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A high-order discontinuous Galerkin approach to the elasto-acoustic problem

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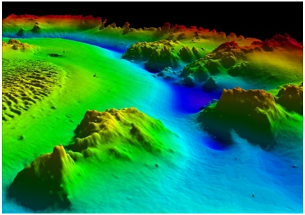
Motivations

Coupled elasto-acoustic wave propagation arises in several scientific and engineering contexts

Radar and sonar detection



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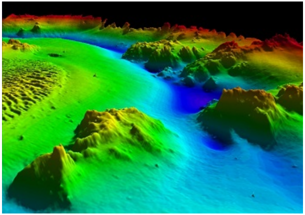
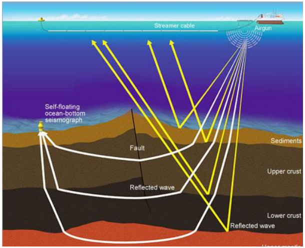


Geophysical exploration



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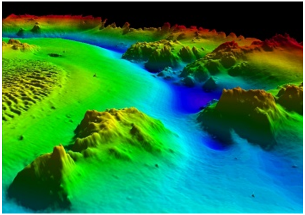

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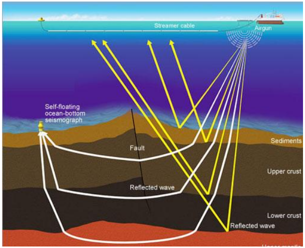

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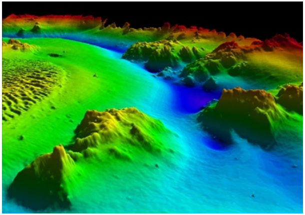

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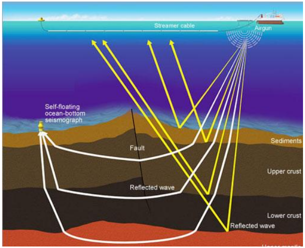
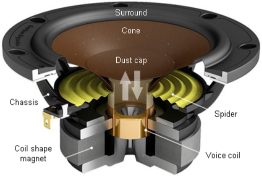
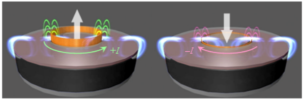


Sound engineering





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Features of the physical model

- Nonlinear coupled problem
- Thin structures and highly heterogeneous media
- Scattered fields at high-frequency/small-wavelength

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Objective
Development and analysis of a **high-order** discontinuous Galerkin method on **polytopal grids** for the coupled **elastic-acoustic** wave propagation problem.

State of the art

Minimal bibliography

- [Komatitsch *et al.*, 2000]: Spectral Elements
- [Fischer and Gaul, 2005]: FEM–BEM coupling, Lagrange multipliers
- [Flemisch *et al.*, 2006]: classical FEM on two independent meshes
- [Brunner *et al.*, 2009]: FEM–BEM comparison
- [Ghattas *et al.*, 2010]: dG, velocity-strain formulation
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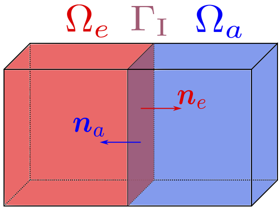
Our contribution

- **Well-posedness** of the coupled problem in the **continuous setting**
- **Detailed analysis** of a dG scheme on **general polytopal meshes**

Elasto-acoustic coupling: governing equations

$$\begin{cases} \rho_e \ddot{\mathbf{u}} - \mathbf{div}\,(\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u})) = \mathbf{f}_e & \text{in } \Omega_e \times (0,T], \\ \mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u})\mathbf{n}_e = \rho_a \dot{\varphi}\,\mathbf{n}_a & \text{on } \Gamma_{\mathrm{I}} \times (0,T], \\ c^{-2}\ddot{\varphi} - \triangle\varphi = f_a & \text{in } \Omega_a \times (0,T], \\ \partial\varphi/\partial\mathbf{n}_a = \dot{\mathbf{u}}\cdot\mathbf{n}_e & \text{on } \Gamma_{\mathrm{I}} \times (0,T], \end{cases}$$



- $\mathbf{u}$ is the elastic displacement, φ is the acoustic potential
- ρ_e and ρ_a are the elastic and acoustic mass densities
- $\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u})$ is the stress tensor (Hooke's law)
- c is the characteristic acoustic velocity

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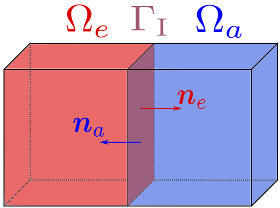

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Interface conditions on Γ_I

- Continuity of the pressure loads (acoustic pressure $p_a = \rho_a \dot{\varphi}$)
- Continuity of the normal component of the velocity field (acoustic velocity $\mathbf{v}_a = -\boldsymbol{\nabla}\varphi$)

Theoretical and numerical analysis

Well-posedness

Theorem

Under suitable regularity hypotheses on initial data and source terms, there is a **unique strong solution** s.t.

$$\mathbf{u} \in C^2([0,T];\mathbf{L}^2(\Omega_e)) \cap C^1([0,T];\mathbf{H}^1_D(\Omega_e)) \cap C^0([0,T];\mathbf{H}^{\triangle}_{\mathbb{C}}(\Omega_e) \cap \mathbf{H}^1_D(\Omega_e)),$$
$$\varphi \in C^2([0,T];L^2(\Omega_a)) \cap C^1([0,T];H^1_D(\Omega_a)) \cap C^0([0,T];H^{\triangle}(\Omega_a) \cap H^1_D(\Omega_a))$$

$$\mathbf{H}^{\triangle}_{\mathbb{C}}(\Omega_e) = \{\mathbf{v} \in \mathbf{L}^2(\Omega_e) : \mathbf{div}\,\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{v}) \in \mathbf{L}^2(\Omega_e)\},$$
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Idea of the proof. Rewrite the problem as: find $\mathcal{U}(t) \in \mathbb{H}$ such that

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}(t) + A\mathcal{U}(t) = \mathcal{F}(t), \quad t \in (0,T],$$
$$\mathcal{U}(0) = \mathcal{U}_0,$$

and prove that A is maximal monotone, i.e., $(A\mathcal{U},\mathcal{U})_{\mathbb{H}} \geqslant 0$ for all $\mathcal{U} \in D(A)$ and that $I + A$ is surjective from $D(A)$ onto $\mathbb{H}$. Then, apply the Hille–Yosida theorem.

Sketch of the proof

Let $\mathcal{U} = (\mathbf{u}, \mathbf{w}, \varphi, \phi)$ and take $\mathbf{w} = \dot{\mathbf{u}}$, $\phi = \dot{\varphi}$. Consider

$$\mathbb{H} = \mathbf{H}^1_D(\Omega_e) \times \mathbf{L}^2(\Omega_e) \times H^1_D(\Omega_a) \times L^2(\Omega_a),$$

with scalar product

$$(\mathcal{U}_1, \mathcal{U}_2)_{\mathbb{H}} = (\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u}_1), \boldsymbol{\varepsilon}(\mathbf{u}_2))_{\Omega_e} + (\rho_e \mathbf{w}_1, \mathbf{w}_2)_{\Omega_e} + (\rho_a \boldsymbol{\nabla}\varphi_1, \boldsymbol{\nabla}\varphi_2)_{\Omega_a} + (c^{-2}\rho_a \phi_1, \phi_2)_{\Omega_a}.$$

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Then, we define the operator $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$A\mathcal{U} = \left(-\mathbf{w},\ -\rho_e^{-1}\mathbf{div}\,\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u}),\ -\phi,\ -c^2\triangle\varphi\right) \quad \forall \mathcal{U} \in D(A),$$

$$D(A) = \Big\{\mathcal{U} \in \mathbb{H} : \mathbf{u} \in \mathbf{H}^{\triangle}_{\mathbb{C}}(\Omega_e),\ \mathbf{w} \in \mathbf{H}^1_D(\Omega_e),\ \varphi \in H^{\triangle}(\Omega_a),\ \phi \in H^1_D(\Omega_a);$$

$$\left(\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u}) + \rho_a\phi\mathbf{I}\right)\mathbf{n}_e = \mathbf{0} \text{ on } \Gamma_{\mathrm{I}},\ (\boldsymbol{\nabla}\varphi + \mathbf{w})\cdot\mathbf{n}_a = 0 \text{ on } \Gamma_{\mathrm{I}}\Big\}.$$

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Finally, let $\mathcal{F} = (\mathbf{0}, \rho_e^{-1}\mathbf{f}_e, 0, c^2 f_a)$.

For $\mathcal{F} \in C^1([0,T];\mathbb{H})$ and $\mathcal{U}_0 \in D(A)$, find $\mathcal{U} \in C^1([0,T];\mathbb{H}) \cap C^0([0,T];D(A))$:

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}(t) + A\mathcal{U}(t) = \mathcal{F}(t), \quad t \in (0,T],$$
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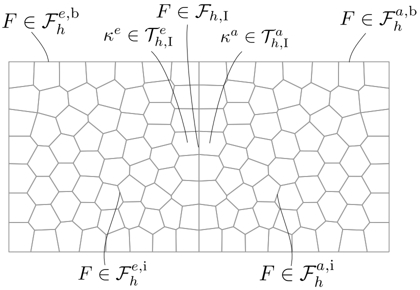
Discrete settings: mesh assumptions



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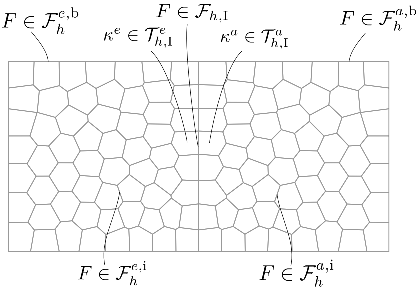

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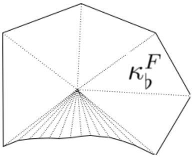

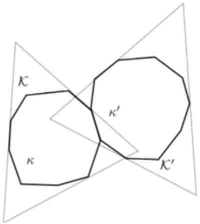

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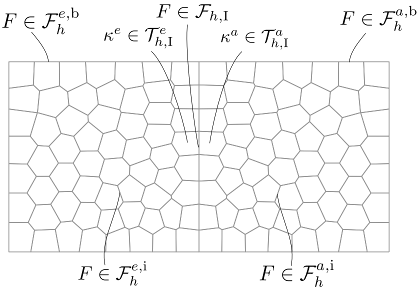

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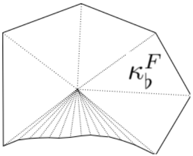

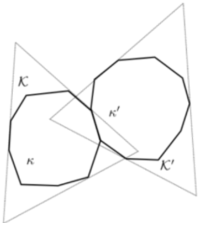

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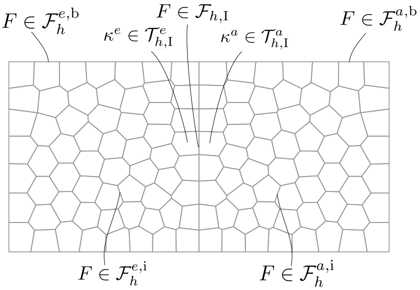

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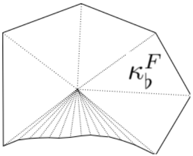

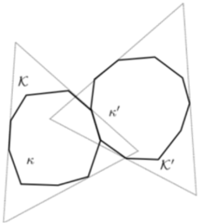

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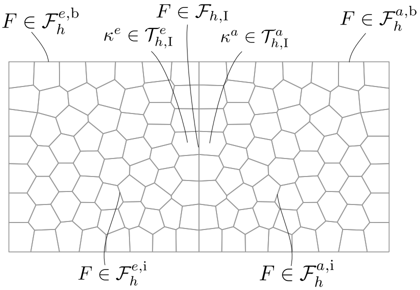

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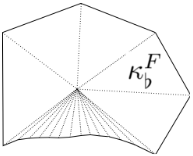

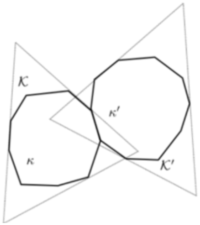

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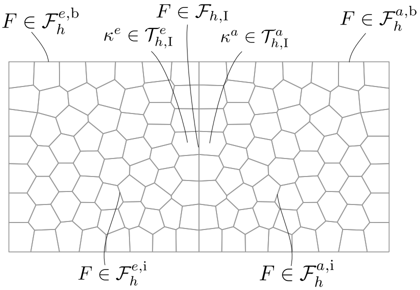

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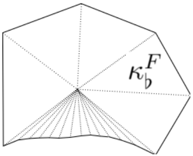

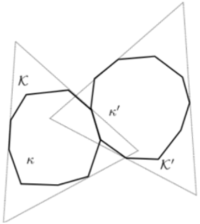

Consequences [Cangiani et al. 17]

- **Trace-inverse inequality** on polytopal elements
- **Approximation results** in $\mathscr{P}_p(\kappa)$

Semi-discrete problem (SIP dG)

$$\boldsymbol{V}_h^e = \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega_e) : \boldsymbol{v}_{h|\kappa} \in [\mathscr{P}_{p_{e,\kappa}}(\kappa)]^d,\ p_{e,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^e\},$$

$$V_h^a = \{\psi_h \in L^2(\Omega_a) : \psi_{h|\kappa} \in \mathscr{P}_{p_{a,\kappa}}(\kappa),\ p_{a,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^a\}$$

Find $(\boldsymbol{u}_h, \varphi_h) \in C^2([0,T]; \boldsymbol{V}_h^e) \times C^2([0,T]; V_h^a)$ s.t., for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{V}_h^e \times V_h^a$,

$$(\rho_e \ddot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (c^{-2}\rho_a \ddot{\varphi}_h(t), \psi_h)_{\Omega_a} + \mathcal{A}_h^e(\boldsymbol{u}_h(t), \boldsymbol{v}_h) + \mathcal{A}_h^a(\varphi_h(t), \psi_h)$$
$$+\mathcal{C}_h^e(\dot{\varphi}_h(t), \boldsymbol{v}_h) + \mathcal{C}_h^a(\dot{\boldsymbol{u}}_h(t), \psi_h) = (\boldsymbol{f}_e(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_a f_a(t), \psi_h)_{\Omega_a}$$

Semi-discrete problem (SIP dG)

$$\boldsymbol{V}_h^e = \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega_e) : \boldsymbol{v}_{h|\kappa} \in [\mathscr{P}_{p_{e,\kappa}}(\kappa)]^d,\ p_{e,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^e\},$$
$$V_h^a = \{\psi_h \in L^2(\Omega_a) : \psi_{h|\kappa} \in \mathscr{P}_{p_{a,\kappa}}(\kappa),\ p_{a,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^a\}$$

Find $(\boldsymbol{u}_h, \varphi_h) \in C^2([0,T]; \boldsymbol{V}_h^e) \times C^2([0,T]; V_h^a)$ s.t., for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{V}_h^e \times V_h^a$,

$$(\rho_e \ddot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (c^{-2}\rho_a \ddot{\varphi}_h(t), \psi_h)_{\Omega_a} + \mathcal{A}_h^e(\boldsymbol{u}_h(t), \boldsymbol{v}_h) + \mathcal{A}_h^a(\varphi_h(t), \psi_h)$$
$$+\mathcal{C}_h^e(\dot{\varphi}_h(t), \boldsymbol{v}_h) + \mathcal{C}_h^a(\dot{\boldsymbol{u}}_h(t), \psi_h) = (\boldsymbol{f}_e(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_a f_a(t), \psi_h)_{\Omega_a}$$

$$\mathcal{A}_h^e(\boldsymbol{u}, \boldsymbol{v}) = (\mathbb{C}\boldsymbol{\varepsilon}_h(\boldsymbol{u}), \boldsymbol{\varepsilon}_h(\boldsymbol{v}))_{\Omega_e} - \langle \{\!\{\mathbb{C}\boldsymbol{\varepsilon}_h(\boldsymbol{u})\}\!\}, [\![\boldsymbol{v}]\!] \rangle_{\mathcal{F}_h^e}$$
$$- \langle [\![\boldsymbol{u}]\!], \{\!\{\mathbb{C}\boldsymbol{\varepsilon}_h(\boldsymbol{v})\}\!\} \rangle_{\mathcal{F}_h^e} + \langle \eta [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!] \rangle_{\mathcal{F}_h^e} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_h^e,$$

Semi-discrete problem (SIP dG)

$$\boldsymbol{V}_h^e = \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega_e) : \boldsymbol{v}_{h|\kappa} \in [\mathscr{P}_{p_{e,\kappa}}(\kappa)]^d,\ p_{e,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^e\},$$

$$V_h^a = \{\psi_h \in L^2(\Omega_a) : \psi_{h|\kappa} \in \mathscr{P}_{p_{a,\kappa}}(\kappa),\ p_{a,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^a\}$$

Find $(\boldsymbol{u}_h, \varphi_h) \in C^2([0,T]; \boldsymbol{V}_h^e) \times C^2([0,T]; V_h^a)$ s.t., for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{V}_h^e \times V_h^a$,

$$(\rho_e \ddot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (c^{-2}\rho_a \ddot{\varphi}_h(t), \psi_h)_{\Omega_a} + \mathcal{A}_h^e(\boldsymbol{u}_h(t), \boldsymbol{v}_h) + \mathcal{A}_h^a(\varphi_h(t), \psi_h)$$
$$+\mathcal{C}_h^e(\dot{\varphi}_h(t), \boldsymbol{v}_h) + \mathcal{C}_h^a(\dot{\boldsymbol{u}}_h(t), \psi_h) = (\boldsymbol{f}_e(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_a f_a(t), \psi_h)_{\Omega_a}$$

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$$- \langle [\![\boldsymbol{u}]\!], \{\!\{\mathbb{C}\boldsymbol{\varepsilon}_h(\boldsymbol{v})\}\!\} \rangle_{\mathcal{F}_h^e} + \langle \eta [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!] \rangle_{\mathcal{F}_h^e} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_h^e,$$

$$\mathcal{A}_h^a(\varphi, \psi) = (\rho_a \boldsymbol{\nabla}_h \varphi, \boldsymbol{\nabla}_h \psi)_{\Omega_a} - \langle \{\!\{\rho_a \boldsymbol{\nabla}_h \varphi\}\!\}, [\![\psi]\!] \rangle_{\mathcal{F}_h^a}$$
$$- \langle [\![\varphi]\!], \{\!\{\rho_a \boldsymbol{\nabla}_h \psi\}\!\} \rangle_{\mathcal{F}_h^a} + \langle \chi [\![\varphi]\!], [\![\psi]\!] \rangle_{\mathcal{F}_h^a} \qquad \forall \varphi, \psi \in V_h^a,$$

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Find $(\boldsymbol{u}_h, \varphi_h) \in C^2([0,T]; \boldsymbol{V}_h^e) \times C^2([0,T]; V_h^a)$ s.t., for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{V}_h^e \times V_h^a$,

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$$+\mathcal{C}_h^e(\dot{\varphi}_h(t), \boldsymbol{v}_h) + \mathcal{C}_h^a(\dot{\boldsymbol{u}}_h(t), \psi_h) = (\boldsymbol{f}_e(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_a f_a(t), \psi_h)_{\Omega_a}$$

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$$\mathcal{C}_h^e(\psi, \boldsymbol{v}) = (\rho_a \psi \boldsymbol{n}_e, \boldsymbol{v})_{\Gamma_{\mathrm{I}}} = \langle \rho_a \psi \boldsymbol{n}_e, \boldsymbol{v} \rangle_{\mathcal{F}_{h,\mathrm{I}}} \qquad \forall (\psi, \boldsymbol{v}) \in V_h^a \times \boldsymbol{V}_h^e,$$

$$\mathcal{C}_h^a(\boldsymbol{v}, \psi) = (\rho_a \boldsymbol{v} \cdot \boldsymbol{n}_a, \psi)_{\Gamma_{\mathrm{I}}} = -\mathcal{C}_h^e(\psi, \boldsymbol{v}) \qquad \forall (\boldsymbol{v}, \psi) \in \boldsymbol{V}_h^e \times V_h^a$$

Penalization functions

The stabilization functions $\eta \in L^\infty(\mathcal{F}_h^e)$ and $\chi \in L^\infty(\mathcal{F}_h^a)$ are defined as follows

$$\eta_{|F} = \begin{cases} \alpha \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left(\dfrac{\overline{\mathbb{C}}_\kappa p_{e,\kappa}^2}{h_\kappa} \right) & \forall F \in \mathcal{F}_h^{e,\mathrm{i}}, \quad F \subseteq \partial\kappa^+ \cap \partial\kappa^-, \\ \dfrac{\overline{\mathbb{C}}_\kappa p_{e,\kappa}^2}{h_\kappa} & \forall F \in \mathcal{F}_h^{e,\mathrm{b}}, \quad F \subseteq \partial\kappa; \end{cases}$$

$$\chi_{|F} = \begin{cases} \beta \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left(\dfrac{\overline{\rho}_{a,\kappa} p_{a,\kappa}^2}{h_\kappa} \right) & \forall F \in \mathcal{F}_h^{a,\mathrm{i}}, \quad F \subseteq \partial\kappa^+ \cap \partial\kappa^-, \\ \dfrac{\overline{\rho}_{a,\kappa} p_{a,\kappa}^2}{h_\kappa} & \forall F \in \mathcal{F}_h^{a,\mathrm{b}}, \quad F \subseteq \partial\kappa. \end{cases}$$

where α and β are positive constants to be properly chosen.

$$\overline{\mathbb{C}}_\kappa = (|\mathbb{C}^{1/2}|_2^2)_{|\kappa} \quad \forall \kappa \in \mathcal{T}_h^e, \qquad \overline{\rho}_{a,\kappa} = \rho_{a|\kappa} \quad \forall \kappa \in \mathcal{T}_h^a.$$

Semi-discrete stability and error estimate

Define the following **energy norm** for $(\mathbf{v}_h, \psi_h) \in C^1([0,T]; \mathbf{V}_h^e) \times C^1([0,T]; V_h^a)$:

$$\|(\mathbf{v}_h, \psi_h)\|_{\mathcal{E}}^2 = \|\rho_e^{1/2}\dot{\mathbf{v}}_h\|_{\Omega_e}^2 + \|\mathbb{C}^{1/2}\boldsymbol{\varepsilon}_h(\mathbf{v})\|_{\Omega_e}^2 + \|\eta^{1/2}[\![\mathbf{v}]\!]\|_{\mathcal{F}_h^e}^2$$
$$+ \|c^{-1}\rho_a^{1/2}\dot{\psi}_h\|_{\Omega_a}^2 + \|\rho_a^{1/2}\boldsymbol{\nabla}_h\psi\|_{\Omega_a}^2 + \|\chi^{1/2}[\![\psi]\!]\|_{\mathcal{F}_h^a}^2$$

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Stability of the semi-discrete formulation

For sufficiently large stabilization parameters α and β, we have

$$\|(\mathbf{u}_h(t), \varphi_h(t))\|_{\mathcal{E}} \lesssim \|(\mathbf{u}_h(0), \varphi_h(0))\|_{\mathcal{E}} + \int_0^t \left(\|\mathbf{f}_e(\tau)\|_{\Omega_e} + \|f_a(\tau)\|_{\Omega_a}\right) \mathrm{d}\tau$$

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Energy-error estimate

Provided $(\mathbf{u}, \varphi) \in C^2([0,T]; \mathbf{H}^m(\Omega_e)) \times C^2([0,T]; H^n(\Omega_a))$, $m \geqslant p_e + 1$, $n \geqslant p_a + 1$,

$$\sup_{t \in [0,T]} \|(\mathbf{u}(t) - \mathbf{u}_h(t), \varphi(t) - \varphi_h(t))\|_{\mathcal{E}} \lesssim C_{\mathbf{u}}(T)\frac{h^{p_e}}{p_e^{m-3/2}} + C_{\varphi}(T)\frac{h^{p_a}}{p_a^{m-3/2}}$$

Proof. Properly use discrete trace inequality to bound interface contributions.

Time discretization

Algebraic semi-discrete problem. Let U and Φ be two vectors containing the unknown expansion coefficients for $\mathbf{u}_h$ and φ_h respectively. Then, one can obtain

$$\begin{cases} \mathsf{M}_e\ddot{\mathsf{U}}(t) + \mathsf{A}_e\mathsf{U}(t) \; + \; \mathsf{C}_e\dot{\Phi}(t) \; = \mathsf{F}_e(t), \quad t \in (0,T], \\ \mathsf{M}_a\ddot{\Phi}(t) + \mathsf{A}_a\Phi(t) \; + \; \mathsf{C}_a\dot{\mathsf{U}}(t) \; = \mathsf{F}_a(t), \quad t \in (0,T], \\ \qquad\qquad + \text{ initial conditions.} \end{cases}$$

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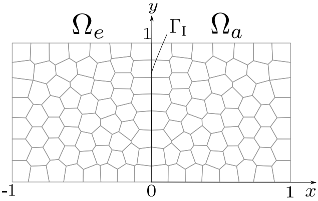
Leap-frog method. Subdivide the time interval $[0,T]$ into N_T subintervals of length Δt. For any $n = 0,...,N_T - 1$ solve

$$\begin{bmatrix} \mathsf{M}_e & \frac{\Delta t}{2}\mathsf{C}_e \\ -\frac{\Delta t}{2}\mathsf{C}_e^\mathsf{T} & \mathsf{M}_a \end{bmatrix} \begin{bmatrix} \mathsf{U}^{n+1} \\ \Phi^{n+1} \end{bmatrix} = \begin{bmatrix} \Delta t^2 \mathsf{F}_e^n \\ \Delta t^2 \mathsf{F}_a^n \end{bmatrix} + \begin{bmatrix} -\mathsf{M}_e & \frac{\Delta t}{2}\mathsf{C}_e \\ -\frac{\Delta t}{2}\mathsf{C}_e^\mathsf{T} & -\mathsf{M}_a \end{bmatrix} \begin{bmatrix} \mathsf{U}^{n-1} \\ \Phi^{n-1} \end{bmatrix} + \begin{bmatrix} 2\mathsf{M}_e - \Delta t^2\mathsf{A}_e & 0 \\ 0 & 2\mathsf{M}_a - \Delta t^2\mathsf{A}_a \end{bmatrix} \begin{bmatrix} \mathsf{U}^n \\ \Phi^n \end{bmatrix}$$

- **Explicit** and **second order accurate** with respect to the time step Δt

Numerical experiments

2D verification: [Mönköla, 2016]

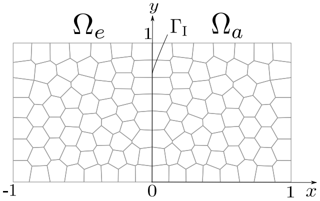


- homogeneous isotropic elastic material
- $T = 0.8\ s$ and $\Delta t = 10^{-4}\ s$
- **analytical solution**:

$$\boldsymbol{u}(x, y; t) = \left(\cos\left(\frac{4\pi x}{c_p}\right), \cos\left(\frac{4\pi x}{c_s}\right)\right)\cos(4\pi t),$$

$$\varphi(x, y; t) = \sin(4\pi x)\sin(4\pi t),$$

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho_e}},\ c_s = \sqrt{\frac{\mu}{\rho_e}}$$



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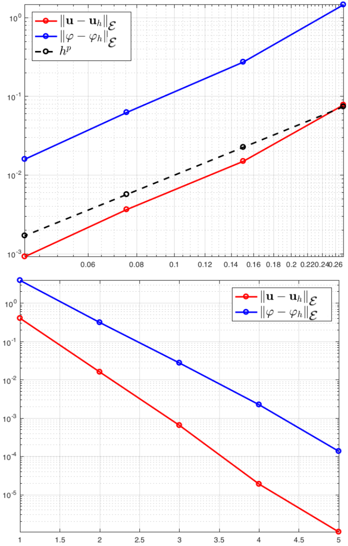
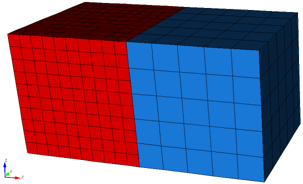


$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathcal{E},e}$ and $\|\varphi - \varphi_h\|_{\mathcal{E},a}$ vs. h (top) and p (bottom) at $T = 0.8$

3D verification



- nonconforming mesh
- for $T = 0.1s$, $\Delta t = 10^{-6}s$
- **analytical solution**

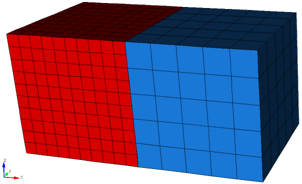
$$u_x(x, y, z; t) = \cos\left(\frac{4\pi x}{c_p}\right)\cos(4\pi t),$$

$$u_y(x, y, z; t) = \cos\left(\frac{4\pi x}{c_s}\right)\cos(4\pi t),$$

$$u_z(x, y, z, t) = \cos\left(\frac{4\pi x}{c_s}\right)\cos(4\pi t),$$

$$\varphi(x, y, z; t) = \sin(4\pi x)\sin(4\pi t).$$

3D verification



- nonconforming mesh
- for $T = 0.1s$, $\Delta t = 10^{-6}s$
- **analytical solution**

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$$u_y(x,y,z;t) = \cos\left(\frac{4\pi x}{c_s}\right)\cos(4\pi t),$$
$$u_z(x,y,z,t) = \cos\left(\frac{4\pi x}{c_s}\right)\cos(4\pi t),$$
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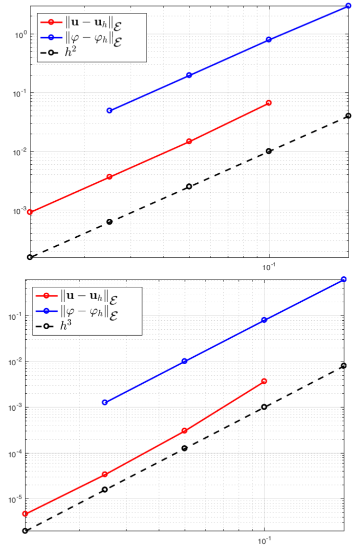
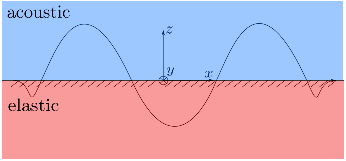


$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathcal{E},e}$ and $\|\varphi-\varphi_h\|_{\mathcal{E},a}$ vs. h, for $p=2$ (top) and $p=3$ (bottom) at $T=0.1$

3D verification: Scholte waves [Wilcox et al. 2010]

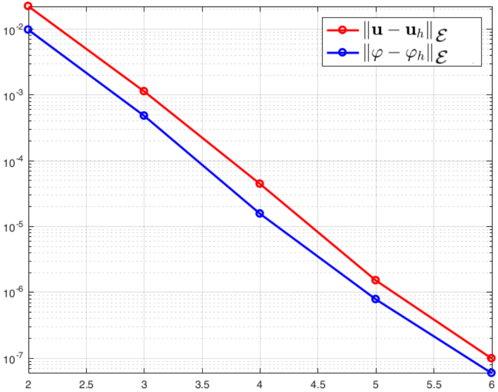


Scholte waves propagate along elasto-acoustic interfaces.
We consider $\Omega_e \cup \Omega_a = (-1, 1)\ m \times (-1, 1)\ m \times (-20, 20)\ m$,
$h_e = h_a = 0.41\ m$, $T = 0.1\ s$, and $\Delta t = 10^{-6}\ s$, with

$$
\begin{aligned}
u_1(x, y, z; t) &= k(B_2 e^{kb_{2p}z} - B_3 b_{2s} e^{kb_{2s}z}) \cos(kx - \omega t), \\
u_2(x, y, z; t) &= 0, \\
u_3(x, y, z; t) &= k(B_2 b_{2p} e^{kb_{2p}z} - B_3 e^{kb_{2s}z}) \sin(kx - \omega t), \quad && z < 0; \\
\varphi(x, y, z; t) &= \omega B_1 e^{-kb_{1p}z} \cos(kx - \omega t), && z > 0.
\end{aligned}
$$

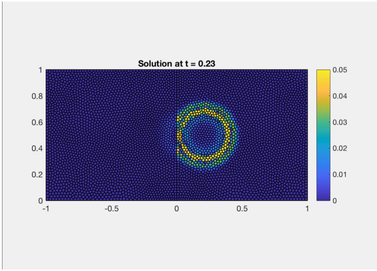
Wave amplitudes B_1, B_2 and B_3 have to satisfy a suitable **eigenvalue problem** of the form $\mathbf{\Lambda B} = \mathbf{0}$ stemming from the transmission conditions on Γ_I, and the speed of a Scholte wave is such that $\det \mathbf{\Lambda} = 0$.

We choose $\lambda = \mu = 1\ N/m^2$ and $\rho_e = 1\ kg/m^3$ for the elastic medium; $c = 1\ m/s$ and $\rho_a = 1\ kg/m^3$ for the acoustic medium. This yields $c_{\text{sch}} = 0.7110017230197\ m/s$, and we choose $B_1 = 0.3594499773037$, $B_2 = 0.8194642725978$, and $B_3 = 1$.

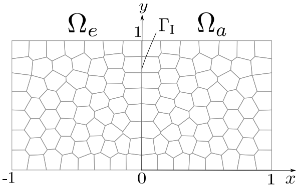


$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathcal{E},e}$ and $\|\varphi - \varphi_h\|_{\mathcal{E},a}$ vs. p at $T = 0.1$

Scattering by a point source



$t \mapsto \|\boldsymbol{u}(\boldsymbol{x};t)\|$ and $t \mapsto |\varphi(\boldsymbol{x};t)|$



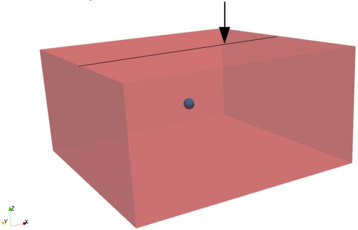
Point source in the acoustic domain (Ricker wavelet):

$$f_a(\boldsymbol{x},t) = -f_0\left(1 - 2\pi^2 f_p^2 (t-t_0)^2\right) e^{-\pi^2 f_p^2 (t-t_0)^2}\,\delta(\boldsymbol{x}-\boldsymbol{x}_0),$$

$$\boldsymbol{x}_0 \in \Omega_a,\ \ t_0 \in (0,T],$$

$$\boldsymbol{x}_0 = (0.2, 0.5), \quad t_0 = 0.1$$

Vertical point source at $\boldsymbol{x}_0 = (200, 0, 300)\, m$



$$\mathbf{f}_e(\boldsymbol{x}, t) = f(t)\mathbf{e}_z \delta(\boldsymbol{x} - \boldsymbol{x}_0),$$

$$f(t) = f_0 \left(1 - 2\pi^2 f_p^2 (t - t_0)^2\right) e^{-\pi^2 f_p^2 (t - t_0)^2},$$

$$t_0 = 0.25\, s,\ \ f_0 = 10^{10}\, N,\ \ f_p = 22\, Hz$$

Discretization parameters

- $h_e = 20\, m, h_a = 5\, m$
- $p_e = 4, p_a = 4$
- $\Delta t = 10^{-5}\, s$

Geometry & Material properties

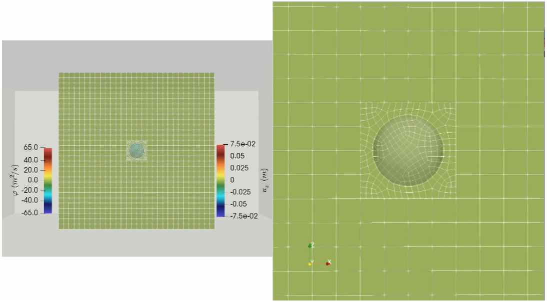
$$\Omega_a = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| < R\},\ \ R = 30\, m$$

$$\Omega_e = (-L_x, L_x) \times (-L_y, L_y) \times (-L_z, L_z) \backslash \overline{\Omega}_a$$

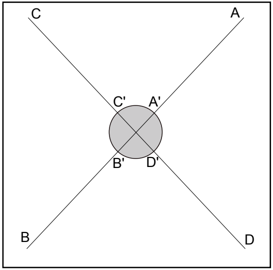
$$L_x = L_y = 600\, m,\ \ L_z = 300\, m$$

Region	$\rho\, (kg/m^3)$	$c_p\, (m/s)$	$c_s\, (m/s)$
Ω_e	2700	3000	1734
Ω_a	1024	300	–

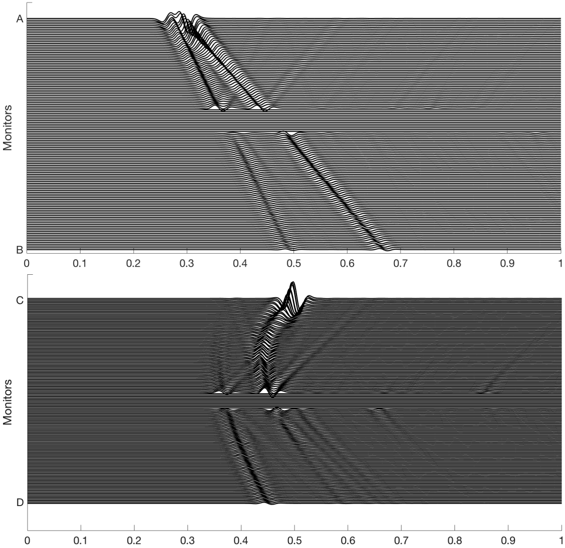
Underground acoustic cavity



Underground acoustic cavity: elastic monitors



$t \mapsto u_z(P,t)$ for monitored elastic points P from A to B, (top) and from C to D (bottom)



Conclusions & perspectives

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- hp-convergence for a dG method was proven on polytopal meshes
- Verfication by 2D and 3D numerical experiments
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Perspectives

- Inferring error estimates for the fully discrete problem
- Consider elastic-nonlinear acoustic models (Westervelt equation)
- Enriching the elastic model by considering a viscoelastic material response

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Thank you for the attention