Non standard virtual element methods for the Helmholtz problem

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Joint work with
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universität wien

Vienna PDE

SFB PDE

FWF Der Wissenschaftsfonds.

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Trefftz finite element methods

Trefftz finite element methods are FEM whose basis functions are solutions to the *homogeneous* PDE in each element of a mesh $\mathcal{T}_h$.

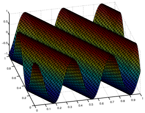
same accuracy with less d.o.f., as compared to standard polynomial FEM

A Trefftz finite element method is defined by the choice of

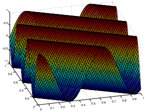
- a family of (operator dependent) discrete Trefftz spaces
- a variational framework that allows to approximate interface and boundary conditions (e.g., least squares, Lagrange multipliers, discontinuous Galerkin)

Trefftz basis functions for Helmholtz

- plane waves $exp(ik\mathbf{d}_\ell \cdot (\mathbf{x} - \mathbf{x}_0)),\ \ell = 1, \ldots, p$

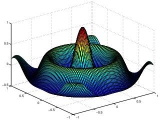


$k = 20,\ \mathbf{d} = [cos(\pi/6), \sin(\pi/6)]$

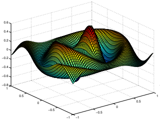


$k = 20,\ \mathbf{d} = [cos(\pi/3), \sin(\pi/3)]$

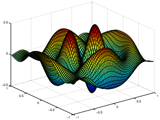
- Fourier-Bessel functions $J_\ell(k|\mathbf{x} - \mathbf{x}_0|)exp(i\ell\vartheta),\ \ell = -q, \ldots, q$



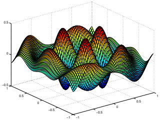
$k = 10,\ \ell = 0$



$k = 10,\ \ell = 1$



$k = 10,\ \ell = 2$

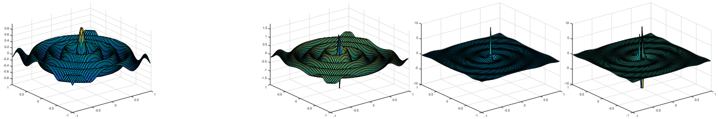


$k = 10,\ \ell = 3$

Trefftz basis functions for Helmholtz

- fundamental solutions $\quad H_0^{(1)}(k|\mathbf{x}-\mathbf{x}_\ell|),\ \ell = 1,\dots,p$

  multipoles $\quad H_\ell^{(1)}(k|\mathbf{x}-\mathbf{x}_0|)exp(i\ell\vartheta),\ \ell = -q,\dots,q$



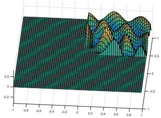
$k = 20,\ \ell = 0$ $\qquad$ $k = 20,\ \ell = 1$ $\qquad$ $k = 20,\ \ell = 2$ $\qquad$ $k = 20,\ \ell = 3$

Trefftz basis functions for Helmholtz

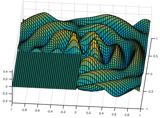
Basis functions for special situations

- near a corner of angle $\pi\alpha$: corner waves

$$J_{\ell/\alpha}(k|\mathbf{x}-\mathbf{x}_0|)\cos(\ell\vartheta/\alpha)$$



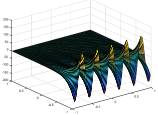
$k = 20,\ \alpha = 1/2$



$k = 20,\ \alpha = 3/2$

- for interface problems: evanescent waves

$$\exp(ik\mathbf{d}_\ell \cdot (\mathbf{x}-\mathbf{x}_0)) \text{ with complex unit vectors } \mathbf{d}_\ell$$



$k = 16,\ \mathbf{d} = [\sqrt{1.1}, 0] + i[0, \sqrt{0.1}]$

[Stojeck, 1998]

Variational frameworks

- least squares[1]
- ultra weak variational formulation[2], discontinuous Galerkin[3,4]
- Lagrange multipliers[5,6]
- variational theory of complex rays[7]
- wave-based method[8]
- BEM-based FEM[9]
- partition of unity[10], virtual partition of unity[11]

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[1] Monk, Wang 1999

[2] Cessenat, Després, 1998-

[3] Buffa, Monk, 2008

[4] Gittelson, Hiptmair, Moiola, Perugia, 2009-

[5] Babuška, Ihlenburg, 1997

[6] Farhat, Harari, Hetmaniuk, 2003-

[7] Ladevèze, 1996-

[8] Desmet, 1998-

[9] Copeland, Langer, Pusch, 2009

[10] Melenk, Babuška, 1995-

[11] Perugia, Pietra, Russo, 2016

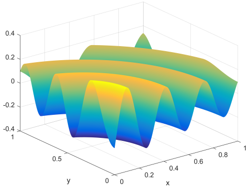
Survey: Hiptmair, Moiola, Perugia, 2016

A numerical experiment

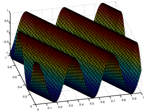
- Helmholtz BVP with exact solution

  $$u(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x} - \mathbf{x}_0|)$$

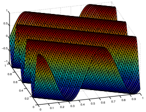
  $$\mathbf{x}_0 = (-0.25, 0),\ \Omega := (0,1)^2$$



- plane wave basis functions



$k = 20$, angle $\pi/6$



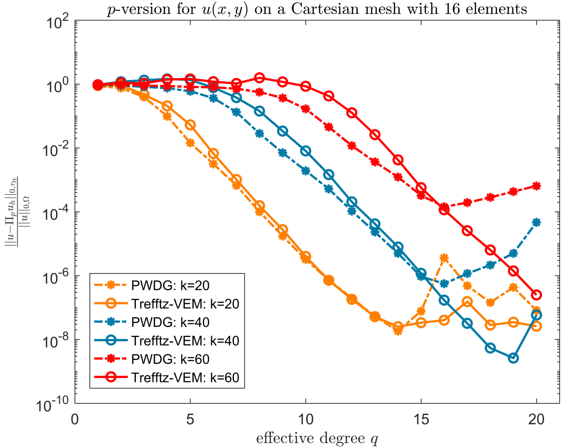
$k = 20$, angle $\pi/3$

- *p*-version results obtained with PW-DG and with a *new* PW virtual element method

[Mascotto, Perugia, Pichler, 2019]

A numerical experiment

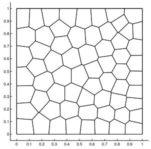
p-version, Cartesian mesh made of 16 elements, L^2-error

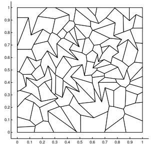


Virtual element method (VEM)

[Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, 2013]

- generalization of FEM to polytopal meshes
- the local basis functions are known explicitly on the boundary but *not in the interior* of each element
- the degrees of freedom are chosen so that H^1 conformity can be imposed
- the local VEM spaces contain subspaces that are known in closed form and possess good approximation properties





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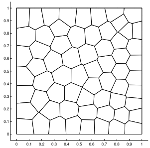
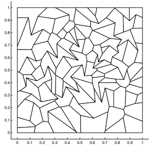




Nonconforming VEM (à la Crouzeix-Raviart)

[Ayuso, Lipnikov, Manzini, 2016], [Cangiani, Manzini, Sutton, 2016]

- nonconforming *Trefftz* VEM for the Laplacian [Mascotto, Perugia, Pichler, 2018]

Model problem

Helmholtz problem with impedance boundary condition

$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega \\ \nabla u \cdot \mathbf{n}_\Omega + \mathrm{i}ku = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygon, $k > 0$ is the wave number.

variational formulation

Find $u \in H^1(\Omega)$:

$$a(u,v) + \mathrm{i}k \int_{\partial\Omega} u\overline{v}\,\mathrm{d}s = \int_{\partial\Omega} g\overline{v}\,\mathrm{d}s \quad \forall v \in H^1(\Omega),$$

where

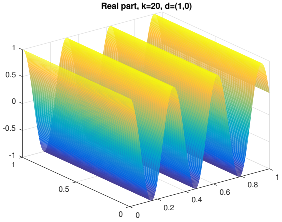
$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x - k^2 \int_\Omega u\overline{v}\,\mathrm{d}x.$$

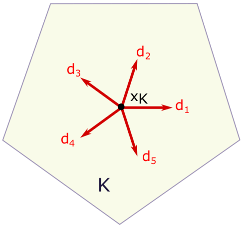
$\mathcal{T}_n := \{K\}$ mesh of polygons; $\{\mathbf{d}_\ell\}_{\ell=1}^{p=2q+1}$ directions; $\mathcal{E}_n$ edges

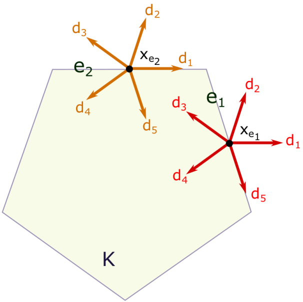
plane waves: $w_\ell(\mathbf{x}) := e^{\mathrm{i}k\mathbf{d}_\ell\cdot\mathbf{x}}, \quad \ell = 1,\dots,p$

- $\mathbb{PW}_p(K) := \mathrm{span}\{w_{\ell|_K}\}_{\ell=1}^p$, $\dim(\mathbb{PW}_p(K)) = p, \quad K \in \mathcal{T}_n$
- $\mathbb{PW}_p(e) := \mathrm{span}\{w_{\ell|_e}\}_{\ell=1}^p$, $\dim(\mathbb{PW}_p(e)) \leq p, \quad e \in \mathcal{E}_n$

$\rightsquigarrow$ *remove redundant directions & include constants* $\rightarrow \mathbb{PW}_p^c(e)$





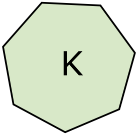


Nonconforming Trefftz-VEM: local spaces

- local **Trefftz**-VE space (for any $K \in \mathcal{T}_n$)

$$V_h(K) := \Big\{ v_h \in H^1(K) \mid \boldsymbol{\Delta v_h + k^2 v_h = 0} \text{ in } K, \\ (\nabla v_h \cdot \mathbf{n}_K + \mathrm{i} k v_h)_{|_e} \in \mathbb{PW}_p^c(e) \quad \forall e \in \mathcal{E}^K \Big\}$$

  - $\mathbb{PW}(K) \subset V_h(K)$
  - functions $v_h \in V_h(K)$ are known *neither* inside K *nor* on ∂K



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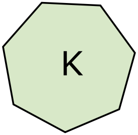
  - $\mathbb{PW}(K) \subset V_h(K)$
  - functions $v_h \in V_h(K)$ are known <u>*neither*</u> inside K <u>*nor*</u> on ∂K

- degrees of freedom (on any $e_r \in \mathcal{E}^K$)

$$\mathrm{dof}_{r,j}(v_h) := \frac{1}{h_{e_r}} \int_{e_r} v_h \overline{w_j^{e_r}} \, \mathrm{d}s \quad \forall w_j^{e_r} \in \mathbb{PW}_p^c(e_r)$$



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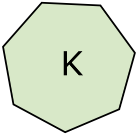
$$\mathrm{dof}_{r,j}(v_h) := \frac{1}{h_{e_r}} \int_{e_r} v_h \overline{w_j^{e_r}} \,\mathrm{d}s \quad \forall w_j^{e_r} \in \mathbb{PW}_p^c(e_r)$$



- local basis functions

$$\mathrm{dof}_{r,j}(\varphi_{s,\ell}) = \delta_{r,s}\delta_{j,\ell}$$

Unisolvence of d.o.f

Provided that k^2 is not a Dirichlet-Laplace eigenvalue on K, the set of d.o.f. is unisolvent. In fact, assume that $v_h \in V_h(K)$ and all its d.o.f. are zero. Then,

$$
\begin{aligned}
&|v_h|^2_{1,K} - k^2\|v_h\|^2_{0,K} - \mathrm{i}k\|v_h\|^2_{0,\partial K} \\
&= \int_K \nabla v_h \cdot \overline{\nabla v_h}\,\mathrm{d}x - k^2 \int_K v_h \overline{v_h}\,\mathrm{d}x - \mathrm{i}k \int_{\partial K} v_h \overline{v_h}\,\mathrm{d}s \\
&= \int_K v_h \underbrace{\overline{(-\Delta v_h - k^2 v_h)}}_{=0}\,\mathrm{d}x + \int_{\partial K} v_h \overline{(\nabla v_h \cdot \mathbf{n}_K + \mathrm{i}k v_h)}\,\mathrm{d}s \\
&= \sum_{e \in \mathcal{E}^K} \int_e v_h \overline{(\nabla v_h \cdot \mathbf{n}_K + \mathrm{i}k v_h)_{|_e}}\,\mathrm{d}s = 0
\end{aligned}
$$

- imaginary part $\to$ $v_h = 0$ on ∂K
- since $\Delta v_h + k^2 v_h = 0$, we conclude

Nonconforming Trefftz-VEM: local spaces

Dirichlet/Neumann boundary conditions

- modification of local VE spaces on Dirichlet/Neumann boundary edges

$$V_h(K) := \Big\{ v_h \in H^1(K) \mid \Delta v_h + k^2 v_h = 0 \text{ in } K, \\ (\nabla v_h \cdot \mathbf{n}_K + \mathrm{i} k v_h)_{|_e} \in \mathbb{PW}_p^c(e) \quad \forall e \in \mathcal{E}^K \setminus (\Gamma_D \cup \Gamma_N) \\ v_{h|_e} \in \mathbb{PW}_p^c(e) \quad \forall e \in \mathcal{E}^K \cap (\Gamma_D \cup \Gamma_N) \Big\}$$

- d.o.f. on Dirichlet/Neumann boundary edges: as for all the other edges

piecewise constant wave numbers

- combination of plane waves and evanescent waves

[Mascotto, Pichler, 2019]

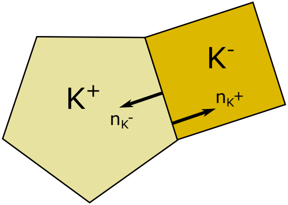
Nonconforming Trefftz-VEM: global spaces

- global nonconforming Sobolev space

$$H^{1,nc}(\mathcal{T}_n) := \left\{ v \in \prod_{K\in\mathcal{T}_n} H^1(K) : \int_e (v_{|_{K^+}} - v_{|_{K^-}})\, \overline{w^e}\, \mathrm{d}s = 0 \quad \forall w^e \in \mathbb{PW}_p^c(e),\ \forall e \in \mathcal{E}_n^I \right\}$$

- global nonconforming Trefftz-VE space

$$V_h := \{ v_h \in H^{1,nc}(\mathcal{T}_n) : v_{h|_K} \in V_h(K) \quad \forall K \in \mathcal{T}_n \}$$



Discrete variational formulation

- local sesquilinear form

$$a^K(u,v) := \int_K \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x - k^2 \int_K u\overline{v}\,\mathrm{d}x$$

discrete variational formulation

Find $u_h \in V_h$:

$$\sum_{K\in\mathcal{T}_n} a^K(u_h, v_h) + \mathrm{i}k \int_{\partial\Omega} u_h \overline{v_h}\,\mathrm{d}s = \int_{\partial\Omega} g\overline{v_h}\,\mathrm{d}s \quad \forall v_h \in V_h.$$

- local sesquilinear form

$$a^K(u,v) := \int_K \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x - k^2 \int_K u \overline{v} \, \mathrm{d}x$$

discrete variational formulation

Find $u_h \in V_h$:

$$\sum_{K \in \mathcal{T}_n} a^K(u_h, v_h) + \mathrm{i}k \int_{\partial\Omega} u_h \overline{v_h} \, \mathrm{d}s = \int_{\partial\Omega} g \overline{v_h} \, \mathrm{d}s \quad \forall v_h \in V_h.$$

Problem: none of these terms is computable

- bulk projector

$$\Pi_p^K : V_h(K) \to \mathbb{PW}_p(K)$$

$$a^K(\Pi_p^K u_h, w^K) = a^K(u_h, w^K) \quad \forall w^K \in \mathbb{PW}_p(K)$$

- L^2 (boundary) edge projector ($e \in \mathcal{E}_n^B$)

$$\Pi_p^{0,e} : V_h(K)_{|_e} \to \mathbb{PW}_p^c(e)$$

$$\int_e (\Pi_p^{0,e} u_h)\overline{w^e}\,\mathrm{d}s = \int_e u_h \overline{w^e}\,\mathrm{d}s \quad \forall w^e \in \mathbb{PW}_p^c(e)$$

Π_p^K and $\Pi_p^{0,e}$ are computable in terms of the d.o.f.
(obvious for $\Pi_p^{0,e}$, by integration by parts for Π_p^K)

for $u_h \in V_h(K)$ and $w^K \in \mathbb{PW}_p(K)$, integration by part and $\Delta w^K + k^2 w^K = 0$ give

$$\begin{aligned} a^K(u_h, w^K) &= \int_K \nabla u_h \cdot \overline{\nabla w^K}\,\mathrm{d}x - k^2 \int_K u_h \overline{w^K}\,\mathrm{d}x \\ &= \sum_{e \in \mathcal{E}^K} \int_e u_h \underbrace{\overline{(\nabla w^K \cdot \mathbf{n}_K)}}_{\in \mathbb{PW}_p^c(e)}\,\mathrm{d}s \end{aligned}$$

Replace each term of the variational formulation with something computable

$$\sum_{K\in\mathcal{T}_n} \underbrace{a^K(u_h, v_h)}_{=:(I)} + \underbrace{\mathrm{i}k \int_{\partial\Omega} u_h \overline{v_h}\,\mathrm{d}s}_{=:(II)} = \underbrace{\int_{\partial\Omega} g\overline{v_h}\,\mathrm{d}s}_{=:(III)} \quad \forall v_h \in V_h$$

(II), (III): $\quad v_h \mapsto \Pi_p^{0,\partial\Omega} v_h$

(I): $\quad a^K(u_h, v_h) = \underbrace{a^K(\Pi_p^K u_h, \Pi_p^K v_h)}_{computable} + \underbrace{a^K((I-\Pi_p^K)u_h, (I-\Pi_p^K)v_h)}_{\approx S^K((I-\Pi_p^K)u_h,(I-\Pi_p^K)v_h)}$

$$S^K(u_h, v_h) = \sum_{e\in\mathcal{E}^K} \sum_{\ell=1}^{\dim(\mathbb{PW}_p^c(e))} a^K(\Pi_p^K \varphi_{e,\ell}, \Pi_p^K \varphi_{e,\ell})\,\mathrm{dof}_{e,\ell}(u_h)\,\overline{\mathrm{dof}_{e,\ell}(v_h)}$$

(*diagonal recipe*, [Beirão da Veiga, Dassi, Russo, 2017])

Nonconforming Trefftz-VEM: formulation

nonconforming Trefftz-VEM

Find $u_h \in V_h$:

$$a_h(u_h, v_h) + \mathrm{i}k \int_{\partial\Omega} u_h \, \overline{(\Pi_p^{0,\partial\Omega} v_h)} \, \mathrm{d}s = \int_{\partial\Omega} g \, \overline{(\Pi_p^{0,\partial\Omega} v_h)} \, \mathrm{d}s \qquad \forall v_h \in V_h,$$

where

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_n} \left[a^K(\Pi_p^K u_h, \Pi_p^K v_h) + S^K((I - \Pi_p^K)u_h, (I - \Pi_p^K)v_h) \right].$$

$\rightsquigarrow$ well-posedness and h-convergence; pollution $\sim$ PWDG

General remark on interpolation error in nonconforming spaces: next slide

Interpolation error in nc-VEM spaces

[Mascotto, Perugia, Pichler, 2018]

- Poisson problem, homogeneous Dirichlet boundary conditions
- $H_p^{1,\mathrm{nc}}(\mathcal{T}_h) = \{v \in H^1(\mathcal{T}_h) : \int_e [\![v]\!]\, q_{p-1} = 0 \quad \forall q_{p-1} \in \mathbb{P}_{p-1}(e), \forall e \in \mathcal{E}_h\}$
- nc-VEM space

$$V(K) = \{v \in H^1(K) :\ \Delta v \in \mathbb{P}_{p-2}(K),\ (\partial_{n_K} v)_{|_e} \in \mathbb{P}_{p-1}(e)\ \forall e \subset \partial K\}$$

$$V_p^{\mathrm{nc}}(\mathcal{T}_h) = \{v \in H_p^{1,\mathrm{nc}}(\mathcal{T}_h) :\ v_{|_K} \in V(K) \quad \forall K \in \mathcal{T}_h\}$$

- $\psi \in H_0^1(\Omega)$, nc-VEM interpolant $\psi^I \in V_p^{\mathrm{nc}}(\mathcal{T}_h)$: for all $K \in \mathcal{T}_h$,

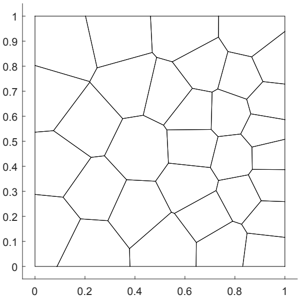
$$\frac{1}{h_e}\int_e (\psi^I - \psi)\, q_{p-1}^e = 0 \qquad \forall q_{p-1}^e \in \mathbb{P}_{p-1}(e),\ \forall e \subset \partial K$$
$$\frac{1}{|K|}\int_K (\psi^I - \psi)\, q_{p-2} = 0 \qquad \forall q_{p-2} \in \mathbb{P}_{p-2}(K)$$

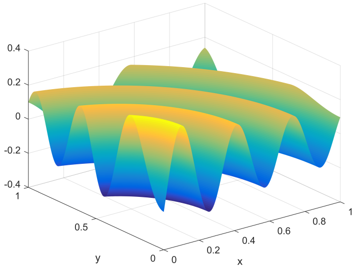
for all $q_p \in \mathbb{P}_p(K)$,

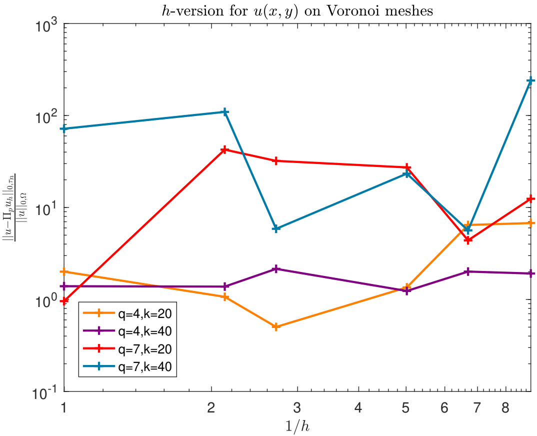
$$
\begin{aligned}
|\psi - \psi^I|^2_{1,K} &= \int_K \nabla(\psi - \psi^I) \cdot \nabla(\psi - \psi^I) \\
&= \int_K \nabla(\psi - \psi^I) \cdot \nabla(\psi - q_p) + \int_K \nabla(\psi - \psi^I) \cdot \nabla(q_p - \psi^I) \\
&= \int_K \nabla(\psi - \psi^I) \cdot \nabla(\psi - q_p) - \underbrace{\int_K (\psi - \psi^I)\Delta(q_p - \psi^I)}_{=0} + \underbrace{\int_{\partial K} (\psi - \psi^I)\partial_{n_K}(q_p - \psi^I)}_{=0} \\
&= \int_K \nabla(\psi - \psi^I) \cdot \nabla(\psi - q_p) \leq |\psi - \psi^I|_{1,K} \, |\psi - q_p|_{1,K}
\end{aligned}
$$

interpolation error in $V_p^{\mathrm{nc}}(\mathcal{T}_h) \leq$ best approximation in $\mathbb{P}_p^{\mathrm{disc}}(\mathcal{T}_h)$

- $\Omega := (0,1)^2$
- Voronoi meshes
- exact solution: $u(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x}-\mathbf{x}_0|), \quad \mathbf{x}_0 = (-0.25, 0),$

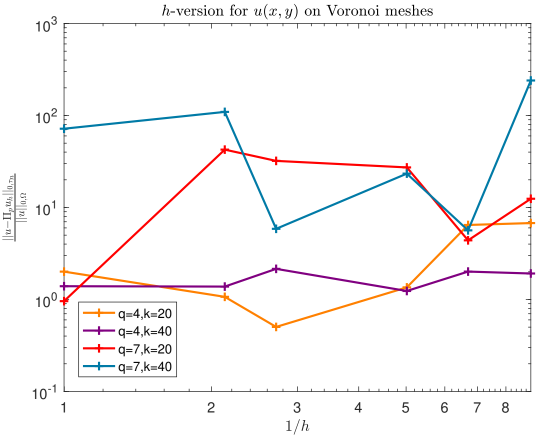






h-version for $u(x,y)$ on Voronoi meshes
10^3
10^2
10^1
10^0
10^{-1}
$\frac{\|u-\Pi_p u_h\|_{0,\tau_h}}{\|u\|_{0,\Omega}}$
1
2
3
4
5
6
7
8
$1/h$
q=4,k=20
q=4,k=40
q=7,k=20
q=7,k=40

h-version

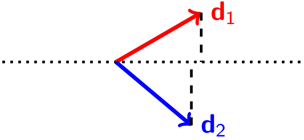


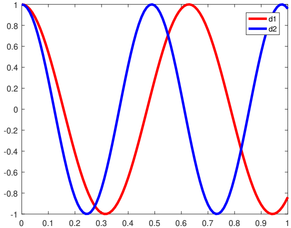
no convergence!

The ill-conditioning is *much more* serious than for other plane wave methods

- primary source: almost singularity of local plane wave mass matrices on boundary edges (in spite of the applied filtering), which need to be inverted for the computation of

$$\mathrm{i}k \int_{\partial\Omega} u_h \, \overline{(\Pi_p^{0,\partial\Omega} v_h)} \, \mathrm{d}s \qquad \text{and} \qquad \int_{\partial\Omega} g \, \overline{(\Pi_p^{0,\partial\Omega} v_h)} \, \mathrm{d}s$$





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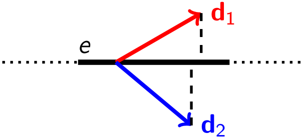

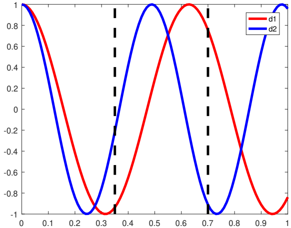

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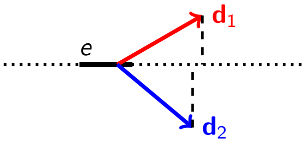
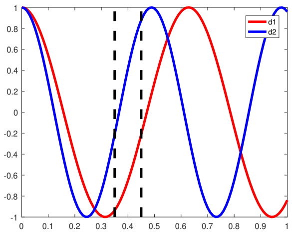




The ill-conditioning is *much more* serious than for other plane wave methods

- primary source: almost singularity of local plane wave mass matrices on boundary edges (in spite of the applied filtering), which need to be inverted for the computation of

$$\mathrm{i}k \int_{\partial\Omega} u_h \overline{(\Pi_p^{0,\partial\Omega} v_h)}\, \mathrm{d}s \qquad \text{and} \qquad \int_{\partial\Omega} g\, \overline{(\Pi_p^{0,\partial\Omega} v_h)}\, \mathrm{d}s$$

- secondary source: ill-conditioning of the (bulk) plane wave basis

Idea: act directly on the edge mass matrices (and for *all* the edges)

Orthogonalization-and-filtering process

Algorithm: Orthogonalization-and-filtering

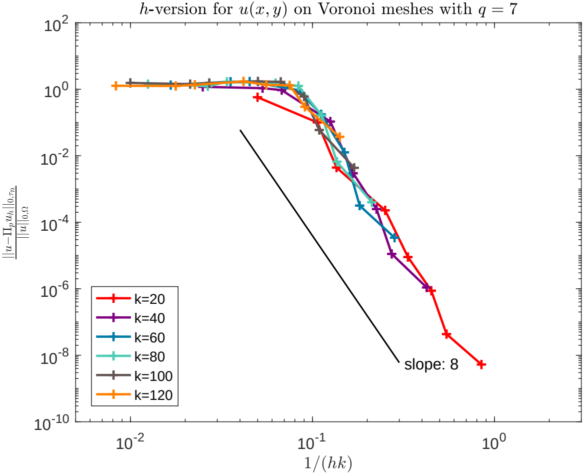
Let $\sigma > 0$ be a given tolerance. For *any* *edge* $e \in \mathcal{E}_n$:

1. assemble the matrix $\boldsymbol{G}_0^e$ with entries $(\boldsymbol{G}_0^e)_{j,\ell} = (w_\ell^e, w_j^e)_{0,e}$.
2. compute the eigendecomposition: $\boldsymbol{G}_0^e \boldsymbol{Q}^e = \boldsymbol{Q}^e \boldsymbol{\Lambda}^e$.
3. remove the columns of $\boldsymbol{Q}^e$ corresponding to the eigenvalues smaller than σ
4. define the new orthonormal edge functions in terms of the old ones

($\sigma = 10^{-13}$ in our experiments)

Remark: On structured meshes, the local orthonormal bases can be computed once for all in the elements of the reference patch.

h-version (revisited)

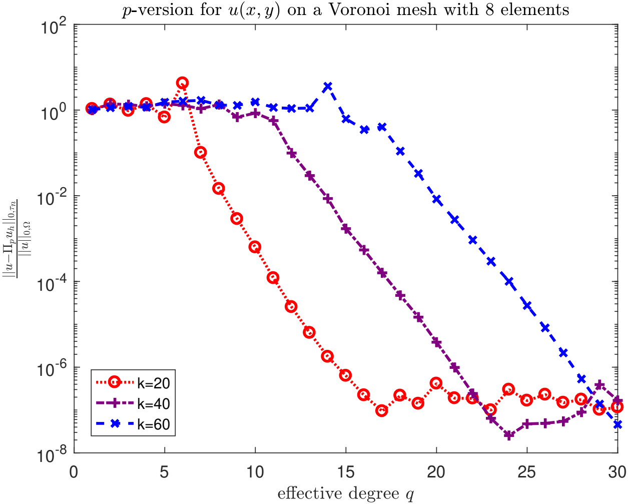


$\rightsquigarrow$ algebraic convergence (reduction of #dofs: up to almost 70%)

- Voronoi meshes
- $k = 20$, $q = 7$

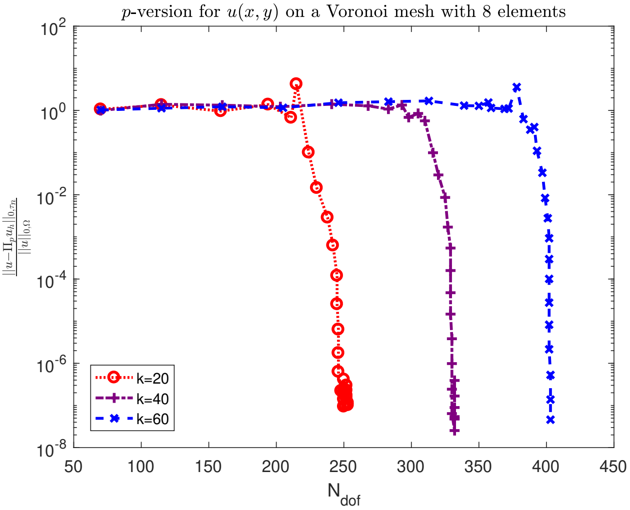
it.	L^2-error	# dofs (practice)	# dofs (theory)	reduct. (%)
0	5.75e-01	131	174	24.7
1	1.00e-01	224	340	34.1
2	4.42e-03	394	672	41.4
3	2.28e-04	695	1392	50.1
4	8.99e-06	1243	2783	55.3
5	8.73e-07	2206	5635	60.1
6	4.33e-08	4002	11353	64.7
7	5.16e-09	7282	22810	68.1

$$u(x,y) := H_0^{(1)}(k|\mathbf{x}-\mathbf{x}_0|), \quad \mathbf{x}_0 = (-0.25, 0), \quad \Omega = (0,1)^2$$

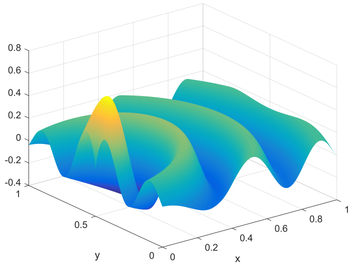


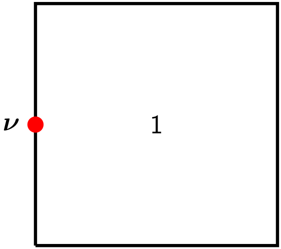
p-version (error vs. number of dofs)

$$u(x, y) := H_0^{(1)}(k|\mathbf{x} - \mathbf{x}_0|), \quad \mathbf{x}_0 = (-0.25, 0), \quad \Omega = (0, 1)^2$$



- $\Omega := (0,1)^2$
- singular solution: $v(x,y) := J_\xi(kr)\cos(\xi\theta)$, $\xi = \frac{2}{3}$
- $v \in H^{\frac{5}{3}-\epsilon}(\Omega)$, for all $\epsilon > 0$ arbitrarily small
- *h*-version: convergence rate $\frac{5}{3}$ in L^2; *p*-version: algebraic conv.
- *hp*-graded meshes ($\mu = 1/3$) $\rightsquigarrow$ exponential convergence





hp-version: a numerical experiment

- $\Omega := (0,1)^2$
- singular solution: $v(x,y) := J_\xi(kr)\cos(\xi\theta)$, $\xi = \frac{2}{3}$
- $v \in H^{\frac{5}{3}-\epsilon}(\Omega)$, for all $\epsilon > 0$ arbitrarily small
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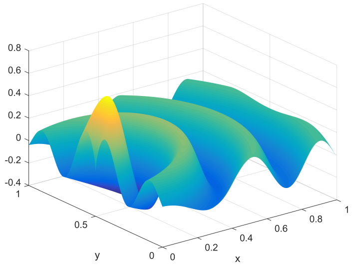

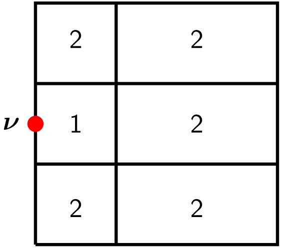

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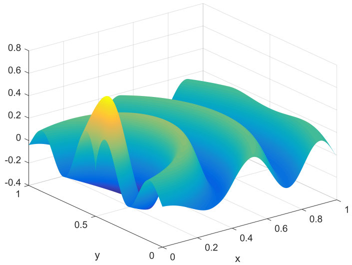
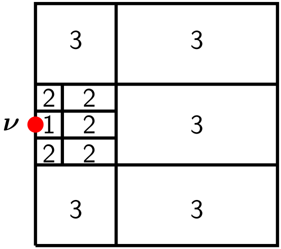
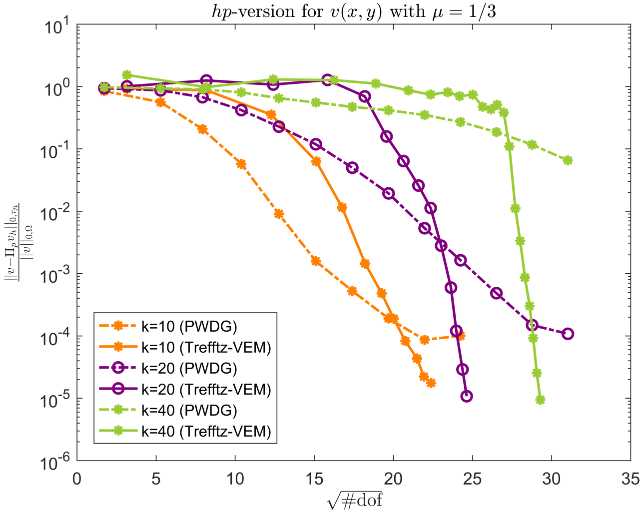




hp mesh and plane wave basis

- $\mathcal{T}_n$ mesh with $n+1$ layers of elements near the singularity $\boldsymbol{\nu}$
- $h_K = \mu^n$ if K is in the 0-th layer

  $h_K = \frac{1-\mu}{\mu}\,\mathrm{dist}(K, \boldsymbol{\nu})$ otherwise
- $p_K = 2q_K + 1$, with $q_K = \ell + 1$, if K is in the ℓ-th layer
- hierarchical sets of directions obtained by suitable removal from a set of $2q_{max} + 1$ evenly spaced directions



hp-version for $v(x,y)$ with $\mu = 1/3$
$\frac{||v-\Pi_p v_h||_{0,\mathcal{T}h}}{||v||_{0,\Omega}}$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
$\sqrt{\#\mathrm{dof}}$
k=10 (PWDG)
k=10 (Trefftz-VEM)
k=20 (PWDG)
k=20 (Trefftz-VEM)
k=40 (PWDG)
k=40 (Trefftz-VEM)

Final remarks

- nc Trefftz-VEM combines the Trefftz framework with the nc-VE technology
  - general polygonal meshes
  - basis functions are not known in closed form but satisfy the homogeneous Helmholtz equation
- for stability reasons, we applied *edgewise* an orthogonalization-and-filtering procedure, which turned out to have positive effects in terms of
  - accuracy vs. number of d.o.f.
  - maximal achievable accuracy
- extension to the 3D case (with Marco Manzini)

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Thank you for your attention!