POEMS 2019, 29-3 May 2019 @ CIRM, Marseille, FR

p-Multilevel solution strategies for HHO methods

F. Bassi[†], **L. Botti**[†], A. Colombo[†], F. Massa[†]

[†]*University of Bergamo, Department of Engineering and Applied Sciences*

Motivation: promote industrialization of high-order methods for CFD

More than second order accurate CFD... using dG and HHO.

2006 (FP6)	2010 (FP7)	2015 (H2020)	2019 (H2020)
ADIGMA	IDIHOM	TILDA	Hi-Fi Turb

A crucial point is the efficiency of the solution strategy.

$\{h{-}p{-}hp\}$-multigrid preconditioners for dG discretizations
[Botti, Colombo, Bassi, JCP, 2018; Botti, Colombo, Crivellini, Franciolini, IJCFD, In press]

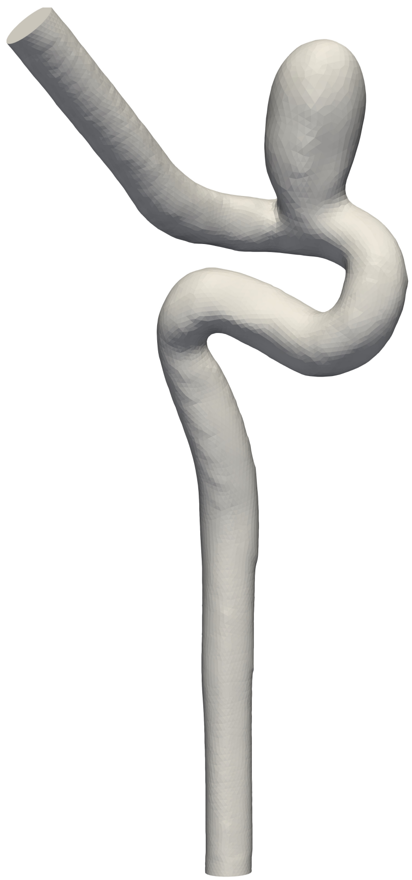
Applications: Incompressible flow problems (hemodynamics, aerodynamics)
Linear (non-linear) incompressible elasticity (blow molding)

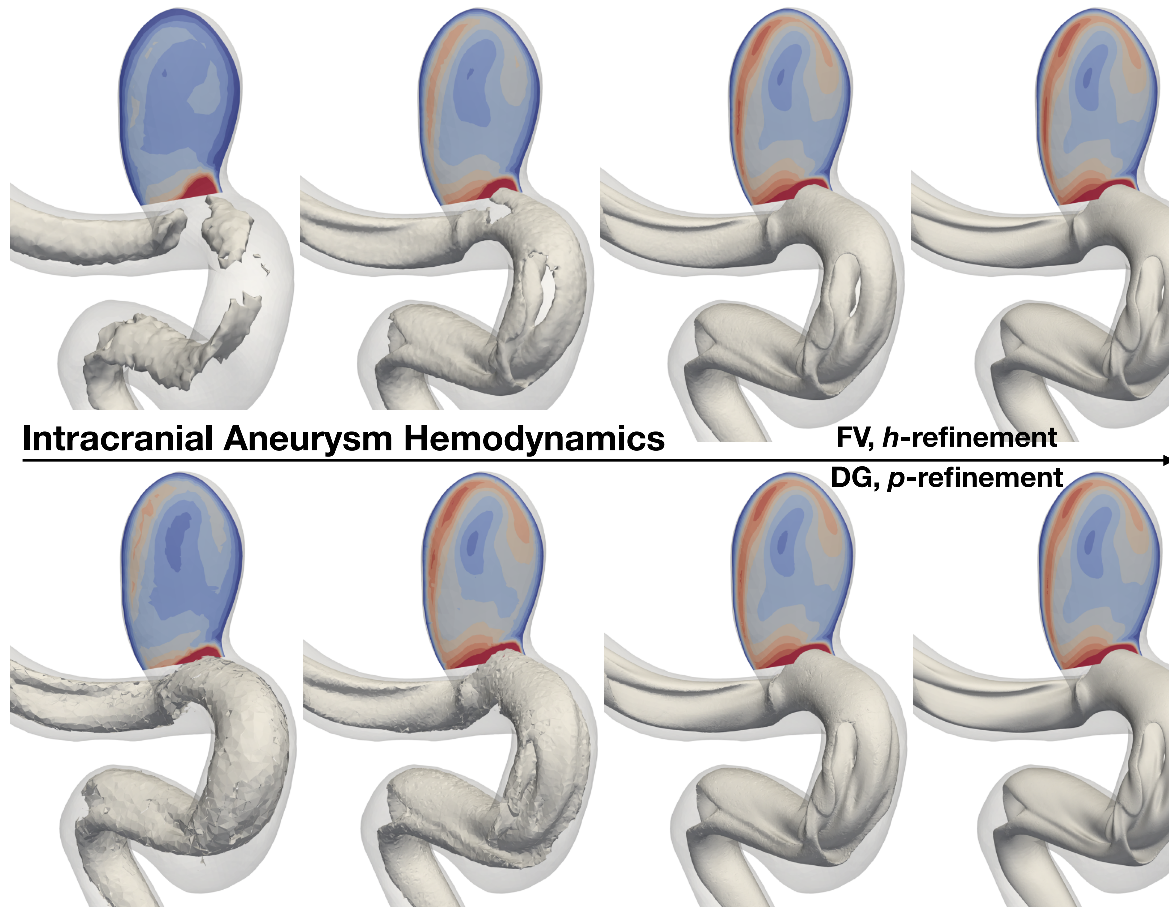
$\{p\}$-**multilevel preconditioners for HHO discretizations**...
[Franciolini, Fidkowski, Crivellini, ECCOMAS 2018 (for HDG);
Antonietti, Mascotto, Verani, ESAIM-M2AN, 2018 (for VEM)]

Computational hemodynamics (dG vs FV, steady, Re=500 [B ea, IJNMBE, 2018])

dG (Bassi ea, JCP, 2006) *p*-refinement: polynomial degrees 1,2,3,4 on a 134k tet grid

FV (ANSYS Fluent) *h*-refinement: 134k, 1.1m, 8.6m and 68.5m grids





Validation: convergence study

	dG error [cm/s]			FV error [cm/s]	
polynomial degree	$E^{\text{dG}_k}_{L^1(\Omega_H)}$	$E^{\text{dG}_k,\text{FV}_4}_{L^1(\Omega_H)}$	mesh index	$E^{\text{FV}_i,\text{dG}_4}_{L^1(\Omega_H)}$	$E^{\text{FV}_i}_{L^1(\Omega_H)}$
$k = 1$	6.73003	6.54353	$i = 1$	13.4277	13.4227
$k = 2$	4.02893	4.12215	$i = 2$	5.58825	5.30604
$k = 3$	0.88863	1.29241	$i = 3$	2.07106	1.47922
$k = 4$	-	0.83734	$i = 4$	0.83734	-
ref. sol.	dG $\mathcal{P}_4(\mathcal{T}_1)$	FV $\mathcal{T}_4$	ref. sol.	dG $\mathcal{P}_4(\mathcal{T}_1)$	FV $\mathcal{T}_4$

Average velocity error on 68.5 cell centroids $C \in \mathcal{C}(\mathcal{T}_4)$.

$$E^{\text{dG}_k}_{L^1(\Omega_H)} := \frac{\sum_{C\in\mathcal{C}} \|\mathbf{v}^{\text{dG}}_{p_k,h_1}(C) - \mathbf{v}^{\text{dG}}_{p_4,h_1}(C)\|}{\text{card}(\mathcal{C})} \qquad E^{\text{dG}_k,\text{FV}_4}_{L^1(\Omega_H)} := \frac{\sum_{C\in\mathcal{C}} \|\mathbf{v}^{\text{dG}}_{p_k,h_1}(C) - \mathbf{v}^{\text{FV}}_{h_4}(C)\|}{\text{card}(\mathcal{C})}$$

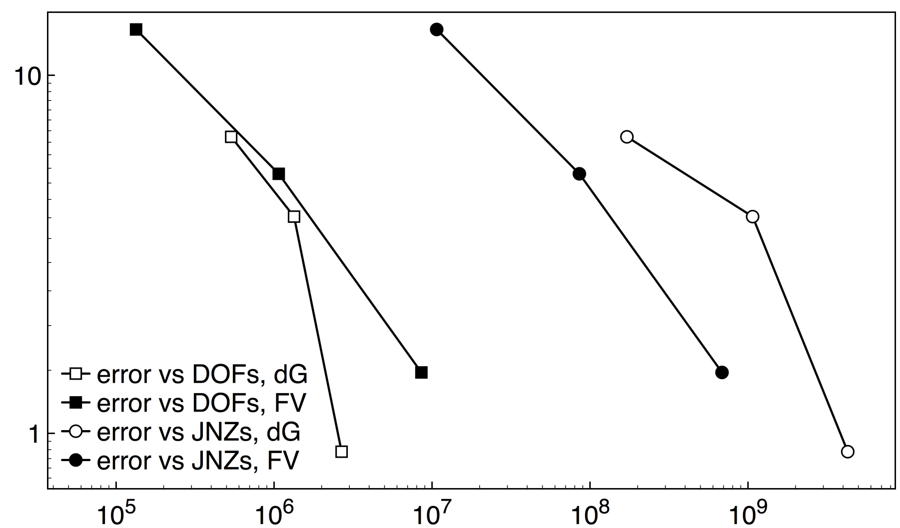
Degrees of freedom and Jacobian non-zeros abacus

polynomial space	dG DOFs	dG JNZs	grid	FV DOFs	FV JNZs
$\mathcal{P}_d^1(\mathcal{T}_1)$	2.14m	171.2m	$\mathcal{T}_1$	535k	10.7m
$\mathcal{P}_d^2(\mathcal{T}_1)$	5.35m	1.070b	$\mathcal{T}_2$	4.28m	85.62m
$\mathcal{P}_d^3(\mathcal{T}_1)$	10.7m	4.280b	$\mathcal{T}_3$	34.2m	684.9m
$\mathcal{P}_d^4(\mathcal{T}_1)$	18.7m	13.11b	$\mathcal{T}_4$	273.9m	5.48b

JNZs(dG/FV)$\simeq$3, **DOFs**(FV/dG)$\simeq$15

dG best accuracy per DOF
FV best accuracy per JNZ

Which one is faster?
Krylov iteration's cost scales linearly with JNZs plus number of Krylov spaces times DOFs.



Degrees of freedom and Jacobian non-zeros abacus

	dG			HHO			FV	
polynomial space	DOFs	JNZs	polynomial space	DOFs	JNZs	grid	DOFs	JNZs
$\mathcal{P}_d^1(\mathcal{T}_1)$	2.14m	171.2m	$\mathcal{P}_{d-1}^1(\mathcal{F}_1)$	2.67m	225m	$\mathcal{T}_1$	535k	10.7m
$\mathcal{P}_d^2(\mathcal{T}_1)$	5.35m	1.070b	$\mathcal{P}_{d-1}^2(\mathcal{F}_1)$	5.20m	813m	$\mathcal{T}_2$	4.28m	85.62m
$\mathcal{P}_d^3(\mathcal{T}_1)$	10.7m	4.280b	$\mathcal{P}_{d-1}^3(\mathcal{F}_1)$	8.58m	2.16b	$\mathcal{T}_3$	34.2m	684.9m
$\mathcal{P}_d^4(\mathcal{T}_1)$	18.7m	13.11b	$\mathcal{P}_{d-1}^4(\mathcal{F}_1)$	12.8m	4.77b	$\mathcal{T}_4$	273.9m	5.48b

[Di Pietro, Krell, JSC, 2018, Botti, Di Pietro, Droniou, JCP, 2019]

JNZs(dG/FV)$\simeq$3, **DOFs**(FV/dG)$\simeq$15
JNZs(HHO/FV)$\simeq$0.9 **DOFs**(FV/HHO)$\simeq$21

	DOFs	JNZs
dG	$(d+1)\,\mathrm{card}(\mathcal{T}_h)\,\dim(\mathbb{P}_d^k)$	$\mathrm{card}(\mathcal{T}_h)\,\left(\overline{\mathrm{card}(\mathcal{F}_T)}+1\right)\,\left((d+1)\,\dim(\mathbb{P}_d^k)\right)^2$
FV	$(d+1)\,\mathrm{card}(\mathcal{T}_h)$	$\mathrm{card}(\mathcal{T}_{h_i})\,\left(\overline{\mathrm{card}(\mathcal{F}_T)}+1\right)\,(d+1)^2$
HHO	$d\,\mathrm{card}(\mathcal{F}_h)\,\dim(\mathbb{P}_{d-1}^k)+\mathrm{card}(\mathcal{T}_h)$	$\mathrm{card}(\mathcal{F}_h)\,\left(2\overline{\mathrm{card}(\mathcal{F}_T)}\right)\,\left(d\,\dim(\mathbb{P}_{d-1}^k)+1\right)^2$

HHO: growing interest in high-order discretizations
growing interest in p-multilevel solution strategies (Poisson, Stokes)

HHO for Poisson: For all $\underline{u}_T, \underline{v}_T \in \underline{U}_T := \mathbb{P}_d^k(T) \times \left\{ \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}$

[Di Pietro, Ern, Lemaire, Comput. Meth. Appl. Mat., 2014]

$$\boxed{a^T(\underline{u}_T, \underline{v}_T) = \int_T \left(\nabla p^{k+1}\underline{u}_T\right) \cdot \left(\nabla p^{k+1}\underline{v}_T\right) + s^T(\underline{u}_T, \underline{v}_T)}$$

Define $p^{k+1} : \underline{U}_T \to \mathbb{P}_d^{k+1}(T)$ such that, $\forall \underline{v}_T \in \underline{U}_T, \forall w_T \in \mathbb{P}_d^{k+1}(T)$

$$\begin{cases} \int_T \nu \left(\nabla p^{k+1}\underline{v}_T\right) \cdot \nabla w_T & = \int_T \nu \nabla v_T \cdot \nabla w_T + \sum_{F\in\mathcal{F}_T} \int_F (v_F - v_T)\, \nu \nabla w_T \cdot \mathbf{n}_{TF} \\ \int_T p^{k+1}\underline{v}_T & = \int_T v_T \end{cases}$$

Defining the intepolation by means of L^2 projections: $\underline{\mathcal{I}}_T^k v = (\pi_T^k v, (\pi_F^K v)_{F\in\mathcal{F}_T})$

It is possible to show that, given $v \in H^1(\Omega)$

$$\int_T (\nabla p^{k+1}\underline{\mathcal{I}}_T^k v - \nabla v) \cdot \nabla w_T = 0, \qquad \forall w_T \in \mathbb{P}_d^{k+1}(T)$$

Potential reconstruction $p^{k+1} : \underline{U}_T \to \mathbb{P}_d^{k+1}(T)$, **for all** $\underline{u}_T, \underline{v}_T \in \underline{U}_T$

$$\int_T \nu \left(\nabla p^{k+1} \underline{v}_T\right) \cdot \nabla w_T = \int_T \nu \nabla v_T \cdot \nabla w_T + \sum_{F \in \mathcal{F}_T} \int_F (v_F - v_T)\, \nu \nabla w_T \cdot \mathbf{n}_{TF}$$

Bases functions choice:

$\{\phi^T\}$ spans $\mathbb{P}_d^k(T)$

$\{\psi^F\}$ spans $\mathbb{P}_{d-1}^k(F)$

$\{\varphi^T\}$ spans $\mathbb{P}_d^{k+1}(T) - \mathbb{P}_d^0(T)$ $\rightarrow$ $\nabla p^{k+1} \underline{v}_T = \nabla \widehat{p}_j \varphi_j = \widehat{p}_j \nabla \varphi_j$

$\{\underline{\phi}\} = \{\phi^T, \psi^{F_1}, ..., \psi^{F_N}\}$ spans $\underline{U}_T$ $\rightarrow$ $\nabla p^{k+1} \underline{\phi}_k = \widehat{P}_{j,k} \nabla \varphi_j$

$$\int_T \nu \widehat{P}_{j,k} \nabla \varphi_j \cdot \nabla \varphi_i = \int_T \nu \nabla \phi_k^T \cdot \nabla \varphi_i + \sum_{F \in \mathcal{F}_T} \int_F \left(\psi_{\tilde{k}}^F - \phi_k^T\right) \nu \nabla \varphi_i \cdot \mathbf{n}_{TF}$$

Note that $\underline{\phi}_k = \begin{cases} \phi_k^T & \text{if } 0 < k < \dim(\mathbb{P}_d^k(T)) \\ \psi_{\tilde{k}}^F & \text{if } k > \dim(\mathbb{P}_d^k(T)) \end{cases}$

In matrix form the potential reconstruction reads $\widehat{P} = K^{-1} B$.

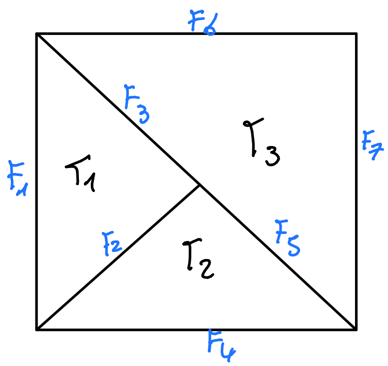
The potential reconstruction reads $\widehat{P} = K^{-1}B$, with

$$B_{i,k} = \left[\underbrace{\begin{bmatrix} \int_T \nu \nabla \phi_k^T \cdot \nabla \varphi_i \\ -\int_{F_1} \phi_k^T \, \nu \nabla \varphi_i \cdot \mathbf{n}_{TF} \\ \dots \\ -\int_{F_N} \phi_k^T \, \nu \nabla \varphi_i \cdot \mathbf{n}_{TF} \end{bmatrix}}_{B_T} \quad \underbrace{\begin{bmatrix} \int_{F_1} \psi_{\tilde{k}}^F \, \nu \nabla \varphi_i \cdot \mathbf{n}_{TF} \end{bmatrix}}_{B_{F_1}} \quad \underbrace{\dots}_{\dots} \quad \underbrace{\begin{bmatrix} \int_{F_N} \psi_{\tilde{k}}^F \, \nu \nabla \varphi_i \cdot \mathbf{n}_{TF} \end{bmatrix}}_{B_{F_N}} \right]$$

The local HHO consistent contribution

$$\begin{aligned} \int_T \left(\nabla p^{k+1} \underline{\phi}_j \right) \cdot \left(\nabla p^{k+1} \underline{\phi}_i \right) &= \int_T \left(\widehat{P}_{l,j} \nabla \varphi_l \right) \cdot \left(\widehat{P}_{m,i} \nabla \varphi_m \right) \\ &= \widehat{P}_{l,j} \left(\int_T \nabla \varphi_l \cdot \nabla \varphi_m \right) \widehat{P}_{m,i} = \widehat{P}_{l,j} K_{l,m} \widehat{P}_{m,i} \\ \widehat{P}^t K \widehat{P} = \widehat{P}^t B = B^t K^{-1} B &= \begin{bmatrix} A_{TT} & A_{TF_1} & \dots & A_{TF_N} \\ A_{F_1T} & A_{F_1F_1} & \dots & A_{F_1F_N} \\ \dots & \dots & \dots & \dots \\ A_{F_NT} & A_{F_NF_1} & \dots & A_{F_NF_N} \end{bmatrix} \end{aligned}$$

Static condensation

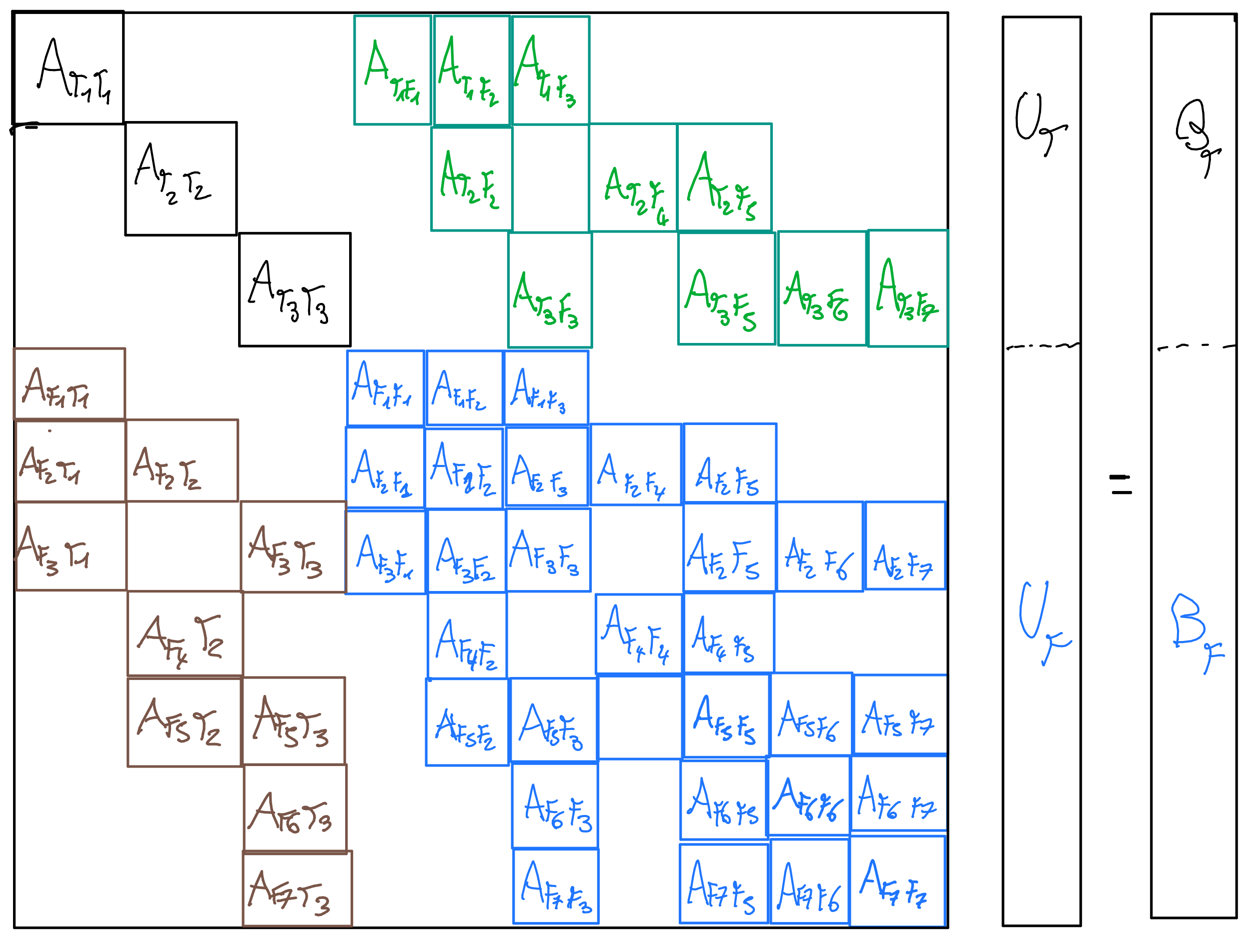


$$A_{TT} U_T + A_{TF} U_F = B_T$$

$$A_{FT} U_T + A_{FF} U_F = B_F$$

$$U_T = A_{TT}^{-1} B_T - A_{TT}^{-1} A_{TF} U_F$$

$$\left(A_{FF} - A_{FT} A_{TT}^{-1} A_{TF}\right) U_F = B_F - A_{FT} A_{TT}^{-1} B_T$$



Multigrid

After static condensation we have a (smaller) system to solve $\boldsymbol{A}_h \boldsymbol{u}_h = \mathbf{b}_h$

MG: speedup the solution process solving coarse problems $\boldsymbol{A}_H \boldsymbol{u}_H = \mathbf{b}_H$

p-MG: coarse problem by polynomial degree reduction:

L coarse problems indexed as $\ell = 0, ..., L$ with $k_{\ell+1} < k_\ell$

$$\underline{U}_T^{k_\ell} := \mathbb{P}_d^{k_\ell}(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^{k_\ell}(F) \right\}, \qquad \rightarrow \boldsymbol{A}_\ell \boldsymbol{u}_\ell = \mathbf{b}_\ell$$

Two ways of building $\boldsymbol{A}_\ell$

non-inherited: $\displaystyle\sum_{T \in \mathcal{T}_h} a_\ell^T(\underline{u}_T, \underline{v}_T) \qquad \forall \underline{u}_T, \underline{v}_T \in \underline{U}_T^{k_\ell}$

inherited: $\displaystyle\sum_{T \in \mathcal{T}_h} a_0^T(\underline{\mathcal{I}}_\ell^0 \underline{u}_T, \underline{\mathcal{I}}_\ell^0 \underline{v}_T) \quad \forall \underline{u}_T, \underline{v}_T \in \underline{U}_T^{k_\ell}$

Prolongation operator $\underline{\mathcal{I}}_\ell^0 : \underline{U}_T^{k_\ell} \rightarrow \underline{U}_T^{k}$ is an injection, note that $\underline{U}_T^{k_\ell} \subset \underline{U}_T^{k_{\ell+1}}$

Inherited p-MG

Inherited operators are computed recursively with Galerkin projections:

$$\boldsymbol{A}_{\ell+1} = \boldsymbol{\mathcal{I}}_{\ell}^{\ell+1} \boldsymbol{A}_{\ell} \, \boldsymbol{\mathcal{I}}_{\ell+1}^{\ell}$$

$\boldsymbol{\mathcal{I}}_{\ell}^{\ell+1}$ and $\boldsymbol{\mathcal{I}}_{\ell+1}^{\ell}$ are the matrix form of restriction and prolongation operators.

Prolongation operator	$\underline{\mathcal{I}}_{\ell+1}^{\ell} \underline{v}_{\ell+1} = \underline{v}_{\ell+1}$	injection
Restriction operator	$\underline{\mathcal{I}}_{\ell}^{\ell+1} \underline{v}_{\ell} = (\pi_T^{k_{\ell+1}} v_T, (\pi_F^{k_{\ell+1}} v_F)_{F \in \mathcal{F}_T})$	L^2 projection

With orthogonal basis functions: **simply shrink the local matrix blocks.**

$$A^T = \begin{bmatrix} A_{TT} & A_{TF_1} & \dots & A_{TF_N} \\ A_{F_1 T} & A_{F_1 F_1} & \dots & A_{F_1 F_N} \\ \dots & \dots & \dots & \dots \\ A_{F_N T} & A_{F_N F_1} & \dots & A_{F_N F_N} \end{bmatrix}$$

$$\widetilde{A}^T = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \widetilde{A}_{F_1 F_1}^{\ell} & \\ & \end{bmatrix} & \\ & \end{bmatrix} & \dots & \widetilde{A}_{F_1 F_N} \\ & \dots & \dots \\ \widetilde{A}_{F_N F_1} & \dots & \widetilde{A}_{F_N F_N} \end{bmatrix}$$

$$\widetilde{A}^T = A_{FF} - A_{FT}(A_{TT})^{-1} A_{TF}$$

Multigrid V-cycle: $\text{MG}_V(\ell, \boldsymbol{b}_\ell, \boldsymbol{u}_\ell)$

if $(\ell = L)$ **then**

$$\overline{\boldsymbol{u}}_\ell = \boldsymbol{A}_\ell^{-1} \boldsymbol{b}_\ell$$

if $(\ell < L)$ **then**

1 Pre-smoothing:

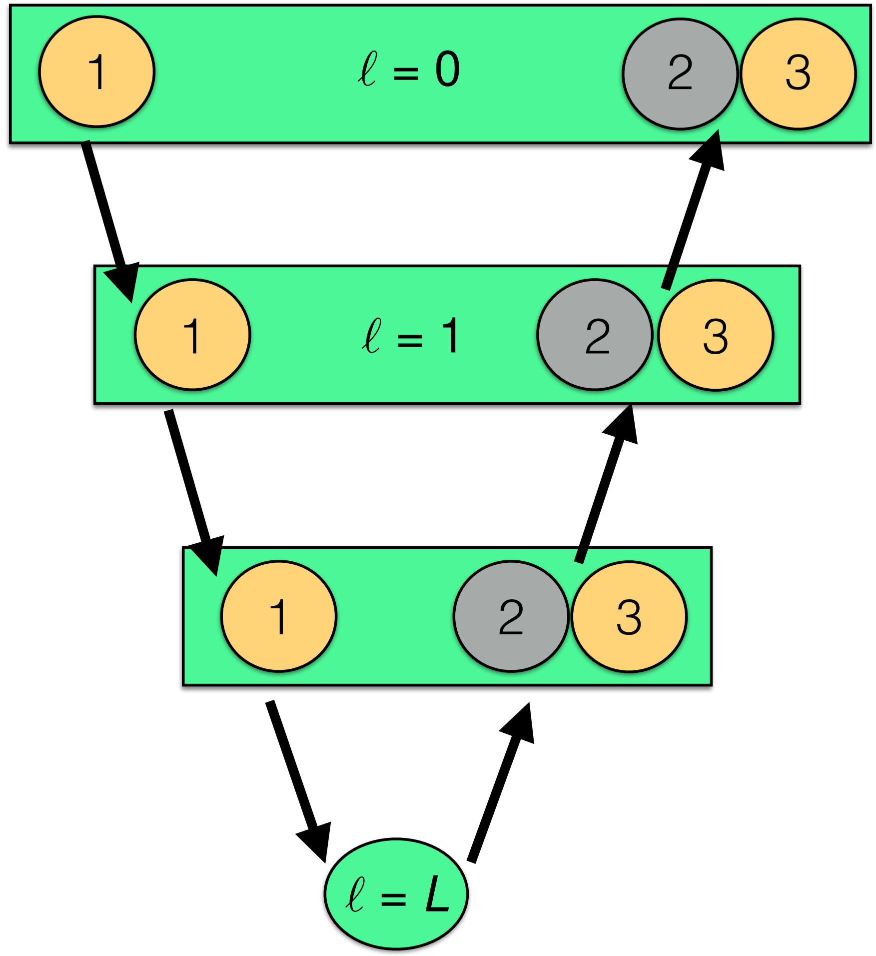
$$\overline{\boldsymbol{u}}_\ell = \text{GMRES}(\boldsymbol{A}_\ell, \boldsymbol{u}_\ell, \boldsymbol{b}_\ell)$$
$$\boldsymbol{d}_{\ell+1} = \mathcal{I}_\ell^{\ell+1}(\boldsymbol{b}_\ell - \boldsymbol{A}_\ell \overline{\boldsymbol{u}}_\ell)$$
$$\boldsymbol{c}_{\ell+1} = \text{MG}_V(\ell + 1, \boldsymbol{d}_{\ell+1}, 0)$$

2 Coarse grid correction:

$$\widehat{\boldsymbol{u}}_\ell = \overline{\boldsymbol{u}}_\ell + \mathcal{I}_{\ell+1}^{\ell} \boldsymbol{c}_{\ell+1}$$

3 Post-smoothing:

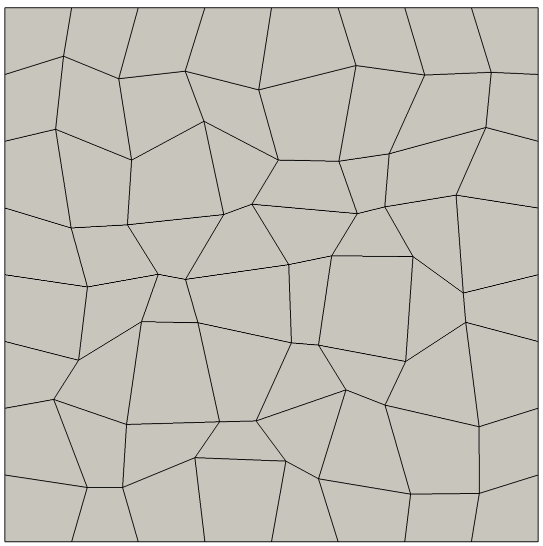
$$\overline{\boldsymbol{u}}_\ell = \text{GMRES}(\boldsymbol{A}_\ell, \widehat{\boldsymbol{u}}_\ell, \boldsymbol{b}_\ell)$$



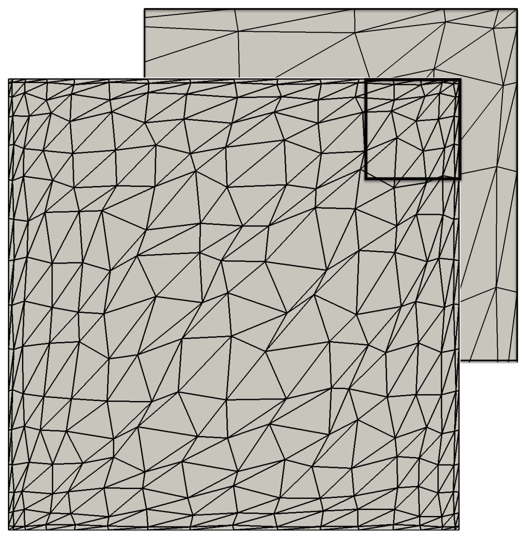
Performance of FGMRES p-MG$_V$ for HHO discretizations of

Poisson Problem: $\begin{cases} -\nabla \cdot \nabla u = g, & \text{in } [-1,1]^d \\ u = u_{ex}, & \text{on } \partial[-1,1]^d \end{cases}$, $\quad u_{ex} = \prod_{i=1,\dots,d} \sin(n\pi x_i)$

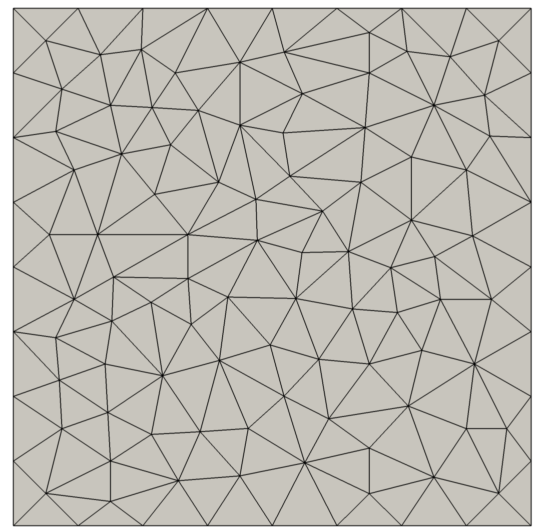
Trapezoidal elements mesh sequence: $32^2, 64^2, 128^2, 256^2$



Distorted graded triangular elems mesh seq: $2{*}(32^2, 64^2, 128^2, 256^2)$



Regular Delaunay triangular elems mesh seq: $2{*}(39^2, 79^2, 158^2, 311^2)$



FGMRES p-MG$_V$ rtol=1e-10: L=2, 1 smit (GMRES ILU), LU solver$_L$

	dG-BR2, $k_\ell = 3,2,1$					HHO $k_\ell = 3,2,1$				
	L2 error		conv. rate		iterations		CPU time			Eff
	u_h	Gu_h	u_h	Gu_h	ITs	ITs$_L$	Sol.	Ass.	Tot.	
trapezoidal elements grid										
1k	4.74e-06	0.000702	-	-	7	1	0.047	0.04	0.088	-
4k	3.03e-07	8.9e-05	3.97	2.98	7	1	0.22	0.16	0.38	92
16k	1.92e-08	1.12e-05	3.98	2.99	7	1	0.98	0.66	1.6	92.8
65k	1.2e-09	1.41e-06	4	3	7	1	4.9	2.6	7.5	87.6
1k	8.46e-07	0.00011	-	-	4	1	0.018	0.1	0.12	-
4k	2.68e-08	6.98e-06	4.98	3.97	4	1	0.11	0.38	0.49	96.6
16k	8.52e-10	4.41e-07	4.98	3.98	4	1	0.68	1.5	2.2	88.4
65k	2.65e-11	2.75e-08	5.01	4	4	1	5.1	6.1	11	78.7
delaunay triangular grid										
3k	2.24e-08	1.05e-05	-	-	11	1	0.19	0.1	0.29	-
13k	1.35e-09	1.29e-06	4	2.99	11	1	0.87	0.43	1.3	89.5
50k	9.03e-11	1.69e-07	3.97	2.98	11	1	4	1.8	5.8	90.2
194k	6.12e-12	2.22e-08	3.97	2.99	11	1	20	6.8	26	87.1
3k	2.25e-09	7.7e-07	-	-	7	1	0.035	0.26	0.3	-
13k	7.34e-11	4.6e-08	4.87	4.01	7	1	0.21	1.1	1.3	93.2
50k	2.15e-12	3e-09	5.17	4.01	7	1	1.3	4.1	5.4	94.5
194k	1.46e-13	2e-10	3.97	3.99	7	1	7.9	16	24	89.9
distorted triangular grid										
2k	8.71e-06	0.00117	-	-	17	1	0.17	0.066	0.24	-
8k	6.68e-07	0.000166	3.7	2.82	21	1	0.88	0.26	1.1	82.6
33k	4.19e-08	2.08e-05	4	3	28	1	5	1	6.1	75.7
131k	2.69e-09	2.63e-06	3.96	2.98	40	1	31	4.2	35	68.7
2k	2.23e-06	0.000278	-	-	9	1	0.025	0.17	0.19	-
8k	7.76e-08	1.88e-05	4.85	3.89	11	1	0.16	0.67	0.83	93.9
33k	2.41e-09	1.18e-06	5.01	4	15	1	1.1	2.7	3.7	88.4
131k	7.62e-11	7.43e-08	4.98	3.99	19	1	7.2	11	18	83.6

FGMRES p-MG$_V$ rtol=1e-10: L=2, 1 smit (GMRES ILU), LU solver$_L$

	dG-BR2 $k_\ell = 3, 2, 1$					HHO $k_\ell = 2, 1, 0$				
	L2 error		conv. rate		iterations		CPU time			Eff
	u_h	Gu_h	u_h	Gu_h	ITs	ITs$_L$	Sol.	Ass.	Tot.	
trapezoidal elements grid										
1k	4.74e-06	0.000702	-	-	7	1	0.047	0.04	0.088	-
4k	3.03e-07	8.9e-05	3.97	2.98	7	1	0.22	0.16	0.38	92
16k	1.92e-08	1.12e-05	3.98	2.99	7	1	0.98	0.66	1.6	92.8
65k	1.2e-09	1.41e-06	4	3	7	1	4.9	2.6	7.5	87.6
1k	2.8e-05	0.0026	-	-	12	1	0.016	0.044	0.06	-
4k	1.76e-06	0.000326	3.99	2.99	12	1	0.076	0.16	0.24	100
16k	1.1e-07	4.08e-05	4	3	13	1	0.41	0.66	1.1	89.3
65k	6.89e-09	5.1e-06	4	3	13	1	2.3	2.6	4.9	87.3
delaunay triangular grid										
3k	2.24e-08	1.05e-05	-	-	11	1	0.19	0.1	0.29	-
13k	1.35e-09	1.29e-06	4	2.99	11	1	0.87	0.43	1.3	89.5
50k	9.03e-11	1.69e-07	3.97	2.98	11	1	4	1.8	5.8	90.2
194k	6.12e-12	2.22e-08	3.97	2.99	11	1	20	6.8	26	87.1
3k	1.74e-07	5.7e-05	-	-	16	1	0.052	0.12	0.17	-
13k	1.04e-08	6.97e-06	4	2.99	17	1	0.25	0.48	0.73	94.9
50k	6.8e-10	8.98e-07	4	3	18	1	1.2	1.9	3.1	94.5
194k	4.49e-11	1.17e-07	4.01	3	18	1	5.1	6.9	12	103
distorted triangular grid										
2k	8.71e-06	0.00117	-	-	17	1	0.17	0.066	0.24	-
8k	6.68e-07	0.000166	3.7	2.82	21	1	0.88	0.26	1.1	82.6
33k	4.19e-08	2.08e-05	4	3	28	1	5	1	6.1	75.7
131k	2.69e-09	2.63e-06	3.96	2.98	40	1	31	4.2	35	68.7
2k	5.01e-05	0.00446	-	-	16	1	0.022	0.073	0.095	-
8k	3.61e-06	0.000613	3.79	2.86	19	1	0.13	0.29	0.42	91.4
33k	2.31e-07	7.77e-05	3.97	2.98	21	1	0.64	1.1	1.8	92.9
131k	1.44e-08	9.71e-06	4	3	27	1	4	4.6	8.6	83.3

FGMRES p-MG$_V$ rtol=1e-12: L=2, 1 smit (GMRES ILU), LU solver$_L$

	dG-BR2 $k_\ell = 6, 3, 1$						HHO $k_\ell = 6, 3, 1$			
	L2 error		conv. rate		iterations		CPU time			Eff
	u_h	Gu_h	u_h	Gu_h	ITs	ITs$_L$	Sol.	Ass.	Tot.	
trapezoidal elements grid										
1k	6.58e-07	0.00014	-	-	11	1	0.43	0.28	0.71	-
4k	5.52e-09	2.33e-06	6.9	5.91	11	1	1.8	1.1	2.9	97.5
16k	4.44e-11	3.73e-08	6.96	5.96	11	1	7.4	4.5	12	98.4
65k	9.67e-13	5.79e-10	5.52	6.01	11	1	31	18	49	97
1k	4e-07	0.000113	-	-	7	1	0.046	0.44	0.486	-
4k	1.68e-09	9.4e-07	7.9	6.91	6	1	0.211	1.76	1.97	98.8
16k	6.71e-12	7.49e-09	7.97	6.97	7	1	1.19	7.03	8.22	95.7
65k	3.9e-13	6.22e-11	4.11	6.91	6	1	6.91	28	34.9	94.3
delaunay triangular grid										
810	2.54e-06	0.000477	-	-	18	1	0.41	0.18	0.59	-
3k	2.75e-08	9.84e-06	6.71	5.75	18	1	1.6	0.7	2.3	102
13k	2.01e-10	1.47e-07	6.97	5.97	19	1	7.3	2.9	10	91.9
50k	1.89e-12	2.39e-09	6.77	5.97	20	1	31	11	43	95.1
810	3.57e-06	0.000901	-	-	10	1	0.019	0.332	0.351	-
3k	1.72e-08	8.51e-06	7.92	6.91	10	1	0.097	1.19	1.29	81.8
13k	6.05e-11	6.09e-08	8.01	7.01	11	1	0.512	4.87	5.39	95.6
50k	1.56e-12	4.92e-10	5.31	6.99	11	1	2.55	19.4	22	73.6
distorted triangular grid										
512	0.000463	0.0418	-	-	27	1	0.36	0.12	0.48	-
2k	2.96e-06	0.000563	7.29	6.22	34	1	1.8	0.46	2.3	83.4
8k	3.39e-08	1.2e-05	6.45	5.56	44	1	9.5	1.8	11	80.5
33k	2.7e-10	1.9e-07	6.97	5.98	55	1	49	7.3	56	81.2
512	0.00076	0.104	-	-	12	1	0.0129	0.197	0.21	-
2k	3.38e-06	0.000948	7.81	6.78	12	1	0.0679	0.782	0.85	98.8
8k	2.02e-08	1.04e-05	7.38	6.52	14	1	0.361	3.12	3.48	97.6
33k	7.4e-11	7.79e-08	8.09	7.05	25	1	2.7	12.5	15.2	91.7

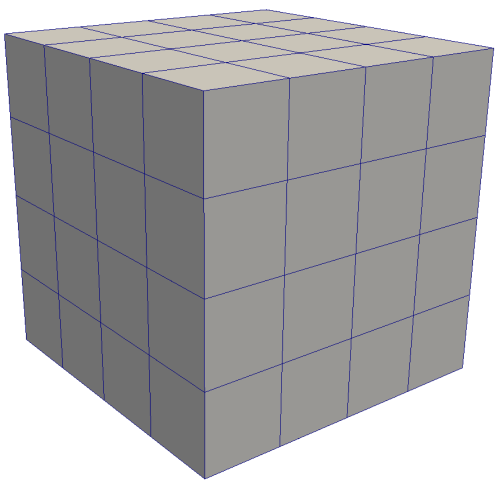
FGMRES p-MG$_V$ rtol=1e-12: L=2, 1 smit (GMRES ILU), LU solver$_L$

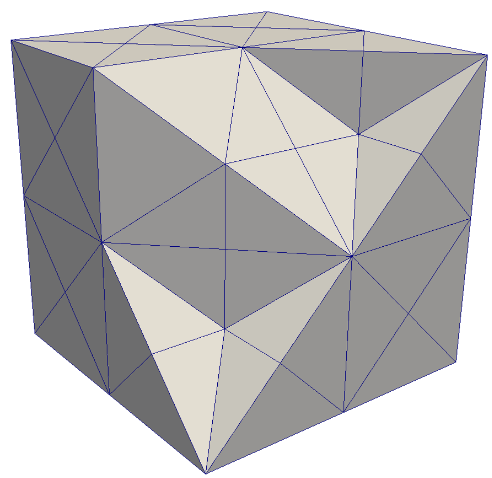
	HHO $k_\ell = 6,3,1$									HHO $k_\ell = 6,3,0$
	L2 error		conv. rate		iterations		CPU time			Eff
	u_h	Gu_h	u_h	Gu_h	ITs	ITs$_L$	Sol.	Ass.	Tot.	
trapezoidal elements grid										
1k	4e-07	0.000113	-	-	7	1	0.046	0.44	0.486	-
4k	1.68e-09	9.4e-07	7.9	6.91	6	1	0.211	1.76	1.97	98.8
16k	6.71e-12	7.49e-09	7.97	6.97	7	1	1.19	7.03	8.22	95.7
65k	3.9e-13	6.22e-11	4.11	6.91	6	1	6.91	28	34.9	94.3
1k	4e-07	0.000113	-	-	17	1	0.0821	0.443	0.525	-
4k	1.68e-09	9.4e-07	7.9	6.91	18	1	0.392	1.8	2.19	95.9
16k	7.32e-12	7.49e-09	7.84	6.97	19	1	1.76	7.02	8.78	99.8
65k	2.48e-12	6.32e-11	1.56	6.89	19	1	7.77	28.1	35.8	98
delaunay triangular grid										
810	3.57e-06	0.000901	-	-	10	1	0.019	0.332	0.351	-
3k	1.72e-08	8.51e-06	7.92	6.91	10	1	0.097	1.19	1.29	81.8
13k	6.05e-11	6.09e-08	8.01	7.01	11	1	0.512	4.87	5.39	95.6
50k	1.56e-12	4.92e-10	5.31	6.99	11	1	2.55	19.4	22	73.6
810	3.57e-06	0.000901	-	-	23	1	0.0364	0.313	0.349	-
3k	1.72e-08	8.51e-06	7.92	6.91	25	1	0.2	1.19	1.39	75.4
13k	6.05e-11	6.09e-08	8.01	7.01	27	1	0.957	4.88	5.84	95.2
50k	1.23e-12	4.92e-10	5.65	6.99	27	1	4.2	19.4	23.6	74.1
distorted triangular grid										
512	0.00076	0.104	-	-	12	1	0.0129	0.197	0.21	-
2k	3.38e-06	0.000948	7.81	6.78	12	1	0.0679	0.782	0.85	98.8
8k	2.02e-08	1.04e-05	7.38	6.52	14	1	0.361	3.12	3.48	97.6
33k	7.4e-11	7.79e-08	8.09	7.05	25	1	2.7	12.5	15.2	91.7
512	0.00076	0.104	-	-	22	1	0.021	0.202	0.223	-
2k	3.38e-06	0.000948	7.81	6.78	26	1	0.124	0.782	0.907	98.5
8k	2.02e-08	1.04e-05	7.38	6.52	32	1	0.675	3.13	3.8	95.4
33k	7.4e-11	7.79e-08	8.09	7.05	38	1	3.4	12.5	15.9	95.4

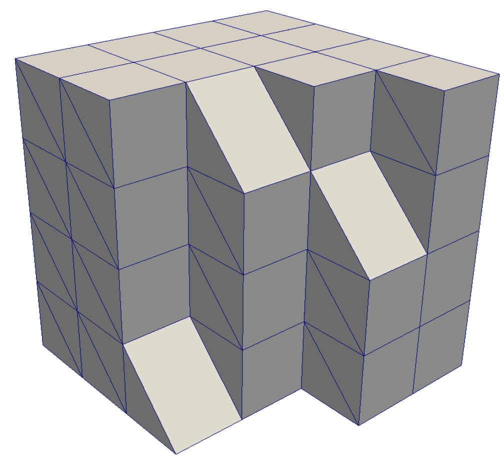
Performance of FGMRES p-MG$_V$ for HHO discretizations

Poisson Problem: $\begin{cases} -\nabla \cdot \nabla u &= g, & \text{in } [-1,1]^d \\ u &= u_{ex}, & \text{on } \partial[-1,1]^d \end{cases}$, $u_{ex} = \prod_{i=1,\dots,d} \sin(n\pi x_i)$

Hexahedral elements	Tetrahedral elements	Prismatic elements
mesh sequence:	mesh sequence:	mesh sequence:
64, 512, 4k, 32k	24, 192, 1536, 12k	128, 1024, 8k, 65k







FGMRES p-MG$_V$ rtol=1e-12: L=2, 1 smit (GMRES ILU), GMRES solver$_L$ rtol = 1.e-3

	dG-BR2 $k_\ell = 3, 2, 1$					HHO $k_\ell = 3, 2, 1$				
	L2 error		conv. rate		iterations		CPU time			Eff
	u_h	Gu_h	u_h	Gu_h	ITs	ITs$_L$	Sol.	Ass.	Tot.	
hexahedral elements grid										
64	0.00083	0.0326	-	-	7	5	0.013	0.024	0.037	-
512	5.32e-05	0.0042	3.97	2.96	8	9	0.131	0.203	0.334	88.2
4k	3.35e-06	0.00053	3.99	2.99	8	17	1.15	1.65	2.8	95.3
32k	2.09e-07	6.61e-05	4	3	7	33	9.27	13.6	22.8	98.1
64	0.00064	0.0194	-	-	5	6	0.0153	0.072	0.087	-
512	1.83e-05	0.0011	5.12	4.1	5	9	0.125	0.575	0.7	99.9
4k	5.42e-07	6.77e-05	5.08	4.06	5	19	1.09	4.6	5.69	98.4
32k	1.65e-08	4.13e-06	5.04	4.03	5	36	10.1	36.8	46.8	97.2
tetrahedral elements grid										
24	0.0061	0.153	-	-	10	5	0.0045	0.0068	0.011	-
192	0.00050	0.0222	3.6	2.78	13	9	0.046	0.055	0.101	89.6
1536	3.2e-05	0.0028	3.98	2.97	14	14	0.481	0.45	0.931	87
12k	2e-06	0.00036	4	2.99	14	22	4.2	3.64	7.83	95
24	0.0146	0.269	-	-	9	5	0.0039	0.022	0.026	-
192	0.00040	0.0155	5.18	4.11	10	9	0.030	0.185	0.215	97.4
1536	1.23e-05	0.00097	5.03	4	10	15	0.286	1.37	1.66	104
12k	3.78e-07	5.99e-05	5.02	4.02	10	27	2.68	11	13.7	97.1
prismatic elements grid										
128	0.00054	0.0247	-	-	8	6	0.023	0.042	0.066	-
1024	3.41e-05	0.00316	3.99	2.97	8	11	0.228	0.353	0.582	90.6
8k	2.14e-06	0.00039	4	2.99	8	21	2.01	2.88	4.9	95
65k	1.34e-07	4.95e-05	4	3	9	19	20.4	23.2	43.5	90
128	0.000354	0.013	-	-	8	9	0.028	0.132	0.16	-
1024	1.07e-05	0.00079	5.04	4.04	8	14	0.26	1.05	1.31	97.8
8k	3.27e-07	4.8e-05	5.04	4.04	8	31	2.48	8.42	10.9	95.9
65k	1e-08	2.95e-06	5.03	4.02	8	57	27.3	67.2	94.5	92.3

FGMRES p-MG$_V$ rtol=1e-12: L=2, 1 smit (GMRES ILU), GMRES solver$_L$ rtol = 1.e-3

	dG-BR2 $k_\ell = 6, 3, 1$				HHO $k_\ell = 6, 3, 1$					
	L2 error		conv. rate		iterations		CPU time			Eff
	u_h	Gu_h	u_h	Gu_h	ITs	ITs$_L$	Sol.	Ass.	Tot.	
hexahedral elements grid										
8	8.84e-05	0.00276	-	-	7	3	0.023	0.098	0.122	-
64	7.63e-07	4.72e-05	6.86	5.87	10	5	0.327	0.852	1.18	82.8
512	6.07e-09	7.53e-07	6.97	5.97	10	9	3.08	7.15	10.2	92.2
4k	4.76e-11	1.18e-08	6.99	5.99	10	17	26.7	57.7	84.4	96.9
8	0.00014	0.00447	-	-	6	3	0.017	0.22	0.237	-
64	5.26e-07	3.46e-05	8.05	7.01	8	6	0.168	1.74	1.91	99.6
512	1.96e-09	2.56e-07	8.07	7.08	8	11	1.37	13.9	15.2	100
4k	7.54e-12	1.96e-09	8.02	7.03	8	20	11.2	111	122	99.6
tetrahedral elements grid										
24	4.2e-05	0.00136	-	-	16	5	0.10	0.24	0.35	-
192	3.4e-07	2.31e-05	6.95	5.88	21	8	1.15	2.04	3.19	87.6
1536	2.72e-09	3.7e-07	6.97	5.97	22	12	10.2	16.6	26.8	95.4
12k	2.14e-11	5.82e-09	6.99	5.99	22	14	84.5	132	217	98.9
24	7.01e-05	0.00312	-	-	11	5	0.029	0.57	0.60	-
192	3.39e-07	2.98e-05	7.69	6.71	14	10	0.29	4.56	4.86	99.3
1536	1.21e-09	2.35e-07	8.13	6.98	14	19	2.42	36.5	38.9	99.9
12k	4.5e-12	1.82e-09	8.07	7.01	14	32	20.1	292	312	99.9
prismatic elements grid										
16	5.14e-05	0.00176	-	-	9	3	0.049	0.18	0.23	-
128	4.5e-07	2.97e-05	6.84	5.88	11	6	0.59	1.53	2.13	86.2
1024	3.62e-09	4.73e-07	6.96	5.97	11	10	5.4	12.8	18.2	93.6
8k	2.85e-11	7.41e-09	6.99	6	11	15	46.1	103	149	97.3
16	7.81e-05	0.00295	-	-	9	4	0.029	0.41	0.44	-
128	2.95e-07	2.35e-05	8.05	6.97	12	10	0.31	3.25	3.56	99.1
1024	1.12e-09	1.82e-07	8.04	7.01	12	18	2.52	26.1	28.6	99.7
8k	4.29e-12	1.41e-09	8.02	7.01	12	36	21	208	229	99.9

HHO for Stokes: $\begin{cases} -\triangle \boldsymbol{u} + \nabla p &= \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 & \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_{ex} & \text{on } \partial\Omega_D. \\ -\nabla \boldsymbol{u} \cdot \mathbf{n} + p\mathbf{n} &= -\nabla \boldsymbol{u}_{ex} \cdot \mathbf{n} + p_{ex}\mathbf{n} & \text{on } \partial\Omega_N. \end{cases}$

For all $\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T \in (\underline{U}_T)^d$ and all $p_T, q_T \in \mathbb{P}^k_d(T)$, the HHO residual reads:

$$\begin{aligned} \mathbf{r}_{QDM} &= \sum_{i=1}^{d} a^T(\underline{u}_{T,i}, \underline{v}_{T,i}) + b(\underline{p}_T, \underline{\boldsymbol{v}}_T) - \int_T \mathbf{f} \cdot \boldsymbol{v}_T \\ r_{CNT} &= \widetilde{b}(\underline{\boldsymbol{u}}_T, q_T) \end{aligned}$$

[Aghili, Boyaval, Di Pietro, CM. Appl. Mat, 2015; Botti, Di Pietro, Droniou, JCP, 2019]

$$\widetilde{b}(\underline{\boldsymbol{u}}_T, q_T): -\int_T \nabla\cdot\boldsymbol{u}_T\, q_T + \sum_{F\in\mathcal{F}_T^{!D}} \int_F (\boldsymbol{u}_T - \boldsymbol{u}_F)\cdot\mathbf{n}_{TF}\, q_T + \sum_{F\in\mathcal{F}_T^{D}} \int_F (\boldsymbol{u}_T - \boldsymbol{u}_{ex})\cdot\mathbf{n}_{TF}\, q_T$$

$$b(p_T, \underline{\boldsymbol{v}}_T): -\int_T p_T \nabla\cdot\boldsymbol{v}_T + \sum_{F\in\mathcal{F}_T^{!\partial}} \int_F p_T(\boldsymbol{v}_T - \boldsymbol{v}_F)\cdot\mathbf{n}_{TF} + \sum_{F\in\mathcal{F}_T^{D}} \int_F p_T \boldsymbol{v}_T\cdot\mathbf{n}_{TF} + \sum_{F\in\mathcal{F}_T^{N}} \int_F p_{ex}\boldsymbol{v}_T\cdot\mathbf{n}_{TF}$$

$$\begin{aligned} a^T(\underline{u}_T, \underline{v}_T) : \quad & \int_T \left(\nabla p^{k+1}\underline{u}_T\right)\cdot\left(\nabla p^{k+1}\underline{v}_T\right) + s^T(\underline{u}_T, \underline{v}_T) \\ & + \sum_{F\in\mathcal{F}_T^{D}} \int_F \left(-\nabla p^{k+1}\underline{u}_T\cdot\mathbf{n}_{TF} v_F + h_F^{-1}(u_F - u_{ex})v_F\right) - \sum_{F\in\mathcal{F}_T^{N}} \int_F \nabla u_{ex}\cdot\mathbf{n}_{TF} v_F \end{aligned}$$

Static condensation (local contribution)

$$\begin{bmatrix} A_{TT} & B_{TT}^{k>0} & A_{TF_i} & B_{TT}^{0} \\ A_{F_iT} & B_{F_iT}^{k>0} & A_{F_iF_i} & B_{F_iT}^{0} \\ \widetilde{B}_{TT}^{k>0} & 0 & \widetilde{B}_{TF_i}^{k>0} & 0 \\ \widetilde{B}_{TT}^{0} & 0 & \widetilde{B}_{TF_i}^{0} & 0 \end{bmatrix} \begin{bmatrix} U_T \\ P^{k>0} \\ U_{F_i} \\ P^0 \end{bmatrix} = \begin{bmatrix} f_T \\ g_T^{k>0} \\ f_{F_i} \\ g_T^0 \end{bmatrix}$$

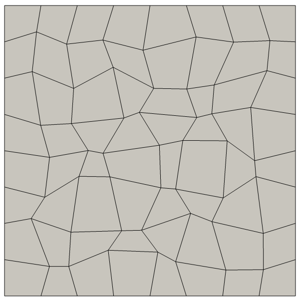
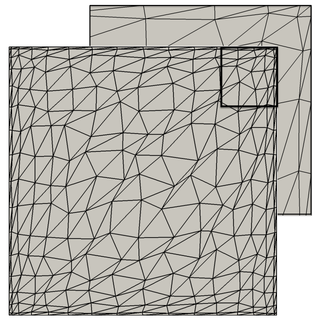
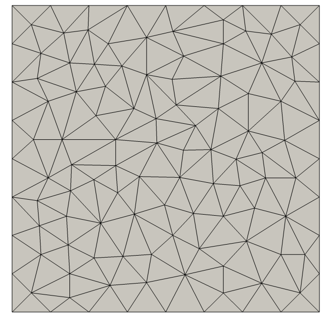
Keep U_F, P_T^0 as globally coupled unknowns.

$$\left(\begin{bmatrix} A_{F_iF_i} & B_{F_iT}^{0} \\ \widetilde{B}_{TF_i}^{0} & 0 \end{bmatrix} - \begin{bmatrix} A_{F_iT} & B_{F_iT}^{k>0} \\ \widetilde{B}_{TT}^{0} & 0 \end{bmatrix} \begin{bmatrix} A_{TT} & B_{TT}^{k>0} \\ \widetilde{B}_{TT}^{k>0} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{TF_i} & B_{TT}^{0} \\ \widetilde{B}_{TF_i}^{k>0} & 0 \end{bmatrix} \right)$$

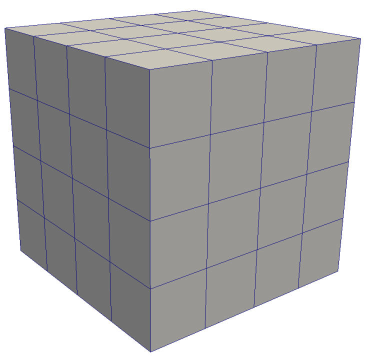
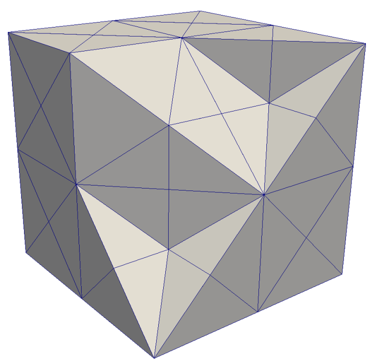
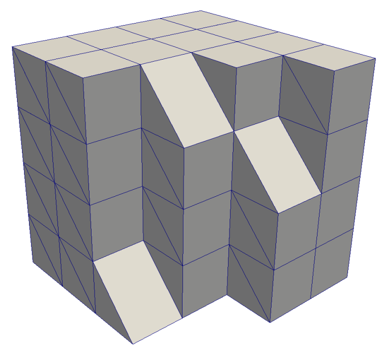
$$\begin{bmatrix} U_{F_i} \\ P_T^0 \end{bmatrix} = \begin{bmatrix} f_{F_i} \\ g_T^0 \end{bmatrix} - \begin{bmatrix} A_{F_iT} & B_{F_iT}^{k>0} \\ \widetilde{B}_{TT}^{0} & 0 \end{bmatrix} \begin{bmatrix} A_{TT} & B_{TT}^{k>0} \\ \widetilde{B}_{TT}^{k>0} & 0 \end{bmatrix}^{-1} \begin{bmatrix} f_T \\ g_T^{k>0} \end{bmatrix}$$

Performance of FGMRES p-MG$_V$ for Stokes HHO discretizations

$$\begin{cases} \mathbf{u} &= \left[-e^x\,(y\ \cos(y) + \sin(y))\,\mathbf{i},\, e^x\,(y\ \sin(y))\,\mathbf{j}\right] \\ p &= 2\,e^x\ \sin(y), \end{cases} \qquad \Omega = [-1, 1]^2$$

$$\begin{cases} \mathbf{u} &= [2\ \sin(\pi\, x)\mathbf{i}, -\pi\, y\ \cos(\pi\, x)\mathbf{j}, -\pi\, z\ \cos(\pi\, x)\mathbf{k}] \\ p &= \sin(\pi\, x)\ \cos(\pi\, y)\ \sin(\pi\, z) \end{cases} \qquad \Omega = [0, 1]^3$$

FGMRES p-MG$_V$, rtol=1e-13: L=2, 1 smit (GMRES ILU), LU solver$_L$

dG [Bassi ea, JCP, 2006] $k_\ell = 3, 2, 1$				HHO [Aghili ea, CMAM, 2015] $k_\ell = 3, 2, 1$							
L2 error				conv. rate			ITs	CPU time			Eff
u_h	Gu_h	p_h	Du_h	u_h	Gu_h	p_h	ITs	Sol.	Ass.	Tot.	
trapezoidal elements grid											
3e-07	5.23e-05	1.09e-05	0.000215	-	-	-	14	0.76	0.085	0.85	-
1.94e-08	6.67e-06	1.42e-06	2.8e-05	3.95	2.97	2.94	14	3.64	0.33	3.97	85.5
1.24e-09	8.46e-07	1.84e-07	3.55e-06	3.97	2.98	2.95	14	18.8	1.35	20.1	79
7.74e-11	1.06e-07	2.29e-08	4.41e-07	4	3	3.01	14	106	5.66	111	72.3
4.74e-09	7.13e-07	4.85e-07	1.68e-06	-	-	-	7	0.15	0.16	0.31	-
1.49e-10	4.54e-08	3.07e-08	1.05e-07	5	3.97	3.98	7	0.73	0.64	1.37	92
4.72e-12	2.88e-09	1.95e-09	6.39e-09	4.98	3.98	3.97	8	3.77	2.53	6.29	87
2.72e-13	1.8e-10	1.22e-10	4e-10	4.12	4	4.01	8	19.5	10.1	29.7	85
delaunay triangular grid											
6.72e-07	9.79e-05	3.6e-05	0.000849	-	-	-	20	0.59	0.055	0.65	-
4.45e-08	1.25e-05	4.84e-06	0.00011	4.02	3.05	2.98	21	2.64	0.20	2.85	91.7
2.64e-09	1.5e-06	5.68e-07	1.3e-05	4.01	3.01	3.04	21	12.4	0.87	13.3	85.9
1.68e-10	1.9e-07	7.12e-08	1.66e-06	4	2.99	3.01	21	59.4	3.63	63	84.1
1.26e-08	1.63e-06	1.42e-06	5.72e-06	-	-	-	15	0.098	0.10	0.20	-
4.16e-10	1.05e-07	9e-08	3.83e-07	5.05	4.07	4.09	15	0.44	0.37	0.81	98
1.23e-11	6.24e-09	5.38e-09	2.31e-08	4.99	4	4	16	2.18	1.52	3.7	88
5.3e-13	3.98e-10	3.44e-10	1.45e-09	4.56	3.99	3.99	16	10.4	6.11	16.5	90

FGMRES p-MG$_V$, rtol=1e-13: L=2, 4 smit (GMRES ILU), LU solver$_L$

dG [Bassi ea, JCP, 2006] $k_\ell = 3, 2, 1$ — HHO [Aghili ea, CMAM, 2015] $k_\ell = 3, 2, 1$

L2 error				conv. rate			ITs	CPU time			Eff
u_h	Gu_h	p_h	Du_h	u_h	Gu_h	p_h	ITs	Sol.	Ass.	Tot.	
distorted triangular grid											
8.01e-06	0.000563	0.000267	0.00558	-	-	-	9	0.42	0.038	0.46	-
4.28e-07	6.32e-05	2.85e-05	0.000585	4.23	3.15	3.23	10	2.16	0.14	2.3	80.8
3.27e-08	8.95e-06	4.27e-06	8.5e-05	3.71	2.82	2.74	12	11.2	0.59	11.8	77.9
1.99e-09	1.11e-06	5.3e-07	1.06e-05	4.04	3.01	3.01	23	85	2.32	87.3	54.1
2.81e-07	1.77e-05	1.47e-05	5.75e-05	-	-	-	11	0.08	0.066	0.14	-
7.48e-09	9.56e-07	8.1e-07	3.27e-06	5.23	4.21	4.18	27	0.82	0.26	1.08	54
distorted quadrilateral grid											
7.47e-07	9.94e-05	2.19e-05	0.000367	-	-	-	7	0.79	0.084	0.88	-
4.75e-08	1.26e-05	2.79e-06	4.64e-05	3.97	2.98	2.97	8	4.19	0.34	4.53	77.9
3.04e-09	1.59e-06	3.97e-07	5.81e-06	3.96	2.99	2.81	8	20	1.39	21.4	84.5
7.33e-10	2.05e-07	5.59e-07	1.03e-06	2.05	2.95	-0.49	8	108	5.48	113	75.6
1.48e-08	1.72e-06	1.12e-06	3.42e-06	-	-	-	5	0.19	0.16	0.35	-
4.73e-10	1.09e-07	7.3e-08	2.25e-07	4.97	3.97	3.94	7	1.1	0.65	1.75	81
2.51e-11	9.18e-09	8.38e-09	2.1e-08	4.24	3.58	3.12	21	10.9	2.63	13.5	52
8.99e-08	0.000217	0.00022	0.0012	-11	-14	-14	45	88.6	10.4	99.1	55

FGMRES p-MG$_V$, rtol=1e-12: L=2, 1 smit (GMRES ILU), GMRES solver$_L$ rtol=1.e-3

dG [Bassi ea, JCP, 2006] $k_\ell = 4, 2, 1$						HHO [Aghili ea, CMAM, 2015] $k_\ell = 4, 2, 1$						
L2 error				conv. rate			ITs		CPU time			Eff
u_h	Gu_h	p_h	Du_h	u_h	Gu_h	p_h	ITs	ITs$_L$	Sol.	Ass.	Tot.	
hexahedral elements grid (up to 4k)												
0.00065	0.0215	0.0074	0.346	-	-	-	10	4	0.05	0.02	0.07	-
2.1e-05	0.0014	0.00054	0.0237	4.95	3.95	3.77	13	9	0.73	0.18	0.91	60.2
6.4e-07	8.7e-05	3.4e-05	0.00149	5.04	3.99	3.98	14	18	7.34	1.5	8.84	82.1
1.9e-08	5.5e-06	2.1e-06	9.3e-05	5.03	4	4.03	14	42	66.2	12.4	78.6	90
0.00069	0.015	0.0131	0.49	-	-	-	8	4	0.06	0.08	0.14	-
8.3e-06	0.00026	0.00024	0.0082	6.37	5.83	5.75	10	15	0.62	0.63	1.26	86.9
1.1e-07	5.4e-06	5.2e-06	0.00015	6.15	5.62	5.55	10	30	5.5	4.96	10.5	96.2
1.7e-09	1.2e-07	1.2e-07	3.2e-06	6.06	5.43	5.37	10	72	54.4	39.1	93.5	89.5
tetrahedral elements grid (up to 12k)												
0.00115	0.0337	0.0136	0.591	-	-	-	18	7	0.21	0.05	0.26	-
3.6e-05	0.00199	0.00077	0.0246	4.99	4.09	4.12	22	13	2.32	0.42	2.75	76
1.1e-06	0.000123	4.8e-05	0.00147	4.93	4.02	4.01	23	17	21	3.44	24.5	89.7
3.8e-08	7.6e-06	3.0e-06	9.2e-05	4.96	4.01	4	24	55	191	27.5	219	89.5
0.00030	0.0068	0.0061	0.255	-	-	-	18	11	0.11	0.16	0.26	-
4.7e-06	0.00022	0.00018	0.0059	5.99	4.97	5.1	23	24	1.36	1.16	2.52	84.5
6.9e-08	6.2e-06	4.4e-06	0.00013	6.09	5.12	5.31	23	48	13.4	9.09	22.5	89.5
1.0e-09	1.8e-07	1.2e-07	3.4e-06	6.05	5.06	5.13	23	108	153	72.2	225	80
prismatic elements grid (up to 8k)												
0.00045	0.0148	0.0067	0.231	-	-	-	13	5	0.11	0.03	0.14	-
1.42e-05	0.00094	0.00042	0.0147	5	3.98	4.01	15	10	1.33	0.32	1.64	70.2
4.3e-07	5.8e-05	2.4e-05	0.00089	5.03	4	4.1	15	17	12.5	2.66	15.1	87
1.3e-08	3.6e-06	1.4e-06	5.4e-05	5.02	4	4.08	15	45	112	21.8	134	90.2
0.00035	0.0074	0.0077	0.51	-	-	-	14	11	0.11	0.14	0.25	-
4.6e-06	0.00016	0.00014	0.0082	6.23	5.52	5.7	16	32	1.33	1.11	2.4	82.9
6.3e-08	3.8e-06	3.2e-06	0.00016	6.2	5.4	5.51	17	59	13.9	8.86	22.8	85.5
9.1e-10	9.8e-08	8.1e-08	3.5e-06	6.1	5.3	5.33	16	193	202	71.1	273	66.8

Conclusions

1. *p*-MG FGMRES performance is satisfactory for HHO discretizations of Poisson problems. Single grid solvers (ILU CG) are outperformed.

2. *p*-MG performance is satisfactory for HHO discretizations of Stokes problems on isotropic meshes.
   p-MG FGMRES is a valuable alternative to LU.

Future works

1. Investigate the convergence degradation on anisotropic meshes.

2. Evaluate other preconditioners.