# Higher order multipoint flux mixed finite element methods on quadrilaterals and hexahedra

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Polytopal Element Methods in Mathematics and Engineering
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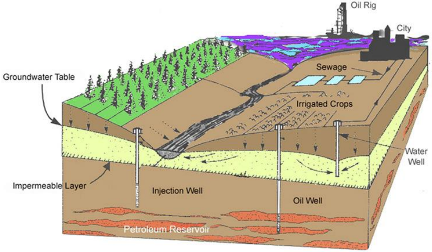
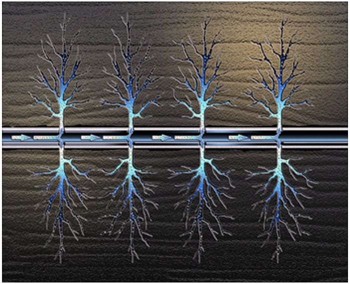
# Outline

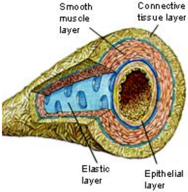
# Applications of coupled flow and mechanics



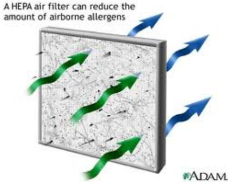

Surface-ground water systems



Hydraulic fracturing




Arterial flows




Industrial filters

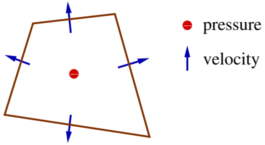
# Discretizations for Darcy flow

Model Problem:

$$\nabla \cdot \mathbf{u} = f, \quad \text{in } \Omega$$
$$\mathbf{u} = -K\nabla p, \quad \text{in } \Omega$$

Mixed Finite Element (MFE) method:

$\mathbf{u}_h \in \mathbf{V}_h \subset H(\mathrm{div}\,;\Omega)$, $p_h \in W_h \subset L^2(\Omega)$:




$$(K^{-1}\mathbf{u}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in \mathbf{V}_h$$
$$(\nabla \cdot \mathbf{u}_h, w) = (f, w), \qquad \forall w \in W_h$$

# Multipoint Flux MFE - Accurate Cell-Centered Scheme [1]

Wheeler - Y., SINUM (2006), Klausen - Winther, Num. Meth. PDEs (2006)

Find $\mathbf{u}_h \in \mathbf{V}_h \subset H(\mathrm{div}\,;\Omega)$ and $p_h \in W_h \subset L^2(\Omega)$,

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v})_Q - (p_h, \nabla\cdot\mathbf{v}) &= 0, \qquad \forall \mathbf{v} \in \mathbf{V}_h \\ (\nabla\cdot\mathbf{u}_h, w) &= (f, w), \qquad \forall w \in W_h \end{aligned}$$

1. Particular finite element spaces:

   The lowest order $\mathrm{BDM}_1$ space

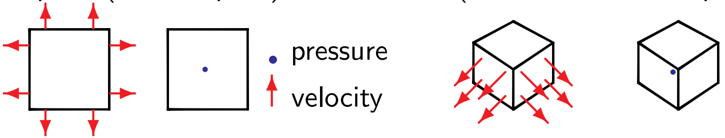
2. Specific numerical quadrature rule:

   Vertex rule for $(K^{-1}\mathbf{u}_h, \mathbf{v})_Q$

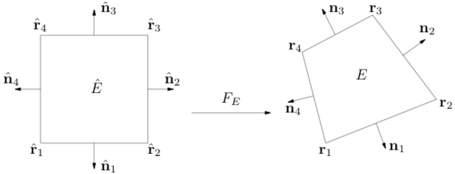
[1] Motivated by MPFA methods - Aavatsmark, Edwards

# Mixed Finite Element Spaces

- 2D square ($\mathrm{BDM}_1$ space) and 3D cube (enhanced $\mathrm{BDDF}_1$ space):




Bilinear mapping




- $DF_E$: Jacobian
- $J_E := \det(DF_E)$

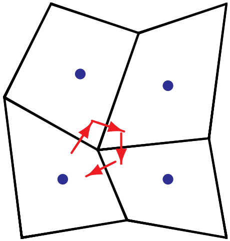
Piola transformation : $\mathbf{v} := \mathcal{P}\hat{\mathbf{v}} = \frac{1}{J_E} DF_E \hat{\mathbf{v}} \circ F_E^{-1}$

# Reduction to a Cell-Centered Stencil

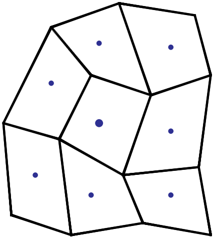
$$(K^{-1}\mathbf{u}_h, \mathbf{v}_h)_E = (\mathcal{M}_E \hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h)_{\hat{E}}\,, \qquad \mathcal{M}_E = \frac{1}{J_E} DF_E^T K^{-1} DF_E$$

Numerical quadrature:

$$(K^{-1}\mathbf{u}_h, \mathbf{v}_h)_{Q,E} = (\mathcal{M}_E \hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h)_{Q,\hat{E}} = \frac{1}{4}\sum_{i=1}^{4} \mathcal{M}_E(\hat{\mathbf{r}}_i)\hat{\mathbf{u}}_h(\hat{\mathbf{r}}_i)\cdot\hat{\mathbf{v}}_h(\hat{\mathbf{r}}_i),$$



Local velocity interaction



Cell-centered pressure stencil

# Accuracy of Multipoint Flux MFE methods

## Theorem (Ingram-Wheeler-Y. 2010)

*For the symmetric MFMFE on smooth quadrilaterals and hexahedra*

$$\|\mathbf{u} - \mathbf{u}_h\| + \|p - p_h\| \leq Ch(|\mathbf{u}|_1 + \|p\|_2)$$

## Theorem (Wheeler-Xue-Y. 2011)

*For the non-symmetric MFMFE on general quadrilaterals and hexahedra*

$$\|\Pi\mathbf{u} - \mathbf{u}_h\| + \|p - p_h\| \leq Ch(|\mathbf{u}|_1 + \|p\|_2)$$

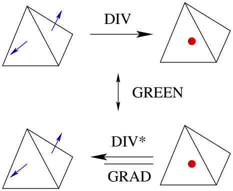
$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{F}_h} \leq Ch(|\mathbf{u}|_1 + \|p\|_2)$$

Face norm: $\|\mathbf{v}\|^2_{\mathcal{F}_h} := \sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} \frac{|E|}{|e|} \|\mathbf{v}\cdot\mathbf{n}_e\|^2_e \approx h \sum_{e\in\partial E} \|\mathbf{v}\cdot\mathbf{n}_e\|^2_e$

# Local flux mimetic finite difference method on polyhedra [2]

**Standard MFD method:**

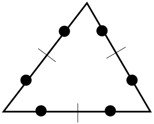
- $(\text{DIV } \mathbf{u}_h)_E = \frac{1}{|E|} \sum_{e \in \partial E} |e| \, u_E^e$
- Define inner products in pressure space $W_h$ and velocity space $\mathbf{V}_h$

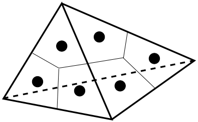



$$[\mathbf{u}_h, \mathbf{v}]_V = [p_h, \operatorname{div} \mathbf{v}]_W, \quad \forall \, \mathbf{v} \in \mathbf{V}_h,$$

$$[\operatorname{div} \mathbf{u}_h, w]_W = [\mathbf{f}, \mathbf{q}]_W, \quad \forall \, w \in W_h.$$

**Local flux MFD method:**





$$[\mathbf{u}, \mathbf{v}]_{V,E} = \sum_{i=1}^{n_E} w_i K_E^{-1} \mathbf{u}_E(\mathbf{r}_i) \cdot \mathbf{v}_E(\mathbf{r}_i).$$

[2]Lipnikov, Shashkov, I.Y., Numer. Math. (2009)

# Mixed elasticity

$$\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{\eta}) = \mathbf{f}$$
$$\boldsymbol{\sigma}(\boldsymbol{\eta}) = \lambda(\operatorname{div}\boldsymbol{\eta})\mathbf{I} + 2\mu\mathbf{D}(\boldsymbol{\eta}), \quad \mathbf{D}(\boldsymbol{\eta}) = (\nabla\boldsymbol{\eta} + \nabla\boldsymbol{\eta}^T)/2$$

Compliance tensor:

$$\mathbf{A}\boldsymbol{\sigma}(\boldsymbol{\eta}) = \mathbf{D}(\boldsymbol{\eta}), \quad \mathbf{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\sigma})\mathbf{I}\right)$$

Weakly symmetric formulation (Arnold, Falk, Winther [2007])

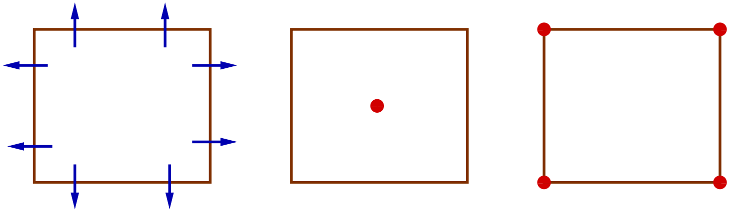
$\mathbb{M} = \mathbb{R}^{d\times d}$, $\mathbb{K} = \mathbb{R}^{d\times d}_{\text{skew}}$

Find $(\boldsymbol{\sigma}, \boldsymbol{\eta}, \mathbf{r}) \in H(\operatorname{div}, \Omega; \mathbb{M}) \times L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{K})$ such that

$$\begin{aligned}
&(\mathbf{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{\eta}) + (\boldsymbol{\tau}, \mathbf{r}) = 0, && \boldsymbol{\tau} \in H(\operatorname{div}, \Omega; \mathbb{M})\\
&(\operatorname{div}\boldsymbol{\sigma}, \boldsymbol{\xi}) = (\mathbf{f}, \boldsymbol{\xi}), && \boldsymbol{\xi} \in L^2(\Omega, \mathbb{V})\\
&(\boldsymbol{\sigma}, \mathbf{t}) = 0, && \mathbf{t} \in L^2(\Omega, \mathbb{K})
\end{aligned}$$

# A multipoint stress mixed finite element method [3]

Mixed finite element spaces (Cockburn, Gopalakrishnan, Guzman [2010], Arnold, Awanou, Qiu [2013])



$\Sigma_h = (\mathrm{BDM}_1)^d$ $\quad V_h = (P_0)^d$ $\quad Q_h = (Q_1)^{d\times d}$

Find $(\boldsymbol{\sigma}_h, \boldsymbol{\eta}_h, \mathbf{r}_h) \in \Sigma_h \times V_h \times Q_h$ such that

$$
\begin{aligned}
&(\mathbf{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau})_Q + (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{\eta}_h) + (\boldsymbol{\tau}, \mathbf{r}_h)_Q = 0, && \boldsymbol{\tau} \in \Sigma_h \\
&(\operatorname{div}\boldsymbol{\sigma}_h, \boldsymbol{\xi}) = (\mathbf{f}, \boldsymbol{\xi}), && \boldsymbol{\xi} \in V_h \\
&(\boldsymbol{\sigma}_h, \mathbf{t})_Q = 0, && \mathbf{t} \in Q_h
\end{aligned}
$$

[3] Ambartsumyan, Khattatov, Nordbotten, I.Y., arXiv:1805.09920 [math.NA] and arXiv:1811.01928 [math.NA]

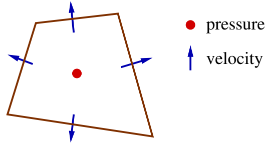
# Higher order MFMFE methods for Darcy flow on quadrilaterals and hexahedra

# A family of enhanced Raviart-Thomas spaces

$\mathcal{Q}^k$: polynomials of degree $\leq k$ in each variable

$\mathrm{RT}_k$ spaces, $k \geq 0$:

$$\hat{\mathbf{V}}_{RT}^k(\hat{E}) = \begin{pmatrix} \mathcal{Q}^k + \mathcal{Q}^k \hat{x} \\ \mathcal{Q}^k + \mathcal{Q}^k \hat{y} \end{pmatrix}, \quad \hat{W}^k(\hat{E}) = \mathcal{Q}^k(\hat{E})$$




**Enhanced spaces, $k \geq 1$:**

$$\hat{\mathbf{V}}^k(\hat{E}) = \hat{\mathbf{V}}_{RT}^{k-1}(\hat{E}) \oplus \tilde{\boldsymbol{\mathcal{B}}}^k(\hat{E}), \quad \tilde{\boldsymbol{\mathcal{B}}}^k(\hat{E}) : \text{curls \& bubbles}$$

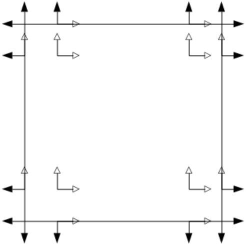
**Lemma**

$$\dim \hat{\mathbf{V}}^k(\hat{E}) = \dim \mathcal{Q}^k(\hat{E})^d, \quad \hat{\nabla} \cdot \hat{\mathbf{V}}^k(\hat{E}) = \hat{W}^{k-1}(\hat{E}), \quad \hat{\mathbf{V}}^k \cdot \hat{\mathbf{n}}_{\hat{e}} \in \mathcal{Q}^k(\hat{e})$$

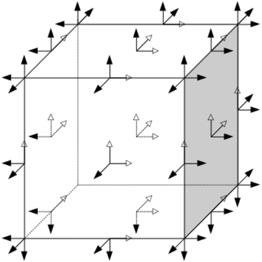
# Degrees of freedom of the enhanced RT spaces

**Lemma**

*A vector in $\hat{\mathbf{V}}^k(\hat{E})$ is uniquely determined by its values at the nodes of the trapezoidal quadrature rule for $k = 1$ and the Gauss-Lobatto quadrature rule of order $k + 1$ for $k \geq 2$.*



$k = 3$ in 2D



$k = 2$ in 3D

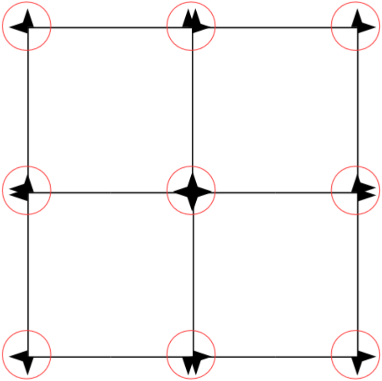
# k-th order multipoint flux MFE method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^k \times W_h^{k-1}$, $k \geq 1$, such that

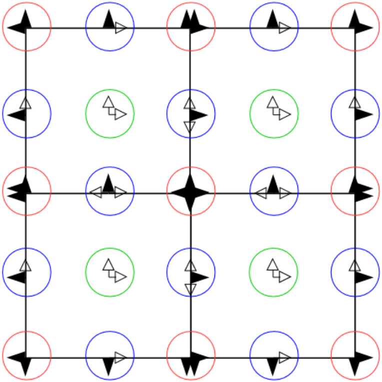
$$\left(K^{-1}\mathbf{u}_h,\, \mathbf{v}\right)_Q - (p_h,\, \nabla \cdot \mathbf{v}) = 0, \quad \mathbf{v} \in \mathbf{V}_h^k$$
$$(\nabla \cdot \mathbf{u}_h,\, w) = (f,\, w), \quad w \in W_h^{k-1}$$

$(\cdot, \cdot)_Q$ denotes the Gauss-Lobatto quadrature rule of order $k+1$
(when $k = 1$ we use trapezoid quadrature rule)

# Localization property of MFMFE



$k = 1$



$k = 2$

# Reduction to a cell-based SPD pressure system

Algebraic system:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix},$$

$A$ is block-diagonal with SPD blocks. Velocity elimination results in SPD pressure system

$$BA^{-1}B^T P = -F$$

# Convergence of the k-th order MFMFE method

**Lemma**

*For any* $\hat{\mathbf{q}} \in \hat{\mathbf{V}}^k(\hat{E})$ *and for any* $k \geq 1$,

$$\left(\hat{\mathbf{q}} - \hat{\Pi}_{RT}^{k-1}\hat{\mathbf{q}},\, \hat{\mathbf{v}}\right)_{\hat{Q},\hat{E}} = 0, \quad \text{for all vectors } \hat{\mathbf{v}} \in \mathcal{Q}^{k-1}(\hat{E}, \mathbb{R}^d).$$

Assume that $\mathcal{T}_h$ consists of $h^2$-parallelograms or $h^2$-parallelepipeds.

**Theorem (Optimal convergence)**

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^k \|\mathbf{u}\|_k$$
$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \leq Ch^k \|\nabla \cdot \mathbf{u}\|_k$$
$$\|p - p_h\| \leq Ch^k \left(\|\mathbf{u}\|_k + \|p\|_k\right)$$

**Theorem (Superconvergence of pressure)**

$$\|\mathcal{Q}_h p - p_h\| \leq Ch^{k+1} \|\mathbf{u}\|_{k+1}$$

# Numerical test in 2D

$$K = \begin{pmatrix} (x+1)^2 + y^2 & \sin(xy) \\ \sin(xy) & (x+1)^2 \end{pmatrix}, \quad p = x^3y^4 + x^2 + \sin(xy)\cos(xy).$$

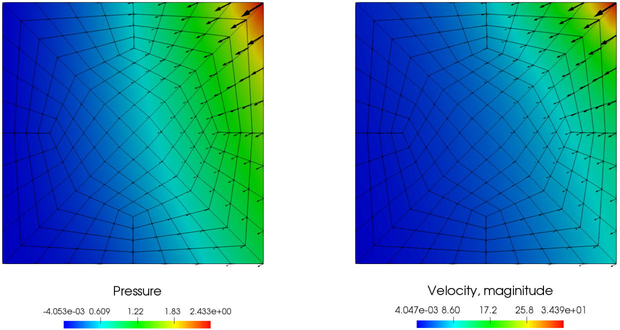



Figure: Computed solution on the third level of refinement

# Relative errors and convergence rates

| | $k = 2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\|\mathbf{u} - \mathbf{u}_h\|$ | | $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ | | $\|p - p_h\|$ | | $\|\mathcal{Q}_h p - p_h\|$ | |
| $h$ | error | rate | error | rate | error | rate | error | rate |
| 1/3 | 8.80E-02 | – | 1.46E-01 | – | 3.20E-02 | – | 5.80E-03 | – |
| 1/6 | 2.36E-02 | 1.9 | 3.74E-02 | 2.0 | 7.90E-03 | 2.0 | 7.73E-04 | 2.9 |
| 1/12 | 6.01E-03 | 2.0 | 9.41E-03 | 2.0 | 1.98E-03 | 2.0 | 1.18E-04 | 2.7 |
| 1/24 | 1.50E-03 | 2.0 | 2.36E-03 | 2.0 | 4.96E-04 | 2.0 | 1.70E-05 | 2.8 |
| 1/48 | 3.74E-04 | 2.0 | 5.89E-04 | 2.0 | 1.24E-04 | 2.0 | 2.30E-06 | 2.9 |
| | $k = 3$ | | | | | | | |
| | $\|\mathbf{u} - \mathbf{u}_h\|$ | | $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ | | $\|p - p_h\|$ | | $\|\mathcal{Q}_h p - p_h\|$ | |
| $h$ | error | rate | error | rate | error | rate | error | rate |
| 1/3 | 1.35E-02 | – | 1.96E-02 | – | 3.16E-03 | – | 4.36E-04 | – |
| 1/6 | 1.69E-03 | 3.0 | 2.44E-03 | 3.0 | 3.95E-04 | 3.0 | 3.33E-05 | 3.7 |
| 1/12 | 2.09E-04 | 3.0 | 3.04E-04 | 3.0 | 4.95E-05 | 3.0 | 2.48E-06 | 3.8 |
| 1/24 | 2.59E-05 | 3.0 | 3.80E-05 | 3.0 | 6.19E-06 | 3.0 | 1.74E-07 | 3.8 |
| 1/48 | 3.22E-06 | 3.0 | 4.75E-06 | 3.0 | 7.73E-07 | 3.0 | 1.17E-08 | 3.9 |
| | $k = 4$ | | | | | | | |
| | $\|\mathbf{u} - \mathbf{u}_h\|$ | | $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ | | $\|p - p_h\|$ | | $\|\mathcal{Q}_h p - p_h\|$ | |
| $h$ | error | rate | error | rate | error | rate | error | rate |
| 1/3 | 1.13E-03 | – | 1.52E-03 | – | 2.46E-04 | – | 2.83E-05 | – |
| 1/6 | 6.84E-05 | 4.1 | 9.24E-05 | 4.0 | 1.52E-05 | 4.0 | 1.00E-06 | 4.8 |
| 1/12 | 4.20E-06 | 4.0 | 5.74E-06 | 4.0 | 9.50E-07 | 4.0 | 3.55E-08 | 4.8 |
| 1/24 | 2.59E-07 | 4.0 | 3.58E-07 | 4.0 | 5.94E-08 | 4.0 | 1.20E-09 | 4.9 |
| 1/48 | 1.61E-08 | 4.0 | 2.25E-08 | 4.0 | 3.71E-09 | 4.0 | 3.98E-11 | 4.9 |

# Numerical test in 3D

$$K = \begin{pmatrix} x^2 + (y+2)^2 & 0 & \cos(xy) \\ 0 & z^2 + 2 & \sin(xy) \\ \cos(xy) & \sin(xy) & (y+3)^2 \end{pmatrix}, \quad p = x^4 y^3 + x^2 + yz^2 + \cos(xy) + \sin(z).$$

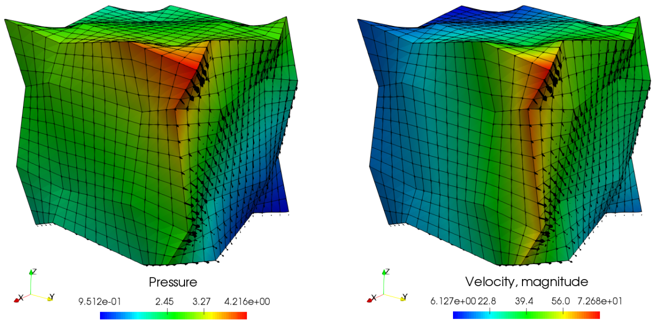
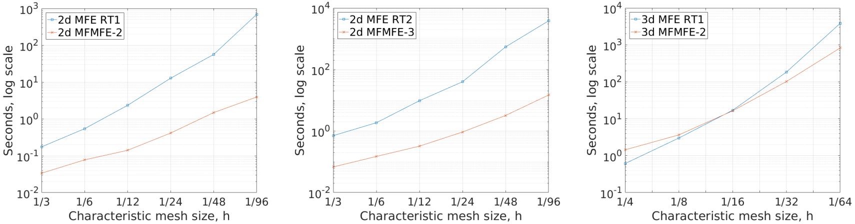



Figure: Computed solution on the third level of refinement

# Relative errors and convergence rates

| | $k = 2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\|\mathbf{u} - \mathbf{u}_h\|$ | | $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ | | $\|p - p_h\|$ | | $\|\mathcal{Q}_h p - p_h\|$ | |
| $h$ | error | rate | error | rate | error | rate | error | rate |
| 1/4 | 7.47E-03 | – | 2.92E-02 | – | 4.97E-03 | – | 1.63E-04 | – |
| 1/8 | 1.82E-03 | 2.0 | 7.24E-03 | 2.0 | 1.24E-03 | 2.0 | 2.23E-05 | 2.9 |
| 1/16 | 4.51E-04 | 2.0 | 1.81E-03 | 2.0 | 3.11E-04 | 2.0 | 3.07E-06 | 2.9 |
| 1/32 | 1.12E-04 | 2.0 | 4.51E-04 | 2.0 | 7.77E-05 | 2.0 | 4.12E-07 | 2.9 |
| 1/64 | 2.80E-05 | 2.0 | 1.13E-04 | 2.0 | 1.94E-05 | 2.0 | 5.38E-08 | 2.9 |
| | $k = 3$ | | | | | | | |
| | $\|\mathbf{u} - \mathbf{u}_h\|$ | | $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ | | $\|p - p_h\|$ | | $\|\mathcal{Q}_h p - p_h\|$ | |
| $h$ | error | rate | error | rate | error | rate | error | rate |
| 1/4 | 5.06E-04 | – | 2.01E-03 | – | 2.03E-04 | – | 3.78E-06 | – |
| 1/8 | 6.37E-05 | 3.0 | 2.46E-04 | 3.0 | 2.54E-05 | 3.0 | 2.56E-07 | 3.9 |
| 1/16 | 7.93E-06 | 3.0 | 3.05E-05 | 3.0 | 3.17E-06 | 3.0 | 1.87E-08 | 3.8 |
| 1/32 | 9.87E-07 | 3.0 | 3.81E-06 | 3.0 | 3.97E-07 | 3.0 | 1.35E-09 | 3.8 |
| 1/64 | 1.21E-07 | 3.0 | 4.88E-07 | 3.0 | 4.96E-08 | 3.0 | 8.83E-11 | 3.9 |

# Numerical results - efficiency

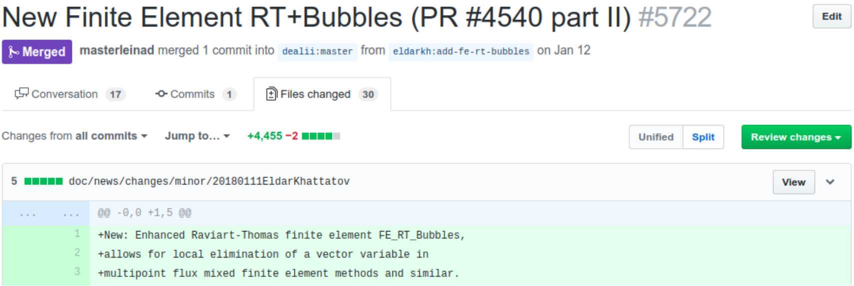



(a) Example 1, $k = 2$ (b) Example 1, $k = 3$ (c) Example 2, $k = 2$

Figure: Time to assemble and solve the linear system

These results were obtained with 8-core Ryzen 1700 CPU and 9.0.0-pre version of deal.II in release configuration.

# Higher order MFMFE methods with deal.II




The implementation of the new finite elements (FE_RT_Bubbles) is available in deal.II: `https://github.com/dealii`

Documented tutorial program for the higher order MFMFE method can be found in deal.II Code Gallery:
`http://dealii.org/developer/doxygen/deal.II/code_gallery_MultipointFluxMixedFiniteElementMethods.html`

# Summary

- General order MFMFE method on quadrilaterals and hexahedra
- A family of curl-enhanced Raviart-Thomas spaces
- Gauss-Lobatto quadrature allows for local flux elimination
- Optimal convergence
- Superconvergence for the pressure at the quadrature points

## Current and Future Work

- Non-symmetric version for general quads and hexahedra
- Extensions to poroelasticity, Stokes, FSI, and FPSI
- Open problem: simplicial grids, polygonal grids

**Reference:** I. Ambartsumyan, E. Khattatov, J. Lee, and I. Yotov, Higher order multipoint flux mixed finite element methods on quadrilaterals and hexahedra, to appear in M3AS; arXiv:1710.06742 [math.NA]