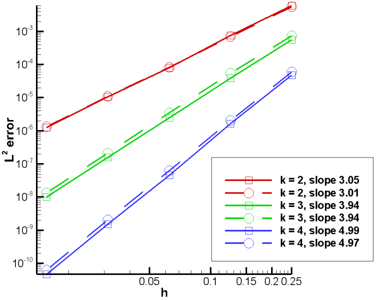
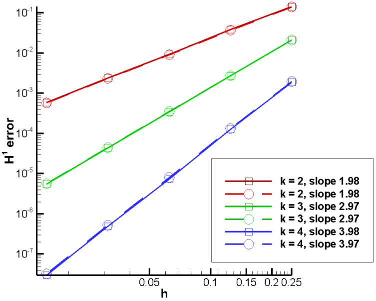


8×8 cells

The first test case: Poisson equation

We solve $-\Delta u = f$ on $\Omega = (0,1)^2$ with homogeneous Dirichlet boundary conditions and the exact solution $u = \sin(\pi x)\sin(\pi y)$

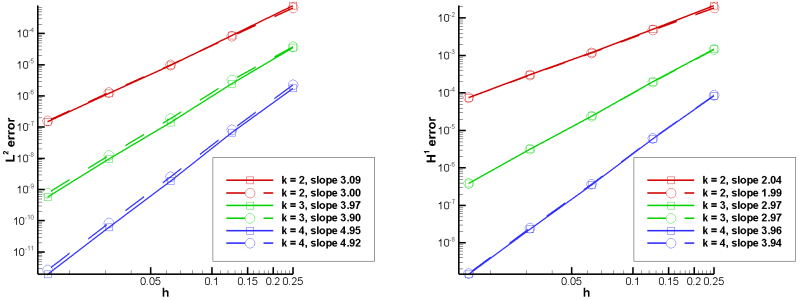




- The solid lines represent the SIP method
- The dashed lines represent the scSIP method

The second test case: non-constant coefficients

We solve $-\operatorname{div}(A\nabla u) = f$ with $A = \begin{pmatrix} 1+x & xy \\ xy & 1+y \end{pmatrix}$ on $\Omega = (0,1)^2$ with non-homogeneous Dirichlet boundary conditions and the exact solution $u = e^{xy}$



- The solid lines represent the SIP method
- The dashed lines represent the scSIP method

An example of assumptions on the mesh that guarantee the interpolation and inverse estimates

M1: $\mathcal{T}_h$ is shape regular in the sense: $\forall T \in \mathcal{T}_h$ there exist two balls $B_T^{in} \subset T \subset B_T$ with radiuses r_T and R_T such that

$$R_T \leq \rho_1 r_T$$

with a regularity parameter $\rho_1 > 1$

M2: $\mathcal{T}_h$ is locally quasi-uniform in the following sense: for any two mesh cells $T, T' \in \mathcal{T}_h$ such that $B_{T'} \cap B_T \neq \varnothing$ there holds

$$\frac{1}{\rho_2} h_{T'} \leq h_T \leq \rho_2 h_{T'}$$

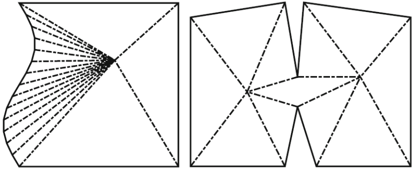
with a parameter $\rho_2 > 1$

M3: The cell boundaries are not too wiggly: for all $T \in \mathcal{T}_h$

$$|\partial T| \leq \rho_3 h_T^{d-1}$$

with a parameter $\rho_3 > 0$.

This is an alternative to the assumptions from Cangiani et al (2017), which are based on the decomposition into the simplexes



Interpolation estimates

- **Local interpolation estimate:** for any $T \in \mathcal{T}_h$, let $v_h \in \mathbb{P}^k(T)$ be s.t. $\int_{B_T} v_h \varphi_h = \int_{B_T} v \varphi_h \quad \forall \varphi_h \in \mathbb{P}^k(T)$
Under Assumptions M1 and M3, we have for any $v \in H^{k+1}(B_T)$

$$|v - v_h|_{H^1(T)} + \frac{1}{h_T} \|v - v_h\|_{L^2(T)} + h_T |v - v_h|_{H^2(T)}$$
$$+ \sqrt{h_T} \|\nabla(v - v_h)\|_{L^2(\partial T)} + \frac{1}{\sqrt{h_T}} \|v - v_h\|_{L^2(\partial T)} \leq C h_T^k |v|_{H^{k+1}(B_T)}$$

- Proof: by a scaling argument

$$\|v - v_h\|_{L^\infty(B_T)} \leq C h_T^{k+1-d/2} |v|_{H^{k+1}(B_T)}$$

and use M1 and M3 ...

- **Global interpolation estimate** obtained by summing over $T \in \mathcal{T}_h$ since the number of intersecting B_T's is uniformly bounded (Assumption M2)

Inverse inequalities

- The needed inverse inequalities follow from

$$\|q_h\|_{L^\infty(T)} \leq \frac{C}{h_T^{d/2}} \|q_h\|_{L^2(T)} \quad \forall q_h \in \mathbb{P}^k(T)$$

- This in turn follows from

$$\|q_h\|_{L^\infty(B_T)} \leq \frac{C}{h_T^{d/2}} \|q_h\|_{L^2(B_T^{in})}$$

- Scaling the ball B_T to a ball of radius 1 B_1 and considering all the possible positions of the inscribed ball, the last inequality can be rewritten as

$$\|q_h\|_{L^\infty(B_1)} \leq C \min_{B^{in} \subset B_1, B^{in} \text{ a ball of radius } \geq \rho_1^{-1}} \|q_h\|_{L^2(B_T^{in})}$$

is valid by equivalence of norms.

Extension to Stokes equations

A straighforward SIP method for $\begin{cases} -\Delta u + \nabla p = f & \text{on } \Omega \\ \text{div}\, u = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

- V_h – discontinuous vector valued $\mathbb{P}^k$ FE on $\mathcal{T}_h$
- Q_h – discontinuous $\mathbb{P}^{k-1}$ FE on $\mathcal{T}_h$
- Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$a(u_{h,} p_h; v_h, q_h) = \int_\Omega f \cdot v_h, \quad \forall (v_h, q_h) \in V_h \times Q_h$$

with

$$\begin{aligned} a_h(u, p; v, q) \quad &= \textstyle\sum_{T \in \mathcal{T}_h} \left(\int_T \nabla u : \nabla v - \int_T p \,\text{div}\, v - \int_T q \,\text{div}\, u \right) \\ &- \textstyle\sum_{E \in \mathcal{E}_h} \int_E (\{n \cdot \nabla u\} \cdot [v] - \{pn\} \cdot [v]) \\ &- \textstyle\sum_{E \in \mathcal{E}_h} \int_E (\{n \cdot \nabla v\} \cdot [u] - \{qn\} \cdot [u]) \\ &+ \textstyle\sum_{E \in \mathcal{E}_h} \frac{\gamma}{h_E} \int_E [u] \cdot [v] + \sum_{E \in \mathcal{E}_h^i} \gamma_p h_E \int_E [p][q] \end{aligned}$$

The local problems

Introduce the differential operator of the Stokes problem

$$\mathcal{L}(u,p) = (-\Delta u + \nabla p, \operatorname{div} u)$$

and let $(u_h^{loc}, p_h^{loc}) \in V_h \times Q_h$ solve on every mesh cell $T \in \mathcal{T}_h$

$$(\mathcal{L}(u_h^{loc}, p_h^{loc}), (v_T, q_T))_T = \int_T f \cdot v_T \quad \forall v_T \in V_h^{loc,T}, q_T \in Q_h^{loc,T}$$

where

- $V_h^{loc,T} = \mathbb{P}^{k-2}(T)$
- $Q_h^{loc,T} = \mathbb{P}_0^{k-1}(T) = \left\{ q \in \mathbb{P}^{k-1}(T) : \int_T q = 0 \right\}$

Existence of the local solutions

The map $\mathcal{L}_{T,h} : \mathbb{P}^k(T) \times \mathbb{P}^{k-1}(T) \to \mathbb{P}^{k-2}(T) \times \mathbb{P}_0^{k-1}(T)$ defined on any $T \in \mathcal{T}_h$ by
$(\mathcal{L}_{T,h}(u,p), (v,q)) = \int_T (-\Delta u + \nabla p) \cdot v + \int_T (\operatorname{div} u) q$
for all $(v,q) \in \mathbb{P}^{k-2}(T) \times \mathbb{P}_0^{k-1}(T)$, is surjective
(proof on the next slide)

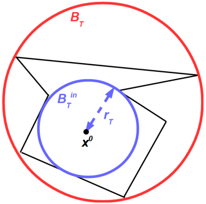
The local problems

Surjectivity of the map

- Let χ_T be the polynomial of degree 2

vanishing on ∂B_T^{in}

$$\chi_T(x) = \left(\sum_{i=1}^{d} (x_i - x_i^0)^2 - r_T^2 \right)$$



- Set the linear map

$$Q \quad : \mathbb{P}^{k-2}(T) \times \mathbb{P}_0^{k-1}(T) \to \mathbb{P}^{k-2}(T) \times \mathbb{P}_0^{k-1}$$
$$Q(v,p) \quad = \left(-\Delta(\chi_T v) + \nabla p,\ \operatorname{div}(\chi_T v) - \frac{1}{|T|} \int_T \operatorname{div}(\chi_T v) \right)$$

- $Ker(Q) = \{0\}$. Indeed, if $Q(v,p) = 0$ then $(\chi_T v, p)$ solve

$$-\Delta(\chi_T v) + \nabla p = 0,\ \operatorname{div}(\chi_T v) = const \text{ on } B_T^{in}, \quad \chi_T v = 0 \text{ on } \partial B_T^{in}$$

In fact, $\operatorname{div}(\chi_T v) = 0$ so that $v = 0, p = 0$

- Q is thus one-to-one

The global problem

- The subspace $X_h' \subset X_h := V_h \times Q_h$

$$X_h' = \{(v_h, q_h) : \mathcal{L}(v_h, q_h) = 0 \text{ on every } T \in \mathcal{T}_h\}$$

- We search for $(u_h', p_h') \in X_h'$ s.t.

$$a_h(u_h', p_h'; v_h, q_h) = \int_\Omega f \cdot v_T - a_h(u_h^{loc}, p_h^{loc}; v_h, q_h)$$

- The structure of X_h':

$$(u_h', p_h') \in X_h' \Leftrightarrow u_h' \in V_h', p_h' = \pi_h(u_h') + \bar{p}_h$$

where

- V_h' is a subspace of V_h
- $\pi_h : V_h' \to \mathbb{P}_0^{k-1}(T)$ given on every $T \in \mathcal{T}_h$ by

$$\pi_h(u_h') = p_h', \qquad \nabla p_h' = \Delta u_h \text{ on every } T \in \mathcal{T}_h$$

$\bar{p}_h \in \overline{Q}_h$ – piecewise constant on $\mathcal{T}_h$

Rewriting the global problem on X_h'

On X_h',

- $\int_T \pi_h(u_h') \,\mathrm{div}\, v_h' = 0$ since $\mathrm{div}\, v_h' = const$ on T
- $a_h(u_h', p_h'; v_h', q_h')$
  $$= \sum_{T\in\mathcal{T}_h} \left(\int_T \nabla u_h' : \nabla v_h' - \int_T \bar{p}_h \,\mathrm{div}\, v_h' - \int_T \bar{q}_h \,\mathrm{div}\, u_h'\right)$$
  $$- \sum_{E\in\mathcal{E}_h} \int_E (\{n\cdot\nabla u_h' - \pi_h(u_h')n\}\cdot[v_h'] - \{\bar{p}_h n\}\cdot[v_h'])$$
  $$- \sum_{E\in\mathcal{E}_h} \int_E (\{n\cdot\nabla v_h' - \pi_h(v_h')n\}\cdot[u_h'] - \{\bar{q}_h n\}\cdot[u_h'])$$
  $$+ \sum_{E\in\mathcal{E}_h} \frac{\gamma}{h_E} \int_E [u_h']\cdot[v_h']$$
  $$+ \sum_{E\in\mathcal{E}_h^i} \gamma_p h_E \int_E [\pi_h(u_h') + \bar{p}_h][\pi_h(v_h') + \bar{q}_h]$$
- Since $\nabla \bar{p}_h = 0$ on every $T \in \mathcal{T}_h$

$$-\sum_{T\in\mathcal{T}_h} \int_T \bar{p}_h \,\mathrm{div}\, v_h' + \sum_{E\in\mathcal{E}_h} \int_E \{\bar{p}_h n\}\cdot[v_h'] = -\frac{1}{2} \sum_{E\in\mathcal{E}_h^i} \int_E [\bar{p}_h n]\cdot[v_h']$$

- so that finally ...

Coercivity of the global problem on X_h'

- For all $(u_h', p_h') \in X_h'$, i.e. $u_h' \in V_h'$, $p_h' = \pi_h(u_h') + \bar{p}_h$, $\bar{p}_h \in \bar{Q}_h$

$$
\begin{aligned}
a_h(u_h', p_h'; u_h', p_h') = {} & \sum_{T \in \mathcal{T}_h} \int_T |\nabla u_h'|^2 \\
& - 2 \sum_{E \in \mathcal{E}_h} \int_E \{n \cdot \nabla u_h' - \pi_h(u_h') n\} \cdot [u_h'] + \sum_{E \in \mathcal{E}_h} \frac{\gamma}{h_E} \int_E |[u_h']|^2 \\
& - \sum_{E \in \mathcal{E}_h^i} \int_E [\bar{p}_h n] \cdot [u_h'] + \sum_{E \in \mathcal{E}_h^i} \gamma_p h_E \int_E |[\pi_h(u_h') + \bar{p}_h]|^2 \\
\geqslant {} & c \, |||\, u_h', p_h' |||^2
\end{aligned}
$$

with

$$
|||u, p|||^2 = \sum_{T \in \mathcal{T}_h} \int_T |\nabla u|^2 + \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \int_E |[u]|^2 + \int_\Omega p^2 + \sum_{E \in \mathcal{E}_h^i} h_E \int_E |[p]|^2
$$

Conclusions and perspectives

- We have presented an interior penalty DG method with static condensation for the diffusion problem which can be cheaper than the methods with skeleton-based DOFs on polygons/polyhedra with many facets

	2D	3D
scSIP	$(2k+1)N_{elements}$	$(k+1)^2 N_{elements}$
HDG, HHO	$\sim kN_{edges}$	$\sim \frac{1}{2}k^2 N_{faces}$

- A lot of things to do and open questions:
  - Convergence order with respect to p $(= k)$
  - Static condensation for other DG methods : incomplete or antisymmetric IP, local DG, ...
  - A proper extension for the Stokes problem (cf. the talk by A. Linke)
  - An extension to (nearly) compressible elasticity (asymptotic preserving in the limit Poisson ratio $\to \frac{1}{2}$)
  - ...