# Coupling Virtual Elements and Finite Volume Methods for Geomechanics

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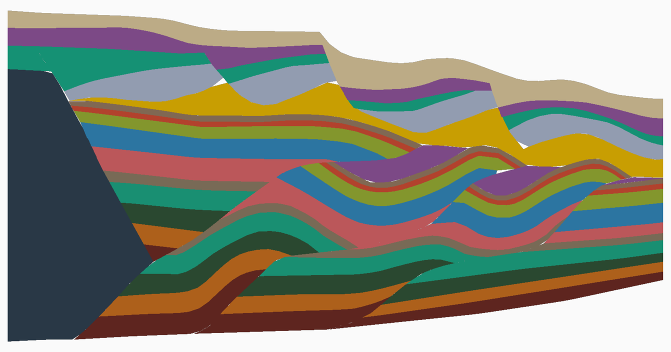
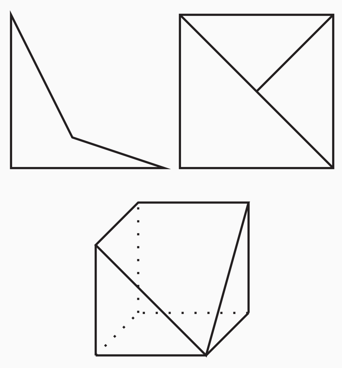
## Geomechanics' challenges

Involves two physics
➜ Requires two spatial discretization methods working on a **single** mesh
Highly deformed media
➜ Methods must work on **poor quality** meshes





## Governing equations

Under the assumptions of quasistatic strains and slightly compressible single-phase flow, the mechanical equilibrium and the fluid mass conservation are coupled through Biot's equations.

$$-\mathbf{div}\left(\overline{\overline{C}}\,\overline{\epsilon}_u - \alpha p \overline{I_d}\right) = \mathbf{f} \qquad \text{with } \overline{\epsilon}_u = \frac{1}{2}\left(\overline{\nabla \mathbf{u}} + \overline{\nabla \mathbf{u}}^T\right)$$
$$\partial_t\left(c_0 p + \alpha \,\mathrm{div}\,(\mathbf{u})\right) + \mathrm{div}\left(-\overline{\kappa}\left(\nabla p - \rho_f \mathbf{g}\right)\right) = q$$

where $\mathbf{u}$ is the solid displacement and $p$ the fluid pressure.
In this work, we assume that the solid has a linear elastic behaviour described by the stiffness tensor $\overline{\overline{C}}$.

| | |
|---|---|
| $\alpha$ | Biot parameter |
| $c_0$ | Constrained specific storage coefficient |
| $\rho_f$ | fluid density |
| $\overline{\kappa}$ | Mobility matrix |

## Spatial discretization

Working in a variationnal framework for the mechanics and in the finite volume framework for the fluid flow,

$$a(\mathbf{u}_h^n, \mathbf{v}_h) - \sum_K \alpha \int_K \mathrm{div}\,(\mathbf{v}_h)\, p_h^n = \int_\Omega \mathbf{f}^n \cdot \mathbf{v}_h$$
$$\int_K (c_0 p_h^n + \alpha \mathrm{div}\,(\mathbf{u}_h^n)) + \Delta t \sum_{f\subset\partial K} \mathcal{F}_{Kf}^n = \int_K \left(\Delta t q^n + c_0 p_h^{n-1} + \alpha \mathrm{div}\,(\mathbf{u}_h^{n-1})\right)$$

Given two bases for the discrete spaces, the equivalent matrix form is given by

$$\begin{bmatrix} \mathcal{A} & -\mathcal{B} \\ \mathcal{B}^T & \mathcal{F} \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ \mathcal{P} \end{bmatrix} = \begin{bmatrix} \mathcal{L}^u \\ \mathcal{L}^p + \mathcal{L}^c \end{bmatrix}$$

where $\mathcal{A}$ depends only on the elasticity problem, $\mathcal{F}$ depends only on flow problem, $\mathcal{B}$ stores the coupling terms $\int p_h^n \mathrm{div}\,(\mathbf{v}_h)$.

We can now choose a discretization method for $\mathcal{A}$ and another for $\mathcal{F}$. Indeed, the scheme can be customized picking the following elements:

## A) A Virtual Element Method

Key idea of VEM[1]: on each element, substitute $a$ with an approximate discrete form $a_h$

- consistent (ensures accuracy)
- stable (ensures coercivity)
- computable from the *dofs*.

To have the consistency and the stability, set

$$a_h^K(\mathbf{u},\mathbf{v}) := a^K(\pi^K\mathbf{u}, \pi^K\mathbf{v}) + h_K^{d-2} \max|\overline{\overline{C}}|\, s^K(\mathbf{u} - \pi^K\mathbf{u}, \mathbf{v} - \pi^K\mathbf{v})$$

where $\pi^K\mathbf{u}$ is a projection to the polynomial part of $\mathbf{u}$ and where $s^K$ stabilizes the non polynomial part.
For the lowest order of accuracy, we can take

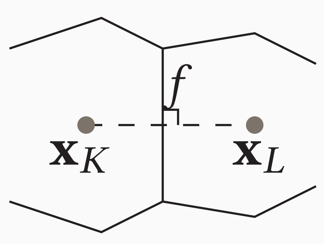
$$\pi^K\mathbf{v}(\mathbf{x}) = \left(\frac{1}{|K|}\int_K \nabla\mathbf{v}\right)(\mathbf{x} - \overline{\mathbf{x}_K}) + \frac{1}{M_K}\sum_{i\in M_K} \mathbf{v}(Vi)$$

which is computable from the *dofs*. With this approximate form, basis functions are *virtually* defined over any general element.

## B) A Finite Volume scheme

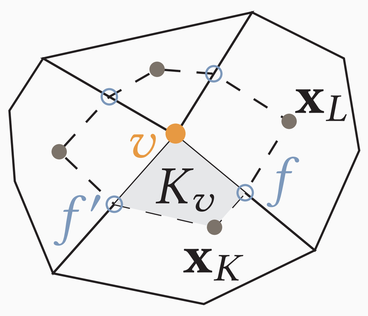
Choosing any FV scheme with cell-centered unknowns allows an easy treatment of the coupling terms. From [2] we highlight:

▷ *Two-Point Flux Approximation*
Use only two points to compute the fluxes $\mathcal{F}_{Kf} = |f|\kappa_{KL} d_{KL}^{-1}(p_K - p_L)$.




♡ Low computational cost
♡ High stability
♡ Requires orthogonality condition $(KL) \perp f$

▷ *Multi-Point Flux Approximation*
Compute half fluxes $\mathcal{F}_{Kf_v}$, then eliminate face unknowns using consistency $\mathcal{F}_{Kf_v} + \mathcal{F}_{Lf_v} = 0$ and finally sum half fluxes to get $\mathcal{F}_{Kf} = \mathcal{F}_{Kf_v} + \mathcal{F}_{Kf_{v'}}$.



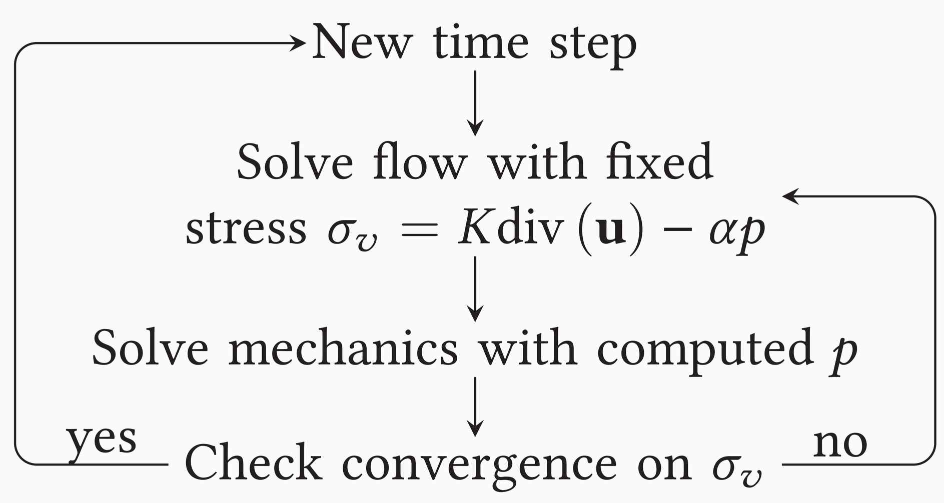

♡ Allows more general meshes
♡ Larger stencil (more costly)
♡ Sometimes unstable

## C) A Solution Strategy

Use either a fully coupled (monolithic) or an iteratively coupled strategy.

▷ *Fully coupled resolution*
Both equations are simultaneously solved.
♡ Unconditionally stable
♡ Huge matrix, no efficient solver available

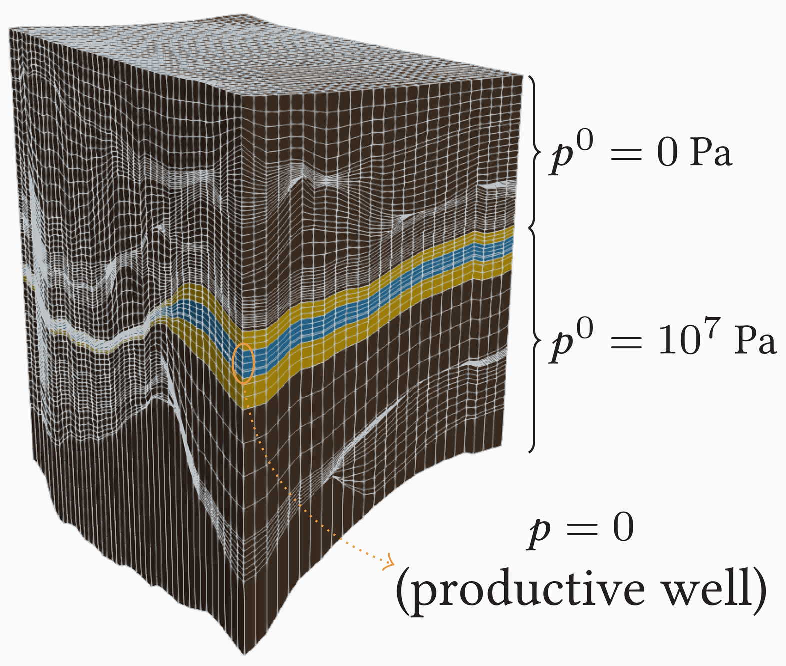
▷ *Iteratively coupled resolution*
Use a splitting [3] such as the fixed-stress split:

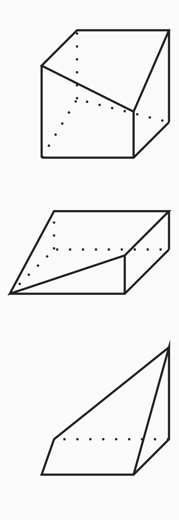



♡ More iterations, needs a convergence criteria
♡ Pick specialized solvers for each subproblem
♡ Krylov-like methods can speed ud convergence

## Numerical illustration

Layered 3D mesh including degenerated cells



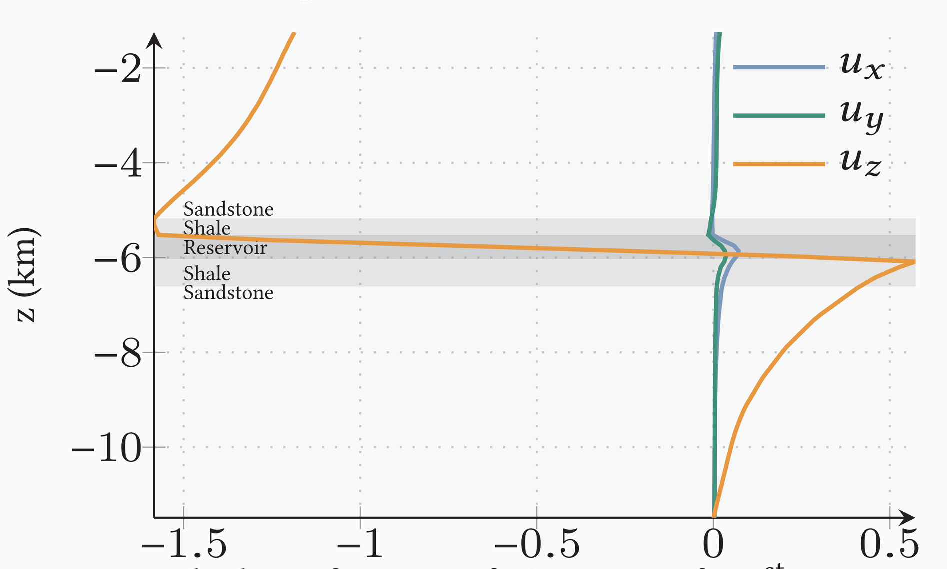



| | $\kappa$ | $\phi$ | $E$ | $\nu$ | $\alpha$ |
|---|---|---|---|---|---|
| Sandstone ■ | $10^{-9}$ | 0.2 | $5\cdot10^9$ | 0.3 | 1 |
| Shale ■ | $10^{-15}$ | 0.1 | $5\cdot10^9$ | 0.3 | 1 |
| Reservoir ■ | $10^{-9}$ | 0.3 | $10^9$ | 0.3 | 1 |

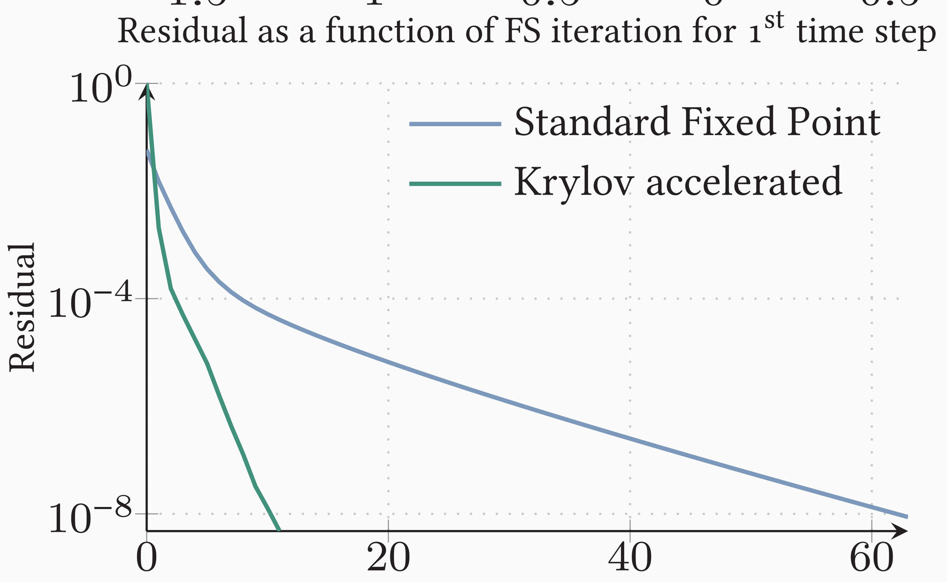
Bottom and lateral faces: $\nabla p \cdot \mathbf{n} = 0, \mathbf{u}\cdot\mathbf{n} = 0$
Top face: $p = 0$ and free stress condition

➜Solve ~300K dofs over 20 time steps (T=1 year) with VEM-MPFA and the fixed-stress strategy:

Relative displacements (in meters) over a vertical line




Residual as a function of FS iteration for 1st time step




## References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[2] J. Droniou. "Finite volume schemes for diffusion equations: Introduction to and review of modern methods". In: *Math. Models & Methods in App. Sciences* 24.8, SI (2014), pp. 1575–1619.

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## Future works

Use domain decomposition methods as an efficient preconditioner to solve $\mathbf{u}$
More general mechanical laws