A high-order conservative remap for DG schemes on curvilinear polygonal meshes

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Content

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Lagrange + Remap

1. **Write equations in a moving frame**
2. **Advance physical fields in time**
3. **Improve mesh quality**
4. **Transfer fields to the improved mesh**
5. **Goto to step 2.**

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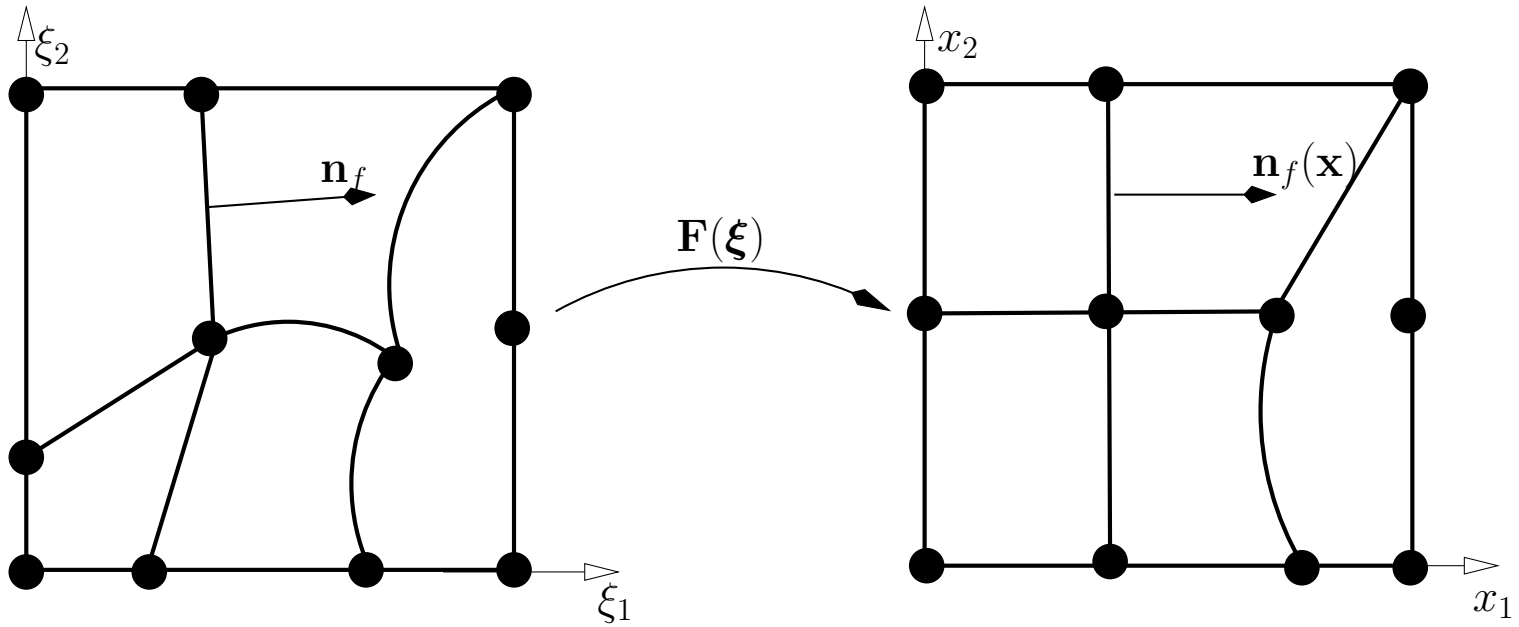
Problem formulation (1/3)

We are looking for algorithms that transfer conservatively DG fields from a Lagrangian (left) to a relaxed (right) mesh:



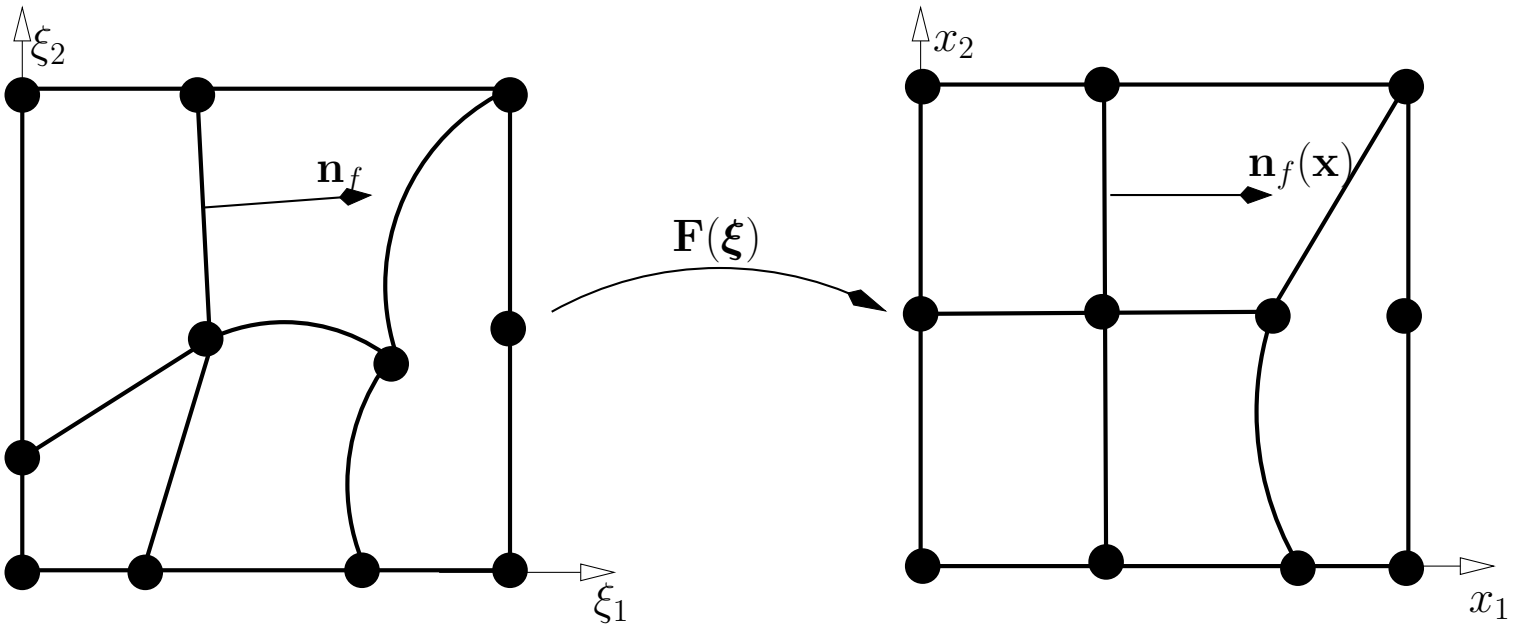
Space-time map:

$$\mathbf{x}(\boldsymbol{\xi}, \tau) = \boldsymbol{\xi} + \tau(\mathbf{F}(\boldsymbol{\xi}) - \boldsymbol{\xi})$$

Remap velocity:

$$\mathbf{u} = \frac{\partial \mathbf{x}}{\partial \tau}$$

Problem formulation (2/3)



$$\mathbf{u} = \frac{\partial \mathbf{x}}{\partial \tau}, \qquad \mathbf{x}(\boldsymbol{\xi}, \tau) = \boldsymbol{\xi} + \tau(\mathbf{F}(\boldsymbol{\xi}) - \boldsymbol{\xi})$$

Assumptions:

1. $\mathbf{u}$ **is known only on the mesh edges.**
2. $\mathbf{u}$ **is a polynomial.**

Problem formulation (3/3)

Consider a scalar field ρ (e.g. density):

$$0 = \frac{\partial \rho}{\partial \tau} = \frac{d\rho}{d\tau} - \mathbf{u} \cdot \nabla \rho$$

Thus, the PDE formulation of the remap problem is

$$\frac{d\rho}{d\tau} = \mathbf{u} \cdot \nabla \rho$$

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Taylor basis

$$\rho_h\big|_c = \rho_0^c + \rho_1^c\,\psi_1^c + \rho_2^c\,\psi_2^c + \rho_3^c\,\psi_3^c + \dots$$

where

$$\begin{aligned}
\psi_0^c(\boldsymbol{\xi}) &= 1 \\
\psi_1^c(\boldsymbol{\xi}) &= a_1^c(\xi_1 - \xi_{1,c}) \\
\psi_2^c(\boldsymbol{\xi}) &= a_2^c(\xi_2 - \xi_{2,c}) \\
\psi_i^c(\boldsymbol{\xi}) &= a_i^c\left[(\xi_1 - \xi_{1,c})^{\alpha_1}(\xi_2 - \xi_{2,c})^{\alpha_2} - b_i^c\,\psi_0^c\right]
\end{aligned}$$

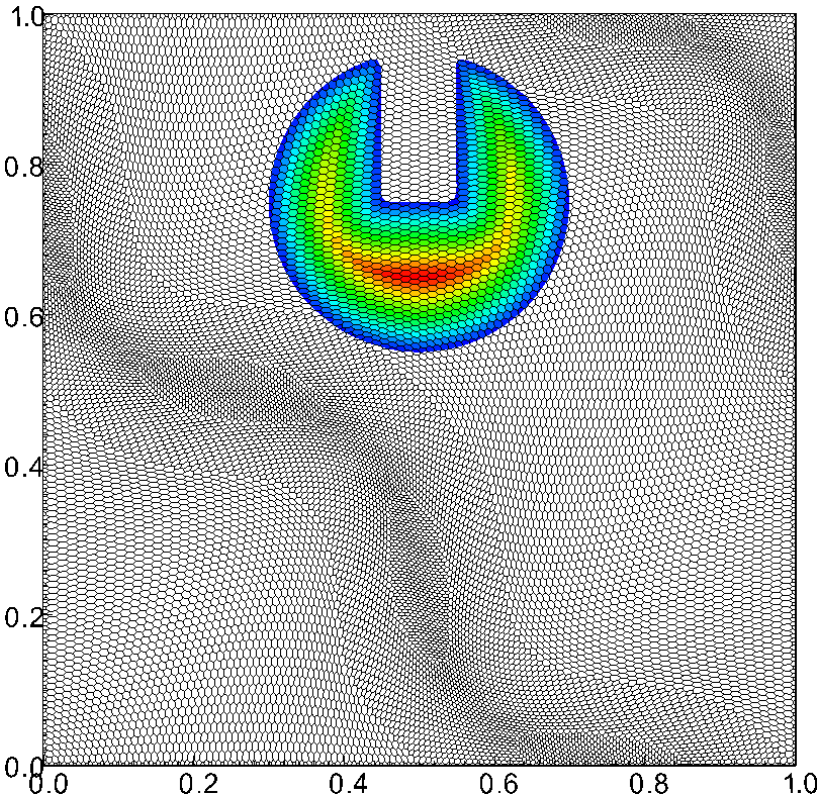
The coefficients a_i^c and b_i^c are defined from

$$\int_c \psi_0^c\,\psi_i^c\,\mathrm{d}\xi = |c|\,\delta_{0,i} \quad \textbf{and} \quad \int_c \left|\psi_i^c\right|^2 \mathrm{d}\xi = |c|$$

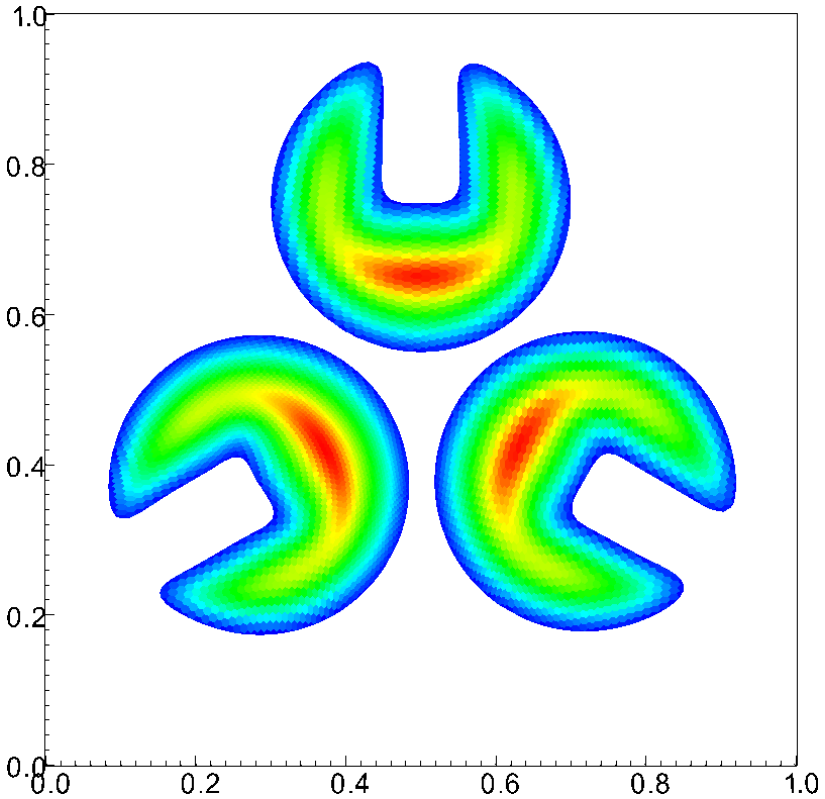
What should we expect? (1/2)

DG shows excellent performance for a similar linear advection equation:

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0, \qquad \mathbf{u} = (y, -x)^T.$$



$t = 0$



$t = 2.1,\ 4.2,\ 2\pi$

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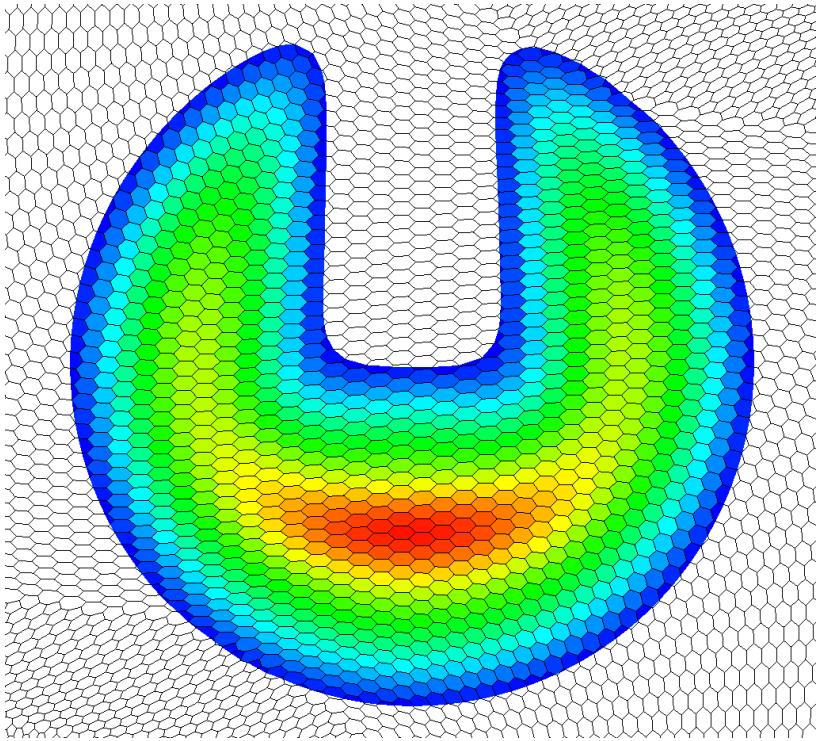
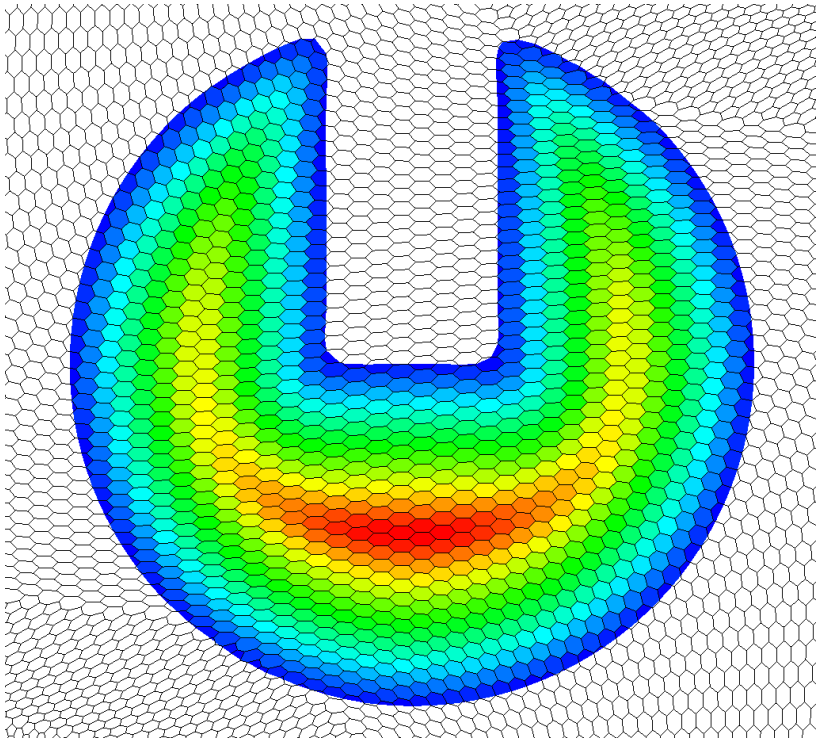
What should we expect (2/2)

High-order DG reduces significantly numerical diffusion which is critical for solutions with sharp features:



DG($\mathcal{P}^1$)



DG($\mathcal{P}^2$)

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Main results for the remap problem

$$\frac{d\rho}{d\tau} = \mathbf{u} \cdot \nabla \rho$$

1. **Locally conservative modal DG schemes**
2. **High-order DG schemes,** $k = 1, 2, 3$
3. **Treatment of curvilinear meshes**
4. **Conservative BJ-type limiter**

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Weak formulation (1/3)

For a (discontinuous) test function ψ, we have

$$\frac{d}{d\tau}\sum_c \int_{c(\tau)} \rho\psi \,\mathrm{d}x = -\sum_c \int_{c(\tau)} (\mathbf{u}\cdot\nabla\psi)\rho \,\mathrm{d}x + \sum_f \int_{f(\tau)} (\mathbf{u}\cdot\mathbf{n})\rho^*[\psi] \,\mathrm{d}x$$

Here '*' is a downwind approximation and we assumed that the basis function is constant on characteristics, i.e.

$$\frac{d\psi}{d\tau} = 0$$

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Weak formulation (2/3)

Change to the reference frame introduces the Jacobian matrix $\mathbb{J}$ and its determinant j:

$$\frac{d}{d\tau}\sum_c \int_{c(0)} \rho\psi\, j\, \mathrm{d}\xi = -\sum_c \int_{c(0)} (\mathbf{u}\cdot j\mathbb{J}^{-t}\nabla_\xi\psi)\rho\, \mathrm{d}\xi + \sum_f \int_{f(0)} (\mathbf{u}\cdot j\mathbb{J}^{-t}\mathbf{n})\rho^*[\psi]\, \mathrm{d}\xi$$

Weak formulation (3/3)

Aprroximation of unknown objects j, $\mathbb{J}$, and $\mathbf{u}$:

$$\frac{d}{d\tau}\sum_c \int_{c(0)} \rho\psi\, \mathcal{I}_{c,1}(j)\,\mathrm{d}\xi = -\sum_c \int_{c(0)} (\mathcal{I}_c(\mathbf{u})\cdot \mathcal{I}_{c,2}(j\mathbb{J}^{-t})\nabla_\xi\psi)\rho\,\mathrm{d}\xi + \sum_f \int_{f(0)} (\mathbf{u}\cdot j\mathbb{J}^{-t}\mathbf{n})\rho^*[\psi]\,\mathrm{d}\xi$$

Simplification: $\mathcal{I}_{c,i}$ are derived from $\mathcal{I}_c$:

$$\mathbb{J}\to\mathbb{J}_h = \mathbf{I} + \tau\nabla_\xi(\mathcal{I}_c(\mathbf{u})), \quad j\to j_h = \det(\mathbb{J}_h)$$

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Semi-explicit DG scheme

Find $\rho_h \in \mathbf{DG}(\mathcal{P}^k)$ s.t. for any $\psi_h \in \mathbf{DG}(\mathcal{P}^k)$ we have

$$\frac{d}{d\tau}\sum_c \int_{c(0)} \rho_h \psi_h j_h \,\mathrm{d}\xi = -\sum_c \int_{c(0)} (\mathcal{I}_c(\mathbf{u}) \cdot j_h \mathbb{J}_h^{-1} \nabla_\xi \psi_h)\rho_h \,\mathrm{d}\xi + \sum_f \int_{f(0)} (\mathbf{u} \cdot j\mathbb{J}^{-1}\mathbf{n}_f)[\psi_h]\rho_h^* \,\mathrm{d}\xi.$$

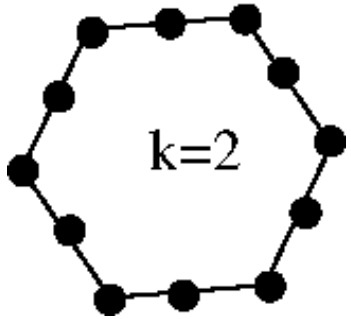
We use a TVD RK3 scheme for time stepping.

Let $\mathrm{u} \cdot \mathrm{n} = 0$ on the boundary. Taking $\psi_h = 1$, we obtain:

$$\frac{d}{d\tau} \sum_c \int_{c(0)} \rho_h j_h \, \mathrm{d}\xi = 0.$$

Virtual approximation space for $\mathbf{u}$

Consider the serendipity virtual element space of order k:



$$\mathcal{V}_k(c) = \Big\{ \mathbf{v} \in C^0(\bar{c}) : \ \mathbf{v}_{|_f} \in \mathcal{P}^k(f) \quad \forall f \in \partial c, \\ \Delta \mathbf{v} \in \mathcal{P}^k(c), \\ \int_c \mathbf{v}\,\mathbf{q}\,\mathrm{d}\xi = \int_c \mathcal{LS}_c(\mathbf{v})\,\mathbf{q}\,\mathrm{d}\xi \quad \forall \mathbf{q} \in \mathcal{P}^k(c) \Big\}$$

- **The least-square operator $\mathcal{LS}_c(\mathbf{v})$ is the L^2 projector for this space.**
- **The virtual space is globally continuous.**

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Elliptic projector in the virtual space

We define the projector $\mathcal{I}_c(\mathbf{u}) \in \mathcal{P}^k(c)$ by

$$\int_c \left(\nabla_\xi \mathcal{I}_c(\mathbf{u}) - \nabla_\xi \mathbf{u}\right) \cdot \nabla_\xi \mathbf{q} \, \mathrm{d}\xi, \quad \forall \mathbf{q} \in \mathcal{P}^k(c)$$

subject to

$$\int_c \mathcal{I}_c(\mathbf{u}) \, \mathrm{d}\xi = \int_c \mathbf{u} \, \mathrm{d}\xi.$$

- $\mathcal{I}_c(\mathbf{u})$ **is the** H^1 **projector for this space**
- **It is computable, see VE/mimetic literature**
- **Quasi-optimality is due to**

$$\|\nabla_\xi \mathcal{I}_c(\mathbf{u})\|_{L^2(c)} \leq \|\nabla_\xi \mathbf{u}\|_{L^2(c)}$$

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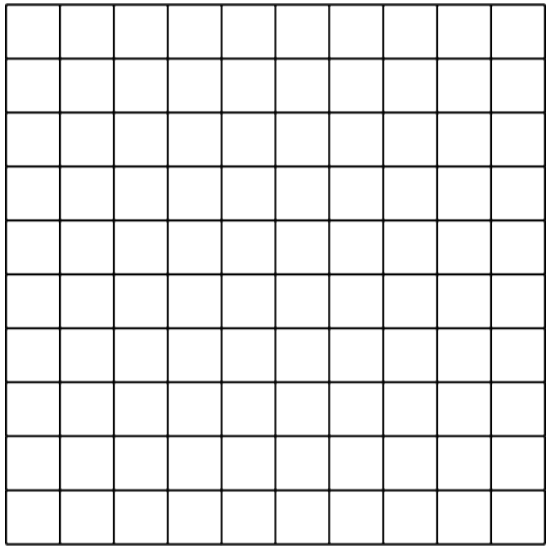
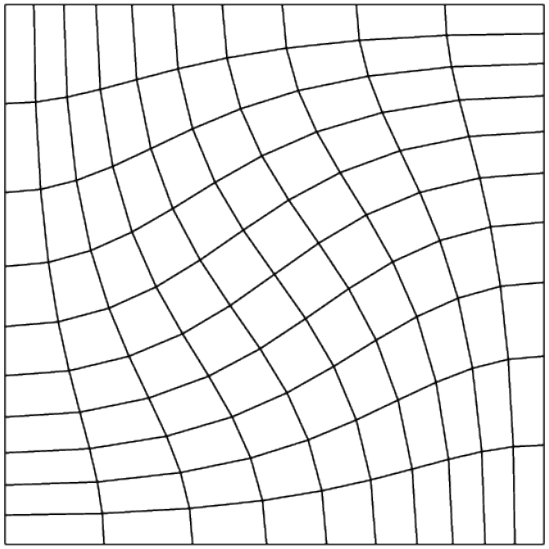
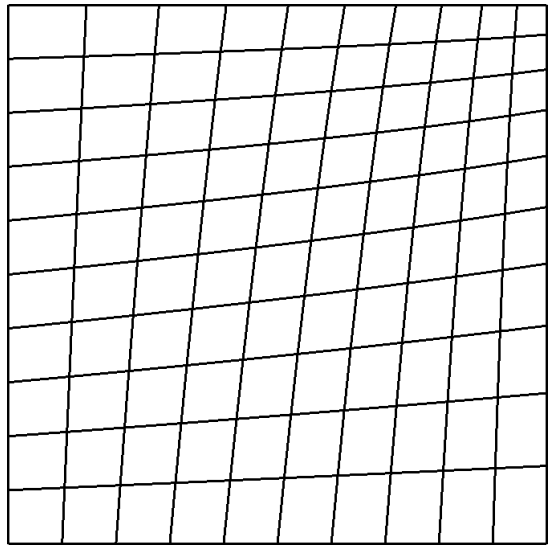
Example of meshes: logically square



Original



TG map



CE map

TG = Taylor-Green vorticial motion

CE = Compression/Expansion

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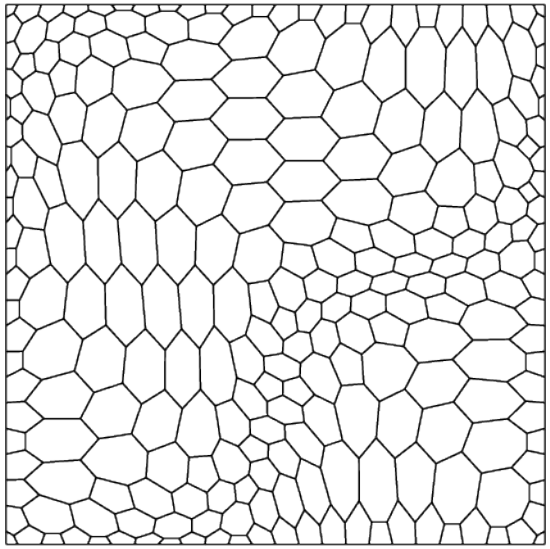
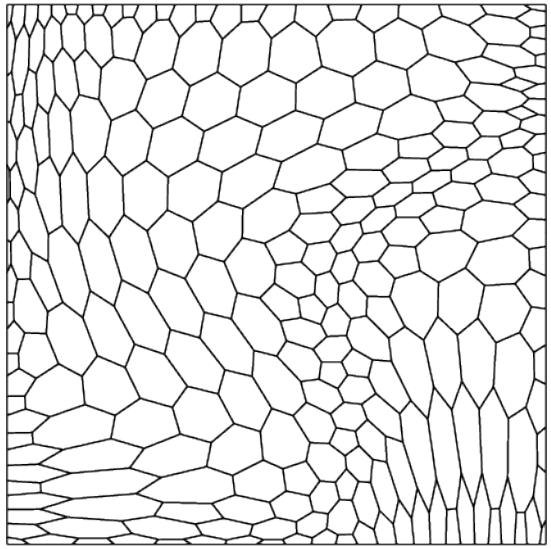
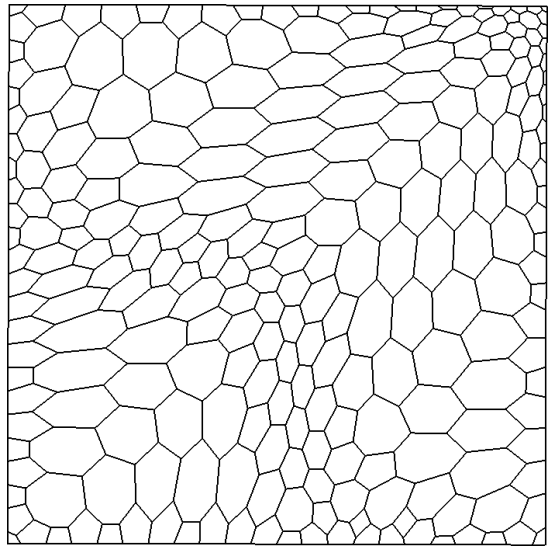
Example of meshes: polygonal



Original



TG map



CE map

TG = Taylor-Green vorticial motion

CE = Compression/Expansion

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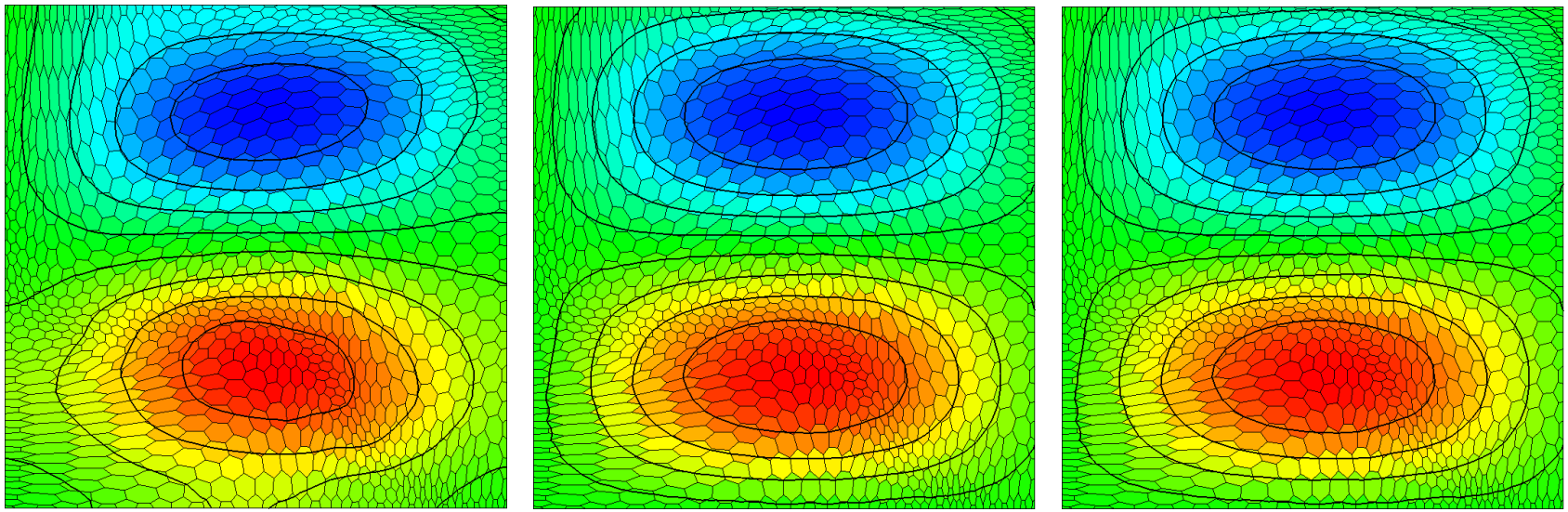
Analytic solution

$$\rho(\xi_1, \xi_2) = \sin(6\xi_1)\sin(3\xi_2)$$



The TG map on polygonal meshes in DG($\mathcal{P}^k$) schemes with $k = 0, 1, 2$

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Convergence (1/2)

$$\rho(\xi_1, \xi_2) = \sin(6\xi_1)\sin(3\xi_2)$$

	DG($\mathcal{P}^0$)		DG($\mathcal{P}^1$)		DG($\mathcal{P}^2$)	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
16^2	1.43e-1	4.31e-1	1.98e-2	7.85e-2	1.57e-3	9.31e-3
32^2	8.13e-2	3.57e-1	4.69e-3	1.94e-2	1.79e-4	1.15e-3
64^2	4.67e-2	2.73e-1	1.14e-3	4.66e-3	2.13e-5	1.38e-4
128^2	2.68e-2	1.99e-1	2.81e-4	1.17e-3	2.61e-6	1.84e-5
rate	0.805	0.372	2.046	2.028	3.077	3.001

TG map on polygonal meshes

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Convergence (2/2)

$$\rho(\xi_1, \xi_2) = \sin(6\xi_1)\sin(3\xi_2)$$

	DG($\mathcal{P}^0$)		DG($\mathcal{P}^1$)		DG($\mathcal{P}^2$)	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
16^2	1.23e-1	4.15e-1	2.03e-2	8.58e-2	1.67e-3	1.40e-2
32^2	6.43e-2	2.25e-1	4.92e-3	2.87e-2	1.94e-4	1.82e-3
64^2	3.35e-2	1.64e-1	1.21e-3	7.24e-3	2.41e-5	2.21e-4
128^2	1.75e-2	1.23e-1	3.01e-4	1.81e-3	3.06e-6	2.77e-5
rate	0.937	0.571	2.025	1.867	3.027	2.998

CE map on polygonal meshes. Factor 200 in CPU performance in favor of DG($\mathcal{P}^1$) Factor 14 in favor of DG($\mathcal{P}^2$)

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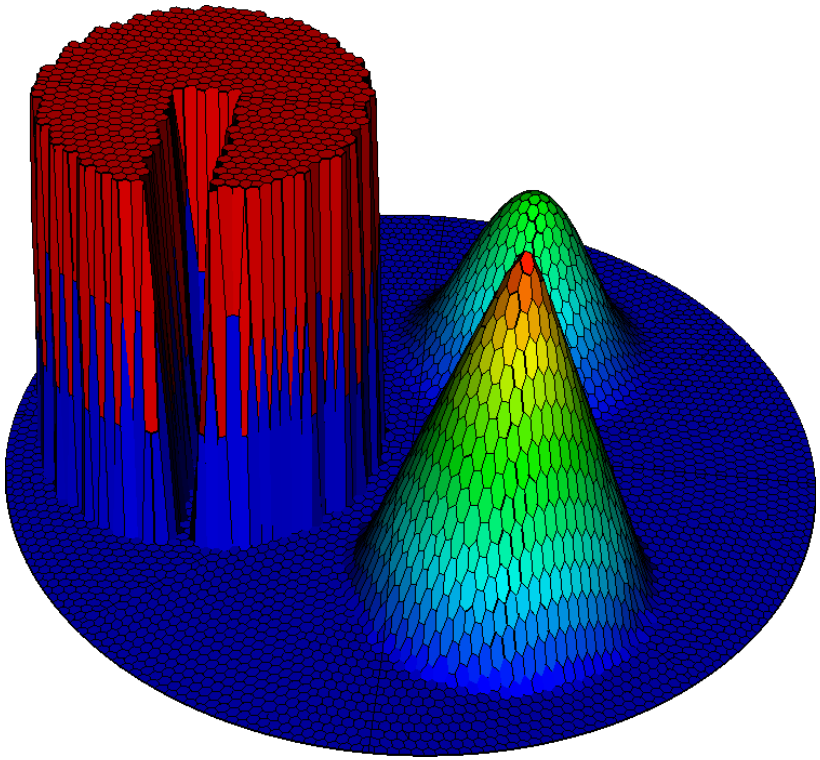
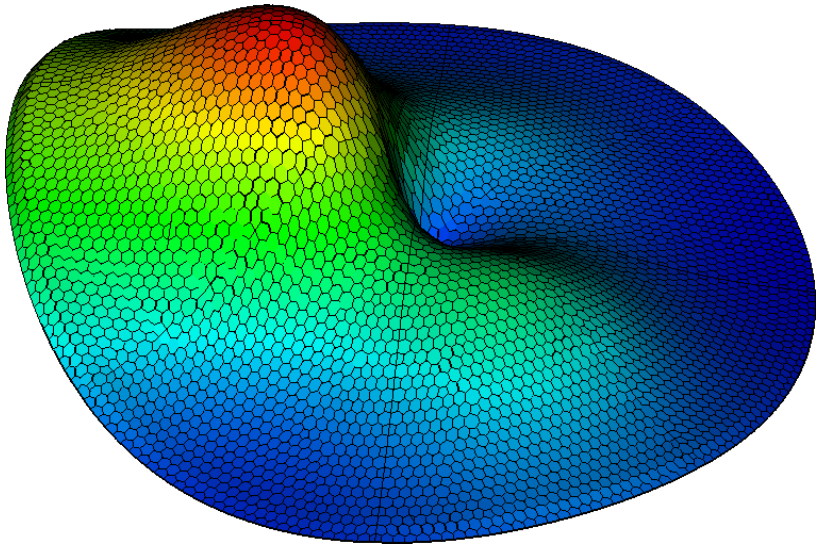
Poor performance of the lowest-order scheme



Initial solution



Unlimited DG($\mathcal{P}^0$)

The lowest-order solution is diffused significantly after 360° rotation of the mesh

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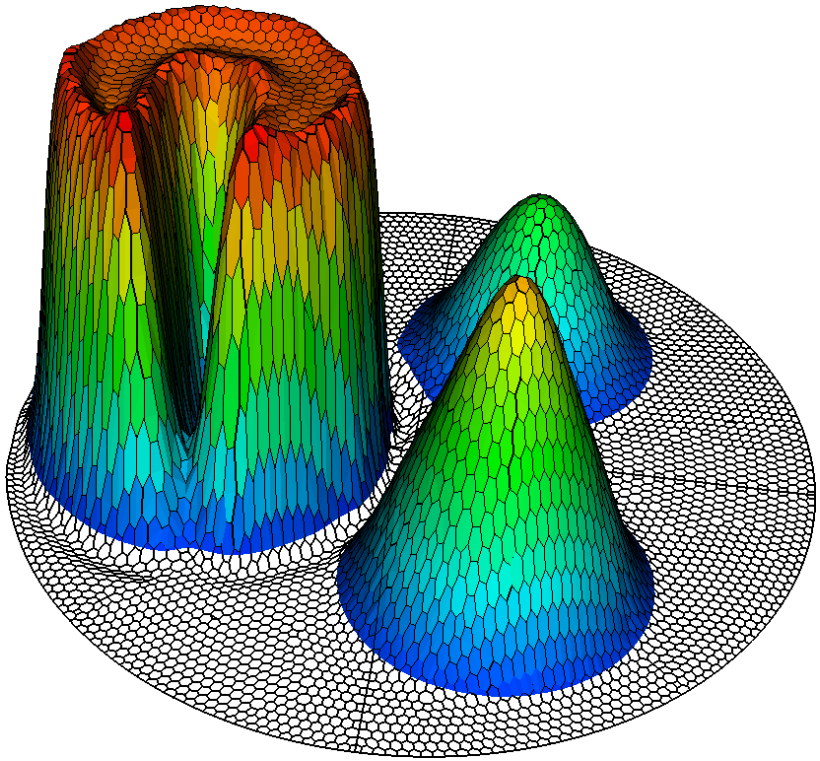
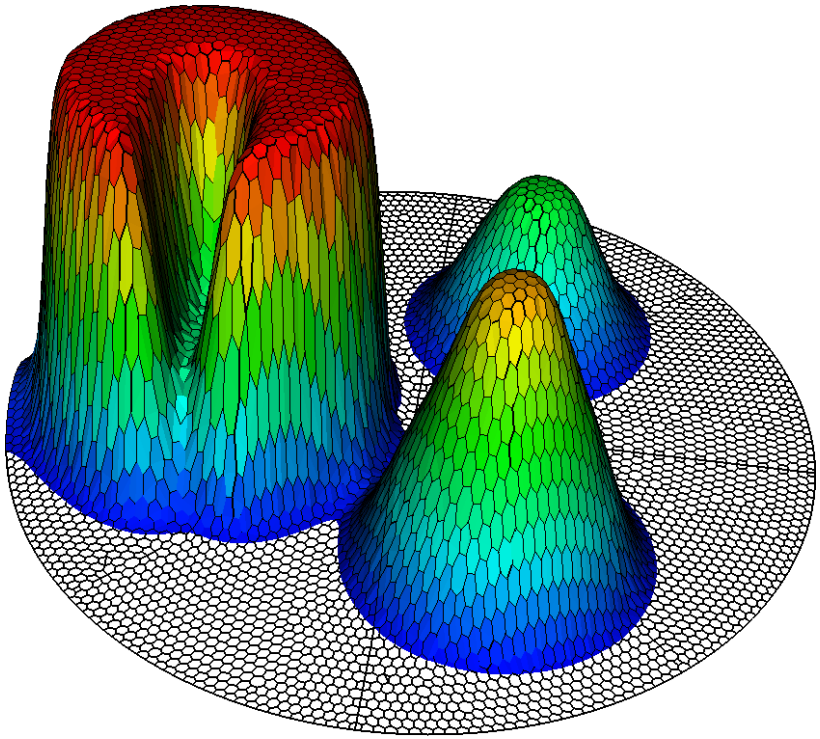
Impact of limiters (1/2)



Unlimited DG($\mathcal{P}^1$)



Limited DG($\mathcal{P}^1$)

3 shapes (cylinder, cone, hump) on the mesh of polygons after 360° rotation of the mesh

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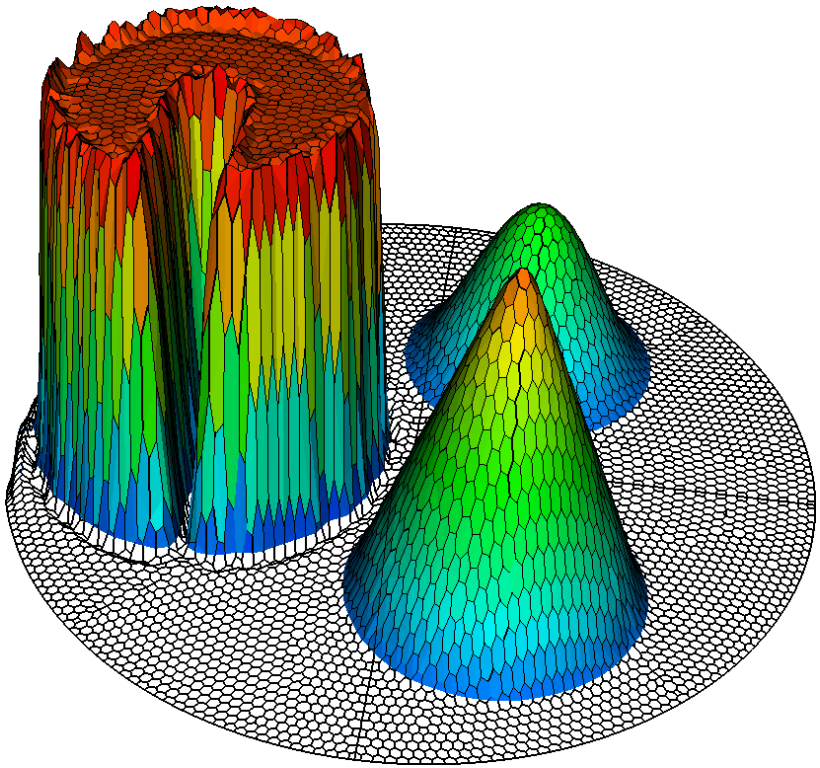
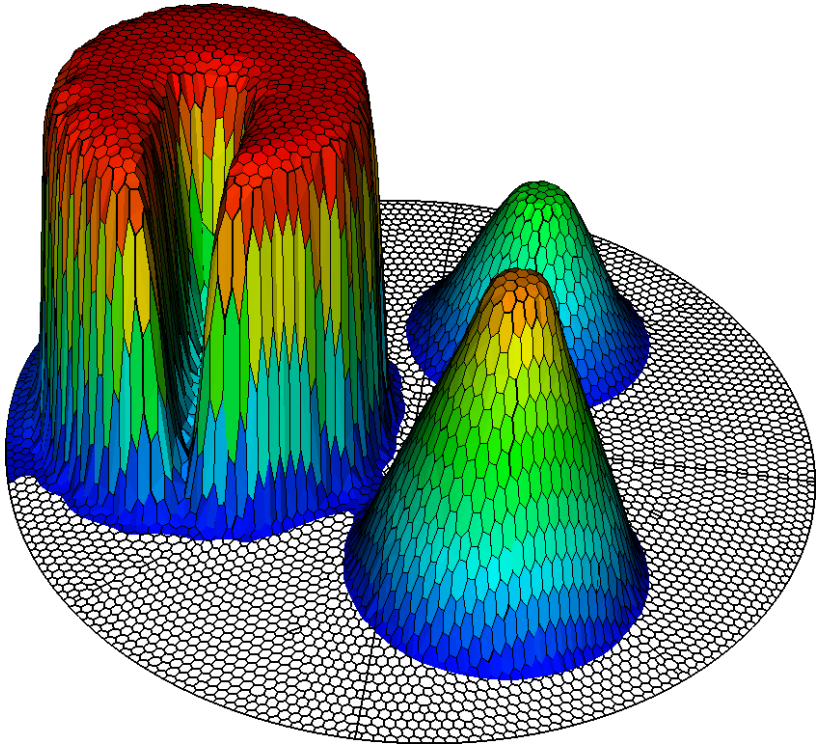
Impact of limiters (2/2)



Unlimited DG($\mathcal{P}^2$)



Limited DG($\mathcal{P}^2$)

3 shapes (cylinder, cone, hump) on the mesh of polygons after 360° rotation of the mesh

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Barth-Jespersen-type limiting strategy

1. **Select a scaling coefficients:**

$$\hat{\rho}_h\big|_c = \rho_0^c + \alpha_c\big(\rho_1^c\,\psi_1^c + \rho_2^c\,\psi_2^c + \rho_3^c\,\psi_3^c + \dots\big)$$

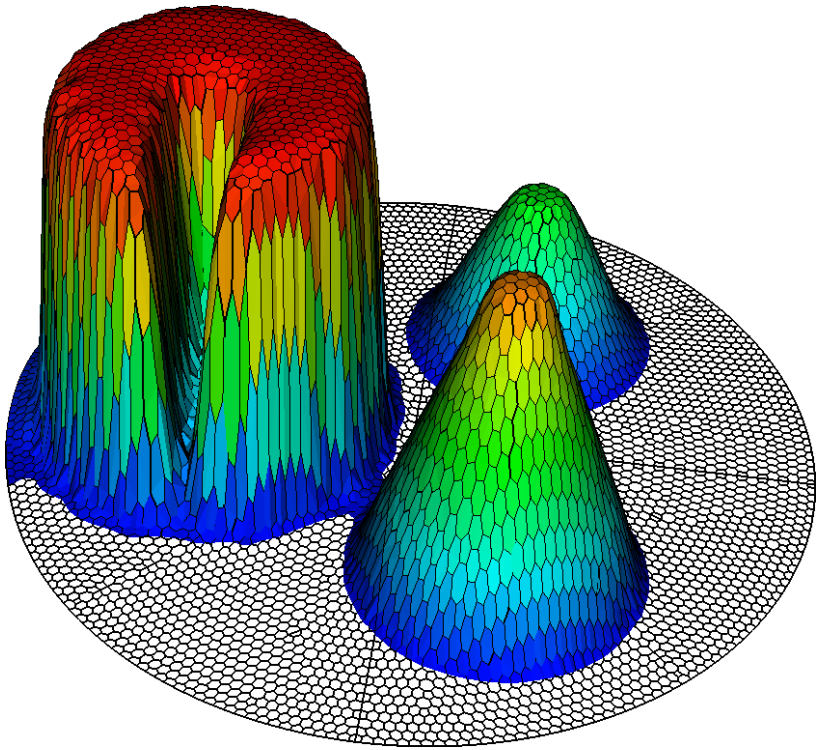
where $0 \le \alpha_c \le 1$ **is such that**

$$\min_{c' \in \mathcal{F}_c}\{\rho_0^{c'}\} \le \hat{\rho}_h(\boldsymbol{\xi}_k) \le \max_{c' \in \mathcal{F}_c}\{\rho_0^{c'}\}, \qquad \forall \boldsymbol{\xi}_k \in \mathcal{G}_c^{k_1}$$

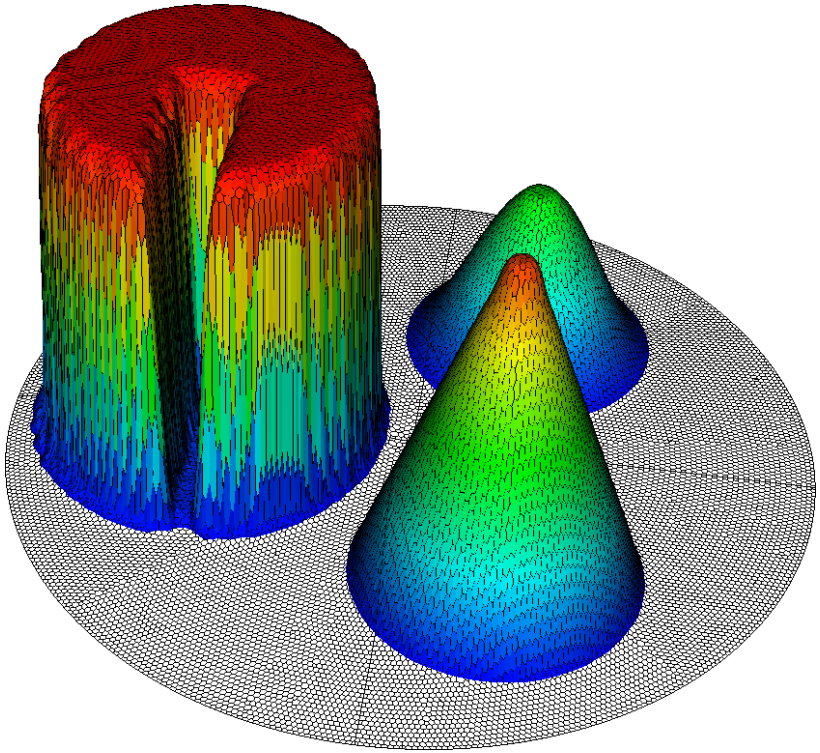
2. **Restore the local conservation:**

$$\hat{\rho}_0^c = \alpha_c\,\rho_0^c + (1-\alpha_c)\frac{1}{|c(\tau)|}\int_{c(0)} \rho_h j_h \,\mathrm{d}V.$$

Impact of resolution



Limited DG($\mathcal{P}^2$)



Limited DG($\mathcal{P}^2$)

Impact of mesh resolution: better plateau and kinks; same roundness of the cylinder brims.

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Summary

- **Developed high-order DG schemes for the remap problem using the virtual element method.**
- **The schemes handle curvilinear polygonal meshes.**
- **Developed a conservative Barth-Jespersen-type limiting strategy for discontinuous functions.**
- **Future work: extend the scheme to 3D.**
- **Future work: shock capturing algorithms.**

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