A unified formulation and analysis of HHO and VE methods

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Setting

- toy problem: $-\triangle u = f$ in $\Omega \subset \mathbb{R}^2$, $u = 0$ on $\partial\Omega$
- polygonal mesh $\mathcal{T}_h$ of Ω fulfilling classical admissibility requirements (no small edge in particular)
- focus on c/nc-VE and HHO methods of arbitrary order $k \geqslant 1$
- skeletal methods: cell DOF can be locally eliminated in terms of skeletal DOF
- VE methods are written in terms of (virtual) functions
- HHO methods are written in terms of DOF
- both paradigms are close: nc-VE and HHO are actually equivalent (up to equivalent cell polynomial degree, choice of stabilization, treatment of the RHS) [Cockburn, Di Pietro, Ern, 16], [Di Pietro, Droniou, Manzini, 18]

Aim of the talk

- there is a difference between VE and HHO when it comes to the analysis
- in standard analyses of VE, the approximation properties of the virtual space appear explicitly in the bound of the scheme error
- this is not the case for HHO
- the aim of this talk is (1) to understand why. . .
- and (2) to propose an alternative analysis of c-VE in broken H^1-seminorm, based on a rewriting of c-VE in terms of DOF (in the vein of HHO), that eludes this virtual contribution. . .
- thus leading to a (3) unified analysis of VE/HHO methods
- we build upon existing works, in particular [Cangiani, Manzini, Sutton, 17] and [Di Pietro, Droniou, 18]

Main notation

- T denotes a generic element of the polygonal mesh $\mathcal{T}_h$
- $\mathcal{F}_T$ denotes the set of edges of T
- $\mathcal{V}_T$ denotes the set of vertices of T

- $\mathbb{P}^l_X$ denotes the space of polynomials of total degree $\leqslant l$ on X
- π^l_X denotes the L^2-orthogonal projector onto $\mathbb{P}^l_X$
- Π^l_X denotes the elliptic projector onto $\mathbb{P}^l_X$

- $\mathbb{P}^l_{\mathcal{F}_T}$ denotes the space of functions v on ∂T s.t. $v_{|F} \in \mathbb{P}^l_F$ for all $F \in \mathcal{F}_T$
- $\mathbb{P}^{l,c}_{\mathcal{F}_T} := \mathbb{P}^l_{\mathcal{F}_T} \cap C^0(\partial T)$

- $H^{1,c}(T) := H^1(T) \cap C^0(\overline{T})$

Outline

Non-conforming case: the HHO viewpoint

Local ingredients in each cell T of the mesh:

- space of DOF: $\underline{V}_T^k := \mathbb{P}_T^{k-1} \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_F^{k-1} \right)$
- polynomial projector: $p_T^k : \underline{V}_T^k \to \mathbb{P}_T^k$ s.t.

$$\begin{cases} \int_T \nabla p_T^k \underline{v}_T \cdot \nabla \theta = -\int_T v_T \Delta\theta + \sum_{F \in \mathcal{F}_T} \int_F v_F \nabla\theta \cdot \boldsymbol{n}_{T,F} & \forall \theta \in \mathbb{P}_T^k \\ \int_T p_T^k \underline{v}_T = \int_T v_T \end{cases}$$

Local bilinear/linear forms on $\underline{V}_T^k \times \underline{V}_T^k / \underline{V}_T^k$:

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla p_T^k \underline{u}_T \cdot \nabla p_T^k \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T), \qquad l_T(\underline{v}_T) := \int_T f v_T$$

The global space of DOF $\underline{V}_{h,0}^k$ is obtained by gluing together the skeletal DOF between adjacent elements (and zeroing out the boundary DOF).

The global bilinear/linear forms a_h / l_h are obtained by summing the local contributions.

The problem reads: find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t. $a_h(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h)$ for all $\underline{v}_h \in \underline{V}_{h,0}^k$.

Non-conforming case: the equivalent nc-VE viewpoint

- local virtual space: $V_T^k := \left\{ v \in H^1(T) \mid \triangle v \in \mathbb{P}_T^{k-1},\ \nabla v \cdot \boldsymbol{n}_T \in \mathbb{P}_{\mathcal{F}_T}^{k-1} \right\}$
- reduction: $\underline{\Sigma}_T^k : V_T^k \to \underline{\mathrm{V}}_T^k$ s.t. $\underline{\Sigma}_T^k v := \left(\pi_T^{k-1} v, \left(\pi_F^{k-1} v\right)_{F \in \mathcal{F}_T} \right)$
- $\underline{\Sigma}_T^k$ is a bijection
- there holds $p_T^k \circ \underline{\Sigma}_T^k = \Pi_T^k$
- equivalent local bilinear form on $V_T^k \times V_T^k$: $a_T(u,v) := \mathrm{a}_T\left(\underline{\Sigma}_T^k u, \underline{\Sigma}_T^k v\right)$
- $a_T(u,v) = \int_T \nabla \Pi_T^k u \cdot \nabla \Pi_T^k v + s_T(u,v)$ with $s_T(u,v) := \mathrm{s}_T\left(\underline{\Sigma}_T^k u, \underline{\Sigma}_T^k v\right)$
- equivalent local linear form on V_T^k: $l_T(v) := \mathrm{l}_T\left(\underline{\Sigma}_T^k v\right) = \int_T f \pi_T^{k-1} v$
- global virtual space: $V_{h,0}^k := \left\{ v_h \in V_{\mathcal{T}_h}^k,\ \pi_F^{k-1}\left([\![v_h]\!]_F\right) \equiv 0\ \forall F \in \mathcal{F}_h \right\}$
- global forms a_h / l_h obtained by sum of local ones
- problem: find $u_h \in V_{h,0}^k$ s.t. $a_h(u_h, v_h) = l_h(v_h)$ for all $v_h \in V_{h,0}^k$
- there holds $\underline{\mathrm{u}}_h = \underline{\Sigma}_h^k u_h$

Conforming case: a DOF-based viewpoint (1/2)

Local ingredients in each cell T of the mesh:

- locally to each edge $F := [\boldsymbol{x}_{\nu_1}, \boldsymbol{x}_{\nu_2}] \in \mathcal{F}_T$
  - space of edge DOF: $\underline{\mathrm{V}}_F^k := \mathbb{P}_F^{k-2} \times \mathbb{R}^2$
  - reconstruction operator: $r_F^k : \underline{\mathrm{V}}_F^k \to \mathbb{P}_F^k$ s.t.

$$\begin{cases} \int_F (r_F^k \underline{\mathrm{v}}_F)' \zeta' = -\int_F \mathrm{v}_F \zeta'' + [\mathrm{v}_{\nu_2} \zeta'(\boldsymbol{x}_{\nu_2}) - \mathrm{v}_{\nu_1} \zeta'(\boldsymbol{x}_{\nu_1})] & \forall \zeta \in \mathbb{P}_F^k \\ r_F^k \underline{\mathrm{v}}_F(\boldsymbol{x}_{\nu_1}) = \mathrm{v}_{\nu_1} & \end{cases}$$

- space of DOF: $\underline{\mathrm{V}}_T^k := \mathbb{P}_T^{k-1} \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_F^{k-2} \times \mathbb{R}^{\mathrm{card}(\mathcal{V}_T)} \right)$
- polynomial projector: $p_T^k : \underline{\mathrm{V}}_T^k \to \mathbb{P}_T^k$ s.t.

$$\begin{cases} \int_T \boldsymbol{\nabla} p_T^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} \theta = -\int_T \mathrm{v}_T \triangle \theta + \sum_{F \in \mathcal{F}_T} \int_F r_F^k \underline{\mathrm{v}}_F \boldsymbol{\nabla} \theta \cdot \boldsymbol{n}_{T,F} & \forall \theta \in \mathbb{P}_T^k \\ \int_T p_T^k \underline{\mathrm{v}}_T = \int_T \mathrm{v}_T & \end{cases}$$

Conforming case: a DOF-based viewpoint (2/2)

Local bilinear/linear forms on $\underline{V}_T^k \times \underline{V}_T^k / \underline{V}_T^k$:

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla p_T^k \underline{u}_T \cdot \nabla p_T^k \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T), \qquad l_T(\underline{v}_T) := \int_T f v_T$$

The global space of DOF $\underline{V}_{h,0}^k$ is obtained by gluing together the skeletal DOF between adjacent elements (and zeroing out the boundary DOF).

The global bilinear/linear forms a_h / l_h are obtained by summing the local contributions.

The problem reads: find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t. $a_h(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h)$ for all $\underline{v}_h \in \underline{V}_{h,0}^k$.

Conforming case: the equivalent c-VE viewpoint

- local virtual space: $V_T^k := \left\{ v \in H^1(T) \mid \triangle v \in \mathbb{P}_T^{k-1},\ v_{|\partial T} \in \mathbb{P}_{\mathcal{F}_T}^{k,c} \right\}$
- reduction: $\underline{\Sigma}_T^k : V_T^k \to \underline{V}_T^k$ s.t. $\underline{\Sigma}_T^k v := \left(\pi_T^{k-1} v, \left(\pi_F^{k-2} v\right)_{F \in \mathcal{F}_T}, \left(v(\boldsymbol{x}_\nu)\right)_{\nu \in \mathcal{V}_T} \right)$
- $\underline{\Sigma}_T^k$ is a bijection
- there holds $p_T^k \circ \underline{\Sigma}_T^k = \Pi_T^k$
- equivalent local bilinear form on $V_T^k \times V_T^k$: $a_T(u, v) := \mathrm{a}_T\left(\underline{\Sigma}_T^k u, \underline{\Sigma}_T^k v\right)$
- $a_T(u, v) = \int_T \boldsymbol{\nabla} \Pi_T^k u \cdot \boldsymbol{\nabla} \Pi_T^k v + s_T(u, v)$ with $s_T(u, v) := \mathrm{s}_T\left(\underline{\Sigma}_T^k u, \underline{\Sigma}_T^k v\right)$
- equivalent local linear form on V_T^k: $l_T(v) := \mathrm{l}_T\left(\underline{\Sigma}_T^k v\right) = \int_T f \pi_T^{k-1} v$
- global virtual space: $V_{h,0}^k := \left\{ v_h \in V_{\mathcal{T}_h}^k \cap C^0(\overline{\Omega}),\ v_{h|\partial\Omega} \equiv 0 \right\} \subset H_0^1(\Omega)$
- global forms a_h / l_h obtained by sum of local ones
- problem: find $u_h \in V_{h,0}^k$ s.t. $a_h(u_h, v_h) = l_h(v_h)$ for all $v_h \in V_{h,0}^k$
- there holds $\underline{\mathrm{u}}_h = \underline{\Sigma}_h^k u_h$

Outline

Non-conforming case

- we extend $\underline{\Sigma}_T^k$ to $H^1(T)$
- we remark that $p_T^k \circ \underline{\Sigma}_T^k : H^1(T) \to \mathbb{P}_T^k$ is still equal to Π_T^k
- we lead the analysis by writing that

$$\|\nabla_h(u - p_h^k \underline{u}_h)\|_{0,\Omega} \leqslant \|\nabla_h(u - \Pi_h^k u)\|_{0,\Omega} + \|\nabla_h p_h^k(\underline{\Sigma}_h^k u - \underline{u}_h)\|_{0,\Omega}$$

- the first term in the RHS is handled using the H^1 approximation properties of Π_h^k
- the second term is such that

$$\|\nabla_h p_h^k(\underline{\Sigma}_h^k u - \underline{u}_h)\|_{0,\Omega} \leqslant \max_{\underline{v}_h \in \underline{V}_{h,0}^k,\ |\underline{v}_h|_{a,h}=1} \left[a_h(\underline{\Sigma}_h^k u, \underline{v}_h) - l_h(\underline{v}_h) \right]$$

- it is bounded by the consistency error of the scheme, and can be estimated using the H^s approximation properties of Π_h^k
- the analysis can be led without explicit reference to the virtual space

Conforming case (1/3)

- we extend $\underline{\Sigma}_T^k$ to $H^{1,c}(T)$
- in that case, $\mathcal{P}_T^k := p_T^k \circ \underline{\Sigma}_T^k : H^{1,c}(T) \to \mathbb{P}_T^k$ is not equal to Π_T^k
- actually, $\mathcal{P}_T^k = \Pi_T^k \circ \mathcal{I}_T^k$, where $\mathcal{I}_T^k : H^{1,c}(T) \to V_T^k$ is the canonical interpolator on the virtual space
- in standard analyses, one splits the error as

$$\begin{aligned}\|\boldsymbol{\nabla}_h(u - p_h^k \underline{\mathrm{u}}_h)\|_{0,\Omega} &\leqslant \|\boldsymbol{\nabla}_h(u - \Pi_h^k u)\|_{0,\Omega} + \|\boldsymbol{\nabla}_h \Pi_h^k(u - \mathcal{I}_h^k u)\|_{0,\Omega} + \|\boldsymbol{\nabla}_h p_h^k(\underline{\Sigma}_h^k u - \underline{\mathrm{u}}_h)\|_{0,\Omega} \\ &\leqslant \|\boldsymbol{\nabla}_h(u - \Pi_h^k u)\|_{0,\Omega} + \|\boldsymbol{\nabla}_h(u - \mathcal{I}_h^k u)\|_{0,\Omega} + \|\boldsymbol{\nabla}_h p_h^k(\underline{\Sigma}_h^k u - \underline{\mathrm{u}}_h)\|_{0,\Omega}\end{aligned}$$

- such a splitting makes the virtual space not that virtual...
- and requires the study of the approximation properties of $\mathcal{I}_h^k$
- in particular, one has to construct a bounded lifting of the traces of virtual functions, which is non-trivial on elements that are not star-shaped (case not covered in standard analyses)
- let us proceed differently and directly consider $\mathcal{P}_h^k$

Conforming case (2/3)

- for any edge $F \in \mathcal{F}_T$, let $\mathcal{I}_F^k := r_F^k \circ \underline{\Sigma}_F^k : C^0(F) \to \mathbb{P}_F^k$
- for any $t \in C^0(F)$, there holds $(\mathcal{I}_F^k t)' = (\Pi_F^k t)'$ and $\mathcal{I}_F^k t(\boldsymbol{x}_{\nu_1}) = t(\boldsymbol{x}_{\nu_1})$
- hence, $\|\mathcal{I}_F^k t\|_{\infty,F} \lesssim \|t\|_{\infty,F}$
- also, $\mathcal{I}_F^k p = p$ for any $p \in \mathbb{P}_F^k$
- there holds, for any $z \in H^{1,c}(T)$,

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla}\mathcal{P}_T^k z \cdot \boldsymbol{\nabla}\theta = -\int_T z \triangle \theta + \sum_{F \in \mathcal{F}_T} \int_F \mathcal{I}_F^k(z_{|F}) \boldsymbol{\nabla}\theta \cdot \boldsymbol{n}_{T,F} & \forall \theta \in \mathbb{P}_T^k \\ \displaystyle\int_T \mathcal{P}_T^k z = \int_T z \end{cases}$$

- from this expression, one can easily prove that, for any $z \in H^2(T)$,

$$\|\mathcal{P}_T^k z\|_{0,T} \lesssim \|z\|_{0,T} + h_T |z|_{1,T} + h_T^2 |z|_{2,T}$$

- combined to the fact that $\mathcal{P}_T^k$ preserves polynomials, this yields H^s approximation properties for $\mathcal{P}_T^k$

Conforming case (3/3)

- with the introduction of $\mathcal{P}_T^k$ and the study of its approximation properties, we can lead the error analysis just as in the non-conforming case:

$$\|\boldsymbol{\nabla}_h(u - p_h^k \underline{\mathrm{u}}_h)\|_{0,\Omega} \leqslant \|\boldsymbol{\nabla}_h(u - \mathcal{P}_h^k u)\|_{0,\Omega} + \|\boldsymbol{\nabla}_h p_h^k(\underline{\Sigma}_h^k u - \underline{\mathrm{u}}_h)\|_{0,\Omega}$$

- the second term in the RHS is here again bounded by the consistency error of the scheme (not that even in the conforming case, the output of the scheme is a nonconforming function), that can be estimated using the H^s approximation properties of $\mathcal{P}_h^k$
- last question: why that in the non-conforming case $\mathcal{P}_T^k = \Pi_T^k$? This is because $\mathcal{I}_T^k = \Pi_V$ with Π_V the elliptic projector onto V_T^k in that case!

Comments and perspectives

- reference for this talk: [SL, preprint hal-01902962]
- no obstruction to the extension to 3D VE
- unified L^2-norm error analysis?
- what about enhanced VE, or serendipity VE?

THANK YOU FOR YOUR ATTENTION
(ESPECIALLY A 1^{st} OF MAY)