A velocity convection operator for unstructured staggered discretizations

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CALIF^{3}S: `https://gforge.irsn.fr/gf/project/califs`

Context: from incompressible to compressible flows

Objective – derive a scheme for Euler (or Navier-Stokes) equations:

$\hookrightarrow$ which is a natural extension of an existing scheme for low Mach number flows: staggered discretization, upwinding with respect to the material velocity, solution of the internal energy balance ...

$\hookrightarrow$ to preserve the positivity of the internal energy: solution of the internal energy balance.

- Euler equations:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0,$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0,$$

$$\partial_t(\varrho E) + \operatorname{div}\big[(\varrho E + p)\boldsymbol{u}\big] = 0,$$

$$p = (\gamma - 1)\ \varrho e, \quad E = \frac{1}{2}\,|\boldsymbol{u}|^2 + e.$$

- Formally, taking the scalar product of the momentum balance equation by $\boldsymbol{u}$ and using the mass balance equation yields the kinetic energy balance equation:

$$\partial_t(\varrho E_c) + \operatorname{div}(\varrho E_c\, \boldsymbol{u}) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0 \quad (\leq 0), \qquad E_c = \frac{1}{2}\,|\boldsymbol{u}|^2.$$

  Subtracting to the total energy balance yields the internal energy balance:

$$\partial_t(\varrho e) + \operatorname{div}(\varrho e \boldsymbol{u}) + p \operatorname{div} \boldsymbol{u} = 0 \quad (\geq 0),$$

  and, from this equation, we get $e \geq 0$.

Context: from incompressible to compressible flows

Objective – derive a scheme for Euler (or Navier-Stokes) equations: staggered discretization, upwinding with respect to the material velocity, solution of the internal energy balance ...

$\hookrightarrow$ but how to ensure the consistency ?

- The scheme must be consistent with the conservative equations (so, the total energy balance), to compute the correct shock solutions.
- so try to take the reverse course ?
  - take the inner product of the discrete momentum balance equation by $\boldsymbol{u}$ to obtain a kinetic energy balance,
  - add to the discrete internal energy balance.
- Needs a discrete local kinetic energy balance.

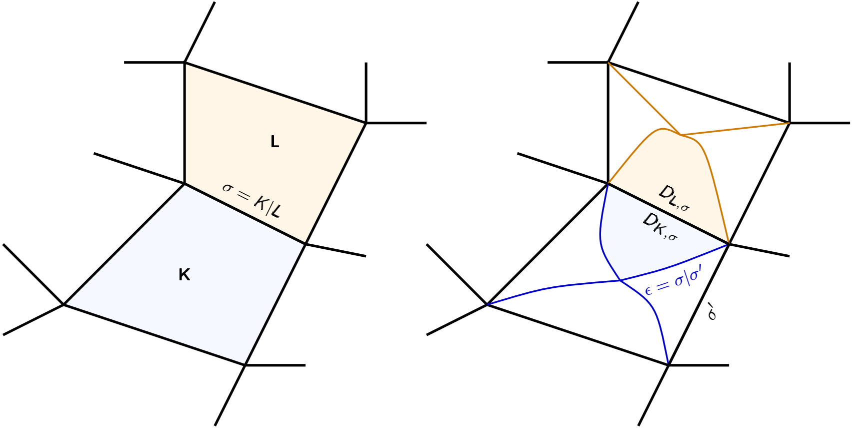
In the incompressible case, a global kinetic energy balance is also an important feature of the scheme (stability, convergence analysis, dissipation properties for Large Eddy Simulation of turbulent flows...).

Staggered schemes for compressible flows: Harlow & Amsden, Wesseling and co-workers, Goudon and co-workers, Després and Dakin...

Outline

1. Space discretization
2. Derivation of a discrete kinetic energy balance
3. Definition of the convection operator
4. A derived convection operator on the primal mesh
5. Weak consistency
   - The time-derivative term
   - The divergence term

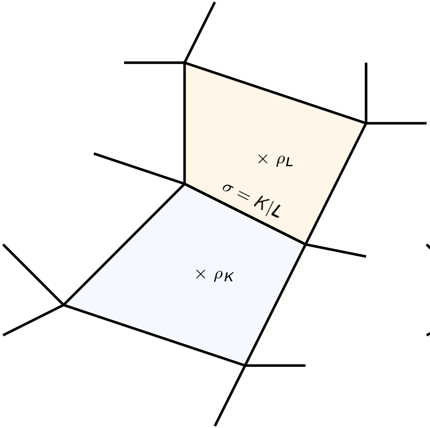
Space discretization (1/2)

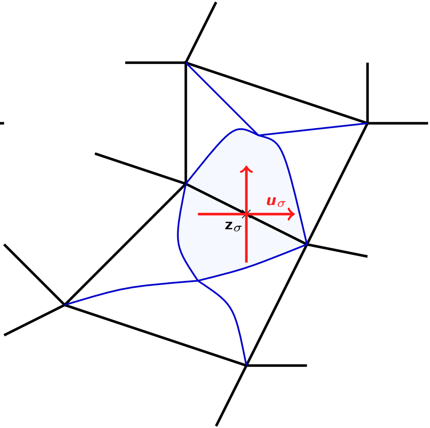


$\mathcal{E}$, $\mathcal{E}(K)$: faces of the primal mesh, faces of the control volume K.

$\bar{\mathcal{E}}$, $\bar{\mathcal{E}}(D_\sigma)$: faces of the dual mesh, faces of the control volume D_σ.

Space discretization (2/2)





Derivation of a discrete kinetic energy balance

- Mass balance over D_σ, $\mathcal{C}(z)_\sigma$:

$$\mathcal{M}_\sigma = \frac{|D_\sigma|}{\delta t}(\rho^{n+1}_{D_\sigma} - \rho^n_{D_\sigma}) + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} = 0,$$

$$\mathcal{C}_\sigma z = \frac{|D_\sigma|}{\delta t}(\rho^{n+1}_{D_\sigma} z^{n+1}_\sigma - \rho^n_{D_\sigma} z^n_\sigma) + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}\; z^{n+1}_\epsilon.$$

- $\partial_t(\rho z) + \mathrm{div}(\rho z \boldsymbol{u}) = \rho\big(\partial_t z + \boldsymbol{u} \cdot \boldsymbol{\nabla} z\big)$:

$$\mathcal{C}_\sigma z = \frac{|D_\sigma|}{\delta t}\rho^n_{D_\sigma}(z^{n+1}_\sigma - z^n_\sigma) + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}\;(z^{n+1}_\epsilon - z^{n+1}_\sigma) + z^{n+1}_\sigma \mathcal{M}_\sigma.$$

- $z\rho\big(\partial_t z + \boldsymbol{u} \cdot \boldsymbol{\nabla} z\big) = \frac{1}{2}\rho\big(\partial_t z^2 + \boldsymbol{u} \cdot \boldsymbol{\nabla} z^2\big)$:

$$z^{n+1}_\sigma \mathcal{C}_\sigma z = \frac{1}{2}\frac{|D_\sigma|}{\delta t}\rho^n_{D_\sigma}\big((z^{n+1}_\sigma)^2 - (z^n_\sigma)^2\big) + \frac{1}{2}\sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}\;\big((z^{n+1}_\epsilon)^2 - (z^{n+1}_\sigma)^2\big) + \mathcal{R}.$$

with $\mathcal{R} \geq 0$ – Tool: $2a(a-b) = a^2 - b^2 + (a-b)^2$.

- $\frac{1}{2}\rho\big(\partial_t z^2 + \boldsymbol{u} \cdot \boldsymbol{\nabla} z^2\big) = \frac{1}{2}\partial_t(\rho z^2) + \frac{1}{2}\mathrm{div}(\rho z^2 \boldsymbol{u})$:

$$z^{n+1}_\sigma \mathcal{C}_\sigma z = \frac{1}{2}\frac{|D_\sigma|}{\delta t}\big(\rho^{n+1}_{D_\sigma}(z^{n+1}_\sigma)^2 - \rho^n_{D_\sigma}(z^n_\sigma)^2\big) + \frac{1}{2}\sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}\;(z^{n+1}_\epsilon)^2 + \mathcal{R}.$$

Definition of the convection operator

- From the mass balance over the primal cells:

$$\frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = 0,$$

- define

$$\rho_{D_\sigma}, \;\; F_{\sigma,\epsilon},$$

- such as the mass balance over the dual cells holds:

$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} = 0.$$

Building a discrete mass balance over the dual cells (1/5)

Let:

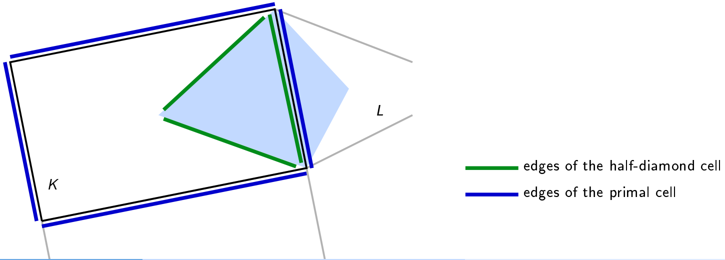
$$\forall \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}, \quad \rho_{D_\sigma} = \underbrace{\frac{|D_{K,\sigma}|}{|D_\sigma|}}_{\xi_K^\sigma} \rho_K + \underbrace{\frac{|D_{L,\sigma}|}{|D_\sigma|}}_{\xi_L^\sigma} \rho_L.$$

$\xi_K^\sigma \geq 0$, and $\displaystyle\sum_{\sigma \in \mathcal{E}(K)} \xi_K^\sigma = 1.$

Assume:

(H1) A discrete mass balance over the half-diamond cells is satisfied, in the following sense:

$$\forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}(K), \qquad F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon = \xi_K^\sigma \Big[\sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'} \Big],$$



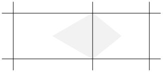
Building a discrete mass balance over the dual cells (2/5)

If:

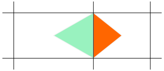
$\forall \sigma = K|L \in \mathcal{E}_{\rm int}$, $\quad |D_\sigma| = \xi_K^\sigma\, |K| + \xi_L^\sigma\, |L|, \quad |D_\sigma|\, \rho_{D_\sigma} = \xi_K^\sigma\, |K|\, \rho_K + \xi_L^\sigma\, |K|\, \rho_L.$

$\forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}(K)$, $\quad F_\sigma + \displaystyle\sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon = \xi_K^\sigma \Big[\sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'} \Big].$

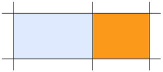
Then:



$$\frac{|D_\sigma|}{\delta t}\, \big[\rho_{D_\sigma} - \rho^*_{D_\sigma}\big] + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_\epsilon =$$



$$\xi_K^\sigma \frac{|K|}{\delta t}\, \big[\rho_K - \rho_K^*\big] + F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon \quad + \quad \xi_L^\sigma \cdots - F_\sigma + \cdots =$$



$$\xi_K^\sigma \Big[\frac{|K|}{\delta t}\, \big[\rho_K - \rho_K^*\big] + \sum_{\sigma \in \mathcal{E}(K)} F_\sigma \Big] \quad + \quad \xi_L^\sigma \Big[\dots \Big] = 0.$$

Building a discrete mass balance over the dual cells (3/5)

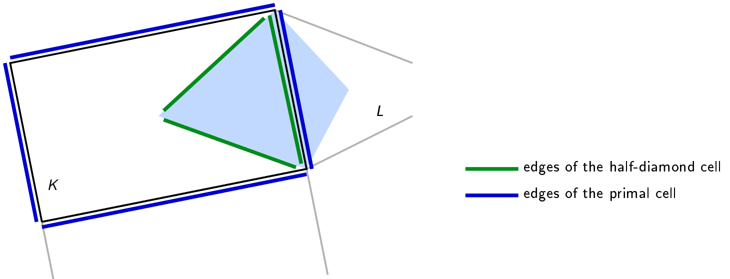
…so we need:

$$\forall \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad |D_\sigma| = \xi_K^\sigma\,|K| + \xi_L^\sigma\,|L|, \quad |D_\sigma|\,\rho_{D_\sigma} = \xi_K^\sigma\,|K|\,\rho_K + \xi_L^\sigma\,|K|\,\rho_L.$$

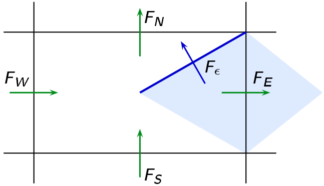
$$\forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}(K), \qquad F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon = \xi_K^\sigma \Big[\sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'} \Big].$$

Let us choose $\xi_K^\sigma = 1/\text{number of faces}$.

- The above system is independent of the cell (in other words, one may choose a unique expression of the dual mass fluxes as a function of the primal ones).
- The dual mesh is only viewed through ξ_K^σ and the sub-cell connectivities, so is completely abstract, and sometimes necessarily non-polygonal.



Building a discrete mass balance over the dual cells (4/5)



- Let $\boldsymbol{w}_K$ be such that:

$$\mathrm{div}(\boldsymbol{w}_K) = cste, \quad \int_\sigma \boldsymbol{w}_K \cdot \boldsymbol{n}_{K,\sigma} = F_{K,\sigma},\ \forall \sigma \in \mathcal{E}(K).$$

Example (1D, and on rectangular (2D and 3D) meshes):

$$\boldsymbol{w}_K = \frac{x_{\sigma'} - x}{x_{\sigma'} - x_\sigma} F_{K,\sigma} + \frac{x - x_\sigma}{x_{\sigma'} - x_\sigma} F_{K,\sigma'}.$$

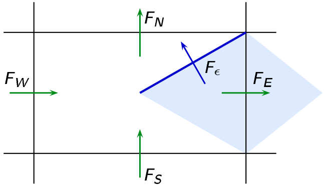
Let:

$$F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w}_K \cdot \boldsymbol{n}_{\sigma,\epsilon},\ \forall \epsilon \in \bar{\mathcal{E}}(D_{K,\sigma}).$$

Then

$$\sum_{\epsilon \in \bar{\mathcal{E}}(D_{K,\sigma})} F_{\sigma,\epsilon} = \int_{D_{K,\sigma}} \mathrm{div}\boldsymbol{w}_K = \frac{|D_{K,\sigma}|}{|K|} \int_K \mathrm{div}\boldsymbol{w}_K = \frac{|D_{K,\sigma}|}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}.$$

Building a discrete mass balance over the dual cells (4/5)



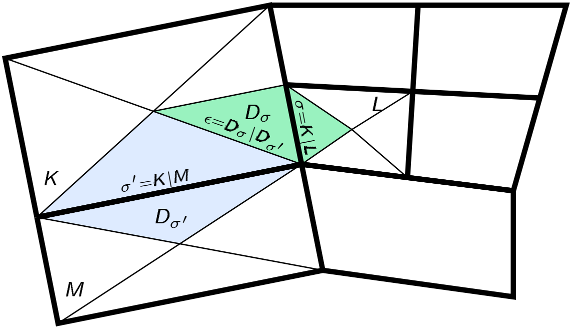
- For Crouzeix-Raviart elements, and for Rannacher-Turek elements on rectangular (2D and 3D) meshes, the construction of mass fluxes uses a constant divergence momentum function having the primal mass fluxes as traces (previous computation).
  This yields expressions of the form:

$$F_\epsilon = \alpha_W^\epsilon F_W + \alpha_N^\epsilon F_N + \alpha_E^\epsilon F_E + \alpha_S^\epsilon F_S.$$

  with constant coefficients α_σ^ϵ (bounded would be sufficient for consistency).
- For general quadrangles, keep the same expression (leads to $\xi_K^\sigma = 1/4$ for $d = 2$ and $\xi_K^\sigma = 1/6$ for $d = 3$) ...

  ...even if the diamond cells may no longer be chosen as cones (it is not possible to split a general quadrangle in four cones of equal volume having the edges as basis).

Building a discrete mass balance over the dual cells (5/5)



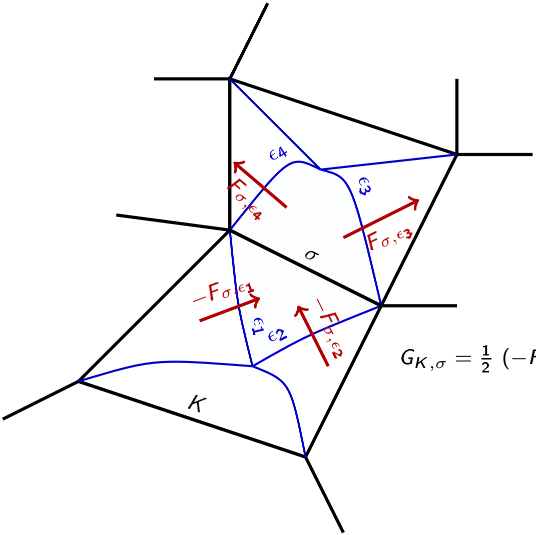
- For locally refined mesh, find a solution to the system (H1):

$$\forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}(K), \qquad F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon = \xi_K^\sigma \Big[\sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'} \Big].$$

  which will lead to the same type of relation, still with constant (for each "topology") coefficients.

Returning to the primal mesh

- A new mass flux through primal cells:



$$G_{K,\sigma} = \frac{1}{2}\left(-F_{\sigma,\epsilon_1} - F_{\sigma,\epsilon_2} + F_{\sigma,\epsilon_3} + F_{\sigma,\epsilon_4}\right)$$

Returning to the primal mesh

For $K \in \mathcal{M}$, let us sum the convection terms over the faces of K and divide by 2 the resulting equation, to get:

$$\mathcal{C}_K^{n+1} z = \frac{1}{\delta t} \sum_{\sigma \in \mathcal{E}(K)} \frac{|D_\sigma|}{2} (\rho_{D_\sigma}^{n+1} z_\sigma^{n+1} - \rho_{D_\sigma}^n z_\sigma^n) + \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n \; z_\epsilon^{n+1}.$$

Define:

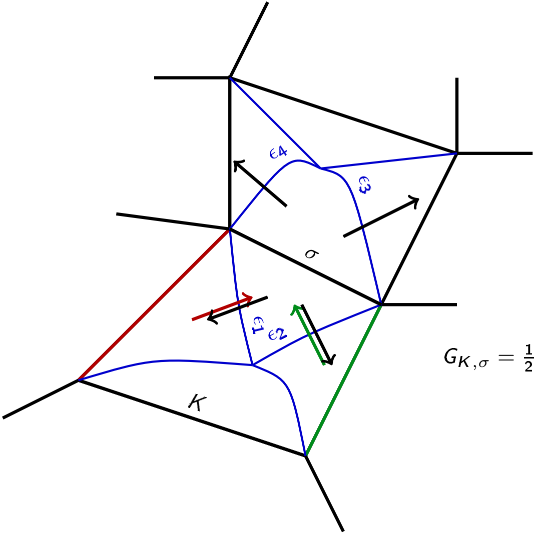
$$|K| \; (\rho z)_K^n = \sum_{\sigma \in \mathcal{E}(K)} \frac{|D_\sigma|}{2} \; \rho_{D_\sigma}^n z_\sigma^n, \tag{1}$$

$$G_{K,\sigma}^{n+1} = -\frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma), \epsilon \subset K} F_{\sigma,\epsilon}^n \; z_\epsilon^{n+1} + \frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma), \epsilon \not\subset K} F_{\sigma,\epsilon}^n \; z_\epsilon^{n+1}. \tag{2}$$

Reordering of the summations:

$$\mathcal{C}_K^{n+1} z = \frac{|K|}{\delta t} \; \left((\rho z)_K^{n+1} - (\rho z)_K^n\right) + \sum_{\sigma \in \mathcal{E}(K)} G_{K,\sigma}^{n+1}.$$

Returning to the primal mesh



$$G_{K,\sigma} = \frac{1}{2}\left(F_{\epsilon_1} + F_{\epsilon_2} + F_{\epsilon_3} + F_{\epsilon_4}\right)$$

Weak consistency

We now suppose given a sequence of meshes $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ and time discretizations $(\mathcal{T}^{(m)})_{m\in\mathbb{N}}$, with $h_{\mathcal{M}^{(m)}}$ and $\delta t_{\mathcal{T}^{(m)}}$ tending to zero as m tends to $+\infty$.
For $m\in\mathbb{N}$, let $\rho^{(m)}$, $\boldsymbol{u}^{(m)}$ and $z^{(m)}$ be discrete functions corresponding to the approximation on the mesh $\mathcal{M}^{(m)}$ and with the time discretization $\mathcal{T}^{(m)}$ of the density, the velocity and z respectively, defined by:

$$\rho^{(m)}(\boldsymbol{x},t)=\sum_{n=0}^{N^{(m)}-1}\ \sum_{K\in\mathcal{M}^{(m)}} \rho_K^n\ \mathcal{X}_K\ \mathcal{X}_{[t_n,t_{n+1})},$$

$$\boldsymbol{u}^{(m)}(\boldsymbol{x},t)=\sum_{n=0}^{N^{(m)}-1}\ \sum_{\sigma\in\mathcal{E}^{(m)}} \boldsymbol{u}_\sigma^n\ \mathcal{X}_{D_\sigma}\ \mathcal{X}_{[t_n,t_{n+1})},$$

$$z^{(m)}(\boldsymbol{x},t)=\sum_{n=0}^{N^{(m)}-1}\ \sum_{\sigma\in\mathcal{E}^{(m)}} z_\sigma^n\ \mathcal{X}_{D_\sigma}\ \mathcal{X}_{[t_n,t_{n+1})},$$

with $\mathcal{X}_K$, $\mathcal{X}_{D_\sigma}$ and $\mathcal{X}_{[t_n,t_{n+1})}$ the characteristic function of K, D_σ and the interval $[t_n,t_{n+1})$.

We suppose that

$$\rho^{(m)}\to\bar\rho,\quad \boldsymbol{u}^{(m)}\to\bar{\boldsymbol{u}},\quad z^{(m)}\to\bar z$$

in $L^1(\Omega\times(0,T))$, and that these sequences are uniformly bounded in $L^\infty(\Omega\times(0,T))$.

Weak consistency

Let $\varphi \in C_c^\infty(\Omega \times [0,T))$ and let us define φ_K^n by

$$\varphi_K^n = \varphi(\boldsymbol{x}_K, t_n), \text{ for } K \in \mathcal{M}^{(m)} \text{ and } 0 \leq n \leq N^{(m)},$$

where $\boldsymbol{x}_K$ stands for an arbitrary point of K.

Let

$$\sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} \mathcal{C}_K^{n+1} z \, \varphi_K^n = T_{\partial t}^{(m)} + T_{\rm div}^{(m)},$$

$$T_{\partial t}^{(m)} = \sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} |K| \, \big((\rho z)_K^{n+1} - (\rho z)_K^n\big) \, \varphi_K^n$$

$$T_{\rm div}^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma \in \mathcal{E}(K)} G_{K,\sigma}^{n+1}.$$

Then

$$T_{\partial t}^{(m)} \to -\int_0^T \int_\Omega \bar\rho \, \bar z \; \partial_t \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_\Omega \rho_0(\boldsymbol{x}) \, z_0(\boldsymbol{x}) \, \varphi(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x}, \;\; T_{\rm div}^{(m)} \to -\int_0^T \int_\Omega \bar\rho \, \bar z \, \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.$$

Weak consistency of the time derivative term

Lemma (Consistency of the time derivative term)

$$T_{\partial t}^{(m)} \to -\int_0^T \int_\Omega \bar\rho\, \bar z\ \partial_t \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \rho_0(\boldsymbol{x})\, z_0(\boldsymbol{x})\, \varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}.$$

Sketch of proof

$$(\rho z)_K = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} \frac{|D_\sigma|}{2}\ \rho_{D_\sigma}\, z_\sigma,$$

Since $\displaystyle\sum_{\sigma\in\mathcal{E}(K)} \frac{|D_\sigma|}{2} \neq |K|$, the function (ρz) oscillates, and don't converge (strongly) to $\bar\rho\bar z$.

However, (ρz) weakly converges to $\bar\rho\bar z$ in L^1. Indeed:

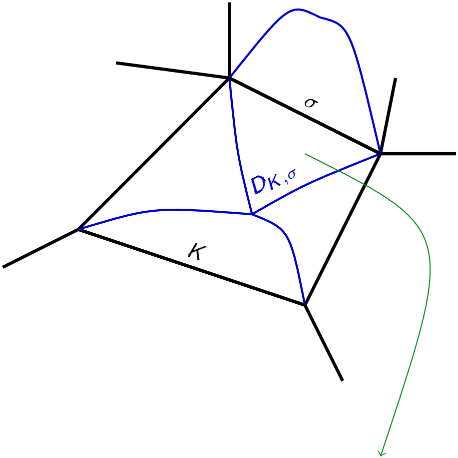
$$\sum_{K\in\mathcal{M}} (\rho z)_K \psi_K = \sum_{\sigma\in\mathcal{E}} (|D_{K,\sigma}|\ \rho_K + |D_{L,\sigma}|\ \rho_L)\ z_\sigma\ \frac{\psi_K+\psi_L}{2},$$

and, with $\psi_{K,\sigma}$ the mean value of ψ over $D_{K,\sigma}$,

$$\sum_{\sigma\in\mathcal{E}} (|D_{K,\sigma}|\ \rho_K\ \psi_{K,\sigma} + |D_{L,\sigma}|\ \rho_L\ \psi_{L,\sigma})\ z_\sigma = \int_\Omega \rho^{(m)} z^{(m)} \psi \,\mathrm{d}\boldsymbol{x},$$

so, by regularity of ψ ...

Then, integrating by parts in time make a discrete time derivative of φ appear which converges to $\partial_t\varphi$ in $L^\infty(\Omega\times(0,T))$ ...



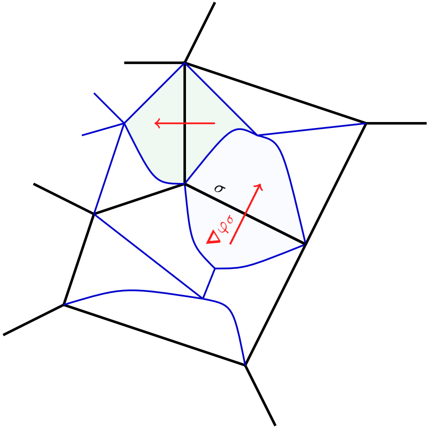
over $D_{K,\sigma}$, $\rho = \rho_K$, $z = z_\sigma$

A weakly convergent gradient (1/2)

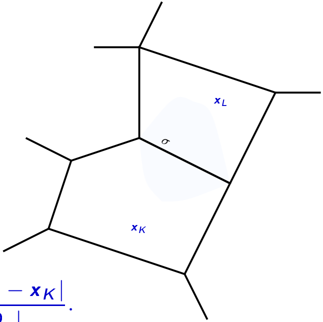
Define

$$\boldsymbol{\nabla}\varphi_\sigma^{\boldsymbol{n}} = \frac{|\sigma|}{|D_\sigma|}\,(\varphi_L^{\boldsymbol{n}} - \varphi_K^{\boldsymbol{n}})\boldsymbol{n}_{K,\sigma}, \quad \boldsymbol{\nabla}_{\mathcal{E},\mathcal{T}}\,\varphi(\boldsymbol{x},t) = \sum_{\boldsymbol{n}=0}^{N^{(\boldsymbol{m})}-1}\ \sum_{\sigma\in\mathcal{E}^{(\boldsymbol{m})}} \boldsymbol{\nabla}\varphi_\sigma^{\boldsymbol{n}}\ \mathcal{X}_{D_\sigma}\ \mathcal{X}_{[\boldsymbol{t_n},\boldsymbol{t_{n+1}})}$$

(Eymard & Gallouët, SINUM, 2000))



A weakly convergent gradient (2/2)



$\theta_{\mathcal{M}}^{\nabla}$ defined by

$$\theta_{\mathcal{M}}^{\nabla} = \max_{\sigma \in \mathcal{E}_{\rm int},\, \sigma = K|L} \frac{|\sigma|\ |\boldsymbol{x}_L - \boldsymbol{x}_K|}{|D_\sigma|}.$$

(characterization of the regularity of the mesh)

Lemma

$(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ sequence of meshes, $\theta_{\mathcal{M}^{(m)}}^{\nabla} \leq \theta^{\nabla}$ for $m \in \mathbb{N}$.
Then the sequence $(\boldsymbol{\nabla}_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\, \varphi)_{m\in\mathbb{N}}$ is bounded in $L^\infty(\Omega \times (0,T))^d$ uniformly with respect to m and converges to $\boldsymbol{\nabla}\varphi$ in $L^\infty(\Omega \times (0,T))^d$ weak $\star$.

How to use this weakly convergent gradient. . .

- Let

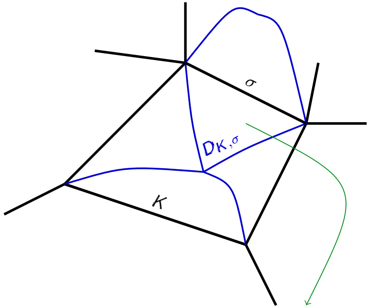
$$\tilde{G}_{K,\sigma} = |\sigma| \left[\frac{|D_{K,\sigma}|}{|D_\sigma|}\rho_K + \frac{|D_{L,\sigma}|}{|D_\sigma|}\rho_L\right] z_\sigma \; \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}.$$

Then

$$\sum_{K\in\mathcal{M}} \varphi_K \sum_{\sigma\in\mathcal{E}(K)} \tilde{G}_{K,\sigma} = \sum_{\sigma\in\mathcal{E}} \Big[|D_{K,\sigma}|\; \rho_K + |D_{L,\sigma}|\rho_L\Big] \; z_\sigma \; \boldsymbol{u}_\sigma \cdot \frac{|\sigma|}{|D_\sigma|}(\varphi_K - \varphi_L)\; \boldsymbol{n}_{K,\sigma}$$

$$= \int_\Omega \rho\, z\, \boldsymbol{u} \cdot \boldsymbol{\nabla}_{\mathcal{E}}\varphi \,\mathrm{d}\boldsymbol{x}.$$

- Unfortunately $G_{K,\sigma} \neq \tilde{G}_{K,\sigma}$.



Convergence to zero of "discrete jumps"

For $u \in L^1(\Omega \times (0,T))$, u_K^{n+1} mean value of u over $K \times (t_n, t_{n+1})$, $[u^n]_\sigma = |u_K^n - u_L^n|$, $[u_K]^n = |u_K^{n+1} - u_K^n|$.

$T_{\mathcal{M},\mathcal{T}}\, u$ defined by:

$$T_{\mathcal{M},\mathcal{T}}\, u = \sum_{n=0}^{N-1} (t_{n+1} - t_n) \sum_{\sigma \in \mathcal{E}_{\rm int}, \sigma = K|L} |D_\sigma| [u^{n+1}]_\sigma + \sum_{n=1}^{N-1} (t_{n+1} - t_n) \sum_{K \in \mathcal{M}} |K| [u_K]^n.$$

$\theta_{\mathcal{M}}$ defined by

$$\theta_{\mathcal{M}} = \max_{K \in \mathcal{M}} \max_{\sigma \in \mathcal{E}_K} \frac{|D_\sigma|}{|K|}.$$

Lemma

$(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ a sequence of meshes such that $\theta_{\mathcal{M}^{(m)}} \le \theta$ for all $m \in \mathbb{N}$. We suppose that the number of faces of a cell $K \in \mathcal{M}^{(m)}$ is bounded by $\mathcal{N}_{\mathcal{E}}$, for all $m \in \mathbb{N}$.

$(u_p)_{p \in \mathbb{N}}$ a sequence of functions of $L^1(\Omega \times (0,T))$ such that $u_p \to u$ in $L^1(\Omega \times (0,T))$ as $p \to +\infty$.

Then $T_{\mathcal{M}^{(m)},\mathcal{T}^{(m)}}\, u_p$ tends to zero when m tends to $+\infty$ uniformly with respect to $p \in \mathbb{N}$.

Weak consistency of the divergence term

Lemma (Consistency of the divergence term)

$$T_{\mathrm{div}}^{(m)} \to -\int_0^T \int_\Omega \bar\rho\, \bar z\, \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Sketch of proof – By a discrete integration by parts with respect to the space, we get something of the form:

$$\begin{aligned} T_{\mathrm{div}}^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)},\ \sigma=K|L} |D_\sigma|\ G_{K,\sigma}^{n+1}\ \frac{1}{|D_\sigma|}(\varphi_K^n - \varphi_L^n) \\ &= \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)},\ \sigma=K|L} |D_\sigma|\ \boldsymbol{G}_\sigma^{n+1} \cdot \ \frac{|\sigma|}{|D_\sigma|}(\varphi_K^n - \varphi_L^n)\, \boldsymbol{n}_{K,\sigma}. \end{aligned}$$

The last term weakly converge to $\boldsymbol{\nabla}\varphi$. Then, struggle with uniform boudedness and the fact that the space translates tend to zero to show that

$$\boldsymbol{G}^{(m)}(\boldsymbol{x},t) = \sum_{n=0}^{N^{(m)}-1} \sum_{\sigma\in\mathcal{E}^{(m)}} \boldsymbol{G}_\sigma^{n+1}\ \mathcal{X}_{D_\sigma}\ \mathcal{X}_{[t_n,t_{n+1})},$$

converges to $\bar\rho\,\bar z\,\bar{\boldsymbol{u}}$ in $L^1(\Omega\times(0,T))$.

Weak consistency of the divergence term

Sketch of proof (continued)

To this purpose, exploit the linear system defining the dual mass fluxes.

$$\forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}(K), \qquad F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_{K,\sigma}) \setminus \{\sigma\}} F_\epsilon = \xi_K^\sigma \Big[\sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'} \Big].$$

A simple subcase, the steady case – In this specific situation,

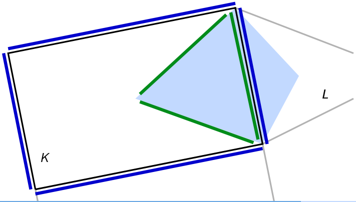
$$\sum_{\epsilon \in \bar{\mathcal{E}}(D_{K,\sigma}) \setminus \{\sigma\}} F_{\sigma,\epsilon} = -F_{K,\sigma} = -|\sigma|\ \rho_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma},$$

so

$$G_{K,\sigma} = |\sigma|\ \rho_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \Big(z_\sigma - \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} z_\epsilon - z_\sigma \Big),$$

and

$$\boldsymbol{G}_\sigma = \rho_\sigma \boldsymbol{u}_\sigma \Big(z_\sigma - \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} z_\epsilon - z_\sigma \Big).$$



edges of the half-diamond cell

edges of the primal cell

Conclusion

- We derived a consistent velocity convection operator which yields a local kinetic energy balance, for staggered discretizations based on (rather) general meshes.
- To obtain a consistent scheme for Euler equations:
  - collect the dissipation terms appearing in the kinetic energy balance,

$$\frac{|D_\sigma|}{\delta t}\,(u_{i,\sigma}^{n+1}-u_{i,\sigma}^{n})\,u_{i,\sigma}^{n+1}=\frac{|D_\sigma|}{\delta t}\,\Big[(u_{i,\sigma}^{n+1})^2-(u_{i,\sigma}^{n})^2+(u_{i,\sigma}^{n+1}-u_{i,\sigma}^{n})^2\Big]$$

    (when refining the mesh, these dissipation terms act as measure born by shocks)
  - compensate them in the internal energy balance.
- Provided that these dissipation terms are non-negative (implicit discretization or explicit discretization under a CFL condition), the scheme preserves the positivity of the internal energy (the density is positive by a simple upwinding of the mass balance).
- Even if solving the internal energy balance, the scheme yields a "conservation equation" for the total energy on the primal mesh.
- Pressure correction or explicit variants.
- Entropy estimates are satisfied by these schemes.