# A Hybrid High-Order method for creeping flows of non-Newtonian fluids

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## Abstract

We illustrate a Hybrid High-Order method for a modified Stokes problem adapted to non-Newtonian fluids. It is a generalization of the Hybrid High-Order methods implemented to the nonlinear elasticity problem in [2] and to the Navier-Stokes problem in [6] based on the works [3, 4]. The space discretization hinges on local reconstruction operators from hybrid polynomial unknowns at elements and faces. The proposed method has several assets: it is able to handle general polyhedral meshes possibly containing nonmatching interfaces, it allows arbitrary approximation orders, it has a dimension-independent implementation, it allows seamless treatment of nonconforming mesh refinement, it offers stability for inf-sup condition. We gives a detailed error estimate in Sobolev-like norms using a discrete Korn's inequality on broken polynomial spaces for the non Hilbertian case. For the sake of simplicity, we will focus on the power-law fluids since their shear stress-strain rate function verifies the assumptions required to obtain the error estimate.

## The Power-Law Stokes Problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, denote a bounded, polyhedral, open star domain with Lipschitz boundary $\partial\Omega$. We consider a power-law fluid occupying $\Omega$ characterized by its shear stress-strain rate function $\sigma : \mathbb{R}^{d\times d}_{\rm s} \to \mathbb{R}^{d\times d}_{\rm s}$ which is defined such that for all $\tau \in \mathbb{R}^{d\times d}_{\rm s}$,

$$\sigma(\tau) = \kappa|\tau|^{r-2}_{d\times d}\tau,$$

where the flow consistency index $\kappa \in (0,+\infty)$ and the flow behavior index $r \in (1,+\infty)$ are two parameters which specify the power-law fluid. Considering that this one is subjected to a volumetric force $f \in L^{r'}(\Omega,\mathbb{R}^d)$, the power-law Stokes problem consists in finding a velocity $u : \Omega \to \mathbb{R}^d$ and a pressure $p : \Omega \to \mathbb{R}$ solution of

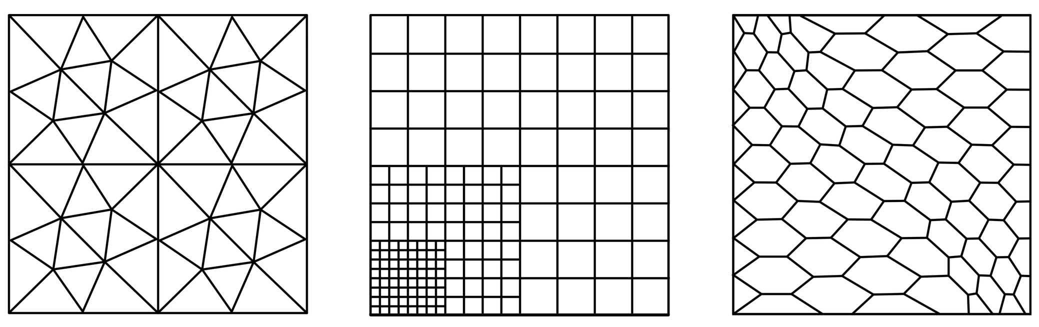
$$\begin{cases} -\nabla\cdot\sigma(\nabla_{\rm s}u) + \nabla p = f & \text{in } \Omega,\\ \nabla\cdot u = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ \int_\Omega p = 0. \end{cases}$$

It is the modified Stokes problem so that the non-Newtonian character of the fluid is taken into account. In fact, the power-law fluid becomes Newtonian if and only if $r = 2$ and in this case, we obtain the Stokes problem.

## Hybrid High-Order

### Features

- Able to handle general polyhedral meshes possibly containing nonmatching interfaces



- Dimension-independent implementation
- Arbitrary order: better accuracy for a fixed mesh or fewer elements for a given precision
- Seamless treatment of nonconforming mesh refinement
- Offers stability for inf-sup condition

### Hybrid spaces

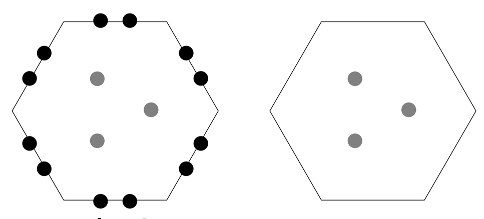
The discrete local and global spaces containing velocity unknowns are respectively defined as

$$\underline{U}^k_T := \mathbb{P}^k(T,\mathbb{R}^d) \times \left( \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^k(F,\mathbb{R}^d) \right), \quad \underline{U}^k_h := \bigtimes_{T\in\mathcal{T}_h} \underline{U}^k_T.$$

For all $\underline{v}_h \in \underline{U}^k_h$, we use the discrete notation $\underline{v}_h := (\underline{v}_T)_{T\in\mathcal{T}_h}$ where $\underline{v}_T := (v_T,(v_F)_{F\in\mathcal{F}_T}) \in \underline{U}^k_T$ for all element $T$ of the mesh $\mathcal{T}_h$. The discrete space containing pressure unknowns is defined by

$$P^k_h := \mathbb{P}^k(\mathcal{T}_h,\mathbb{R}) \cap L^{r'}_0(\Omega,\mathbb{R}).$$

$k = 1$



velocity pressure

We define on the zero boundary subspace $\underline{U}^k_{h,0}$ the norm $\|\cdot\|_{\varepsilon,r,h}$ such that for all $\underline{v}_h \in \underline{U}^k_{h,0}$,

$$\|\underline{v}_h\|_{\varepsilon,r,h} := \left( \sum_{T\in\mathcal{T}_h} \left( \|\nabla_{\rm s}v_T\|^r_{L^r(T,\mathbb{R}^{d\times d}_{\rm s})} + \sum_{F\in\mathcal{F}_T} h_F^{1-r}\|v_F - v_T\|^r_{L^r(F,\mathbb{R}^d)} \right) \right)^{\frac{1}{r}}.$$

We have a discrete Korn's inequality: there exists $C_{\rm K} > 0$ independent of $h$ such that, for all $\underline{v}_h \in \underline{U}^k_{h,0}$,

$$\|v_h\|_{L^r(\Omega,\mathbb{R}^d)} + |v_h|_{W^{1,r}(\mathcal{T}_h,\mathbb{R}^d)} \le C_{\rm K}\|\underline{v}_h\|_{\varepsilon,r,h}.$$

### Discrete operators

We denote $\underline{I}^k_h : W^{1,1}(\Omega,\mathbb{R}^d) \ni v \mapsto (\pi^k_T v, (\pi^k_F v)_{F\in\mathcal{F}_T})_{T\in\mathcal{T}_h} \in \underline{U}^k_h$ the interpolation operator. For all element $T \in \mathcal{T}_h$, we define

- the local symmetric gradient operator $G^k_{{\rm s},T} : \underline{U}^k_T \to \mathbb{P}^k(T,\mathbb{R}^{d\times d}_{\rm s})$ such that, for all $\underline{v}_T \in \underline{U}^k_T$,

$$\int_T G^k_{{\rm s},T}\underline{v}_T : \tau = \int_T \nabla_{\rm s}v_T : \tau + \sum_{F\in\mathcal{F}_T}\int_F (v_F - v_T)\cdot(\tau n_{TF}) \quad \forall\tau \in \mathbb{P}^k(T,\mathbb{R}^{d\times d}_{\rm s}),$$

- the local divergence operator $D^k_T : \underline{U}^k_T \to \mathbb{P}^k(T,\mathbb{R})$ by $D^k_T = \mathrm{tr}(G^k_{{\rm s},T})$,
- the local symmetric reconstruction operator $r^{k+1}_{{\rm s},T} : \underline{U}^k_T \to \mathbb{P}^{k+1}(T,\mathbb{R}^d)$ such that for all $\underline{v}_T \in \underline{U}^k_T$,

$$\begin{cases} \int_T (\nabla_s r^{k+1}_{{\rm s},T}\underline{v}_T - G^k_{s,T}\underline{v}_T) : \nabla_s w = 0 \quad \forall w \in \mathbb{P}^{k+1}(T,\mathbb{R}^d),\\ \int_T r^{k+1}_{{\rm s},T}\underline{v}_T = \int_T v_T, \quad \int_T \nabla_{ss}r^{k+1}_{{\rm s},T}\underline{v}_T = \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F (n_{TF}\otimes v_F - v_F \otimes n_{TF}). \end{cases}$$

### Discrete problem

We define the stabilization function $s_T : \underline{U}^k_T \times \underline{U}^k_T \to \mathbb{R}$ such that for all $\underline{v}_T, \underline{w}_T \in \underline{U}^k_T$,

$$s_T(\underline{w}_T,\underline{v}_T) := \sum_{F\in\mathcal{F}_T} h_F^{1-r}\int_F |\Delta^k_{TF}\underline{w}_T|^{r-2}\Delta^k_{TF}\underline{w}_T \cdot \Delta^k_{TF}\underline{v}_T$$

where $\Delta^k_{TF}\underline{v}_T := \pi^k_F(r^{k+1}_{{\rm s},T}\underline{v}_T - v_F) - \pi^k_T(r^{k+1}_{{\rm s},T}\underline{v}_T - v_T)$ for all $\underline{v}_T \in \underline{U}^k_T$ and all $F \in \mathcal{F}_T$. We define the function $a_h : \underline{U}^k_h \times \underline{U}^k_h \to \mathbb{R}$ such that for all $\underline{w}_h, \underline{v}_h \in \underline{U}^k_h$,

$$a_h(\underline{w}_h,\underline{v}_h) := \sum_{T\in\mathcal{T}_h}\left( \int_T \sigma(G^k_{{\rm s},T}\underline{w}_T) : G^k_{{\rm s},T}\underline{v}_T + \kappa(r-1)s_T(\underline{w}_T,\underline{v}_T) \right).$$

The function $a_h$ has coercivity, boundedness and consistency properties. Indeed, for all $\underline{v}_h, \underline{w}_h \in \underline{U}^k_{h,0}$ and all $w \in W^{1,r}_0(\Omega,\mathbb{R}^d) \cap W^{k+2,r}(\mathcal{T}_h,\mathbb{R}^d)$, there exist $C_{\rm cv}, C_{\rm gr}, C_{\rm a1}(w), C_{\rm a2}(w) > 0$ independent of $h$ such that,

$$a_h(\underline{v}_h,\underline{v}_h) \ge C_{\rm cv}\|\underline{v}_h\|^r_{\varepsilon,r,h}, \quad a_h(\underline{w}_h,\underline{v}_h) \le C_{\rm gr}\|\underline{w}_h\|^{r-1}_{\varepsilon,r,h}\|\underline{v}_h\|_{\varepsilon,r,h},$$

$$\sup_{\underline{v}_h\in\underline{U}^k_{h,0}, \|\underline{v}_h\|_{\varepsilon,r,h}=1}\left| \int_\Omega v_h\cdot(\nabla\cdot\sigma(\nabla_{\rm s}w)) + a_h(\underline{I}^k_h w,\underline{v}_h) \right| \le C_{\rm a1}(w)h^{k+1} + C_{\rm a2}(w)h^{(k+1)(r-1)}.$$

We define the bilinear form $b_h : \underline{U}^k_h \times \mathbb{P}^k(\mathcal{T}_h,\mathbb{R}) \to \mathbb{R}$ such that, for all $(\underline{v}_h, q_h) \in \underline{U}^k_h \times \mathbb{P}^k(\mathcal{T}_h,\mathbb{R})$,

$$b_h(\underline{v}_h,q_h) := -\sum_{T\in\mathcal{T}_h}\int_T D^k_T\underline{v}_T\,(q_h)_{|T}.$$

We can prove an inf-sup condition and a consistency property for the function $b_h$. Indeed, for all $q_h \in P^k_h$ and all $q \in W^{1,r'}(\Omega,\mathbb{R}) \cap W^{k+1,r'}(\mathcal{T}_h,\mathbb{R})$, there exists $C_{\rm b}(q) > 0$ independent of $h$ such that,

$$\|q_h\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim \sup_{\underline{v}_h\in\underline{U}^k_{h,0}, \|\underline{v}_h\|_{\varepsilon,r,h}=1} b_h(\underline{v}_h,q_h), \qquad \sup_{\underline{v}_h\in\underline{U}^k_{h,0}, \|\underline{v}_h\|_{\varepsilon,r,h}=1}\left| \int_\Omega \nabla_h q\cdot v_h - b_h(\underline{v}_h,\pi^k_h q) \right| \le C_{\rm b}(q)h^{k+1}.$$

The discrete problem reads: Find $(\underline{u}_h, p_h) \in \underline{U}^k_{h,0} \times P^k_h$ such that,

$$\begin{cases} a_h(\underline{u}_h,\underline{v}_h) + b_h(\underline{v}_h,p_h) = \int_\Omega f\cdot v_h & \forall\underline{v}_h \in \underline{U}^k_{h,0},\\ -b_h(\underline{u}_h,q_h) = 0 & \forall q_h \in P^k_h. \end{cases}$$

### Error estimate

**Theorem:** Let $(u,p) \in W^{1,r}_0(\Omega,\mathbb{R}^d) \times L^{r'}_0(\Omega,\mathbb{R})$ and $(\underline{u}_h,p_h) \in \underline{U}^k_{h,0} \times P^k_h$ the solutions of the continuous and discrete problem. Assume the regularity $(u,p) \in W^{k+2,r}(\mathcal{T}_h,\mathbb{R}^d) \times (W^{1,r'}(\Omega,\mathbb{R}) \cap W^{k+1,r'}(\mathcal{T}_h,\mathbb{R}))$. Then, setting $\underline{e}_h := \underline{u}_h - \underline{I}^k_h u$ and $\epsilon_h := p_h - \pi^k_h p$, there exists $C_1(u,p), ..., C_5(u,p) > 0$ independent of $h$ such that

$$\|\underline{e}_h\|_{\varepsilon,r,h} \le C_1(u,p)h^{k+1} + C_2(u,p)h^{(k+1)\min(\frac{1}{r-1},r-1)},$$

$$\|\epsilon_h\|_{L^{r'}(\Omega,\mathbb{R})} \le C_3(u,p)h^{k+1} + C_4(u,p)h^{(k+1)\min(\frac{1}{r-1},r-1)} + C_5(u,p)h^{(k+1)(r-1)\max(1,r-1)}.$$

We can deduce that, the discrete velocity converges with an asymptotic order of

$$O^k_{\rm vel} := \begin{cases} \frac{k+1}{r-1} & \text{if } r \ge 2,\\ (k+1)(r-1) & \text{if } r < 2, \end{cases}$$

and the pressure converges with an asymptotic order of

$$O^k_{\rm pre} := \begin{cases} \frac{k+1}{r-1} & \text{if } r \ge 2,\\ (k+1)(r-1)^2 & \text{if } r < 2. \end{cases}$$

## Numerical Results

We demonstrate the asymptotic order of convergence for the velocity given by the above theorem. We solve the non-linear elasticity problem on the unit square $\Omega = (0,1)^2$ with, for all $(x,y) \in \Omega$,

$$u(x,y) = (\cos(x)e^y, \sin(x)e^y).$$

The parameter $\kappa$ is taken equal to 1 and we vary the flow behavior index $r \in (1,+\infty)$.

$k = 1$

| $r = \frac{5}{4}$ $O^k_{\rm vel} = \frac{1}{2}$ | | | | $r = 2$ $O^k_{\rm vel} = 2$ | | | | $r = 3$ $O^k_{\rm vel} = 1$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ndofs | nnz | $\|\underline{e}_h\|_{\varepsilon,r,h}$ | EOC | ndofs | nnz | $\|\underline{e}_h\|_{\varepsilon,r,h}$ | EOC | ndofs | nnz | $\|\underline{e}_h\|_{\varepsilon,r,h}$ | EOC |
| 268 | 4848 | 5.39e-05 | – | 268 | 4848 | 1.59e-03 | – | 268 | 4848 | 4.63e-02 | – |
| 1148 | 21936 | 6.36e-06 | 3.08 | 1148 | 21936 | 4.31e-04 | 1.89 | 1148 | 21936 | 1.79e-02 | 1.38 |
| 4732 | 92592 | 6.73e-07 | 3.24 | 4732 | 92592 | 1.08e-04 | 2.00 | 4732 | 92592 | 7.24e-03 | 1.30 |
| 18548 | 366864 | 7.80e-08 | 3.11 | 18548 | 366864 | 2.83e-05 | 1.93 | 18548 | 366864 | 2.96e-03 | 1.29 |
| 76096 | 1513728 | 7.89e-09 | 3.30 | 76096 | 1513728 | 7.00e-06 | 2.02 | 76096 | 1513728 | 1.17e-03 | 1.34 |

$k = 2$

| $r = \frac{5}{4}$ $O^k_{\rm vel} = \frac{3}{4}$ | | | | $r = 2$ $O^k_{\rm vel} = 3$ | | | | $r = 3$ $O^k_{\rm vel} = \frac{3}{2}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ndofs | nnz | $\|\underline{e}_h\|_{\varepsilon,r,h}$ | EOC | ndofs | nnz | $\|\underline{e}_h\|_{\varepsilon,r,h}$ | EOC | ndofs | nnz | $\|\underline{e}_h\|_{\varepsilon,r,h}$ | EOC |
| 268 | 4848 | 1.24e-07 | – | 268 | 4848 | 4.14e-05 | – | 268 | 4848 | 6.73e-03 | – |
| 1148 | 21936 | 5.02e-09 | 4.63 | 1148 | 21936 | 5.47e-06 | 2.92 | 1148 | 21936 | 1.56e-03 | 2.11 |
| 4732 | 92592 | 1.52e-10 | 5.05 | 4732 | 92592 | 6.83e-07 | 3.00 | 4732 | 92592 | 4.16e-04 | 1.91 |
| 18548 | 366864 | 6.34e-12 | 4.58 | 18548 | 366864 | 9.12e-08 | 2.90 | 18548 | 366864 | 1.07e-04 | 1.95 |
| 76096 | 1513728 | 2.15e-13 | 4.88 | 76096 | 1513728 | 1.12e-08 | 3.03 | 76096 | 1513728 | 2.68e-05 | 2.01 |

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