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MATHEMATICAL JUSTIFICATION OF THE HYDROSTATIC APPROXIMATION IN THE PRIMITIVE EQUATIONS OF GEOPHYSICAL FLUID DYNAMICS*

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Abstract. Geophysical fluids all exhibit a common feature: their aspect ratio (depth to horizontal width) is very small. This leads to an asymptotic model widely used in meteorology, oceanography, and limnology, namely the hydrostatic approximation of the time-dependent incompressible Navier–Stokes equations. It relies on the hypothesis that pressure increases linearly in the vertical direction. In the following, we prove a convergence and existence theorem for this model by means of anisotropic estimates and a new time-compactness criterium.

Key words. Navier–Stokes equations, shallow domains, geophysical fluid dynamics, hydrostatic approximation, singular perturbation, compactness criterium, asymptotic analysis

AMS subject classifications. 35Q30, 35B40, 76D05, 34C35

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1. Introduction.

Atmospheric flow in meteorology, water flow in oceanography, and limnology are all described by the Navier–Stokes equations. Due to the fact that the aspect ratio

$$\epsilon = \frac{\text{characteristic depth}}{\text{characteristic width}}$$

is very small in most geophysical domains, asymptotic models have been used; see, e.g., [9, 15, 22]. One such model is the primitive equations model; see, e.g., [11, 12], wherein the unknown flow variables are velocity, pressure, temperature, and salinity (in the case of an ocean). Besides, most geophysical fluids are stratified (i.e., density is a known function of the temperature (and salinity, if any)) and have a free surface. We shall not investigate these features in this paper, leaving it, rather, for forthcoming work.

Instead we shall focus on the assumption that the pressure is hydrostatic, i.e., increases linearly with respect to the depth, as in the static case. This law agrees well with experiment (as first observed by Blaise Pascal around 1650; see [14])) and is frequently taken as a hypothesis in geophysical fluid dynamics. We justify this assumption by means of asymptotic analysis (taking ϵ as the small parameter). Our derivation is made possible by the use of anisotropic eddy viscosities, namely $\nu = (\nu_x, \nu_y, \nu_z)$, relying on the fact that the ratio between the horizontal and vertical scales leads to very different sizes for the horizontal and vertical eddies (see [9, 15]). Specifically, if we assume that $\nu = (\nu_1, \nu_2, \epsilon^2 \nu_3)$ with $\nu_i = \mathrm{O}(1)$ for $i = 1, 2, 3$, then we will see that weak solutions of the Navier–Stokes equations converge to a weak solution of a limit problem with hydrostatic pressure.

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The stationary case has already been studied (see [4] for the linear problem and [5] for the nonlinear one), whereas the linear time-dependent case was solved in [1]. The main task of this paper is then to solve the nonlinear time-dependent case. Our result was announced in [2], whereas numerical simulations stemming from it were discussed in [3].

Fluid flow in thin domains (flat, curved, and with various boundary conditions) has been extensively studied; see [7, 13, 16, 20, 21]. In these works, an isotropic viscosity is used, and the depth is constant. By averaging along the vertical direction, two-dimensional (2D) limit models are obtained, together with existence and global regularity results.

Our approach is different, because we neither eliminate the vertical velocity by averaging nor assume the depth of the domain to be constant. By making use of different horizontal and vertical eddy viscosities, we are able to derive a three-dimensional (3D) limit nonlinear model. Let us emphasize that the anisotropic viscosity hypothesis is fundamental for the derivation of the primitive equations: in the stationary case, keeping an isotropic viscosity, the asymptotic model is linear, with vanishing horizontal diffusion; see [6].

The paper is organized as follows. In section 2, we present the physical model and the scaling leading to the primitive equations. We state the main theorem in section 3. The functional setting and weak formulation are described in section 4. In the next section, we state and prove a time-compactness result, which we shall use in the proof of the main theorem in section 6. Finally, in section 7, we comment on the convergence of the pressure and the orders of magnitude of the vertical velocity with respect to the aspect ratio.

2. Equations governing the flow and scaling. Let us consider an incompressible homogeneous fluid filling a thin domain defined by

$$\Omega_\epsilon = \left\{(x,y,z) \in \mathbb{R}^3;\ (x,y) \in \omega, -\epsilon\, h(x,y) < z < 0\right\},$$

where ω is an open bounded Lipschitz domain in $\mathbb{R}^2$ and $h : \overline{\omega} \to \mathbb{R}$ is a nonnegative lipschitzian application, which is arbitrary provided that Ω_ϵ is lipschitzian. In particular, h may vanish, contrary to [12, 9], but in order that the domain Ω_ϵ has no cusps, the slope must not vanish on the shores.[1] We denote by $\Gamma_s = \overline{\omega} \times \{0\}$ the fluid surface and by $\Gamma_b^\epsilon = \partial\Omega_\epsilon \setminus \Gamma_s$ the basin bottom. The fluid flow in Ω_ϵ is generated by the wind traction on the surface Γ_s, influenced by the Coriolis and centrifugal forces and governed by the Navier–Stokes equations, in which we take different eddy viscosities according to the direction; see [5, 9, 15]. Finally, we take the density as identically equal to one. In a geophysical rotating frame (z pointing upwards, x east, and y north), the initial-boundary value problem reads as follows.

Find $\mathbf{v} = (v_1, v_2, v_3)$ (velocity) and q (pressure), such that

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \Delta_\nu \mathbf{v} + \nabla q + 2\mathbf{w}\times\mathbf{v} = \mathbf{g} \quad \text{in } \Omega_\epsilon \times (0,\mathrm{T}), \tag{2.1}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega_\epsilon \times (0,T), \tag{2.2}$$

$$\mathbf{v} = 0 \quad \text{on } \Gamma_b^\epsilon \times (0,T), \tag{2.3}$$

$$\nu_z \partial_z v_1 = \tau_1, \quad \nu_z \partial_z v_2 = \tau_2, \quad v_3 = 0 \quad \text{on } \Gamma_s \times (0,T), \tag{2.4}$$

$$\mathbf{v}(\cdot, t=0) = \mathbf{v}_0 \quad \text{in } \Omega_\epsilon. \tag{2.5}$$

[1] This is a technical hypothesis. One could probably dispense with it due to the specific shape of the domain.

In (2.1), $\nabla = (\partial_x, \partial_y, \partial_z)$ denotes the gradient vector, and Δ_ν denotes the anisotropic Laplacian defined by $\Delta_\nu = \nu_x \partial^2_{xx} + \nu_y \partial^2_{yy} + \nu_z \partial^2_{zz}$ with $\nu = (\nu_x, \nu_y, \nu_z)$ the eddy kinematic viscosity vector. Moreover, $\mathbf{w} = f\,(0, \cos(l(y)), \sin(l(y)))$ represents the earth rotation angular speed (f the module and $l(y)$ the latitude), $2\mathbf{w} \times \mathbf{v}$ represents the Coriolis acceleration ($\times$ denotes the cross-product in $\mathbb{R}^3$), and $\mathbf{g}$ represents the force due to gravity (which also includes the centrifugal effect). It is well known (cf. [15, p. 18]) that $\mathbf{g}$ is a potential, i.e., $\mathbf{g} = \nabla\varphi$. It is customary to incorporate the gravity potential in the pressure term; thus we set

$$p = q - \varphi.$$

Equation (2.2) represents the incompressibility condition, and (2.3) represents the no-slip condition on the bottom.

In (2.4), τ_i, $i = 1, 2$, stand for the horizontal tractions exerted by the wind on the (fixed) surface Γ_s of the fluid, and $w = 0$ on Γ_s comes from the rigid lid hypothesis. In (2.5), $\mathbf{v}_0 = (v_{01}, v_{02}, v_{03})$ designates the initial velocity.

Remark. We have neglected the earth's curvature, and hence our analysis is valid only locally, e.g., for lakes; for seas or oceans, spherical coordinates should be used [12], although this can be somewhat cumbersome.

As usual in asymptotic analysis, we perform a vertical scaling to make the domain independent of ϵ, that is,

$$x = x_1, \quad y = x_2, \quad z = \epsilon\, x_3,$$

so that $\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3;\ (x_1, x_2) \in \omega,\ -h(x_1, x_2) < x_3 < 0\right\}$ is the new fixed domain.

The corresponding kinematic scaling is

$$v_1 = u_1^\epsilon, \quad v_2 = u_2^\epsilon, \quad v_3 = \epsilon\, u_3^\epsilon, \quad p = p^\epsilon, \tag{2.6}$$

so that $\mathbf{u}^\epsilon = (u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)$ is the new unknown velocity and p^ϵ is the new pressure.

It is necessary to scale the mechanical quantities accordingly. First, it is only natural to assume $v_{01} = u_{01}, v_{02} = u_{02}$, and $v_{03} = \epsilon u_{03}$, where u_{0i} does not depend on ϵ, $i = 1, 2, 3$. Next we assume $\nu_x = \nu_1$, $\nu_y = \nu_2$, and $\nu_z = \epsilon^2 \cdot \nu_3$, where ν_1, ν_2, ν_3 are constants. As mentioned in the introduction, in oceanography the vertical eddy viscosity is usually very small compared to the horizontal one. We refer to [5] for a mathematical discussion of this assumption, and here we content ourselves with one heuristic comment. Basically, a kinematic viscosity has the dimension L^2/T, where L (resp., T) is a typical length (resp., time) scale so that ν_x and ν_y have the dimension L_H^2/T, whereas ν_z has the dimension L_V^2/T, where L_H (resp., L_V) denotes a typical horizontal (resp., vertical) length scale. It follows that the ratio ν_z/ν_x and $\nu_z/\nu_y = \mathrm{O}(\epsilon^2)$.[2]

Now (2.4) becomes

$$\nu_3 \partial_3 u_i^\epsilon = \tau_i^\epsilon/\epsilon, \quad i = 1, 2.$$

We see that in order to end up with an O(1)-wind force on the rescaled domain, we have to assume that $\tau_i^\epsilon = \epsilon \cdot \theta_i$, $i = 1, 2$, where the θ_i are functions independent of ϵ.

[2]We do not delude ourselves with this sketchy argument. As far as we know, up to now there has been no rigorous derivation of any eddy viscosity model.

Remark. This last assumption can also be motivated by dimensional analysis, as follows. From $\tau_i = \nu_z \partial_z v_i$, one derives that τ_i has the dimension of

$$\frac{L_V^2}{T} \cdot \frac{1}{L_V} \cdot \frac{L_H}{T} = \epsilon \frac{L_H^2}{T^2} = O(\epsilon).$$

With the above considerations, problem (2.1)–(2.5) transforms into the following anisotropic Navier–Stokes equations:

$$\partial_t u_1^\epsilon + \mathbf{u}^\epsilon \cdot \nabla u_1^\epsilon - \Delta_\nu u_1^\epsilon - \alpha\, u_2^\epsilon + \epsilon\, \beta\, u_3^\epsilon + \partial_1 p^\epsilon = 0 \quad \text{in } \Omega \times (0,T), \tag{2.7}$$

$$\partial_t u_2^\epsilon + \mathbf{u}^\epsilon \cdot \nabla u_2^\epsilon - \Delta_\nu u_2^\epsilon + \alpha\, u_1^\epsilon + \partial_2 p^\epsilon = 0 \quad \text{in } \Omega \times (0,T), \tag{2.8}$$

$$\epsilon^2 \left\{ \partial_t u_3^\epsilon + \mathbf{u}^\epsilon \cdot \nabla u_3^\epsilon - \Delta_\nu u_3^\epsilon \right\} - \epsilon\, \beta\, u_1^\epsilon + \partial_3 p^\epsilon = 0 \quad \text{in } \Omega \times (0,T), \tag{2.9}$$

$$\operatorname{div} \mathbf{u}^\epsilon = 0 \quad \text{in } \Omega \times (0,T), \tag{2.10}$$

$$\mathbf{u}^\epsilon = 0 \quad \text{on } \Gamma_b \times (0,T), \tag{2.11}$$

$$\nu_3 \partial_3 u_1^\epsilon = \theta_1, \quad \nu_3 \partial_3 u_2^\epsilon = \theta_2, \quad u_3^\epsilon = 0 \quad \text{on } \Gamma_s \times (0,T), \tag{2.12}$$

$$\mathbf{u}^\epsilon(\cdot, t=0) = \mathbf{u}_0 \quad \text{in } \Omega. \tag{2.13}$$

Now $\nabla = (\partial_1, \partial_2, \partial_3)$, $\Delta_\nu = \nu_1 \partial_{11}^2 + \nu_2 \partial_{22}^2 + \nu_3 \partial_{33}^2$, $\Gamma_b = \partial\Omega \setminus \Gamma_s$, $\alpha = 2f \sin(l(x_2))$, and $\beta = 2f \cos(l(x_2))$.

If we assume that $\mathbf{u}^\epsilon = O(1)$, then neglecting the ϵ^2 and ϵ terms in the first and third momentum equation, (2.7) and (2.9), we formally get the hydrostatic Navier–Stokes equations, also called the primitive equations:

$$\partial_t u_1 + \mathbf{u} \cdot \nabla u_1 - \Delta_\nu u_1 - \alpha\, u_2 + \partial_1 p = 0 \quad \text{in } \Omega \times (0,\mathrm{T}), \tag{2.14}$$

$$\partial_t u_2 + \mathbf{u} \cdot \nabla u_2 - \Delta_\nu u_2 + \alpha\, u_1 + \partial_2 p = 0 \quad \text{in } \Omega \times (0,\mathrm{T}), \tag{2.15}$$

$$\partial_3 p = 0 \quad \text{in } \Omega \times (0,\mathrm{T}), \tag{2.16}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times (0,T), \tag{2.17}$$

$$u_1 = u_2 = u_3 n_3 = 0 \quad \text{on } \Gamma_b \times (0,T), \tag{2.18}$$

$$\nu_3 \partial u_1 = \theta_1, \quad \nu_3 \partial_3 u_2 = \theta_2, \quad u_3 = 0 \quad \text{on } \Gamma_s \times (0,T), \tag{2.19}$$

$$u_i(\cdot, t=0) = u_{0i} \quad \text{in } \Omega,\ i = 1,2. \tag{2.20}$$

Remark. The boundary condition (2.18) differs from its counterpart (2.11) because u_3 is less regular than u_1, u_2 as we shall see below. Also, the initial condition (2.20) does not involve u_3, the time derivative of which is missing in the hydrostatic model. The problem is not in the Cauchy–Kowalevska form.[3]

Remark. If u_3 were to be computed directly from (2.17), which is a *first* order equation, it is not obvious at all that it would fulfill the *two* boundary conditions on the bottom (2.18) and the surface (2.19).

3. Main theorem. Let T be a fixed positive duration. We make the natural assumption of a wind of finite energy: θ_1, $\theta_2 \in L^2(0,T; H^{-1/2}(\Gamma_s))$. Our main result is the following theorem.

THEOREM 3.1. *Let* $\mathbf{u}_0 \in L^2(\Omega)^3$, *with* $\operatorname{div} \mathbf{u}_0 = 0$, $\mathbf{u}_0 \cdot \mathbf{n} = 0$ *on* $\partial\Omega$, *and* $\theta_1, \theta_2 \in L^2(0,T; H^{-1/2}(\Gamma_s))$; *there exists a weak solution* $\mathbf{u}$ *of the hydrostatic Navier–Stokes equations* (2.14)–(2.20), *obtained as a limit of weak solutions* $\mathbf{u}^\epsilon$ *of the anisotropic Navier–Stokes equations* (2.7)–(2.13), *as the aspect ratio* ϵ *tends to zero.*

[3] Meteorologists say that u_3 is no longer a prognostic variable (see [11, 12]).

The proof relies on a priori estimates in anisotropic spaces (Propositions 6.1 and 6.2), which are sufficient to take the limit in the linear terms (see [1]), whereas for the nonlinear terms, we establish a new time-compactness criterium (Theorem 5.1), which enables us to get strong convergence of the horizontal velocities; see Lemma 6.3. This theorem states essentially that a small perturbation of an L^p-equicontinuous family still possesses a strong convergent subsequence. Let us emphasize that this seemingly technical refinement is by no means superfluous. Indeed, the usual compactness estimate fails: as $(u_1^\epsilon, u_2^\epsilon, \epsilon^2 u_3^\epsilon)$ is not divergence free, even if it is easy from (2.7)–(2.9) to control $\partial_t(u_1^\epsilon, u_2^\epsilon, \epsilon^2 u_3^\epsilon)$ in some dual space of divergence free velocities, it is not possible to apply the Aubin–Lions lemma to get compactness.

Another major difficulty of the proof is the lack of regularity of the vertical velocity, which is determined only by the incompressibility equation (2.10).

Remark. It is possible to handle a general force (f_1, f_2, f_3) in problem (2.14)–(2.20), by simply adding $\mathbf{f} = (f_1, f_2, \frac{f_3}{\epsilon})$ to (2.1), in order to end up with (f_1, f_2, f_3) in (2.7)–(2.9).

4. Weak formulation and anisotropic spaces. We need the following Hilbert spaces:

$$H_b^1(\Omega) = \overline{C_b^\infty(\Omega)}^{H^1(\Omega)} = \left\{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_b\right\}$$

(where $C_b^\infty(\Omega) = \left\{\varphi \in C^\infty(\bar{\Omega});\ \varphi = 0 \text{ in some neighborhood of } \Gamma_b\right\}$),

$$\mathbf{V} = \left\{\mathbf{v} \in H_b^1(\Omega) \times H_b^1(\Omega) \times H_0^1(\Omega);\ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\right\},$$

$$H(\partial_3, \Omega) = \left\{v \in L^2(\Omega);\ \partial_3 v \in L^2(\Omega)\right\}$$

(endowed with the norm $\|v\|^2_{H(\partial_3,\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\partial_3 v\|^2_{L^2(\Omega)}$),

$$H_0(\partial_3, \Omega) = \overline{C_0^\infty(\Omega)}^{H(\partial_3,\Omega)} = \{v \in H(\partial_3, \Omega);\ v\, n_3 = 0 \ \text{ on } \partial\Omega\}$$

(n_3 is the third component of the normal exterior vector on $\partial\Omega$, and $v\, n_3$ is understood in the $H^{-1/2}(\partial\Omega)$ sense (see [19] for these spaces)),

$$\mathbf{W} = \left\{\mathbf{u} \in H_b^1(\Omega) \times H_b^1(\Omega) \times H_0(\partial_3, \Omega);\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega\right\}.$$

Let us denote that $u_H = (u_1, u_2)$, $\theta_H = (\theta_1, \theta_2)$, $b(u_H) = \alpha\,(-u_2, u_1)$, and $\nabla_\nu = (\nu_1^{1/2}\partial_1, \nu_2^{1/2}\partial_2, \nu_3^{1/2}\partial_3)$. The scalar product in $L^2(\Omega)^d$, or the duality $L^p(\Omega), L^{p'}(\Omega)$, is denoted by $(\cdot,\cdot)$, and the duality $H^{-1/2}(\Gamma_s)H^{1/2}(\Gamma_s)$, is denoted by $\langle\cdot,\cdot\rangle_{\Gamma_s}$.

The weak form of the hydrostatic Navier–Stokes equations (2.14)–(2.20) is then as follows.

Find $\mathbf{u} = (u_H, u_3) \in L^2(0,T;\mathbf{W})$, with $u_H \in L^\infty(0,T;L^2(\Omega)^2)$, such that

$$\begin{aligned}\int_0^T &-(u_H, \partial_t v_H) - (u_H, (\mathbf{u}\cdot\nabla)v_H) + (b(u_H), v_H) + (\nabla_\nu u_H, \nabla_\nu v_H)\\ &= -(u_{0H}, v_H(0)) + \int_0^T \langle\theta_H, v_H\rangle_{\Gamma_s}\end{aligned} \tag{4.1}$$

for all $\mathbf{v} = (v_H, v_3) \in H^1(0,T;\mathbf{W})$, with $v_H(T) = 0$ and $\partial_3 v_H \in L^\infty(0,T;L^3(\Omega)^2)$.

Remark. Notice that a weak solution of the Navier–Stokes equations verifies the following regularity:

$$\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;L^2(\Omega)^3)$$

(cf. [8, 10, 18]). Now the lack of regularity of u_3 makes it necessary to change $\mathbf{V}$ to $\mathbf{W}$. Moreover, in general, $u_3 \notin L^\infty(0,T;L^2(\Omega))$.

Remark. The regularity $L^\infty(0,T;L^3(\Omega)^2)$ is required for $\partial_3 v_H$ to give a meaning to $\int_0^T (u_H, u_3\partial_3 v_H)\,\mathrm{d}t$. The regularity $L^2(0,T;L^\infty(\Omega)^2)$ or any interpolated one $L^{2/a}(0,T;L^{3/(1-a)}(\Omega)^2)$ with $0 \le a \le 1$ can also be considered.

5. Compactness by perturbation. We give a compactness criterium, new to our knowledge, which generalizes the well-known translation criterium of Riesz–Fréchet–Kolmogorov, extended to the vectorial case by Simon [17]. In the following, $\tau_h f(t)$ denotes $f(t+h)$.

THEOREM 5.1. *Let $T > 0$, and let the Banach spaces $\mathbf{X} \overset{compact}{\hookrightarrow} \mathbf{B} \hookrightarrow \mathbf{Y}$. Let $(f_\epsilon)_{\epsilon>0}$ be a family of functions of $L^p(0,T;\mathbf{X})$, $1 \le p \le \infty$, with the extra condition $(f_\epsilon)_{\epsilon>0} \subset \mathcal{C}(0,T;\mathbf{Y})$ if $p = \infty$, such that*

(H1) *$(f_\epsilon)_{\epsilon>0}$ is bounded in $L^p(0,T;\mathbf{X})$,*

(H2) *$\|\tau_h f_\epsilon - f_\epsilon\|_{L^p(0,T-h;\mathbf{Y})} \le \varphi(h) + \psi(\epsilon)$ with*

$$\begin{cases} \lim_{h\to 0} \varphi(h) = 0, \\ \lim_{\epsilon\to 0} \psi(\epsilon) = 0. \end{cases}$$

Then the family $(f_\epsilon)_{\epsilon>0}$ possesses a cluster point in $L^p(0,T;\mathbf{B})$ and also in $\mathcal{C}(0,T;\mathbf{B})$ if $p = \infty$, as $\epsilon \to 0$.

Proof. It is enough to prove that, for every sequence $(\epsilon_n)_n$ such as $\epsilon_n > 0$ and $\epsilon_n \to 0$, the family $(f_{\epsilon_n})_n$ is relatively compact in $L^p(0,T;\mathbf{B})$. We apply Theorem 5 of Simon [17, p. 84] to the sequence $(f_{\epsilon_n})_n$, while observing that hypothesis (H2) implies that

$$\|\tau_h f_{\epsilon_n} - f_{\epsilon_n}\|_{L^p(0,T-h;\mathbf{Y})} \to 0 \quad \text{as } h \to 0$$

uniformly with respect to n. Indeed, (H2) implies that

$$\forall n,\ \|\tau_h f_{\epsilon_n} - f_{\epsilon_n}\|_{L^p(0,T-h;\mathbf{Y})} \le \varphi(h) + \psi(\epsilon_n).$$

Let $\epsilon > 0$ and then $\exists\, N$, such that for all $n \ge N$, $\psi(\epsilon_n) \le \epsilon/2$. On the other hand, $\exists\, \delta > 0$, such that for all $h : 0 \le h < \delta$, $\varphi(h) \le \epsilon/2$. Therefore, we get the estimate

$$\forall\, n \ge N \text{ and } \forall\, h : 0 \le h < \delta, \quad \|\tau_h f_{\epsilon_n} - f_{\epsilon_n}\|_{L^p(0,T-h;\mathbf{Y})} \le \epsilon.$$

In addition, for each $k \le N$, $\exists\, \delta_k > 0$, such that for all $h : 0 \le h < \delta_k$

$$\|\tau_h f_{\epsilon_k} - f_{\epsilon_k}\|_{L^p(0,T-h;\mathbf{Y})} \le \epsilon.$$

This follows from the L^p-continuity by translation of an L^p function for $p < \infty$ and for $p = \infty$; this is precisely a hypothesis.

Defining $\eta = \min\{\delta, \delta_1, \ldots, \delta_N\}$, we obtain the desired uniform estimate

$$\forall\, h : 0 \le h < \eta, \quad \|\tau_h f_{\epsilon_n} - f_{\epsilon_n}\|_{L^p(0,T-h;\mathbf{Y})} \le \epsilon \quad \forall n.$$

The family $(f_{\epsilon_n})_n$ fulfills the hypotheses of Simon's theorem. ∎

6. Proof of the main theorem. For simplicity in the notation, from now on, unless we specify otherwise, we will denote $\mathbf{u} = \mathbf{u}^\epsilon$ as a weak solution of the anisotropic Navier–Stokes equations (2.7)–(2.13).

6.1. Energy estimates. The usual energy inequality (cf. [10]) for the Navier–Stokes equations gives, for a.e. $t \in [0,T]$,

$$\begin{aligned}&\|u_H(t)\|_{L^2}^2 + \epsilon^2 \|u_3(t)\|_{L^2}^2 + \int_0^t \{\|\nabla_\nu u_H(\tau)\|_{L^2}^2 + \epsilon^2 \|\nabla_\nu u_3(\tau)\|_{L^2}^2\}\, d\tau \\ &\quad \le \|u_{0H}\|_{L^2}^2 + \epsilon^2 \|u_{03}\|_{L^2}^2 + \int_0^t \langle \theta_H, u_H\rangle_{\Gamma_s}.\end{aligned}$$

Hence we obtain as in the isotropic Navier–Stokes system (cf. [1]) the following proposition.

PROPOSITION 6.1. *The sequences $u_1, u_2, \epsilon u_3$ are bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.*

For the vertical velocities, we prove the following.

PROPOSITION 6.2. *The sequences u_3 and $\partial_3 u_3$ are bounded in $L^2(0,T;L^2(\Omega))$; i.e., u_3 is bounded in $L^2(0,T;H_0(\partial_3,\Omega))$.*

Proof. As $\operatorname{div}\mathbf{u} = 0$, $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$ is bounded in $L^2(0,T;L^2(\Omega))$. Moreover, the Poincaré inequality in the vertical direction, owing to $u_3 = 0$ on Γ_s, yields

$$\|u_3\|_{L^2} \le h_{\max} \|\partial_3 u_3\|_{L^2}, \quad \text{where } h_{\max} = \max_{\overline{\omega}} h.$$

Therefore, we have proved the proposition. ∎

6.2. Fractional time derivatives in horizontal spaces. First, we define the auxiliary Hilbert spaces

$$B_H = \overline{P_H \mathcal{U}}^{(L^2)^2}, \quad W_H = \overline{P_H \mathcal{U}}^{(H^1)^2}, \quad \text{and} \quad Y_H = \overline{P_H \mathcal{U}}^{(H^2)^2},$$

where

$$\mathcal{U} = \left\{\varphi \in C_b^\infty(\Omega)^2 \times C_0^\infty(\Omega);\ \operatorname{div}\varphi = 0\right\}$$

and P_H is the projection

$$P_H : (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto (x_1, x_2) \in \mathbb{R}^2.$$

Then, from the Sobolev–Rellich embeddings, one deduces easily that

$$Y_H \hookrightarrow W_H \hookrightarrow B_H \equiv B_H' \hookrightarrow W_H' \hookrightarrow Y_H', \tag{6.1}$$

where all are dense and compact embeddings. Here and henceforth, X' denotes the dual space of X.

Now, we have the following lemma.

LEMMA 6.3. *The estimate $\|\tau_h u_H - u_H\|_{L^\infty(0,T-h;Y_H')} \le C(h^{1/4} + \epsilon)$ holds.*

Proof. The spatial weak form of the Navier–Stokes equation (2.7)–(2.13) is

$$\begin{aligned}&\frac{d}{dt}(u_H, v_H) - (u_H, (\mathbf{u}\cdot\nabla)v_H) + (b(u_H), v_H) + (\nabla_\nu u_H, \nabla_\nu v_H) \\ &\quad + \epsilon^2 \left\{\frac{d}{dt}(u_3, v_3) + (\mathbf{u}\cdot\nabla u_3, v_3) + (\nabla_\nu u_3, \nabla_\nu v_3)\right\} \\ &\quad + \epsilon\left\{(\beta u_3, v_1) - (\beta u_1, v_3)\right\} = \langle \theta_H, v_H\rangle_{\Gamma_s} \quad \text{in } \mathcal{D}'(0,T) \\ &\quad \forall\, \mathbf{v} = (v_H, v_3) \in \mathbf{V}.\end{aligned} \tag{6.2}$$

Letting $v_H \in Y_H$, there is a null divergence lifting $\mathbf{v} = (v_H, v_3) \in H^2_b(\Omega)^2 \times H^1_0(\partial_3, \Omega)$ such that

$$\|v_3\|_{H^1} + \|\partial_3 v_3\|_{H^1} \leq C\|v_H\|_{Y_H}. \tag{6.3}$$

Here, the spaces $H^2_b(\Omega)$ and $H^1_0(\partial_3, \Omega)$ are the natural extensions of the spaces $H^1_b(\Omega)$ and $H_0(\partial_3, \Omega)$:

$$H^2_b(\Omega) = \overline{C^\infty_b(\Omega)}^{H^2(\Omega)} = \left\{ v \in H^2(\Omega);\ v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_b \right\},$$

$$H^1(\partial_3, \Omega) = \left\{ v \in H^1(\Omega);\ \partial_3 v \in H^1\Omega) \right\},$$

$$H^1_0(\partial_3, \Omega) = \overline{C^\infty_0(\Omega)}^{H^1(\partial_3, \Omega)} = \left\{ v \in H^1(\partial_3, \Omega);\ v = \partial_3 v = 0 \ \text{ on } \partial\Omega \right\}.$$

Indeed, as $v_H \in Y_H$, there exists a sequence $(\varphi^n_H, \varphi^n_3) \in \mathcal{U}$ such that $\varphi^n_H \to v_H$ in $H^2(\Omega)^2$. Then $\partial_3\varphi^n_3 = -\partial_1\varphi^n_1 - \partial_2\varphi^n_2$ is a Cauchy sequence in $H^1(\Omega)$, and by vertical Poincaré inequality, φ^n_3 is also a Cauchy sequence in $H^1(\Omega)$. Therefore, φ^n_3, being a Cauchy sequence in $H^1(\partial_3, \Omega)$, converges to a function v_3, which provides the desired lifting function. The continuous dependence (6.3) results from the above construction.

Now we take this $\mathbf{v} = (v_H, v_3)$ as a test function in (6.2) and integrate over $(t, t+h)$; i.e.,

$$(\tau_h u_H(t) - u_H(t), v_H) + \epsilon^2 (\tau_h u_3(t) - u_3(t), v_3) = \int_t^{t+h} g^\epsilon(s)\, \mathrm{ds}, \tag{6.4}$$

where

$$\begin{aligned} g^\epsilon(s) \quad = \quad & (u_H, (\mathbf{u}\cdot\nabla)v_H) - \epsilon^2 (\mathbf{u}\cdot\nabla u_3, v_3) - (b(u_H), v_H) - (\nabla_\nu u_H, \nabla_\nu v_H) \\ & -\epsilon (\nabla_\nu(\epsilon u_3), \nabla_\nu v_3) - \epsilon \{(\beta u_3, v_1) - (\beta u_1, v_3)\} + \langle \theta_H, v_H\rangle_{\Gamma_s}. \end{aligned}$$

Now we prove that

$$\|g^\epsilon\|_{L^{4/3}(0,T)} \leq C\|v_H\|_{Y_H}. \tag{6.5}$$

To this end, we estimate every piece of g^ϵ. For the nonlinear terms, we have

$$(u_H, (\mathbf{u}\cdot\nabla)v_H) \leq \|u_H\|_{L^3}\|\mathbf{u}\|_{L^2}\|\nabla v_H\|_{L^6} \leq C\,\|u_H\|_{L^3}\|\mathbf{u}\|_{L^2}\|v_H\|_{Y_H}$$

and

$$\epsilon^2 (\mathbf{u}\cdot\nabla u_3, v_3) \leq \|\epsilon\mathbf{u}\|_{L^3}\|\nabla(\epsilon u_3)\|_{L^2}\|v_3\|_{L^6} \leq C\,\|\epsilon\mathbf{u}\|_{L^3}\|\epsilon u_3\|_{H^1}\|v_H\|_{Y_H}.$$

By interpolation between $L^\infty(0,T;L^2)$ and $L^2(0,T;L^6)$, u_H is bounded in $L^4(0,T;L^3)$; i.e., $\|u_H\|_{L^3}$ is bounded in $L^4(0,T)$. As $\|\mathbf{u}\|_{L^2}$ is bounded in $L^2(0,T)$, we have $(u_H, (\mathbf{u}\cdot\nabla)v_H)$ bounded in $L^{4/3}(0,T)$. Similarly, as $\|\epsilon\mathbf{u}\|_{L^3}$ is bounded in $L^4(0,T)$ and $\|\epsilon u_3\|_{H^1}$ is bounded in $L^2(0,T)$, we have $\epsilon^2 (\mathbf{u}\cdot\nabla u_3, v_3)$ bounded in $L^{4/3}(0,T)$. The linear terms of g^ϵ are handled easily by the Cauchy–Schwarz inequality:

$$\begin{aligned} (b(u_H), v_H) &\leq \|u_H\|_{L^2}\|v_H\|_{L^2} && \text{bounded in } L^\infty(0,T), \\ (\nabla_\nu u_H, \nabla_\nu v_H) &\leq \|u_H\|_{H^1}\|v_H\|_{H^1} && \text{bounded in } L^2(0,T), \\ \epsilon (\nabla_\nu(\epsilon u_3), \nabla_\nu v_3) &\leq \epsilon\,\|\epsilon u_3\|_{H^1}\|v_3\|_{H^1} && \text{bounded in } L^2(0,T), \\ \epsilon\beta \{(u_3, v_1) - (u_1, v_3)\} &\leq 2f\|\epsilon\mathbf{u}\|_{L^2}\|\mathbf{v}\|_{L^2} && \text{bounded in } L^\infty(0,T), \\ \langle \theta_H, v_H\rangle_{\Gamma_s} &\leq C\,\|\theta_H\|_{H^{-1/2}(\Gamma_s)}\|v_H\|_{H^1} && \text{bounded in } L^2(0,T). \end{aligned}$$

Therefore, taking into account (6.3), according to all previous bounds, (6.5) holds. Next, applying the Hölder inequality to (6.5), we see that

$$\int_t^{t+h} |g^\epsilon(s)|\,\mathrm{ds} \le C\,h^{1/4}\|v_H\|_{Y_H}.$$

On the other hand,

$$|\epsilon^2\,(\tau_h u_3(t) - u_3(t), v_3)| \le \epsilon\,\{\|\tau_h(\epsilon u_3)(t)\|_{L^2} + \|\epsilon u_3(t)\|_{L^2}\}\|v_3\|_{L^2} \le \epsilon\,C\,\|v_H\|_{Y_H}$$

by virtue of Proposition 6.1.

These last two estimates together with (6.4) yield the required result. □

6.3. Convergence. Here we come back to the notation $\mathbf{u}^\epsilon$. The space-time weak form of the anisotropic Navier–Stokes equations (2.7)–(2.13) is as follows.

Find $\mathbf{u}^\epsilon = (u_H^\epsilon, u_3^\epsilon) \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;L^2(\Omega)^3)$ such that

$$\begin{aligned}
&\int_0^T -(u_H^\epsilon, \partial_t v_H) - (u_H^\epsilon, (\mathbf{u}^\epsilon\cdot\nabla)v_H) + (b(u_H^\epsilon), v_H) + (\nabla_\nu u_H^\epsilon, \nabla_\nu v_H)\\
&+\epsilon^2\int_0^T -(u_3^\epsilon, \partial_t v_3) + (\mathbf{u}^\epsilon\cdot\nabla u_3^\epsilon, v_3) + (\nabla_\nu u_3^\epsilon, \nabla_\nu v_3)\\
&+\epsilon\,\beta\int_0^T (u_3^\epsilon, v_1) - (u_1^\epsilon, v_3) = -(u_{0H}, v_H(0)) - \epsilon^2(u_{03}, v_3(0)) + \int_0^T \langle\theta_H, v_H\rangle_{\Gamma_s}\\
&\forall\,\mathbf{v} = (v_H, v_3) \in H^1(0,T;\mathbf{V}),\ \text{with } \mathbf{v}(T) = 0.
\end{aligned} \tag{6.6}$$

The purpose of the following is to take the limit as $\epsilon \to 0$ in (6.6) to come to (4.1). By Propositions 6.1 and 6.2, it follows that $\mathbf{u}^\epsilon$ is bounded in $L^2(0,T;\mathbf{W})$ and u_H^ϵ is bounded in $L^\infty(0,T;B_H)$, allowing us to extract a subsequence, still denoted by $\mathbf{u}^\epsilon$, such that

$$\mathbf{u}^\epsilon = (u_H^\epsilon, u_3^\epsilon) \rightharpoonup \mathbf{u} = (u_H, u_3) \quad \text{in } L^2(0,T;\mathbf{W})\,\text{weak},$$

$$u_H^\epsilon \overset{\star}{\rightharpoonup} u_H \quad \text{in } L^\infty(0,T;B_H)\,\text{weak}-\star.$$

These weak convergences are enough to take the limit in the linear terms of (6.6) (cf. [1]). In particular, the terms of $\mathrm{O}(\epsilon)$ associated with the Coriolis acceleration vanish as ϵ tends to zero. Indeed,

$$\epsilon\,\beta\int_0^T (u_3^\epsilon, v_1) - (u_1^\epsilon, v_3) \le \epsilon\,2f\int_0^T \|\mathbf{u}^\epsilon\|_{L^2}\|\mathbf{v}\|_{L^2}$$

$$\le\ \epsilon\,2f\,\|\mathbf{u}^\epsilon\|_{L^2(0,T;L^2)}\|\mathbf{v}\|_{L^2(0,T;L^2)} \le\ \epsilon\,C\,\|\mathbf{v}\|_{L^2(0,T;L^2)} \le\ C\,\epsilon.$$

On the other hand, combining (6.1), Proposition 6.1, and Lemma 6.3, we can apply Theorem 5.1 for $p=\infty$ and the spaces $B_H \overset{compact}{\hookrightarrow} W_H' \hookrightarrow Y_H'$. Therefore, there exists a subsequence $u_H^\epsilon \to u_H$ in $\mathcal{C}(0,T;W_H')$ strong. Thus we get the weak time-continuity $u_H \in \mathcal{C}(0,T;W_H')$, so that the initial condition (2.20) makes sense for the horizontal velocities. On the other hand, the term of $0(\epsilon^2)$ related to the initial condition for the vertical velocity vanishes as ϵ tends to zero. Indeed,

$$-\epsilon^2(u_{03}, v_3(0)) \le \epsilon^2\|u_{03}\|_{L^2}\|v_3(0)\|_{L^2} \le \epsilon^2\|\mathbf{u}_0\|_{L^2}\|v_3\|_{\mathcal{C}(0,T;L^2)} \le\ C\,\epsilon^2. \tag{6.7}$$

Now the nonlinear terms fall into four types:

$$\text{(I)} \quad \epsilon^2 \int_0^T (u_i^\epsilon \partial_i u_3^\epsilon, v_3)\,\mathrm{dt}, \quad 1 \le i \le 2,$$

$$\text{(II)} \quad \epsilon^2 \int_0^T (u_3^\epsilon \partial_3 u_3^\epsilon, v_3)\,\mathrm{dt},$$

$$\text{(III)} \quad \int_0^T (u_i^\epsilon u_j^\epsilon, \partial_j v_i)\,\mathrm{dt}, \quad 1 \le i,\ j \le 2,$$

$$\text{(IV)} \quad \int_0^T (u_i^\epsilon u_3^\epsilon, \partial_3 v_i)\,\mathrm{dt}, \quad 1 \le i \le 2.$$

Type (I) term:

$$\epsilon^2 \int_0^T (u_i^\epsilon \partial_i u_3^\epsilon, v_3)\,\mathrm{dt} \le \epsilon \int_0^T \|u_i^\epsilon\|_{L^6} \|\partial_i(\epsilon u_3^\epsilon)\|_{L^2} \|v_3\|_{L^3}$$

$$\le C\,\epsilon\, \|u_i^\epsilon\|_{L^2(0,T;H^1)} \|\partial_i(\epsilon u_3^\epsilon)\|_{L^2(0,T;L^2)} \|v_3\|_{\mathcal{C}(0,T;H^1)} \le C\,\epsilon.$$

Type (II) term:

$$\epsilon^2 \int_0^T (u_3^\epsilon \partial_3 u_3^\epsilon, v_3)\,\mathrm{dt} \le \epsilon \int_0^T \|\epsilon u_3^\epsilon\|_{L^6} \|\partial_3 u_3^\epsilon\|_{L^2} \|v_3\|_{L^3}$$

$$\le C\,\epsilon\, \|\epsilon u_3^\epsilon\|_{L^2(0,T;H^1)} \|\partial_3 u_3^\epsilon\|_{L^2(0,T;L^2)} \|v_3\|_{\mathcal{C}(0,T;H^1)} \le C\,\epsilon.$$

Consequently, the type (I) and (II) terms are $O(\epsilon)$ and vanish as ϵ tends to zero.

To handle the type (III) and (IV) terms, we need some strong convergences. From compactness by perturbation (Theorem 5.1 for $p = 2$ and the spaces $W_H \overset{compact}{\hookrightarrow} B_H \hookrightarrow Y_H'$), there exists a subsequence, still denoted by u_H^ϵ, such that

$$u_H^\epsilon \to u_H \quad \text{in } L^2(0,T;L^2(\Omega)^2) \equiv L^2((0,T)\times\Omega)^2 \text{ strong}.$$

By Proposition 6.1, we have u_H^ϵ bounded in $L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;L^6(\Omega)^2)$, which by interpolation ensures that

$$u_H^\epsilon \text{ is bounded in } L^{10/3}(0,T;L^{10/3}(\Omega)^2) \equiv L^{10/3}((0,T)\times\Omega)^2.$$

By the interpolation inequality again, for all $q : 2 \le q < 10/3$ there exists $\alpha : 0 < \alpha \le 1$ such that

$$\|u_H^\epsilon - u_H\|_{L^q} \le \|u_H^\epsilon - u_H\|_{L^2}^{\alpha} \|u_H^\epsilon - u_H\|_{L^{10/3}}^{1-\alpha}.$$

Thus

$$u_H^\epsilon \to u_H \quad \text{in } L^q((0,T)\times\Omega) \text{ strong} \quad \forall\, q : 2 \le q < 10/3. \tag{6.8}$$

Type (III) term: By the Hölder inequality and (6.8), we have

$$u_i^\epsilon u_j^\epsilon \to u_i u_j \quad \text{in } L^r((0,T)\times\Omega) \text{ strong} \quad \forall\, r : 1 \le r < 5/3$$

and for all $i, j = 1, 2$. On the other hand, by interpolation between $L^\infty(0,T;L^2)$ and $L^2(0,T;L^6)$, u_i^ϵ (and u_j^ϵ) is bounded in $L^{8/3}(0,T;L^4)$, and hence $u_i^\epsilon u_j^\epsilon$ is bounded in $L^{4/3}(0,T;L^2)$, and finally,

$$u_i^\epsilon u_j^\epsilon \rightharpoonup u_i u_j \quad \text{in } L^{4/3}(0,T;L^2) \text{ weak}, \quad 1 \le i, \quad j \le 2.$$

In particular, we get

$$\int_0^T (u_i^\epsilon u_j^\epsilon, \partial_j v_i) \to \int_0^T (u_i u_j, \partial_j v_i), \quad 1 \le i, \quad j \le 2.$$

Indeed, $v_i \in \mathcal{C}(0,T;H^1)$ so that

$$\partial_j v_i \in L^\infty(0,T,L^2) \subset L^4(0,T,L^2) \equiv (L^{4/3}(0,T,L^2))'.$$

Type (IV) term: We have

$$u_3^\epsilon \rightharpoonup u_3 \quad \text{in } L^2(0,T;L^2) \text{ weak}.$$

So by the Hölder inequality and (6.8),

$$u_i^\epsilon u_3^\epsilon \rightharpoonup u_i u_3 \quad \text{in } L^s(0,T;L^s) \text{ weak} \quad \forall\, s : 1 \le s < 5/4$$

and for all $i = 1, 2$. On the other hand, it is easy to see that $u_i^\epsilon u_3^\epsilon$ is bounded in $L^{8/7}(0,T;L^{4/3})$, and hence

$$u_i^\epsilon u_3^\epsilon \rightharpoonup u_i u_3 \quad \text{in } L^{8/7}(0,T;L^{4/3}) \text{ weak}, \quad 1 \le i \le 2.$$

Now we shall have to slightly increase the regularity of the test functions of (4.1) to finish the limit process in the Type (IV) terms. For instance, assuming the additional regularity for the test functions $\partial_3 v_i \in L^8(0,T;L^4)$, we get

$$\int_0^T (u_i^\epsilon u_3^\epsilon, \partial_3 v_i) \to \int_0^T (u_i u_3, \partial_3 v_i), \quad 1 \le i \le 2.$$

In conclusion, the limit function $\mathbf{u}$ is a solution of the variational formulation (4.1) for all $\mathbf{v} = (v_H, v_3) \in H^1(0,T,\mathbf{V})$ with $v_H(T) = 0$ and $\partial_3 v_H \in L^8(0,T;L^4)$. Finally, we can argue by density, taking advantage of the regularity of each term of (4.1), and obtain that (4.1) holds for all $\mathbf{v} = (v_H, v_3) \in H^1(0,T,\mathbf{W})$ with $v_H(T) = 0$ and $\partial_3 v_H \in L^\infty(0,T;L^3)$; hence the proof of Theorem 3.1 is finished.

7. Concluding remarks.

7.1. Convergence of the pressure. By using the De Rham lemma [18] in the formulation (6.6) (resp., (4.1)), we can recover the potentials p^ϵ (resp., p) as distributions

$$\nabla p^\epsilon = \begin{pmatrix} -\partial_t u_1^\epsilon - \mathbf{u}^\epsilon \cdot \nabla u_1^\epsilon + \Delta_\nu u_1^\epsilon + \alpha u_2^\epsilon - \epsilon\beta u_3^\epsilon \\ -\partial_t u_2^\epsilon - \mathbf{u}^\epsilon \cdot \nabla u_2^\epsilon + \Delta_\nu u_2^\epsilon - \alpha u_1^\epsilon \\ -\epsilon^2 \left\{\partial_t u_3^\epsilon + \mathbf{u}^\epsilon \cdot \nabla u_3^\epsilon - \Delta_\nu u_3^\epsilon\right\} + \epsilon\beta u_1^\epsilon \end{pmatrix}, \tag{7.1}$$

respectively,

$$\nabla p = \begin{pmatrix} -\partial_t u_1 - \mathbf{u} \cdot \nabla u_1 + \Delta_\nu u_1 + \alpha u_2 \\ -\partial_t u_2 - \mathbf{u} \cdot \nabla u_2 + \Delta_\nu u_2 - \alpha u_1 \\ 0 \end{pmatrix}. \tag{7.2}$$

Moreover, (7.1) is also verified in $H^{-1}(0,T;H^{-1}(\Omega)^3)$ (i.e., in the dual space of $H_0^1(0,T;H_0^1(\Omega)^3)$), whereas (7.2) holds in $H^{-1}(0,T;W^{-1,3/2}(\Omega)^3)$ (i.e., in the dual space of $H_0^1(0,T;W_0^{1,3}(\Omega)^3)$). Proceeding as in subsection 6.3, we may derive that

$$\partial_i p^\epsilon \overset{\star}{\rightharpoonup} \partial_i p \quad \text{in } H^{-1}(0,T;W^{-1,3/2}(\Omega)), \quad i=1,2,$$

and

$$\|\partial_3 p^\epsilon\|_{H^{-1}(0,T;H^{-1}(\Omega))} \leq C\epsilon.$$

In particular, we have the strong convergence of $\partial_3 p^\epsilon$ to $\partial_3 p$.

Remark. The strong convergence of $\partial_3 p^\epsilon$ takes place in a better space than the weak convergence of $\partial_i p^\epsilon$, $i=1,2$. In some sense, this means that the validity of the hydrostatic approximation is less demanding than the validity of the horizontal momentum equations.

Remark. The above convergences can be slightly improved with respect to time. They remain true, replacing the space $H^{-1}(0,T;W^{-1,3/2}(\Omega))$ (resp., $H^{-1}(0,T;H^{-1}(\Omega))$) with the space $W^{-1,\infty}(0,T;W^{-1,3/2}(\Omega))$ (resp., $W^{-1,\infty}(0,T;H^{-1}(\Omega))$).

7.2. Orders of magnitude of the vertical velocity in the original domain. The purpose of this last subsection is to interpret the previous results in the original domain Ω_ϵ. Consequently, we are going to consider $\mathbf{v}=(v_1,v_2,v_3)$, the weak solution in Ω_ϵ of problem (2.1)–(2.5), related to $\mathbf{u}^\epsilon=(u_1^\epsilon,u_2^\epsilon,u_3^\epsilon)$, a weak solution of problem (2.7)–(2.13) in Ω; see (2.6). First, it is important to notice that the true vertical velocity $v_3=\epsilon u_3^\epsilon$ is small with respect to the horizontal velocities v_i, $i=1,2$. Indeed, taking into account the estimates in Proposition 6.1 for u_i^ϵ, $i=1,2$, and Proposition 6.2 for u_3^ϵ, scaling off Ω to Ω_ϵ, we obtain

$$\frac{\|v_3\|_{L^2(0,T;L^2(\Omega_\epsilon))}}{\|v_i\|_{L^2(0,T;L^2(\Omega_\epsilon))}} = 0(\epsilon), \quad i=1,2.$$

By the same argument, we obtain

$$\frac{\|\partial_z v_3\|_{L^2(0,T;L^2(\Omega_\epsilon))}}{\|\partial_z v_i\|_{L^2(0,T;L^2(\Omega_\epsilon))}} = 0(\epsilon), \quad i=1,2.$$

This phenomenon is actually observed in most geophysical flows, which, therefore, are quasi-horizontal. It is striking that the vertical velocity goes to zero even if the initial vertical velocity is not assumed to be small. Looking at (6.7) in the proof of convergence, we need only that $\epsilon^2\|u_{03}\|_{L^2(\Omega)} \to 0$, that is, $\|v_{03}\|_{L^2(\Omega_\epsilon)}/\|v_{0i}\|_{L^2(\Omega_\epsilon)} = O(\epsilon^{-\alpha})$, $\alpha<1$.

Whereas, for the horizontal gradient, we cannot avail ourselves of Proposition 6.2, and we obtain only

$$\frac{\|\nabla_{x,y}\, v_3\|_{L^2(0,T;L^2(\Omega_\epsilon))}}{\|\nabla_{x,y}\, v_i\|_{L^2(0,T;L^2(\Omega_\epsilon))}} = 0(1), \quad i=1,2.$$

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