Virtual Elements for magneto-static problems

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Outline

The continuous problem

$\Omega \subset \mathbb{R}^3$ (simply connected) computational domain
given $\mathbf{j} \in (L^2(\Omega))^3$(with $\mathrm{div}\,\mathbf{j} = 0$), and $\mu = \mu(x) \geq \mu_0 > 0$

$$\begin{cases} \text{find } \mathbf{H} \in H(\mathbf{curl};\Omega) \text{ and } \mathbf{B} \in H(\mathrm{div};\Omega) \text{ such that:} \\ \mathbf{curl}\,\mathbf{H} = \mathbf{j} \text{ and } \mathrm{div}\,\mathbf{B} = 0, \text{ with } \mathbf{B} = \mu\mathbf{H}, \text{ in } \Omega \\ \text{with the boundary conditions } \mathbf{H} \wedge \mathbf{n} = 0 \text{ on } \partial\Omega \end{cases}$$

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Among the various formulations we chose (see Kikuchi 89)

$$\begin{cases} \text{find } \mathbf{H} \in H_0(\mathbf{curl};\Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \mathbf{curl}\,\mathbf{H} \cdot \mathbf{curl}\,\mathbf{v}\,\mathrm{d}\Omega + \int_\Omega \nabla p \cdot \mu\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega \mathbf{j} \cdot \mathbf{curl}\,\mathbf{v}\,\mathrm{d}\Omega \quad \forall \mathbf{v} \in H_0(\mathbf{curl};\Omega) \\ \displaystyle\int_\Omega \nabla q \cdot \mu\mathbf{H}\,\mathrm{d}\Omega = 0 \quad \forall q \in H_0^1(\Omega). \end{cases}$$

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For existence and uniqueness we need:

Inf-Sup $\quad \forall q \in H^1_0(\Omega)\ \exists \mathbf{v}\in H_0(\mathbf{curl};\Omega) : \dfrac{\int_\Omega \nabla q\cdot\mathbf{v}}{\|\mathbf{v}\|_{H_0(\mathbf{curl};\Omega)}} \geq \beta \|\nabla q\|_{L^2(\Omega)}$

Ell-Ker $\quad \displaystyle\int_\Omega |\mathbf{curl}\mathbf{v}|^2 \geq \alpha \|\mathbf{v}\|^2_{H_0(\mathbf{curl};\Omega)} \quad \forall \mathbf{v} \text{ with } \mathrm{div}\mathbf{v} = 0$

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They both hold true since the following sequence is exact:

$$\mathbb{R} \xrightarrow{\mathrm{i}} H^1(\Omega) \xrightarrow{\mathbf{grad}} H(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\mathrm{o}} 0$$

Unique solution $(\mathbf{H}, p)$ with $p \equiv 0$, $\mathbf{curl}\,\mathbf{H} = \mathbf{j}$, $\mathrm{div}\mu\mathbf{H} = 0$.

Towards the discrete problem

Given a decomposition $\mathcal{T}_h$ of Ω into polyhedra P, we need to define spaces

$V^{\mathrm{node}} \subset H^1_0(\Omega)$, $V^{\mathrm{edge}} \subset H_0(\mathbf{curl};\Omega)$, $V^{\mathrm{face}} \subset H(\mathrm{div};\Omega)$, and $V^{\mathrm{vol}} \subset L^2(\Omega)$

such that:

- they form an exact sequence

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- They have good approximation properties

Towards the discrete problem

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- They have good approximation properties

1) the discrete spaces will be defined element-wise on each polyhedron P, and then glued as in the standard Finite Element procedure.

2) we will start by defining the traces of these spaces on the faces of each polyhedron, that is, on a generic polygon.

A 2D space $\widetilde{V}^{\text{node}}(F) \subset H^1(F)$

Let F be a polygon. We define the nodal space as:

$$\widetilde{V}^{\text{node}}(F) := \Big\{ q \in C^0(\overline{F}) : \; q_{|e} \in \mathbb{P}_1(e) \, \forall e \in \partial F, \; \Delta q = 0 \Big\}.$$

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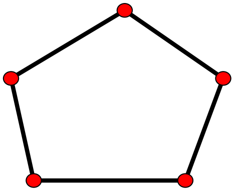

Degrees of freedom

● : values at the vertices (imply global continuity when gluing spaces on adjacent polygons)

(easy to check unisovence of the d.o.f.s)

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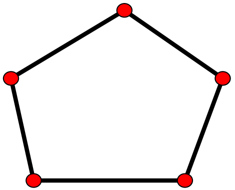

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- the functions in $\widetilde{V}^{\mathrm{node}}(F)$ are known on ∂F but not inside
- $\mathbb{P}_1(F) \subset \widetilde{V}^{\mathrm{node}}(F)$ (good for approximation)

What can we compute in $\widetilde{V}^{\text{node}}(F)$?

The functions in $\widetilde{V}^{\text{node}}(F)$ are not known inside F. How can we compute relevant quantities needed in the approximation?

We can compute the average of ∇q:

$$\int_F \nabla q \mathrm{d}F = \int_{\partial F} q\mathbf{n}\,\mathrm{d}s \qquad \forall q \in \widetilde{V}^{\text{node}}(F)$$

What about the average of q?

$$\int_F q\,\mathrm{d}x = ??$$

$$\boxed{\frac{1}{2}\int_F q\,\mathrm{div}\mathbf{x_F}\mathrm{d}F = \frac{1}{2}\Big(-\int_F \nabla q \cdot \mathbf{x_F}\mathrm{d}F + \int_{\partial F} q\mathbf{x_F}\cdot\mathbf{n}\,\mathrm{d}s\Big)}$$

where $\mathbf{x_F} = \mathbf{x} - \mathbf{b}_F$, with $\mathbf{b}_F =$ barycenter of F.

A NEW 2D SPACE $V^{\rm node}(F) \subset H^1(F)$

$$V^{\rm node}(F) := \Big\{ q \in C^0(\overline{F}) : \ q_{|e} \in \mathbb{P}_1(e) \, \forall e \in \partial F, \ \Delta q \in \mathbb{P}_0,$$
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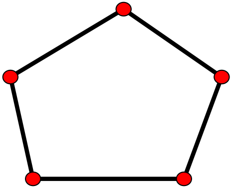

Degrees of freedom

● : values at the vertices

Note that still $\mathbb{P}_1(F) \subset V^{\mathrm{node}}(F)$!

(easy to check unisovence of the d.o.f.s)

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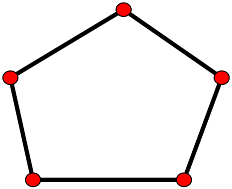

Degrees of freedom

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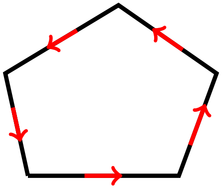
$$\boxed{\frac{1}{2}\int_F q \,\mathrm{div}\mathbf{x_F} \mathrm{d}F = \frac{1}{2}\Big(\int_F \nabla q \cdot \mathbf{x_F} \mathrm{d}F + \int_{\partial F} q \mathbf{x_F} \cdot \mathbf{n} \,\mathrm{d}s \Big)}$$

A 2D SPACE $\widetilde{V}^{\rm edge}(F) \subset H(\mathrm{rot}, F)$

$$\widetilde{V}^{\rm edge}(F) := \Big\{ \mathbf{v} |\, \mathrm{div}\mathbf{v} = 0,\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_0(F),\ \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_0(e)\, \forall e \in \partial F \Big\}.$$



Degrees of freedom

$\rightarrow$: value of the tangential component (imply global continuity of the tangential component when gluing spaces on adjacent polygons)

(easy to check unisolvence)

- the tangential components are known
- $[\mathbb{P}_0(F)]^2 \subseteq \widetilde{V}^{\rm edge}(F)$ and also $N_0^{1st}(F) \subseteq \widetilde{V}^{\rm edge}(F)$

Recall: $N_0^{1st}(F) = span\{(1,0),(0,1),(y,-x)\}$

NOTE: for $\mathbf{v} \in N_0^{1st}(F)$ we have $\int_F \mathbf{v} \cdot \mathbf{x_F} \mathrm{d}F = 0$

A new 2D space $V^{\mathrm{edge}}(F) \subset H(\mathrm{rot}, F)$

$$V^{\mathrm{edge}}(F) := \Big\{ \mathbf{v} |\ \mathrm{div}\mathbf{v} \in \mathbb{P}_0(F),\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_0(F),\ \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_0(e)\ \forall e \in \partial F,$$
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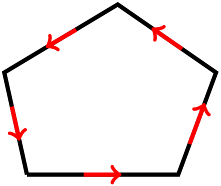


Degrees of freedom

$\longrightarrow$: value of the tangengial component

still $N_0^{1st}(F) \subset V^{\mathrm{edge}}(F)$

Integrals against linear polynomials

$$V^{\mathrm{edge}}(F) := \Big\{ \mathbf{v} |\ \mathrm{div}\mathbf{v} \in \mathbb{P}_0(F),\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_0(F),\ \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_0(e)\ \forall e \in \partial F,$$
$$\int_F \mathbf{v} \cdot \mathbf{x_F} \mathrm{d}F = 0 \Big\}$$

Observe that any $\mathbf{p}_1 \in [\mathbb{P}_1(F)]^2$ can be written as

$$\mathbf{p}_1 = \mathrm{rot}\, p_2 + p_0 \mathbf{x_F}$$

Hence, $\forall \mathbf{v} \in V^{\mathrm{edge}}(F)$ we can compute

$$\int_F \mathbf{v} \cdot \mathbf{p}_1 = \int_F \mathbf{v} \cdot (\mathrm{rot}\, p_2 + p_0 \mathbf{x_F})$$
$$\underbrace{\int_F \mathrm{rot}\mathbf{v}\, p_2}_{\text{computable}} + \underbrace{\int_{\partial F} (\mathbf{v} \cdot \mathbf{t}) p_2}_{\text{computable}} + p_0 \underbrace{\int_F \mathbf{v} \cdot \mathbf{x_F}}_{0}$$

The 2D exact sequence

Exact sequence $\quad \mathbb{R} \xrightarrow{\text{i}} V^{\text{node}}(F) \xrightarrow{\textbf{grad}} V^{\text{edge}}(F) \xrightarrow{\text{rot}} \mathbb{P}_0(F) \xrightarrow{\text{o}} 0$

$$V^{\text{node}}(F) := \Big\{ q \in C^0(\overline{F}) : \; q_{|e} \in \mathbb{P}_1(e) \, \forall e \in \partial F, \; \Delta q \in \mathbb{P}_0, \text{ and } \int_F \nabla q \cdot \mathbf{x_F} \mathrm{d}F = 0 \Big\}.$$

D.O.F: Vertex values (uniquely identify q on ∂F)

$$V^{\text{edge}}(F) := \Big\{ \mathbf{v} | \; \text{div}\mathbf{v} \in \mathbb{P}_0(F), \; \text{rot}\mathbf{v} \in \mathbb{P}_0(F), \; \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_0(e) \, \forall e \in \partial F, \int_F \mathbf{v} \cdot \mathbf{x_F} \mathrm{d}F = 0 \Big\}.$$

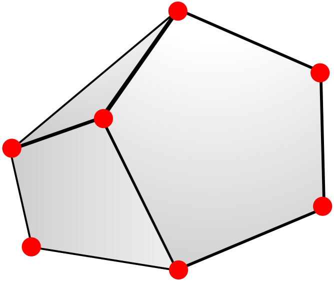
D.O.F: Midpoint tangent values (uniquely identify $\mathbf{v} \cdot \mathbf{t}$ on ∂F)

The 3D space $V^{\mathrm{node}}(\mathrm{P}) \subset \mathrm{H}^1(\mathrm{P})$

Let P be a generic polyhedron of the decomposition of Ω.
The *nodal* space is:

$$V^{\mathrm{node}}(\mathrm{P}) := \{\mathrm{q} \in \mathrm{C}^0(\overline{\mathrm{P}}) : \ \mathrm{q}_{|\mathrm{F}} \in \mathrm{V}^{\mathrm{node}}(\mathrm{F}) \ \forall \mathrm{F} \in \partial\mathrm{P}, \Delta\mathrm{q} = 0\}$$

- clearly $\mathbb{P}_1(\mathrm{P}) \subseteq \mathrm{V}^{\mathrm{node}}(\mathrm{P})$



Degrees of freedom

• : value at the vertices

$\Longrightarrow$

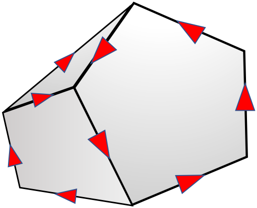
global space $V_h^{\mathrm{node}}(\Omega) \subset H^1(\Omega)$

THE 3D SPACE $V^{\mathrm{edge}}(\mathrm{P}) \subset \mathrm{H}(\mathbf{curl};\mathrm{P})$

The *edge* space is:

$$V^{\mathrm{edge}}(\mathrm{P}) := \{\mathbf{v} \in H(\mathbf{curl};\mathrm{P}) : \ (\mathbf{v}_{|\mathrm{F}})_{\mathrm{tang}} \in \mathrm{V}^{\mathrm{edge}}(\mathrm{F}) \ \forall \mathrm{F} \in \partial\mathrm{P},$$
$$\mathbf{v}\cdot\mathbf{t} \text{ continuous on each edge } e \in \partial\mathrm{P}$$
$$\mathrm{div}\mathbf{v} = 0, \ \mathbf{curl}(\mathbf{curl}\mathbf{v}) \in [\mathbb{P}_0(\mathrm{P})]^3,$$
$$\int_{\mathrm{P}} (\mathbf{curl}\mathbf{v})\cdot(\mathbf{x}_{\mathrm{P}} \wedge \mathbf{p}_0) = 0 \ \forall \mathbf{p}_0 \in [\mathbb{P}_0(\mathrm{P})]^3\}$$

- clearly $[\mathbb{P}_0(\mathrm{P})]^3 \subseteq \mathrm{V}^{\mathrm{edge}}(\mathrm{P})$, and $N_0^{1st}(\mathrm{P}) \equiv \mathbf{p}_0 + \mathbf{x}_{\mathrm{P}} \wedge \mathbf{q}_0 \subset \mathrm{V}^{\mathrm{edge}}(\mathrm{P})$



Degrees of freedom
value of the tangential component (constant) on each edge

$\Longrightarrow$

global space $V_h^{\mathrm{edge}}(\Omega) \subset H(\mathbf{curl};\Omega)$

A local projection on constant vector fields

Out of the above d.o.f. we can compute the $(L^2(\mathrm{P}))^3$-orthogonal projection Π_0 from $V^{\mathrm{edge}}(\mathrm{P})$ to $(\mathbb{P}_0(\mathrm{P}))^3$.

A local projection on constant vector fields

Out of the above d.o.f. we can compute the $(L^2(\mathrm{P}))^3$-orthogonal projection Π_0 from $V^{\mathrm{edge}}(\mathrm{P})$ to $(\mathbb{P}_0(\mathrm{P}))^3$.
Indeed, since $\mathbf{p_0} = \mathbf{curl}(\mathbf{x}_\mathrm{P} \wedge \boldsymbol{q_0})$ with $\boldsymbol{q}_0 = -\frac{1}{2}\boldsymbol{p}_0$,

$$\int_\mathrm{P} \Pi_0\mathbf{v}\cdot\mathbf{p_0}\mathrm{dP} := \int_\mathrm{P} \mathbf{v}\cdot\mathbf{p_0}\mathrm{dP} = \int_\mathrm{P} \mathbf{v}\cdot\mathbf{curl}(\mathbf{x}_\mathrm{P}\wedge\boldsymbol{q_0})\mathrm{dP}$$

$$= \int_\mathrm{P} \mathbf{curl}\mathbf{v}\cdot(\mathbf{x}_\mathrm{P}\wedge\boldsymbol{q}_0)\mathrm{dP} + \int_{\partial\mathrm{P}} (\mathbf{v}\wedge\mathbf{n})\cdot(\mathbf{x}_\mathrm{P}\wedge\boldsymbol{q}_0)\,\mathrm{d}S$$

$$= \qquad 0 \qquad + \int_{\partial\mathrm{P}} \Big(\mathbf{n}\wedge(\mathbf{x}_\mathrm{P}\wedge\boldsymbol{q}_0)\Big)\cdot\mathbf{v}\,\mathrm{d}S$$

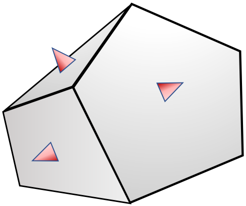
$$= \sum_F \int_F \Big(\mathbf{n}\wedge(\mathbf{x}_\mathrm{P}\wedge\boldsymbol{q}_0)\Big)^\tau\cdot\mathbf{v}^\tau\mathrm{d}F$$

The 3D space $V^{\mathrm{face}}(\mathrm{P}) \subset \mathrm{H}(\mathrm{div};\mathrm{P})$

The *face* space is:

$$V^{\mathrm{face}}(\mathrm{P}) := \{\mathbf{w} \in H(\mathrm{div};\mathrm{P}) : \ (\mathbf{w}_{\mathrm{F}} \cdot \mathbf{n}_{\mathrm{F}}) \in \mathbb{P}_0(\mathrm{F}) \ \forall \mathrm{F} \in \partial\mathrm{P},$$
$$\mathrm{div}\mathbf{w} \in \mathbb{P}_0(\mathrm{P}), \ \mathbf{curl}\mathbf{w} \in [\mathbb{P}_0(\mathrm{P})]^3,$$
$$\int_{\mathrm{P}} \mathbf{w} \cdot (\mathbf{x}_{\mathrm{P}} \wedge \mathbf{p}_0) = 0 \ \forall \mathbf{p}_0 \in [\mathbb{P}_0(\mathrm{P})]^3\}$$

- clearly $[\mathbb{P}_0(\mathrm{P})]^3 \subseteq \mathrm{V}^{\mathrm{face}}(\mathrm{P})$, and $RT_0(\mathrm{P}) \equiv \mathbf{p}_0 + \mathbf{x}_{\mathrm{P}} \mathrm{q}_0 \subseteq \mathrm{V}^{\mathrm{face}}(\mathrm{P})$



Degrees of freedom
value of the normal component
(constant) on each face

$\Longrightarrow$

global space $V_h^{\mathrm{face}}(\Omega) \subset H(\mathrm{div};\Omega)$

The global spaces

• $\mathcal{T}_h$ = decomposition of Ω into polyhedra P, μ constant on each P
The global spaces are defined as in FEM:

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$$V_h^{\mathrm{node}} := \{q \in H_0^1(\Omega) : \ q_{|\mathrm{P}} \in V^{\mathrm{node}}(\mathrm{P}) \ \forall \mathrm{P} \in \mathcal{T}_{\mathrm{h}}\}$$

$$V_h^{\mathrm{edge}} := \{\mathbf{v} \in H_0(\mathbf{curl};\Omega) : \ \mathbf{v}_{|\mathrm{P}} \in V^{\mathrm{edge}}(\mathrm{P}) \ \forall \mathrm{P} \in \mathcal{T}_{\mathrm{h}}\}$$

$$V_h^{\mathrm{face}} := \{\mathbf{w} \in H(\mathrm{div};\Omega) : \ \mathbf{w}_{|\mathrm{P}} \in V^{\mathrm{face}}(\mathrm{P}) \ \forall \mathrm{P} \in \mathcal{T}_{\mathrm{h}}\}$$

$$V_h^{\mathrm{vol}} := \{\varphi \in L^2(\Omega) : \ \varphi_{|\mathrm{P}} \in \mathbb{P}_0(\mathrm{P}) \ \forall \mathrm{P} \in \mathcal{T}_{\mathrm{h}}\}$$

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$$V_h^{\text{vol}} := \{\varphi \in L^2(\Omega) : \; \varphi_{|\mathrm{P}} \in \mathbb{P}_0(\mathrm{P}) \; \forall \mathrm{P} \in \mathcal{T}_\mathrm{h}\}$$

One can prove [Beirão da Veiga, Brezzi, Dassi, M, Russo, CMAME 2018]

Exact sequence

The sequence

$$\mathbb{R} \xrightarrow{\text{i}} V_h^{\text{node}} \xrightarrow{\mathbf{grad}} V_h^{\text{edge}} \xrightarrow{\mathbf{curl}} V_h^{\text{face}} \xrightarrow{\mathrm{div}} V_h^{\text{vol}} \xrightarrow{\text{o}} 0$$

is exact

Discrete problem. We would like to write:

$$\begin{cases} \text{given } \mathbf{j} \in H(\mathrm{div};\Omega) \quad (\text{with } \mathrm{div}\mathbf{j} = 0 \text{ in } \Omega), \qquad \text{and } \mu = \mu(\mathbf{x}) \geq \mu_0 > 0, \\ \text{find } \mathbf{H}_h \in V_h^{\mathrm{edge}} \text{ and } p_h \in V_h^{\mathrm{node}} \text{ such that:} \\ \displaystyle\int_\Omega \mathbf{curl}\mathbf{H}_h \cdot \mathbf{curl}\mathbf{v}\,\mathrm{d}\Omega + \int_\Omega \nabla p_h \cdot \mu\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega \mathbf{j}\cdot\mathbf{curl}\mathbf{v}\,\mathrm{d}\Omega \quad \forall \mathbf{v} \in V_h^{\mathrm{edge}} \\ \displaystyle\int_\Omega \nabla q \cdot \mu\mathbf{H}_h\,\mathrm{d}\Omega = 0 \quad \forall q \in V_h^{\mathrm{node}}. \end{cases}$$

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Instead we will write

$$\begin{cases} \text{find } \mathbf{H}_h \in V_h^{\mathrm{edge}} \text{ and } p_h \in V_h^{\mathrm{node}} \text{ such that:} \\ [\mathbf{curl}\mathbf{H}_h, \mathbf{curl}\mathbf{v}]_{\mathrm{face}} + [\nabla p_h, \mu\mathbf{v}]_{\mathrm{edge}} = [\mathbf{j}_I, \mathbf{curl}\mathbf{v}]_{\mathrm{face}} \quad \forall \mathbf{v} \in V_h^{\mathrm{edge}} \\ [\nabla q, \mu\mathbf{H}_h]_{\mathrm{edge}} = 0 \quad \forall q \in V_h^{\mathrm{node}}. \end{cases}$$

after defining a suitable $\mathbf{j}_I \in V_h^{\mathrm{face}}$ and approximate L^2-scalar products.

Scalar product in V_h^{edge}

We saw that in each element P we can project onto constants.

Then we can define an edge scalar product $[\mathbf{v},\mathbf{w}]_{\mathrm{P}}^{\mathrm{edge}} \simeq \int_{\mathrm{P}} \mathbf{v}\cdot\mathbf{w}\mathrm{dP}$:

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where $s_{\mathrm{P}}(\mathbf{v},\mathbf{w})$ is a symmetric and positive definite bilinear form.
For instance:

$$s_{\mathrm{P}}(\mathbf{v},\mathbf{w}) = |\mathrm{P}| \sum_{\mathrm{i}=1}^{\#\mathrm{edges}} \mathrm{DOF}_{\mathrm{i}}(\mathbf{v})\mathrm{DOF}_{\mathrm{i}}(\mathbf{w})$$

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(Note: the face scalar product is handled analogously)

Consistency and Stability

CONSISTENCY: For all P, and for all $\mathbf{v} \in V^{\mathrm{edge}}(\mathrm{P})$ and $\mathbf{p}_0 \in [\mathbb{P}_0(\mathrm{P})]^3$

$$[\mathbf{v}, \mathbf{p}_0]_{\mathrm{P}}^{\mathrm{edge}} = \int_{\mathrm{P}} \Pi^0 \mathbf{v} \Pi^0 \mathbf{p}_0 \mathrm{dP} + \mathrm{s_P}(\mathbf{v} - \Pi^0 \mathbf{v}, \mathbf{p}_0 - \Pi^0 \mathbf{p}_0)$$

$$= \int_{\mathrm{P}} \Pi^0 \mathbf{v} \cdot \mathbf{p}_0 \mathrm{dP} = (\mathbf{v}, \mathbf{p}_0)_{0,\mathrm{P}}$$

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$$= \int_{\mathrm{P}} \Pi^0\mathbf{v} \cdot \mathbf{p}_0 \mathrm{dP} = (\mathbf{v}, \mathbf{p}_0)_{0,\mathrm{P}}$$

STABILITY: under suitable mesh assumptions

$$c_*\|\mathbf{v}\|_{0,\mathrm{P}}^2 \leq s_{\mathrm{P}}(\mathbf{v}, \mathbf{v}) \leq c^*\|\mathbf{v}\|_{0,\mathrm{P}}^2 \qquad \forall \mathbf{v} \in V^{\mathrm{edge}}(\mathrm{P})$$

for some constants $c^* \geq c_* > 0$ independent of h_{P}. Thus,

$$c_*\|\mathbf{v}\|_{0,\mathrm{P}}^2 \leq [\mathbf{v}, \mathbf{v}]_{\mathrm{P}}^{\mathrm{edge}} \leq c^*\|\mathbf{v}\|_{0,\mathrm{P}}^2 \qquad \forall \mathbf{v} \in V^{\mathrm{edge}}(\mathrm{P})$$

The discrete problem

Given a decomposition$\mathcal{T}_h$ of Ω into polyhedra, the final discrete problem is

$$\begin{cases} \text{find } \mathbf{H}_h \in V_h^{\text{edge}} \text{ and } p_h \in V_h^{\text{node}} \text{ such that:} \\ [\mathbf{curl}\mathbf{H}_h, \mathbf{curl}\mathbf{v}]_{\text{face}} + [\nabla p_h, \mu\mathbf{v}]_{\text{edge}} = [\mathbf{j}_I, \mathbf{curl}\mathbf{v}]_{\text{face}} \quad \forall \mathbf{v} \in V_h^{\text{edge}} \\ [\nabla q, \mu\mathbf{H}_h]_{\text{edge}} = 0 \quad \forall q \in V_h^{\text{node}}. \end{cases}$$

where

- the face and edge scalar products are built as shown above
- $\mathbf{j}_I$ is the standard DOF-interpolant of $\mathbf{j}$ in V_h^{face}

The exact sequence property guarantees existence-uniqueness of the solution $(\mathbf{H}_h, p_h)$ with $p_h = 0$.

Convergence results

Let:

$$\|\mathbf{v}\|_{0,\Omega}^2 := \int_\Omega \mu|\mathbf{v}|^2 \quad \forall \mathbf{v} \in [L^2(\Omega)]^2$$

and assume that

- all the elements are (uniformly) star-shaped with respect to a ball of radius $\geq \gamma h_{\rm P}$, for some positive γ
- every face is star-shaped with respect to a ball of radius $\geq \gamma h_{\rm P}$, and every edge has length $\geq \gamma h_{\rm P}$

Theorem

The following estimate holds:

$$\|\mathbf{H} - \mathbf{H}_h\|_{0,\Omega} + \|\mathbf{curl}(\mathbf{H} - \mathbf{H}_h)\|_{0,\Omega} \leq C\, h \Big(\sum_{\rm P} |\mathbf{H}|_{1,{\rm P}}^2 + |\mathbf{j}|_{1,{\rm P}}^2 \Big)^{1/2}$$

Numerical results ($\mu = 1$)

PROBLEM 1 The domain is a truncated octahedron, and the exact solution is

$$\mathbf{H}(\mathbf{x},\,\mathbf{y},\,\mathbf{z}) := \frac{\mathbf{1}}{\pi}\begin{pmatrix} \sin(\pi y) - \sin(\pi z) \\ \sin(\pi z) - \sin(\pi x) \\ \sin(\pi x) - \sin(\pi y) \end{pmatrix}$$

The data **j** and $\mathbf{H} \wedge \mathbf{n}$ are set accordingly.

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The data $\mathbf{j}$ and $\mathbf{H} \wedge \mathbf{n}$ are set accordingly.

PROBLEM 2 $\Omega = [0,1]^3$, and the solution is

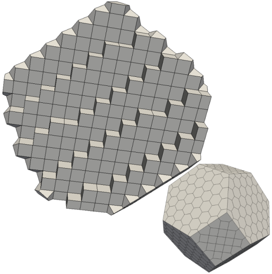
$$\mathbf{H}(\mathbf{x},\mathbf{y},\mathbf{z}) := \mathbf{curl}\,(\zeta(\mathbf{x},\mathbf{y},\mathbf{z}),\zeta(\mathbf{x},\mathbf{y},\mathbf{z}),\zeta(\mathbf{x},\mathbf{y},\mathbf{z}))$$

where

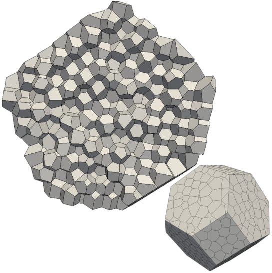
$$\zeta(x,y,z) := (x^2 - x)(y^2 - y)(z^2 - z)$$

The data $\mathbf{j}$ is set in accordance to the solution. The boundary conditions are "of Neumann type" $\mu\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

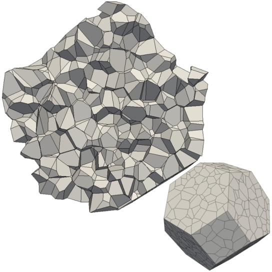
VORONOI MESH FAMILIES



Structured: structured seed distribution



Centroidal: each element seed corresponds to the element barycenter



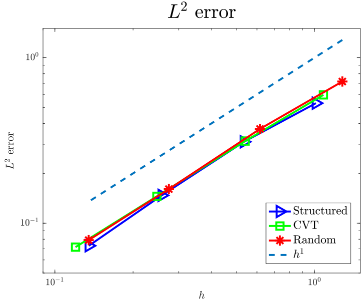
Random: random seed distribution

Convergence graphs

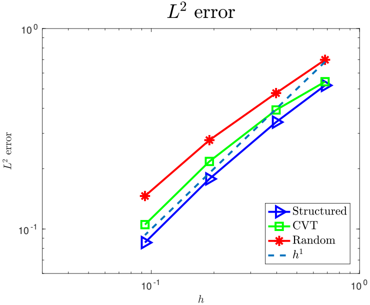
We compute the L^2-relative error on $\mathbf{H}$ as

$$\frac{||\mathbf{H} - \Pi_0 \mathbf{H}_h||_{0,\Omega}}{||\mathbf{H}||_{0,\Omega}}$$

PROBLEM 1

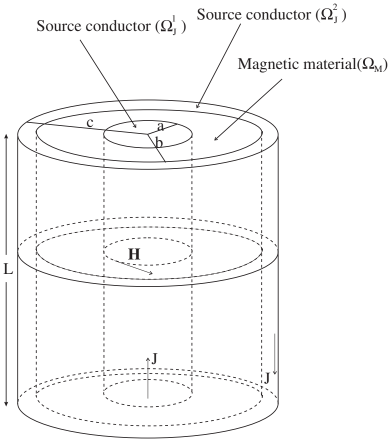


PROBLEM 2



The multiplier p_h vanishes up to machine precision

A simple benchmark (with known solution)



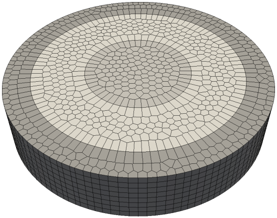
- constant electric current (of same intensity) in the two conductors
- permeability:

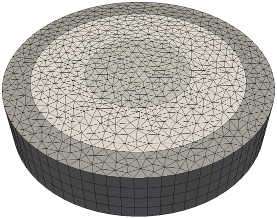
$$\mu = \begin{cases} \mu_0 & \text{in } \Omega_J^1 \cup \Omega_J^2 \\ 10^4 \mu_0 & \text{in } \Omega_M \end{cases}$$

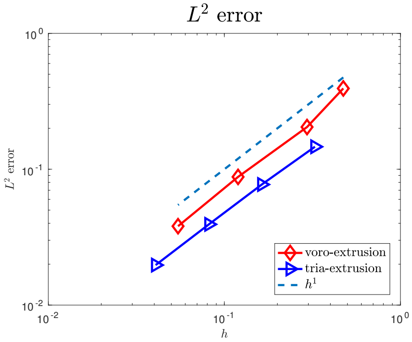
- boundary conditions $\mu H \cdot \mathbf{n} = 0$

[C. T. A. Jhonk, 88]

A simple benchmark problem (known solution)







A FAMILY OF NÉDÉLEC SECOND KIND VEM

Local spaces on the faces of polyhedra

Let $k \geq 1$. For each face f of P, the *edge* space on f is defined as

$$V_k^{\mathrm{edge}}(f) := \left\{ \mathbf{v} \in [L^2(f)]^2 : \mathrm{div}\mathbf{v} \in \mathbb{P}_k(f),\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_{k-1}(f),\ \mathbf{v}\cdot\mathbf{t}_e \in \mathbb{P}_k(e)\, \forall e \subset \partial f \right\}$$

with the degrees of freedom

- on each $e \subset \partial f$, the moments $\int_e (\mathbf{v} \cdot \mathbf{t}_e) p_k \,\mathrm{d}s \quad \forall p_k \in \mathbb{P}_k(e)$,
- the moments $\int_f \mathbf{v} \cdot \boldsymbol{x}_f\, p_k \,\mathrm{d}f \quad \forall p_k \in \mathbb{P}_k(f)$,
- $\int_f \mathrm{rot}\mathbf{v}\ p_{k-1}^0 \,\mathrm{d}f \quad \forall p_{k-1}^0 \in \mathbb{P}_{k-1}^0(f)$ (only for $k > 1$)),

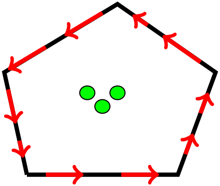
where $\boldsymbol{x}_f = \mathbf{x} - \mathbf{b}_f$, with $\mathbf{b}_f =$ barycenter of f.

• Note: with the serendipity version the d.o.f. $\int_f \mathbf{v} \cdot \boldsymbol{x}_f\, p_k \,\mathrm{d}f$ can be reduced

• Note: $N_k^{2nd}(f) \subset V_k^{\mathrm{edge}}(f)$

Example of d.o.f. for $k = 1$

Original VEM



Degrees of freedom

→ : value of the tangential component

● $= \int_f \mathbf{v} \cdot \boldsymbol{x}_f \; p_1 \, \mathrm{d}f$

Example of d.o.f. for $k = 1$

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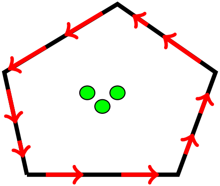

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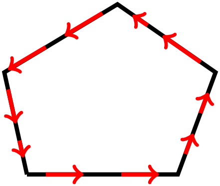
Serendipity VEM



Degrees of freedom

→ : value of the tangential component

$N_1^{2nd}(f) \subset V_1^{\mathrm{edge}}(f)$

A family of Nédélec second kind VEM

For each face f of P, the *nodal* space of order $k+1$ is defined as

$$V_{k+1}^{\mathrm{node}}(f) := \Big\{ q \in H^1(f) :\ q_{|e} \in \mathbb{P}_{k+1}(e)\, \forall e \subset \partial f,\ \Delta q \in \mathbb{P}_k(f) \Big\},$$

with the degrees of freedom

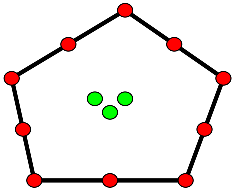
- for each vertex ν the value $q(\nu)$,
- for each edge e the moments $\int_e q\, p_{k-1}\, \mathrm{d}s \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(e)$,
- $\int_f (\nabla q \cdot \boldsymbol{x}_f)\, p_k\, \mathrm{d}f \quad \forall p_k \in \mathbb{P}_k(f)$.

• Note: with the serendipity version the d.o.f. $\int_f (\nabla q \cdot \boldsymbol{x}_f)\, p_k\, \mathrm{d}f$ can be reduced

• Note: $\mathbb{P}_{k+1}(f) \subset V_{k+1}^{\mathrm{node}}(f)$

Example of d.o.f. for $k = 1$

Original VEM

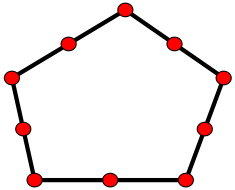


Degrees of freedom

● : values at vertices and midpoints

● $= \int_f (\nabla q \cdot \boldsymbol{x}_f)\, p_1$

Serendipity VEM



Degrees of freedom

● : values at vertices and midpoints

$\mathbb{P}_2(f) \subset V_2^{\mathrm{node}}(f)$

Local spaces on polyhedra

Let P be a polyhedron, simply connected with all its faces simply connected and convex.

$$V_k^{\mathrm{edge}}(\mathrm{P}) := \Big\{\mathbf{v} \in [\mathrm{L}^2(\mathrm{P})]^3 \;:\mathrm{div}\mathbf{v} \in \mathbb{P}_{\mathrm{k}-1}(\mathrm{P}),\ \mathbf{curl}(\mathbf{curl}\mathbf{v})) \in [\mathbb{P}_{\mathrm{k}}(\mathrm{P})]^3,$$
$$\mathbf{v}^\tau_{|f} \in V_k^{\mathrm{edge}}(\mathrm{f})\ \forall \text{ face } \mathrm{f} \subset \partial\mathrm{P},\ \mathbf{v}\cdot\mathbf{t}_{\mathrm{e}} \text{ continuous on each edge } \mathrm{e} \subset \partial\mathrm{P}\Big\},$$

$$V_{k+1}^{\mathrm{node}}(\mathrm{P}) := \Big\{\mathrm{q} \in \mathrm{C}^0(\mathrm{P}) : \mathrm{q}_{|\mathrm{f}} \in \mathrm{V}_{\mathrm{k}+1}^{\mathrm{node}}(\mathrm{f}) \quad \forall \text{ face } \mathrm{f} \subset \partial\mathrm{P},\ \Delta\,\mathrm{q} \in \mathbb{P}_{\mathrm{k}-1}(\mathrm{P})\Big\},$$

$$V_{k-1}^{\mathrm{face}}(\mathrm{P}) := \Big\{\mathbf{w} \in [\mathrm{L}^2(\mathrm{P})]^3 : \mathrm{div}\mathbf{w} \in \mathbb{P}_{\mathrm{k}-1},\ \mathbf{curl}\,\mathbf{w} \in [\mathbb{P}_{\mathrm{k}}]^3,\ \mathbf{w}\cdot\mathbf{n}_{\mathrm{f}} \in \mathbb{P}_{\mathrm{k}-1}(\mathrm{f})\ \forall \mathrm{f}\Big\}.$$

Internal d.o.f. in $V_k^{\rm edge}(\mathrm{P})$:

- $\displaystyle\int_{\mathrm{P}}(\mathbf{v}\cdot\mathbf{x}_{\mathrm{P}})p_{k-1}\,\mathrm{dP}\quad\forall \mathrm{p_{k-1}}\in\mathbb{P}_{\mathrm{k-1}}(\mathrm{P}),$
- $\displaystyle\int_{\mathrm{P}}(\mathbf{curl v})\cdot(\mathbf{x}_{\mathrm{P}}\wedge\mathbf{p}_k)\,\mathrm{dP}\quad\forall \mathbf{p}_{\mathrm{k}}\in[\mathbb{P}_{\mathrm{k}}(\mathrm{P})]^3.$

We can compute the $[L^2(\mathrm{P})]^3$-projection Π_k^0 from $V_k^{\rm edge}(\mathrm{P})$ to $[\mathbb{P}_k(\mathrm{P})]^3$. Hence we define a μ-dependent scalar product

$$[\mathbf{v},\mathbf{w}]_{edge}=(\mu\Pi_k^0\mathbf{v},\Pi_k^0\mathbf{w})_{0,\mathrm{P}}+h_{\mathrm{P}}\mu_0\sum_i(dof_i(I-\Pi_k^0)\mathbf{v}),dof_i(I-\Pi_k^0)\mathbf{w}),$$

Stability there exist two positive constants α_*,α^* independent of h_{P}:

$$\alpha_*\mu_0\|\mathbf{v}\|^2_{0,\mathrm{P}}\le\|\mathbf{v}\|^2_{edge}\le\alpha^*\mu_1\|\mathbf{v}\|_{0,\mathrm{P}}\qquad\forall\mathbf{v}\in V_k^{\rm e}(\mathrm{P}).$$

Consistency:

$$[\mathbf{v},\mathbf{p}_k]_{edge}=\int_{\mathrm{P}}\mu\Pi_k^0\mathbf{v}\cdot\mathbf{p}_k\mathrm{d}F\qquad\forall\mathbf{v}\in V_k^{\rm edge}(\mathrm{P}),\ \forall\mathbf{p}_{\mathrm{k}}\in[\mathbb{P}_{\mathrm{k}}(\mathrm{P})]^3.$$

Internal d.o.f. in $V_{k+1}^{\mathrm{node}}(\mathrm{P})$:

- the moments $\int_{\mathrm{P}} \nabla q \cdot \mathbf{x}_{\mathrm{P}}\, p_{k-1} \mathrm{dP} \quad \forall \mathrm{p_{k-1}} \in \mathbb{P}_{\mathrm{k-1}}(\mathrm{P})$.

These, together with the d.o.f. on the faces, allow to compute $L^2(\mathrm{P})$-projection from $V_{k+1}^{\mathrm{node}}(\mathrm{P})$ to $\mathbb{P}_{k-1}(\mathrm{P})$.

For $V_{k-1}^{\mathrm{face}}(\mathrm{P})$ we have the degrees of freedom

- $\forall$ face f: $\int_f (\mathbf{w}\cdot\mathbf{n}_f)p_{k-1}\,\mathrm{d}f \quad \forall p_{k-1}\in\mathbb{P}_{k-1}(f)$,
- $\int_{\mathrm{P}} \mathbf{w}\cdot(\mathbf{grad}\,p_{k-1})\mathrm{dP} \quad \forall \mathrm{p_{k-1}}\in\mathbb{P}_{\mathrm{k-1}}(\mathrm{P}),$ for $\mathrm{k}>1$
- $\int_{\mathrm{P}} \mathbf{w}\cdot(\mathbf{x}_{\mathrm{P}}\wedge\mathbf{p}_k)\mathrm{dP} \quad \forall \mathbf{p}_{\mathrm{k}}\in[\mathbb{P}_{\mathrm{k}}(\mathrm{P})]^3.$

From the above d.o.f we can compute the $[L^2(\mathrm{P})]^3$-projection Π_s^0 from $V_{k-1}^{\mathrm{face}}(\mathrm{P})$ to $[\mathbb{P}_s(\mathrm{P})]^3$ with $s\le k+1$.

$$\|\mathbf{v}\|_{face}^2 := \|\Pi_{k-1}^0\mathbf{v}\|_{0,\mathrm{P}}^2 + h_{\mathrm{P}}\sum_f \|(I-\Pi_{k-1}^0)\mathbf{v}\cdot\mathbf{n}_f\|_{0,f}^2 \simeq \|\mathbf{v}\|_0^2$$

The global spaces

$$V_{k+1}^{\text{node}} \equiv V_{k+1}^{\text{node}}(\Omega) := \left\{ q \in H_0^1(\Omega) \text{ such that } q_{|\mathrm{P}} \in V_{k+1}^{\text{node}}(\mathrm{P}) \; \forall \mathrm{P} \in \mathcal{T}_{\mathrm{h}} \right\},$$

$$V_k^{\text{edge}} \equiv V_k^{\text{edge}}(\Omega) := \left\{ \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \text{ such that } \mathbf{v}_{|\mathrm{P}} \in V_k^{\text{edge}}(\mathrm{P}) \forall \mathrm{P} \in \mathcal{T}_{\mathrm{h}} \right\},$$

$$V_{k-1}^{\text{face}} \equiv V_{k-1}^{\text{f}}(\Omega) := \left\{ \mathbf{w} \in H_0(\mathrm{div}; \Omega) \text{ such that } \mathbf{w}_{|\mathrm{P}} \in V_{k-1}^{\text{face}}(\mathrm{P}) \forall \mathrm{P} \in \mathcal{T}_{\mathrm{h}} \right\},$$

Exact sequence

The sequence

$$\mathbb{R} \xrightarrow{\mathrm{i}} V_{k+1}^{\text{node}} \xrightarrow{\textbf{grad}} V_k^{\text{edge}} \xrightarrow{\textbf{curl}} V_{k-1}^{\text{face}} \xrightarrow{\mathrm{div}} V_{k-1}^{\text{vol}} \xrightarrow{\mathrm{o}} 0$$

is exact

The discrete problem

$$\begin{cases} \text{find } \mathbf{H}_h \in V_k^{\text{edge}} \text{ and } p_h \in V_{k+1}^{\text{node}} \text{ such that:} \\ [\mathbf{curl}\mathbf{H}_h, \mathbf{curl}\mathbf{v}]_{V_{k-1}^{\text{face}}} + [\nabla p_h, \mathbf{v}]_{e,\mu} \; = \; [\mathbf{j}_I, \mathbf{curl}\mathbf{v}]_{V_{k-1}^{\text{face}}} \quad \forall \mathbf{v} \in V_k^{\text{edge}} \\ [\nabla q, \mathbf{H}_h]_{e,\mu} \; = \; 0 \quad \forall q \in V_{k+1}^{\text{node}}. \end{cases}$$

Theorem

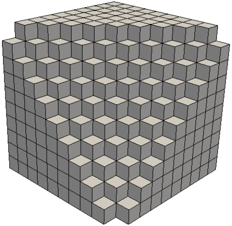
The discrete problem has a unique solution, and we have

$$\|\mathbf{H} - \mathbf{H}_h\|_{0,\Omega} \le C \left(\|\mathbf{H} - \mathbf{H}_I\|_{0,\Omega} + \|\Pi_k^0 \mathbf{H} - \mathbf{H}\|_{0,\Omega} + \|\mu\mathbf{H} - \Pi_k^0(\mu\mathbf{H})\|_{0,\Omega} \right),$$

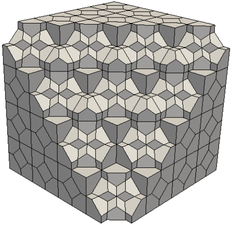
with C a constant depending on μ but independent of the mesh size. Moreover,

$$\|\mathbf{curl}(\mathbf{H} - \mathbf{H}_h)\|_{0,\Omega} = \|\mathbf{j} - \mathbf{j}_I\|_{0,\Omega}.$$

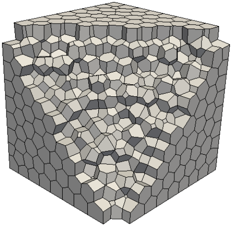
$\Omega = [0, 1]^3$. Example of meshes



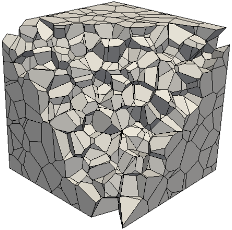
Cube



Nine



CVT



Random

Numerical results

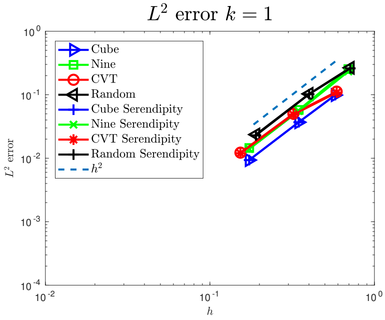
Problem 1: $\Omega = [0,1]^3, \mu = 1$. Exact solution

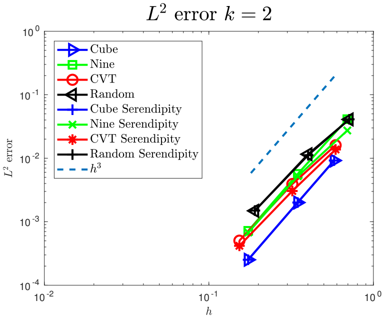
$$\mathbf{H}(x,\, y,\, z) := \frac{1}{\pi}\begin{pmatrix} \sin(\pi y) - \sin(\pi z) \\ \sin(\pi z) - \sin(\pi x) \\ \sin(\pi x) - \sin(\pi y) \end{pmatrix}$$

We compute the error

$$\frac{\|\mathbf{H} - \Pi_k^0 \mathbf{H}_h\|_{0,\Omega}}{\|\mathbf{H}\|_{0,\Omega}}.$$

CONVERGENCE CURVES





L^2-error for standard and serendipity approach: case $k = 1$ and $k = 2$.

Numerical results

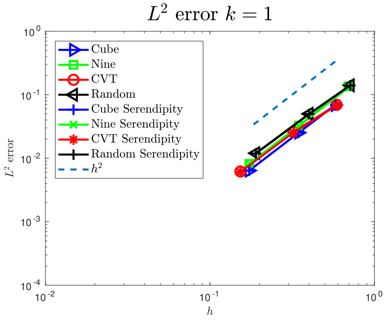
Problem 2: $\Omega = [0,1]^3, \mu(x,\ y,\ z) := 1 + x + y + z$. Exact solution

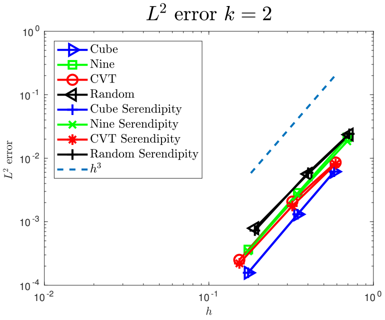
$$\mathbf{H}(x,\ y,\ z) := \frac{1}{(1+x+y+z)} \begin{pmatrix} \sin(\pi y) \\ \sin(\pi z) \\ \sin(\pi x) \end{pmatrix}$$

We compute the error

$$\frac{\|\mathbf{H} - \Pi_k^0 \mathbf{H}_h\|_{0,\Omega}}{\|\mathbf{H}\|_{0,\Omega}}.$$

Convegence curves





L^2-error for standard and serendipity approach: case $k = 1$ and $k = 2$

Conclusions

- We presented a lowest-order Virtual Element for magnetostatic problems which can be seen as the extension to polyhedral decompositions of the lowest-order Nédélec element of first type
- The element proved robust to element distortions
- A whole family of elements of the Nédélec second type has been constructed ([Beirão da Veiga, Brezzi, Dassi, M., Russo, SINUM 2018])