QUANTUM COADJOINT ACTION AND THE $6j$-SYMBOLS OF $U_q sl_2$

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Dedicated to my advisor, Claude Hayat-Legrand, on the occasion of her 65th birthday.

Abstract. We review the representation theory of the quantum group $U_\varepsilon sl_2\mathbb{C}$ at a root of unity ε of odd order, focusing on geometric aspects related to the 3-dimensional quantum hyperbolic field theories (QHFT). Our analysis relies on the quantum coadjoint action of De Concini-Kac-Procesi, and the theory of Heisenberg doubles of Poisson-Lie groups and Hopf algebras. We identify the $6j$-symbols of generic representations of $U_\varepsilon sl_2\mathbb{C}$, the main ingredients of QHFT, with a bundle morphism defined over a finite cover of the algebraic quotient $PSL_2\mathbb{C}//PSL_2\mathbb{C}$, of degree two times the order of ε. It is characterized by a non Abelian 3-cocycloid identity deforming the fundamental five term relation satisfied by the classical dilogarithm functions, that relates the volume of hyperbolic 3-polyhedra under retriangulation, and more generally, the simplicial formulas of Chern-Simons invariants of 3-manifolds with flat $sl_2\mathbb{C}$-connections.

1. Introduction

After more than twenty years of outstanding efforts, the geometry of quantum groups, especially the relationships between their representation theories at roots of unity and the underlying Lie groups, remains a prominent matter of quantum topology. It stands, for instance, in the background of the geometric realization problem of the combinatorially defined state spaces of the Reshetikhin-Turaev TQFT ([BHMV], [T]), the deformation quantization of character varieties of 3-manifolds via skein modules [BFK, PS], or the asymptotic expansion of quantum invariants ([Prob, Ch. 7], [MN]).

In recent years, new and completely unexpected interactions between the *non restricted* quantum group $U_q = U_q sl_2\mathbb{C}$ and 3-dimensional hyperbolic geometry have been revealed by the volume conjecture [Ka], and the subsequent development of the quantum hyperbolic field theories (QHFT) [BB1, BB2, BB3] and quantum Teichmüller theory [BL]. The global picture is that many of the fondamental invariants of hyperbolic (ie. $PSL_2\mathbb{C}$) geometry, like the volume of hyperbolic manifolds, and more generally (some of the) Chern-Simons invariants of 3-manifolds with flat $sl_2\mathbb{C}$-connections, should be determined by the semi-classical limits of these quantum algebraic objects.

Like any topological quantum field theory, the QHFT are symmetric tensor functors defined on a category of 3-dimensional bordisms. For QHFT, these are equipped with additional structures given by refinements of holonomy representations in $PSL_2\mathbb{C}$, and links up to isotopy. The QHFT are built from "local" basic datas, universally encoded by the moduli space of isometry classes of "flattened" (a kind a framing) hyperbolic ideal 3-simplices, and the associativity constraint, or $6j$*-symbols*, of the cyclic representations of a Borel subalgebra

Date: January 31, 2010 and, in revised form, XXX..

2010 *Mathematics Subject Classification.* Primary 17B37, 57R56; Secondary 14L24.

Key words and phrases. quantum groups, Poisson-Lie groups, coadjoint action, geometric invariant theory, dilogarithm functions, TQFT, invariants of 3-manifolds.

This work was supported by the grant ANR-08-JCJC-0114-01 of the French Agence Nationale de la Recherche.

$U_\varepsilon b$ of U_ε at a primitive root of unity ε of odd order. Working with such a moduli space allows one to define the QHFT also for 3-dimensional bordisms equipped with holonomies having singularities along the links, like cusped hyperbolic manifolds (the link being "at infinity"), and to get surgery formulas. For cylindrical bordisms the QHFT are related by this way to the local version of quantum Teichmüller theory ([BB3],[Bai, BBL]).
Similarly, the Chern-Simons gauge theory for flat $sl_2\mathbb{C}$-connections, which was originally derived from the integral complex-valued invariant "volume" 3-form of $SL_2\mathbb{C}$ via secondary characteristic class theory, defines a functor by means of the same moduli space as the QHFT, using Neumann's simplicial formulas [N] in place of quantum state sums, where a dilogarithm function formally corresponds to the cyclic $6j$-symbols of $U_\varepsilon b$ ([BB2], see also [Ma]). In fact, both maps are 3-cocycloids in a natural way, the latter on the category of cyclic $U_\varepsilon b$-modules with (partially defined) tensor product, and the former on the group $PSL_2\mathbb{C}$ with discrete topology, via Neumann's isomorphism of $H_3(BPSL_2\mathbb{C}^\delta;\mathbb{Z})$ with a certain extension of the Bloch group.
In this paper we consider this interplay of Abelian vs. non Abelian cohomological structures, which certainly concentrates a key part of the quantization procedure relating the Chern-Simons theory for $PSL_2\mathbb{C}$ to the QHFT. We describe in detail the non Abelian part of the story, that is, how the simple U_ε-modules "fiber" over $PSL_2\mathbb{C}$, and the $6j$-symbols of *regular* U_ε-modules (rather than the cyclic $U_\varepsilon b$-modules). By the way we indicate common features and discrepancies with the $6j$-symbols of the *color modules* of the *restricted* quantum group $\bar{U}_\varepsilon$.
We point out also that the cyclic $6j$-symbols of $U_\varepsilon b$, or (basic) *matrix dilogarithms*, coincide with the regular $6j$-symbols of U_ε. More precisely, we define a bundle $\Xi^{(2)}$ of regular U_ε-modules over a covering of degree n^2 (n being the order of ε) of a smooth subset of a Poisson-Lie group H^2 dual to $PSL_2\mathbb{C}^2$, endowed with an action of an infinite dimensional Lie group derived from the quantum coadjoint action of De Concini-Kac-Procesi, originally defined for U_ε. We have (see Theorem 6.13, 6.14 and 6.16 for precise statements):

Theorem 1.1. *The regular $6j$-symbols of U_ε and the matrix dilogarithms coincide and define a bundle morphism $\mathcal{R}\colon \Xi^{(2)} \longrightarrow \Xi^{(2)}$ equivariant under the quantum coadjoint action.*

Since the quantum coadjoint action lifts the adjoint action of $PSL_2\mathbb{C}$ via an unramifield 2-fold covering $H \to PSL_2\mathbb{C}^0$ of the big cell of $PSL_2\mathbb{C}$, it will follow that $\mathcal{R}$ descends to a morphism of a vector bundle of rank n^2 over a $2n$-fold covering of the algebraic quotient $PSL_2\mathbb{C}//PSL_2\mathbb{C}$.
The remarkable dependence of the matrix dilogarithms on cross-ratios of 4-tuples of points on $\mathbb{P}^1$, which was previously known by direct computation and allowed the QHFT to be defined on the moduli space of flattened hyperbolic ideal tetrahedra, is a consequence of Theorem 1.1. This alternative description is presented in Section 6.4.
In order to achieve our goals we have to review quite a lot of material, starting from the basic properties of U_ε (Section 2), and developing in detail its representation theory and quantum coadjoint action (Section 4 and 3, respectively). The $6j$-symbols of the color modules of $\bar{U}_\varepsilon$ and of the regular modules of U_ε are defined in Section 5. Theorem 1.1 is proved in Section 6. There we make a crucial use of fundamental results of Semenov-Tian-Shansky [STS], Weinstein-Xu [WX], and Lu [Lu1] on Poisson-Lie groups and their doubles and quantizations. There should be no obstruction to extend Theorem 1.1 to the quantum groups $U_\varepsilon\mathfrak{g}$ of arbitrary complex simple Lie algebras $\mathfrak{g}$. The corresponding quantum coadjoint action theory is described in [DCK], [DCKP], [DCP], and [DCPRR]. Recent works of Geer and Patureau-Mirand [GP] show that the QHFT setup extends to "relative homotopy quantum field theories", including TQFT associated to the categories of finite dimensional weight $U_\varepsilon\mathfrak{g}$-modules.

A challenging problem is to relate them to the three-dimensional Chern-Simons theory for flat $\mathfrak{g}$-connections.
Finally, let us note that a similar approach can be used to describe "holonomy" R-matrices for the regular $U_\varepsilon\mathfrak{g}$-modules, in the spirit of [KaRe].
We will often meet notions from the theory of Poisson-Lie groups. The reader will find the needed material in standard textbooks, like [CP], [ES] and [KS].
This paper is based on my lectures at the workshop "Interactions Between Hyperbolic Geometry, Quantum Topology and Number Theory" held at Columbia University (June 3rd - 19th, 2009). I am grateful to the organizers for their hospitality and wonderful working conditions during my stay.

2. The quantum group $U_q sl_2$

2.1. **Definition & PBW basis.** We fix our ground ring to be $\mathbb{C}$, and denote by q a complex number such that $q \neq -1, 0, 1$. When q is constrained to be a root of unity we denote it by ε, and we assume that ε has odd order $n \geq 3$.

Definition 2.1. The quantum group $U_q = U_q sl_2$ is the algebra generated over $\mathbb{C}$ by elements E, F, K and K^{-1}, with defining relations $KK^{-1} = K^{-1}K = 1$ and

$$KEK^{-1} = q^2E \ , \ KFK^{-1} = q^{-2}F \ , \ [E,F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{2.1}$$

The algebra U_q is a *Hopf algebra* with coproduct $\Delta\colon U_q \to U_q \otimes U_q$, antipode $S\colon U_q \to U_q$ and counit $\eta\colon U_q \to \mathbb{C}$ defined on generators by

$$\begin{gathered} \Delta(K) = K \otimes K \\ \Delta(E) = E \otimes 1 + K \otimes E, \ \Delta(F) = 1 \otimes F + F \otimes K^{-1} \\ S(K) = K^{-1}, \ S(E) = -K^{-1}E, \ S(F) = -FK \\ \eta(K) = 1, \ \eta(E) = \eta(F) = 0. \end{gathered} \tag{2.2}$$

To make sense of this definition, let us just recall here that being a Hopf algebra means that Δ and η are morphisms of algebras satisfying the *coassociativity* and *counitality* constraints

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \quad , \quad (\eta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \eta) \circ \Delta = \mathrm{id}. \tag{2.3}$$

Here the algebra structure of $U_q \otimes U_q$ is by componentwise multiplication, and U_q is identified with $U_q \otimes \mathbb{C}$ in the canonical way. The antipode S is the inverse of the identity for the convolution product, that is, it satisfies

$$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta 1, \tag{2.4}$$

where $\mu\colon U_q \otimes U_q \to U_q$ is the product. In particular, this implies $S(1) = 1$, $\eta \circ S = \eta$, $S(xy) = S(y)S(x)$ for all x, $y \in U_q$, and

$$(S \otimes S)\Delta = \tau \circ \Delta \circ S, \tag{2.5}$$

where $\tau(u \otimes v) = v \otimes u$ is the flip map ([Ks, Ch. III]).

Remark 2.2. A *simply-connected* version of U_q is often considered in the litterature (see eg. [DCP, §9]). It is obtained by adding to the generators a square root of K acting by conjugation on E and F by multiplication by q and q^{-1}, respectively.

Exercise 2.3. Define an algebra U'_q with generators E, F, K and K^{-1} satisfying the same relations as in U_q except the last one in (2.1), and with a further generator L such that

$$\begin{gathered} [E,F] = L \ , \ (q - q^{-1})L = K - K^{-1} \\ [L,E] = q(EK + K^{-1}E) \ , \ [L,F] = -q^{-1}(FK + K^{-1}F). \end{gathered} \tag{2.6}$$

Show that we have isomorphisms $U_q \cong U'_q$, $U'_1 \cong U[K]/(K^2-1)$, $U \cong U'_1/(K-1)$, where $U = Usl_2$ is the universal enveloping algebra of sl_2.

This exercise shows that, as suggested by the notation, U_q is a genuine deformation depending on the complex parameter q of the universal enveloping algebra of sl_2. Like the latter, we can think of U_q as a ring of polynomials (in non commuting variables). In particular, we have the following fundamental result (see [J, Th. 1.5-1.8] or [Ks, Th. VI.1.4] for a proof):

Theorem 2.4. *1) (PBW basis) The monomials $F^tK^sE^r$, where t, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, make a linear basis of U_q.*
2) The algebra U_q has no zero divisors and is given a grading by stipulating that each monomial $F^tK^sE^r$ is homogeneous of degree $r-t$.

Note that the relations (2.1) are homogeneous of degree 1, -1 and 0, respectively. Products of monomials can be written in the basis $\{F^tK^sE^r\}_{r,t\in\mathbb{N},s\in\mathbb{Z}}$ by using the two first commutation relations in (2.1), together with

$$E^rF^s = \sum_{i=0}^{\min(r,s)} F^{s-i}h_iE^{r-i}, \quad r,s \in \mathbb{N}, \tag{2.7}$$

where the h_i are Laurent polynomials in $\mathbb{C}[K,K^{-1}]$ given by

$$h_i = \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} s \\ i \end{bmatrix} [i]! \prod_{j=1}^{i} [K; i+j-(r+s)] \,.$$

Here we assume that the product is 1 for $i=0$, and we put

$$[K;l] = \frac{Kq^l - K^{-1}q^{-l}}{q-q^{-1}}. \tag{2.8}$$

We use also the standard notations for q-integers, q-factorials, and q-binomial coefficients

$$[l] = \frac{q^l - q^{-l}}{q-q^{-1}} \ , \ [l]! = [l]\,[l-1]\ldots[1] \ , \ \begin{bmatrix} l \\ m \end{bmatrix} = \frac{[l]!}{[m]!\,[l-m]!}, \quad l \in \mathbb{Z}, \tag{2.9}$$

with $[0]! = 1$ by convention. Note that $[l] = q^{l-1} + \ldots + q^{1-l}$, so $[l] \in \mathbb{Z}[q,q^{-1}]$. Also,

$$\begin{bmatrix} l \\ 0 \end{bmatrix} = \begin{bmatrix} l \\ l \end{bmatrix} = 1 \quad \text{and} \quad [l] \text{ divides } \begin{bmatrix} l \\ m \end{bmatrix} \text{ if } 1 < m < l \text{ and } l \text{ is odd.} \tag{2.10}$$

The elements (2.8) appear in all computations involving commutators. In the sequel we will use (2.7) when $r=1$ and s is arbitrary, or r is arbitrary and $s=1$:

$$EF^s = F^sE + [s]F^{s-1}[K;1-s] \ , \ FE^r = E^rF - [r]E^{r-1}[K;r-1]. \tag{2.11}$$

Another important identity is the *q-binomial formula*, which holds for any u, v such that $vu = q^2uv$ [Ks, Ch. VI (1.9)]:

$$(u+v)^r = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} q^{j(r-j)}u^jv^{r-j}. \tag{2.12}$$

In particular, it implies that $\begin{bmatrix} l \\ m \end{bmatrix} \in \mathbb{Z}[q,q^{-1}]$.

Exercise 2.5. (a) Prove (2.11) by induction, using the relations (2.1).
(b) Show that the second relation in (2.11) follows from the first one by applying the *Cartan automorphism* of the algebra U_q, defined by

$$\omega(E) = F \ , \ \omega(F) = E \ , \ w(K) = K^{-1}. \tag{2.13}$$

2.2. **Center.** Theorem 2.4 gives a lot of informations on the center Z_q of U_q. Indeed, denote by U_d the degree d piece of U_q. For any $u \in U_d$ we have

$$KuK^{-1} = q^{2d}u. \tag{2.14}$$

Hence, when q is not a root of unity, U_d is the eigenspace of ad_K with eigenvalue q^{2d}. In particular, $Z_q \subset U_0$.
When $q = \varepsilon$ is a root of unity of odd order $n \geq 3$, Z_ε is much bigger. Let us show that it is generated by E^n, F^n and $Z_\varepsilon \cap U_0$. Since the relations (2.1) are homogeneous, the homogeneous parts of a central element are central. Hence it is enough to prove our claim for a central element $u \in U_d$. By (2.14), if $u \in U_d$ is central then n divides d. Now, if $d = ln$ for some $l \in \mathbb{Z} \setminus \{0\}$ with $l > 0$ (resp. $l < 0$), U_d is spanned by the monomials $F^tK^sE^{t+ln}$ (resp. $F^{t+ln}K^sE^t$). Hence $u = u'E^{ln}$ (resp. $u = F^{ln}u'$) for some $u' \in U_0$. The relations

$$\begin{array}{c} K^lE = q^{2l}EK^l \ , \ K^lF = q^{-2l}FK^l \\ KE^l = q^{2l}E^lK \ , \ KF^l = q^{-2l}F^lK \end{array} \ , \quad l \in \mathbb{N} \tag{2.15}$$

imply that $K^{\pm n} \subset Z_\varepsilon$. Together with $[n] = 0$ and (2.11) for $s = n$ and $r = n$, they give also E^n, $F^n \subset Z_\varepsilon$. Since U_ε has no zero divisors, we get that any $u \in U_d \cap Z_\varepsilon$ can be decomposed as a product of central elements E^{ln} and F^{ln} with a central element in U_0.

Definition 2.6. The subalgebra Z_0 of the center Z_ε of U_ε is generated by E^n, F^n and $K^{\pm n}$.

The algebra Z_0 plays a key role in the structure of $U_\varepsilon sl_2$. In particular, the next theorem shows that any element of Z_ε is a root of a monic polynomial with coefficients in Z_0.

Theorem 2.7. *1) The algebra U_ε is a free Z_0-module of rank n^3, with basis the monomials $F^tK^sE^r$, $0 \leq t, s, r \leq n-1$.*
2) The center Z_ε of U_ε is a finitely generated algebra, and is integrally closed over Z_0.

Proof. The first claim follows directly from Theorem 2.4 (note in particular that E^n, F^n and $K^{\pm n}$ are independent). It implies that U_ε is a noetherian Z_0-module, and so the Z_0-submodule Z_ε is finitely generated. Hence Z_ε is integrally closed over Z_0 [AMcD, Th. 5.3]. Then, by Hilbert's basis theorem Z_ε is finitely generated as an algebra. □

In another direction, we will see in the next section that Z_0 is the basic link between U_ε and the Lie group $PSL_2\mathbb{C}$. This depends on the fact that it is a (commutative) *Hopf* algebra. To prove this, let us introduce for future reference a prefered set $\{x, y, z\}$ of generators of Z_0, given by

$$x = -(\varepsilon - \varepsilon^{-1})^n E^n K^{-n} \ , \ y = (\varepsilon - \varepsilon^{-1})^n F^n \ , \ z = K^n. \tag{2.16}$$

We also put

$$e = (\varepsilon - \varepsilon^{-1})^n E^n \ , \ f = -(\varepsilon - \varepsilon^{-1})^n F^n K^n. \tag{2.17}$$

Note that

$$x = T(y) \ , \ f = T(e), \ x = S(e) \ , \ y = S(f)$$

where S is the antipode of U_ε, and T the *braid group automorphism* of the algebra U_ε, defined by

$$T(E) = -FK \ , \ T(F) = -K^{-1}E \ , \ T(K) = K^{-1} \ , \ T(K^{-1}) = K. \tag{2.18}$$

Lemma 2.8. *The subalgebra Z_0 of U_ε is a Hopf subalgebra.*

Proof. We have to show that $\Delta(Z_0) \subset Z_0 \otimes Z_0$ and $S(Z_0) \subset Z_0$. Because these maps are morphisms of algebras, it is enough to prove this on the generators x, y and z of Z_0. We claim that

$$\begin{gathered}\Delta(z) = z \otimes z \\ \Delta(x) = x \otimes z^{-1} + 1 \otimes x,\ \Delta(y) = 1 \otimes y + y \otimes z^{-1} \\ S(z) = z^{-1},\ S(x) = -zx,\ S(y) = -yz \\ \eta(z) = 1,\ \eta(x) = \eta(y) = 0.\end{gathered} \tag{2.19}$$

The proof of the formulas for S is immediate. As for Δ, they are consequences of (2.10), the q-binomial identity (2.12), and the fact that $[n] = 0$ when $q = \varepsilon$. □

So far, we have only considered in (2.14) the invariance of the center of U_q under conjugation by K. In order to describe the effect of the conjugation action by E and F, let us introduce the automorphisms γ_l of $\mathbb{C}[K, K^{-1}]$ given by

$$\gamma_l(K) = q^l K, \quad l \in \mathbb{Z}. \tag{2.20}$$

By (2.1), for all $h \in \mathbb{C}[K, K^{-1}]$ we have

$$hE = E\gamma_2(h) \quad \text{and} \quad hF = F\gamma_{-2}(h).$$

It is easy to check that $u = \sum_{i \geq 0} F^i h_i E^i \in U_0$ satisfies $uE = Eu$ and $uF = Fu$ in U_q if and only if for all i,

$$h_i - \gamma_{-2}(h_i) = [i+1][K; -i]h_{i+1}. \tag{2.21}$$

Since $\mathbb{C}[K, K^{-1}]$ is an integral domain, h_{i+1} is thus determined inductively by h_0 if $[j+1] \neq 0$ for all $0 \leq j \leq i$. This condition is satisfied for all i when q is not a root of unity, and for $i < n-1$ when $q = \varepsilon$. Hence the projection map

$$\begin{array}{cccc}\pi : & U_0 & \longrightarrow & \mathbb{C}[K, K^{-1}] \\ & \sum_{r=0}^{\infty} F^r h_r E^r & \longmapsto & h_0\end{array} \tag{2.22}$$

is injective over $Z_q \subset U_0$ when q is not a root of unity, and injective over $Z_\varepsilon \cap U_0'$ when $q = \varepsilon$, where

$$U_0' = \left\{ \sum_{i=0}^{n-1} F^i h_i E^i \mid h_0, \ldots, h_{n-1} \in \mathbb{C}[K, K^{-1}] \right\} \subset U_0. \tag{2.23}$$

In fact, $\mathrm{Ker}(\pi) = FU_0E$ is a two-sided ideal in U_0, so π is a homomorphism of algebras, the *Harish-Chandra homomorphism*. It is easy to check that Z_ε is generated by F^n, E^n and $Z_\varepsilon \cap U_0'$. On another hand, (2.21) is solved by

$$h_0 = \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}\ ,\ h_1 = 1\ ,\ h_r = 0 \quad \text{if } r \geq 2,$$

which defines the *Casimir element*

$$\Omega = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} \ \in Z_q. \tag{2.24}$$

Note that (see (2.20))

$$\gamma_{-1} \circ \pi(\Omega) = \frac{K + K^{-1}}{(q - q^{-1})^2}.$$

We will see in Section 4 that Z_q is actually generated by Ω, and $Z_\varepsilon \cap U_0'$ by Ω and $K^{\pm n}$ (see Theorem 4.9; for an alternative approach based on induction on i in (2.21), see [J, Prop. 2.20]). More precisely:

Theorem 2.9. *(1) If q is not a root of unity, Z_q is a polynomial algebra over $\mathbb{C}$ generated by Ω, and $\gamma_{-1}\circ\pi : Z_q \to \mathbb{C}[K+K^{-1}]$ is an isomorphism.*
(2) If $q=\varepsilon$ is a root of unity of odd order $n\geq 3$, then Z_ε is the algebra over $\mathbb{C}$ generated by E^n, F^n, $K^{\pm n}$ and Ω with relation

$$\prod_{j=0}^{n-1}(\Omega - c_j) = E^nF^n + \frac{K^n+K^{-n}-2}{(\varepsilon-\varepsilon^{-1})^{2n}} \tag{2.25}$$

where

$$c_j = \frac{\varepsilon^{j+1}+\varepsilon^{-j-1}}{(\varepsilon-\varepsilon^{-1})^2}.$$

Statement (2) is part of [DCK, Th. 4.2]. It shows that Z_ε is a finite extension of Z_0 built on the image $\mathbb{C}[\Omega]$ of the injective homomorphism

$$(\gamma_{-1}\circ\pi)^{-1}\colon \mathbb{C}[K+K^{-1}] \longrightarrow Z_\varepsilon. \tag{2.26}$$

The domain $\mathbb{C}[K+K^{-1}]$ is the fixed point set of $\mathbb{C}[K,K^{-1}]$ under the involution $s(K^{\pm 1}) = K^{\mp 1}$, which can be identified with a generator of the Weyl group of $PSL_2\mathbb{C}$.

3. The quantum coadjoint action for U_ε

Consider the set $\mathrm{Spec}(Z_\varepsilon)$ of algebra homomorphisms from the center Z_ε of U_ε to $\mathbb{C}$. An element of $\mathrm{Spec}(Z_\varepsilon)$ is called a *central character* of U_ε. The set $\mathrm{Spec}(Z_\varepsilon)$ has a very rich Poisson geometry related to the adjoint action of $PSL_2\mathbb{C}$, that we are going to describe. It will be used in Section 4.

Since Z_ε is finitely generated, by Hilbert's nullstellensatz $\mathrm{Spec}(Z_\varepsilon)$ is an affine algebraic set. The inclusion $Z_0\subset Z_\varepsilon$ induces a regular (restriction) map

$$\tau\colon \mathrm{Spec}(Z_\varepsilon) \longrightarrow \mathrm{Spec}(Z_0). \tag{3.1}$$

We have seen in Proposition 2.7 that Z_ε is integrally closed over Z_0. Hence τ is *finite*: any $y\in \mathrm{Im}(\tau)$ has a finite number of preimages. From this property it follows that τ is surjective [Sh, §5.3].

3.1. A Poisson-Lie group structure on Spec(Z_0).

We need some explicit formulas. Denote by T, $U_\pm$ and $PB_\pm$ the subgroups of $PSL_2\mathbb{C} = SL_2\mathbb{C}/(\pm 1)$ of diagonal, upper/lower unipotent and upper/lower triangular matrices up to sign, with their natural structures of affine algebraic groups induced by the adjoint action on $sl_2\mathbb{C}$. The action

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} : w\mapsto \frac{aw+b}{cw+d},\ w\in\mathbb{P}^1$$

identifies $PSL_2\mathbb{C}$ with the group $\mathrm{Aut}(\mathbb{P}^1)$ of automorphisms of the Riemann sphere, and T with the subgroup of maps $w\mapsto zw$, $z\in\mathbb{C}^*$. Consider the group

$$H = \{(tu_+, t^{-1}u_-)\ | t\in T, u_\pm\in U_\pm\}\subset PB_+\times PB_-.$$

By evaluating on the generators x, y and z of Z_0 (see (2.16)) we get a map

$$\begin{array}{rccc} \psi': & \mathrm{Spec}(Z_0) & \longrightarrow & \mathbb{C}^2\times\mathbb{C}^* \\ & g & \longmapsto & (x_g, y_g, z_g). \end{array} \tag{3.2}$$

For all $g\in \mathrm{Spec}(Z_0)$, consider the automorphisms of $\mathbb{P}^1$ given in ψ'-coordinates by

$$\psi_1(g)\colon w\mapsto z_g w - x_g z_g\ ,\ \psi_2(g)\colon w\mapsto w/(-z_g y_g w + z_g),\quad w\in\mathbb{P}^1.$$

Under the above isomorphism of $\mathrm{Aut}(\mathbb{P}^1)$ with $PSL_2\mathbb{C}$, they correspond respectively to

$$\pm\begin{pmatrix} \sqrt{z_g} & -x_g\sqrt{z_g} \\ 0 & 1/\sqrt{z_g} \end{pmatrix} \in PB_+ \quad , \quad \pm\begin{pmatrix} 1/\sqrt{z_g} & 0 \\ -y_g\sqrt{z_g} & \sqrt{z_g} \end{pmatrix} \in PB_-$$

for any choice of square root $\sqrt{z_g}$. Define

$$\begin{array}{rccl} \psi: & \mathrm{Spec}(Z_0) & \longrightarrow & H \\ & g & \longmapsto & (\psi_1(g), \psi_2(g)). \end{array}$$

Clearly, ψ is an isomorphism of algebraic varieties. It gives H a Poisson-Lie bracket as follows. By using the PBW basis $\{F^tK^sE^r\}_{r,t\in\mathbb{N},s\in\mathbb{Z}}$ we can identify U_ε as a linear subspace of U_q, considered as a family of algebras over $\mathbb{C}$ with varying parameter q. Hence, for any given element $u \in U_\varepsilon$ we can specify a *lift* $\tilde{u} \in U_q$, such that

$$u = \tilde{u} \ \mathrm{mod}(q^n - q^{-n})U_q.$$

We can write this as $u = \lim_{q\to\varepsilon} \tilde{u}$. Then, for all $a \in Z_0$ and $u \in U_\varepsilon$, put

$$D_a(u) = \lim_{q\to\varepsilon} \frac{[\tilde{a}, \tilde{u}]}{n(q^n - q^{-n})}.$$

Note that

$$D_a(u) = \frac{1}{(\varepsilon - \varepsilon^{-1})} \lim_{q\to\varepsilon} \left[\frac{\tilde{a}}{n[n]}, \tilde{u}\right] = \frac{1}{(\varepsilon - \varepsilon^{-1})^n} \lim_{q\to\varepsilon} \left[\frac{\tilde{a}}{[n]!}, \tilde{u}\right] \tag{3.3}$$

where we use the formula $[n-1]! = n(\varepsilon - \varepsilon^{-1})^{1-n}$.

Proposition 3.1. *(i) The map* $\psi\colon \mathrm{Spec}(Z_0) \to H$ *is an isomorphism of algebraic groups.*
(ii) The maps $D_a\colon U_\varepsilon \to U_\varepsilon$ *are well-defined derivations of* U_ε *preserving* Z_0 *and* Z_ε*. At the generators* $e = -xz$ *and* z *in* (2.17) *we have:*

$$D_e(F) = \frac{(\varepsilon - \varepsilon^{-1})^{n-1}}{n}[K;1]E^{n-1} \ , \ D_e(K^{\pm1}) = \mp\frac{(\varepsilon - \varepsilon^{-1})^n}{n}K^{\pm1}E^n \tag{3.4}$$

$$D_z(E) = \frac{1}{n}K^nE \ , \ D_z(F) = -\frac{1}{n}FK^n. \tag{3.5}$$

where $[K;l]$ *is given by* (2.8) *with* $q = \varepsilon$*.*

Proof. (i) This follows from classical duality arguments in Hopf algebra theory (see [Mo, § 9]). Because Z_0 is a commutative Hopf algebra (Lemma 2.8), the algebraic set $\mathrm{Spec}(Z_0)$ has a canonical group structure identifying Z_0 with the algebra of regular functions on $\mathrm{Spec}(Z_0)$. It is defined dually by stipulating that for any $u \in Z_0$ and $g, h \in \mathrm{Spec}(Z_0)$,

$$u(gh) = \mu\circ\Delta(u)(f,g) \ , \ u(g^{-1}) = S(u)(g) \ , \ u(\mathbb{1}) = \eta(u). \tag{3.6}$$

Here, $\mathbb{1} \in \mathrm{Spec}(Z_0)$ is the identity and μ the product of Z_0. The associativity of the product of $\mathrm{Spec}(Z_0)$ follows from the coassociativity of Δ (see (2.3)). The inverse is well-defined by $u(\mathbb{1}) = \eta(u)$ and the computation

$$u(gg^{-1}) = \mu\circ(\mathrm{id}\otimes S)\circ\Delta(u)(g,g) = \eta(u)1(g) = \eta(u),$$

where $1 \in Z_0$ is identified with the constant function on $\mathrm{Spec}(Z_0)$ with value 1, and we use (2.4) in the second equality. By (2.19) and (3.6) we get in the ψ'-coordinates (x_g, y_g, z_g) of the point g of $\mathrm{Spec}(Z_0)$:

$$\begin{gathered} x_{gh} = x_h + x_gz_h^{-1} \ , \ y_{gh} = y_h + y_gz_h^{-1} \ , \ z_{gh} = z_gz_h \\ x_{g^{-1}} = -z_gx_g \ , \ y_{g^{-1}} = -y_gz_g \ , \ z_{g^{-1}} = z_g^{-1} \\ x_{\mathbf{1}} = 0 \ , \ y_{\mathbf{1}} = 0 \ , \ z_{\mathbf{1}} = 1. \end{gathered} \tag{3.7}$$

Now we have

$$\begin{aligned}\psi_1(g)\psi_1(h)(w) =& z_g(z_h w - z_h x_h) - z_g x_g\\ =& z_g z_h w - z_g z_h (x_h + x_g z_h^{-1}),\end{aligned}$$

which is equal to $\psi_1(gh)$ by (3.7). A similar computation gives $\psi_2(g)\psi_2(h)(w) = \psi_2(gh)$. Hence ψ is a group homomorphism.

(ii) The maps D_a are well-defined because $a \in Z_\varepsilon$ implies $[\tilde{a}, \tilde{u}] \in (q^n - q^{-n})U_q$. They are derivations of U_ε because the commutator $[\ ,\ ]$ of U_q satisfies the Leibniz rule. Clearly, they map Z_ε to Z_ε, and also Z_0 to Z_0 since for all $u \in Z_0$, $D_a(u)$ is a sum of monomials in Z_0. All this can also be check by direct computation, as follows. By (2.7), the products of basis elements of U_q read as

$$(F^t K^s E^r)(F^c K^b E^a) = \sum_{i=0}^{\min(r,c)} q^{-2((r-i)b+s(c-i))} F^{c+t-i} h_i K^{s+b} E^{r+a-i}.$$

Suppose that r, s and t are (possibly 0) multiples of n, say $r = r'n$, $s = s'n$ and $t = t'n$. Then, all the h_i except $h_0 = 1$ are divided by $[ln]$ for some l, and so vanish at $q = \varepsilon$. Since the coefficient of $F^{c+t}K^{s+b}E^{r+a}$ in the commutator $[F^t K^s E^r, F^c K^b E^a]$ is

$$q^{-2n(r'b+s'c)} - q^{-2n(s'a+bt')} = q^{-n(b(r'+t')+s'(a+c)}[n(s'(a-c) + b(t'-r'))],$$

and $\lim_{q\to\varepsilon}[nl]/[n] = l$ for any integer l, the limit (3.3) is well-defined.

The formula for $D_e(F)$ follows from a straightforward computation using (2.11) and (2.15). Also we have

$$\begin{aligned}D_e(K^{\pm 1}) =& \lim_{q\to\varepsilon} \frac{[(q-q^{-1})^n E^n, K^{\pm 1}]}{n(q^n - q^{-n})}\\ =& \lim_{q\to\varepsilon} \frac{(q-q^{-1})^n}{n} \frac{q^{\mp 2n} - 1}{q^n - q^{-n}} K^{\pm 1} E^n\\ =& \mp \frac{(\varepsilon - \varepsilon^{-1})^n}{n} K^{\pm 1} E^n.\end{aligned}$$

We get D_z by similar computations. □

For all u, $v \in Z_0$ and $f \in \mathrm{Spec}(Z_0)$, put

$$\{u, v\}(f) = D_u(v)(f).$$

By Proposition 3.1, for all $a \in Z_0$, $D_a \colon Z_0 \to Z_0$ defines an algebraic vector field on $\mathrm{Spec}(Z_0)$, and $\{\ ,\ \}$ is a bivector satisfying the Jacobi identity. Hence:

Corollary 3.2. *The bivector $\{\ ,\ \}$ is a Poisson bracket on Spec(Z_0).*

Exercise 3.3. (a) Check that

$$\begin{gathered}\{e, x\} = -zx^2\ ,\ \{e, y\} = z - z^{-1}\ ,\ \{e, z\} = xz^2\\ \{f, x\} = z^{-1} - z\ ,\ \{f, y\} = y^2 z\ ,\ \{f, z\} = -yz^2\\ \{y, x\} = 1 - xy - z^{-2}\ ,\ \{z, x\} = zx\ ,\ \{z, y\} = -yz.\end{gathered}$$

(b) Deduce that the Poisson bracket $\psi_*\{\ ,\ \}$ on H does not depend on the order n of ε.

(c) Show that for any algebra automorphism ϕ of U_ε we have

$$\phi D_a \phi^{-1}(u) = D_{\phi(a)}(u). \tag{3.8}$$

Deduce that the group of algebra automorphisms of U_ε induces a group of Poisson automorphisms of $(\mathrm{Spec}(Z_0), \{\ ,\ \})$. By using the braid group automorphism (2.18), compute the value of the derivation D_f on E, F and $K^{\pm 1}$.

(d) Show that the derivations $D_{\tilde{a}}$ obtained from the D_a by allowing the lift $\tilde{a}$ to be arbitrary generate a Lie algebra $\mathcal{L}$. Show that $\mathcal{L}$ fits into an exact sequence of Lie algebras $0 \longrightarrow \mathcal{L}^0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}' \longrightarrow 0$, where $\mathcal{L}^0$ (resp. $\mathcal{L}'$) is the Lie algebra of inner derivations of U_ε (resp. of derivations of Z_0 induced by $\{\ ,\ \}$) .

Remark 3.4. *(Semi-classical geometry, I)* By Exercise 3.3 the Poisson bracket $\psi_*\{\ ,\ \}$ is canonically associated to U_q. Conversally, $(H, \psi_*\{\ ,\ \})$ is *dual* to the standard Poisson-Lie structure on $PSL_2\mathbb{C}$, and the *rational quantization* of the latter is U_q ([DCKP, §7], [DCP, §11, 14 & 19]).

Denote by $PSL_2\mathbb{C}^0 = U_-TU_+$ the *big cell* of $PSL_2\mathbb{C}$; it is the open subset consisting of matrices up to sign with non vanishing upper left entry. We have an unramified 2-fold covering

$$\begin{array}{rccc} \sigma : & H & \longrightarrow & PSL_2\mathbb{C}^0 \\ & (t^{-1}u_-, tu_+) & \longmapsto & u_-^{-1}t^2u_+. \end{array} \tag{3.9}$$

Consider the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.10}$$

They correspond to the Chevalley generators of $sl_2\mathbb{C}$ in its fundamental representation. Since σ is an unramified covering, the associated (complex) left invariant vector fields on $PSL_2\mathbb{C}$ lift to vector fields $\underline{e}$, $\underline{f}$ and $\underline{h}$ on H. By using Proposition 3.1 we consider them as vector fields on Spec(Z_0).
Recall the generators e, f and z of Z_0, which define functions on Spec(Z_0). Denote by $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) the Lie algebras of vector fields on Spec(Z_0) generated by $\underline{e}$ and $\underline{f}$ (resp. by D_e and D_f).

Proposition 3.5. *We have*

$$D_e = z\underline{f}\ ,\ D_f = -z\underline{e}\ ,\ D_z = z\underline{h}/2 \tag{3.11}$$

and

$$[D_e, D_f] = z^2\underline{h} - z^2x\underline{e} + z^2y\underline{f}. \tag{3.12}$$

Hence the linear spans of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ coincide at every point of Spec(Z_0).

Proof. Since all vector fields are algebraic, it is enough to check (3.11) on functions on Spec$(Z_0) \cong H$ lifting functions on $PSL_2\mathbb{C}^0$, and we can restrict to coordinates of the map σ. Put

$$u_- = \pm \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} ,\ u_+ = \pm \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} ,\ t = \pm \begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1/\sqrt{z} \end{pmatrix}$$

and

$$M = \sigma(tu_+, t^{-1}u_-) = u_-^{-1}t^2u_+ = \pm \begin{pmatrix} z & -zx \\ zy & -zxy + z^{-1} \end{pmatrix}.$$

By Exercise 3.3 (a) we have

$$D_eM = \pm \begin{pmatrix} z^2x & 0 \\ z^2xy + z^2 - 1 & -z^2x \end{pmatrix}.$$

Let $a \in sl_2\mathbb{C}$. Denote by a_* the left invariant vector field on $PSL_2\mathbb{C}$ associated to a, and $\underline{a}$ its pull-back to Spec(Z_0) via σ. By definition, $\underline{a}(M) = a_*(M)$, and for all $g \in PSL(2,\mathbb{C})^0$,

$$a_*(M)(g) = \frac{\mathrm{d}}{\mathrm{dt}} \left(M(ge^{at})\right)_{t=0} = \frac{\mathrm{d}}{\mathrm{dt}} \left(R^*_{e^{at}}M\right)_{t=0}(g),$$

where $R_{e^{at}}$ denotes right translation by e^{at}. On another hand, for any vector field X on $PSL_2\mathbb{C}$ we have $R_{h*}X = \mathrm{Ad}_{h^{-1}}(X)$. Since the adjoint representation is faithful, dually $(R_h^*\rho)(g) = \mathrm{Ad}_h(\rho(g))$ for any linear representation ρ of $PSL_2\mathbb{C}$. Hence

$$a_*(M)(g) = \mathrm{ad}_a(M)(g) = [a, M(g)].$$

It is immediate to check that

$$[\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, M] = D_e M.$$

Hence $D_e = z\underline{f}$. The formulas for D_f and D_z can be proved in the same way. We obtain (3.12) by using $[\underline{e}, \underline{f}] = \underline{h}$, the formulas in Exercise 3.3 (a), and the fact that

$$[fU, gV] = fg[U,V] + fU(g)V - gV(f)U$$

for any vector fields U, V and functions f, g. □

3.2. **Flows on U_ε.** The infinitesimal action of the Lie algebra $\tilde{\mathfrak{g}}$ generated by D_e and D_f can be integrated to an infinite dimensional group of automorphisms of the algebra U_ε. In fact, in order to make sense of this we have to allow holomorphic series in the generators of Z_0 as coefficients.

More precisely, denote by $\hat{Z}_0$ the algebra of power series in the generators x, y, z and z^{-1} of Z_0 which converge to a holomorphic function for all values of (x, y, z) in $\mathbb{C}^2 \times \mathbb{C}^*$. Set

$$\hat{U}_\varepsilon = U_\varepsilon \otimes_{Z_0} \hat{Z}_0\ ,\ \hat{Z}_\varepsilon = Z_\varepsilon \otimes_{Z_0} \hat{Z}_0. \tag{3.13}$$

Proposition 3.6. *For all values of $t \in \mathbb{C}$ the series $\exp(tD_e)$, $\exp(tD_f)$ and $\exp(tD_z)$ converge to automorphisms of $\hat{U}_\varepsilon$ preserving $\hat{Z}_0$ and $\hat{Z}_\varepsilon$.*

Proof. By (3.5) the statement is clearly true for $\exp(tD_z)$. To conclude it is enough to check that the series $\exp(tD_e)$ when applied to the generators K and F converge to an element of $\hat{U}_e$; the result for $\exp(tD_f)$ follows from this by using $T(e) = f$ (see (2.17)) and Exercise 3.3 (c). Now, by (3.4)-(3.5) we have

$$\exp(tD_e)K = \exp(-te/n)K$$

and

$$(D_e)^l(F) = -\frac{(\varepsilon - \varepsilon^{-1})^{n-2}}{n^l} e^{l-1}((-1)^l K\varepsilon + K^{-1}\varepsilon^{-1})E^{n-1}.$$

Hence

$$\exp(tD_e)F = F - (\varepsilon - \varepsilon^{-1})^{n-2}\left(\frac{e^{-te/n}-1}{e}K\varepsilon + \frac{e^{te/n}-1}{e}K^{-1}\varepsilon^{-1}\right)E^{n-1}.$$

The stability of $\hat{Z}_0$ and $\hat{Z}_\varepsilon$ follows from that of Z_0 and Z_ε under the derivations D_a (see Proposition 3.1). □

Definition 3.7. The subgroup $\mathcal{G}$ of $\mathrm{Aut}(\hat{U}_\varepsilon)$ is generated by the 1-parameter groups $\exp(tD_e)$ and $\exp(tD_f)$, $t \in \mathbb{C}$.

Since $\mathcal{G}$ leaves $\hat{Z}_0$ and $\hat{Z}_\varepsilon$ invariant, it acts dually by holomorphic transformations on the varieties $\mathrm{Spec}(Z_0)$ and $\mathrm{Spec}(Z_\varepsilon)$:

$$\forall g \in \mathcal{G}, u \in \hat{Z}_\varepsilon, \chi \in \mathrm{Spec}(Z_\varepsilon), \quad (g.u)(\chi) = u(g^{-1}.\chi).$$

By Proposition 3.5, the $\mathcal{G}$-action on $\mathrm{Spec}(Z_0)$ lifts the conjugation action of $PSL_2\mathbb{C}$. Also, it follows from (3.12) that the $\mathcal{G}$-orbits are both open and closed in the preimage of any conjugacy class, and so are connected. Since any element of $PSL_2\mathbb{C}$ is conjugate to one in a given Borel subgroup, any conjugacy class in $PSL_2\mathbb{C}$ intersects $PSL_2\mathbb{C}^0$ in a non empty smooth connected variety. Hence we have [DCKP, Prop. 6.1]:

Theorem 3.8. *(i) For any conjugacy class Γ in $PSL_2\mathbb{C}$, the connected components of the variety $\sigma^{-1}\Gamma$ are orbits of $\mathcal{G}$ in Spec(Z_0).*
(ii) The set $\sigma^{-1}(\{\mathrm{id}\})$ coincides with the fixed point set of $\mathcal{G}$ in Spec(Z_0).

Remark 3.9. *(Semi-classical geometry, II)* (i) By construction, the orbits of the $\mathcal{G}$-action on Spec(Z_0) are symplectic leaves for the Poisson bracket $\{\ ,\ \}$. A result of M. Semenov-Tian-Shansky implies that they coincide under ψ with the orbits of the dressing action of $PSL_2\mathbb{C}$ on H ([KS, Ch. 1.3], [CP, Ch. 1.5]) (see Remark 3.4). However there does not seem to have any homomorphism $\mathcal{G} \to PSL_2\mathbb{C}$ relating these two actions on H.
(ii) The action of $\mathcal{G}$ on Spec(Z_0) is called the *quantum coadjoint action* [DCK], because it coincides at the tangent space of fixed points with the coadjoint action of $PSL_2\mathbb{C}$ on its dual Lie algebra.

4. Representation theory of U_ε

In this section we use the quantum coadjoint action of $\mathcal{G}$ on Spec(Z_0), and its extension to $\hat{U}_e$, to describe the representation theory of U_ε in geometric terms. The main results are Theorem 4.11 and Theorem 4.12.

To begin with, recall that two U_ε-modules V and W are *isomorphic* if there exists a linear isomorphism $\theta : V \to W$ commuting with the action of U_ε. A U_ε-module is *simple* if it has no proper non trivial submodule. In what follows, all modules are *left* modules.
It is classical that any simple U_ε-module is a finite dimensional vector space [CP, p. 339], and that the action of the center Z_ε is by scalars. More precisely:

Lemma 4.1. *Any central element $u \in Z_q$ acts on a finite dimensional simple U_q-module V by multiplication by a scalar.*

Proof. Since $\mathbb{C}$ is algebraically closed and V is finite dimensional, the linear operator $u_V \in \mathrm{End}(V)$ associated to u has an eigenvalue λ. Because u is central, u_V commutes with the action of U_q. Hence the kernel of $u_V - \lambda.\mathrm{id}$ must be the whole of V, for otherwise it would be a proper non zero submodule. □

By the lemma we can associate to any simple U_ε-module V the central character $\chi_V : Z_\varepsilon \to \mathbb{C}$ in Spec(Z_ε) such that $uv = \chi_V(u)v$ for all $u \in Z_\varepsilon$ and $v \in V$. Clearly any two isomorphic U_ε-modules define the same central character. Hence, denoting by Rep(U_ε) the set of isomorphism classes of simple U_ε-modules, we have a *central character map*

$$\Xi : \begin{array}{ccc} \mathrm{Rep}(U_\varepsilon) & \longrightarrow & \mathrm{Spec}(Z_\varepsilon) \\ V & \longmapsto & \chi_V. \end{array} \tag{4.1}$$

For any $\chi \in$ Spec(Z_ε), consider the two-sided ideal $\mathcal{I}^\chi$ of U_ε generated by its kernel, $\mathrm{Ker}(\chi) = \{z - \chi(z).\mathrm{id} | z \in Z_\varepsilon\}$. Define an algebra

$$U_\varepsilon^\chi = U_\varepsilon / \mathcal{I}^\chi. \tag{4.2}$$

By Theorem 2.7 and the fact that the degree of a homogeneous element of Z_ε is always a multiple of n, the algebra U_ε^χ is finite dimensional over $\mathbb{C}$ and non zero. It is also a U_ε-module by the left regular representation on cosets, and clearly any simple submodule of U_ε^χ is in $\Xi^{-1}(\chi)$. This proves:

Lemma 4.2. *The central character map Ξ is surjective.*

In order to clarify the properties of Ξ we are going to describe the set Rep(U_ε) explicitly.

4.1. **The simple U_ε-modules.** Consider the following two families of U_ε-modules:

- the modules $V_r^\pm$ of dimension $r+1$, where $0 \leq r \leq n-1$, with a basis $v_0, \ldots, v_r$ such that
$$Kv_j = \pm\varepsilon^{r-2j} v_j \quad , \quad Fv_j = \begin{cases} v_{j+1}, & \text{if } j < r \\ 0, & \text{if } j = r \end{cases}$$
$$Ev_j = \begin{cases} \pm[j][r-j+1]v_{j-1}, & \text{if } j > 0 \\ 0, & \text{if } j = 0. \end{cases}$$
- the modules $V(\lambda, a, b)$ of dimension n, where $\lambda \in \mathbb{C}^*$ and $a,\ b \in \mathbb{C}$, with a basis $v_0, \ldots, v_{n-1}$ such that
$$Kv_j = \lambda\varepsilon^{-2j} v_j \quad , \quad Fv_j = \begin{cases} v_{j+1}, & \text{if } j < n-1 \\ bv_0, & \text{if } j = n-1 \end{cases}$$
$$E \cdot v_j = \begin{cases} av_{n-1}, & \text{if } j = 0 \\ \left(ab + [j]\frac{\lambda\varepsilon^{1-j} - \lambda^{-1}\varepsilon^{j-1}}{\varepsilon - \varepsilon^{-1}}\right) v_{j-1}, & \text{if } j > 0. \end{cases}$$

That these formulas actually define actions of U_ε is a direct consequence of (2.1) and (2.11). We will see in Section 5 that the modules V_r^- and V_r^+ are essentially equivalent (we have an isomorphism $V_r^- = V_0^- \otimes V_r^+$). The two families of modules $V_r^\pm$ and $V(\lambda, a, b)$ intersect exactly at $V_{n-1}^\pm = V(\pm\varepsilon^{-1}, 0, 0)$. The action of U_ε on the modules $V_r^\pm$ is often expressed in the litterature in a different form, by using the "balanced" basis $m_j = v_j/[j]!$ rather than the v_j.

4.1.1. *Highest weight U_ε-modules.* The modules $V_r^\pm$ and $V(\lambda, 0, b)$ have the important common property to be highest weight modules.

Recall that a U_ε-module is a *highest weight module* of highest weight $\lambda \in \mathbb{C}^*$ if it is generated by a non zero vector v such that $Ev = 0$ and $Kv = \lambda v$; v is called a *highest weight vector.* It is canonical up to a scalar factor, since K is diagonal in the basis $F^i v$. In fact, there is a *universal* U_ε-module of highest weight λ, the (infinite-dimensional) *Verma module* $M(\lambda)$ defined by
$$M(\lambda) = U_\varepsilon/(U_\varepsilon \cdot E + U_\varepsilon \cdot (K - \lambda)).$$
Equivalently, $M(\lambda)$ has for highest weight vector the coset v_0 of $1 \in U_\varepsilon$, and has the basis $v_0, v_1, v_2, \ldots$ where v_i is the coset of F^i and the action of U_ε is given by
$$Kv_j = \lambda\varepsilon^{-2j} v_j \quad , \quad Fv_j = v_{j+1}$$
$$Ev_j = \begin{cases} [j]\frac{\lambda\varepsilon^{1-j} - \lambda^{-1}\varepsilon^{j-1}}{\varepsilon - \varepsilon^{-1}} v_{j-1}, & \text{if } j > 0 \\ 0, & \text{if } j = 0. \end{cases}$$
The linear independence of the v_i follows from Theorem 2.4. From the formulas we see that the modules $V_r^\pm$ and $V(\lambda, 0, b)$ are highest weight U_ε-modules of highest weights $\pm\varepsilon^r$ and λ, respectively, and that
$$V_r^\pm = M(\pm\varepsilon^r)/M_r\ ,\ V(\lambda, 0, b) = M(\lambda)/U_\varepsilon \cdot (v_n - bv_0),$$
where M_r is the submodule of $M(\pm\varepsilon^r)$ spanned by the v_i with $i > r$. Note that $U_\varepsilon \cdot (v_n - bv_0)$ is spanned by all $F^i(v_n - bv_0) = v_{i+n} - bv_i$, since $[n] = 0$ implies $Ev_n = Ev_0 = 0$, and $K(v_n - bv_0) = \lambda(v_n - bv_0)$.

The modules $V_r^\pm$ and $M(\lambda)$ extend to modules $V_{r,q}^\pm$ and $M(\lambda)_q$ over U_q for all values of q such that $q^2 \neq 1$, by replacing ε by q in all formulas. The U_q-modules $V_{r,q}^\pm$ are defined in any dimension, hence with no bound on r, and still satisfy $V_{r,q}^\pm = M(\pm q^r)_q/M_r$. They are simple modules, and any simple U_q-module is isomorphic to some $V_{r,q}^\pm$ ([J, Prop. 2.6], [Ks, Th. VI.3.5]). In fact, $M(\lambda)_q$ is simple if and only if $\lambda \neq \pm q^l$ for all integers $l \geq 0$. When

$\lambda = \pm q^l$, the v_i for $i > l$ span a submodule of $M(\lambda)_q$ isomorphic to $M(q^{-2(l+1)}\lambda)$, and this is the only non trivial proper submodule of $M(\lambda)_q$ [J, Prop. 2.5].

4.1.2. *Symmetries.* We have just claimed that the modules $M(\lambda)_q$ are not simple when $\lambda = \pm q^l$ for some integer $l \geq 0$. This follows from the fact that the vector v_{l+1} satisfies

$$Ev_{l+1} = \pm[l+1]\frac{q^l q^{1-(l+1)} - q^{-l}q^{(l+1)-1}}{q-q^{-1}}v_l = 0$$

and

$$Kv_{l+1} = \pm q^l q^{-2(l+1)}v_{l+1} = \pm q^{-l-2}v_{l+1}.$$

Hence we have an injection

$$M(\pm q^{-l-2})_q \longrightarrow M(\pm q^l)_q \tag{4.3}$$

mapping the highest weight vector v_0 of $M(\pm q^{-l-2})_q$, which is a simple U_q-module since $-l-2 \leq 0$, to v_{l+1}.
In particular, by specializing at $q = \varepsilon$ and taking the quotient by $U_\varepsilon \cdot (v_n - bv_0)$, we get that $V(\pm\varepsilon^r, 0, 0)$ is simple if and only if $r = n-1$, and there is for every $0 \leq r \leq n-2$ a (non split) short exact sequence of U_ε-modules:

$$0 \longrightarrow V^{\pm}_{n-r-2} \longrightarrow V(\pm\varepsilon^r, 0, 0) \longrightarrow V^{\pm}_r \longrightarrow 0. \tag{4.4}$$

The inclusion $V^{\pm}_{n-r-2} \to V(\pm\varepsilon^r, 0, 0)$ maps the highest weight vector $v_0 \in V^{\pm}_{n-r-2}$ to $v_{r+1} \in V(\pm\varepsilon^r, 0, 0)$.

Exercise 4.3. *(Application: the image of the Harish-Chandra homomorphism.)* Recall that π maps a central element $z = \sum_{i\geq 0} F^i h_i E^i$ of degree 0 to the Laurent polynomial $\pi(z) = h_0 \in \mathbb{C}[K, K^{-1}]$ (see (2.22)).
(a) Prove a statement analogous to Lemma 4.1 for finite dimensional highest weight U_q-modules V of highest weight $\lambda \in \mathbb{C}^*$ (possibly, not simple): show that when q is not a root of unity (resp. when $q = \varepsilon$), for all $v \in V$ and $z \in Z_q$ (resp. $Z_\varepsilon \cap U_0$) we have

$$zv = \pi(z)(\lambda)\ v. \tag{4.5}$$

(b) Deduce from (4.3) and (4.5) that when q is not a root of unity, for all $z \in Z_q$ and integer $l \geq 0$ we have

$$\pi(z)(\pm q^{l-1}) = \pi(z)(\pm q^{-l-1}).$$

Similarly, deduce from (4.4) and (4.5) that for all $z \in Z_\varepsilon \cap U_0$ and $1 \leq r \leq n-1$ we have

$$\pi(z)(\pm\varepsilon^{r-1}) = \pi(z)(\pm\varepsilon^{-r-1}).$$

(c) Show that the monomials $K^{in}(K+K^{-1})^j$ and K^l, where $i \in \mathbb{Z}$, $0 \leq j < n$, and $0 < l < n$, form a linear basis of $\mathbb{C}[K, K^{-1}]$.
(d) Deduce from (b) and (c) that when q is not a root of unity (resp. when $q = \varepsilon$), the homomorphism $\gamma_{-1} \circ \pi$ maps Z_q (resp. $Z_\varepsilon \cap U_0$) to $\mathbb{C}[K + K^{-1}]$ (resp. the polynomial subalgebra $\mathbb{C}[K^n, K^{-n}, K + K^{-1}]$ of $\mathbb{C}[K, K^{-1}]$).

By the discussion preceding Theorem 2.9, we know that $\gamma_{-1} \circ \pi$ is injective over Z_q (resp. $Z_\varepsilon \cap U_0'$) when q is not a root of unity (resp. when $q = \varepsilon$), and maps the Casimir element Ω to a scalar multiple of $K + K^{-1}$. Also, Z_ε is generated by E^n, F^n and $Z_\varepsilon \cap U_0'$, and $\gamma_{-1} \circ \pi(Z_\varepsilon \cap U_0') = \gamma_{-1} \circ \pi(Z_\varepsilon \cap U_0)$. Then, by Exercise 4.3 we have:

Proposition 4.4. *(i) When q is not a root of unity, we have an isomorphism*

$$\gamma_{-1}\circ\pi\colon Z_q \xrightarrow{\cong} \mathbb{C}[K+K^{-1}].$$

(ii) When $q=\varepsilon$, we have an isomorphism

$$\gamma_{-1}\circ\pi\colon Z_\varepsilon\cap U_0' \xrightarrow{\cong} \mathbb{C}[K^n,K^{-n},K+K^{-1}]$$

mapping $\mathbb{C}[\Omega]$ to $\mathbb{C}[K+K^{-1}]$. Moreover, Z_ε is generated by E^n, F^n, $K^{\pm n}$ and Ω.

4.1.3. *Classification.* The following result can be proved by elementary methods [J, §2.11-2.13].

Theorem 4.5. *The U_ε-modules $V_r^\pm$, $0\leq r\leq n-1$, and $V(\lambda,a,b)$ are simple, and any non zero simple U_ε-module is isomorphic to one of them.*

Remark 4.6. When $b\neq 0$ the module $V(\lambda,a,b)$ uniquely determines b but not λ and a. In fact, we get an isomorphic module by replacing v_0 by any other v_i, that is, λ by $\lambda\varepsilon^{-2i}$ and a by $a+[i](\lambda\varepsilon^{1-i}-\lambda^{-1}\varepsilon^{i-1})(\varepsilon-\varepsilon^{-1})^{-1}b^{-1}$.

By Proposition 4.4, any central character $\chi=\chi_V\in\operatorname{Spec}(Z_\varepsilon)$ is determined by its values at the generators x, y, z and Ω of Z_ε. Put

$$(x_\chi,y_\chi,z_\chi,c_\chi)=(\chi(x),\chi(y),\chi(z),\chi(\Omega))\in\mathbb{C}^4. \tag{4.6}$$

We have:

- if $V=V_r^\pm$, then

$$x_\chi=0\ ,\ y_\chi=0\ ,\ z_\chi=\pm 1\ ,\ c_\chi=\pm\frac{\varepsilon^{r+1}+\varepsilon^{-1-r}}{(\varepsilon-\varepsilon^{-1})^2}; \tag{4.7}$$

- If $V=V(\lambda,a,b)$, then

$$x_\chi=-(\varepsilon-\varepsilon^{-1})^n\lambda^{-n}a\prod_{j=1}^{n-1}\left(ab+[j]\frac{\lambda\varepsilon^{1-j}-\lambda^{-1}\varepsilon^{j-1}}{\varepsilon-\varepsilon^{-1}}\right) \tag{4.8}$$

$$y_\chi=(\varepsilon-\varepsilon^{-1})^n b\ ,\ z_\chi=\lambda^n\ ,\ c_\chi=ab+\frac{\lambda\varepsilon+\lambda^{-1}\varepsilon^{-1}}{(\varepsilon-\varepsilon^{-1})^2}. \tag{4.9}$$

Set

$$c_r^\pm=\pm\frac{\varepsilon^{r+1}+\varepsilon^{-1-r}}{(\varepsilon-\varepsilon^{-1})^2}.$$

From (4.4) or by a direct computation, we get

$$c_r^\pm=c_{n-r-2}^\pm,\quad 0\leq r\leq n-2. \tag{4.10}$$

Hence there are $n-1$ distinct values $c_r^\pm$ of the Casimir element, achieved at $r=0,1,\ldots,(n-3)/2$.
It will be useful to distinguish among central characters and U_ε-modules by using the map

$$\varphi\colon \operatorname{Rep}(U_\varepsilon)\xrightarrow{\Xi}\operatorname{Spec}(Z_\varepsilon)\xrightarrow{\tau}\operatorname{Spec}(Z_0)\xrightarrow{\sigma} PSL_2\mathbb{C}^0. \tag{4.11}$$

Recall the big cell decomposition $PSL_2\mathbb{C}^0=U_-TU_+$. Set

$$\mathcal{D}=\{\chi_r^\pm\in\operatorname{Spec}(Z_\varepsilon)\mid (x_{\chi_r^\pm},y_{\chi_r^\pm},z_{\chi_r^\pm},c_{\chi_r^\pm})=(0,0,\pm1,c_r^\pm),\ 0\leq r\leq (n-3)/2\}.$$

By (4.7)-(4.10) we have

$$\Xi^{-1}(\chi_r^\pm)=\{V_r^\pm,V_{n-r-2}^\pm\}\ ,\ \varphi^{-1}(\mathrm{Id})=\mathcal{D}. \tag{4.12}$$

Definition 4.7. A simple U_ε-module V and its central characters $\Xi(V)$ and $\tau\circ\Xi(V)$ are called:

- *diagonal* (resp. *triangular*) if $\varphi(V)\in T$ (resp. $\varphi(V)\in PB_\pm = U_-T$ or TU_+);
- *regular* (resp. *singular*) if $\varphi(V)\neq \mathrm{Id}$ (resp. $\varphi(V)=\mathrm{Id}$);
- *regular semisimple* if the conjugacy class of $\varphi(V)$ intersects $T\setminus \mathrm{Id}$.
- *cyclic* if E^n and F^n act as non zero scalars.

So, a point $\chi\in\mathrm{Spec}(Z_\varepsilon)$ is regular if $\chi\notin\mathcal{D}$. Regular semisimple characters are sent by π to loxodromic elements of $PSL_2\mathbb{C}$; the regular diagonal U_ε-modules are the $V(\lambda,0,0)$s with $\lambda^n\neq\pm1$.
The cyclic simple U_ε-modules have the property that any two eigenvectors of K can be obtained one from each other by applying some power of E or F; matrix realizations are immediately derived from the formulas of $V(\lambda,a,b)$ when $ab\neq 0$. They are complementary to the highest weight modules in $\mathrm{Rep}(U_\varepsilon)$, since a module on which F^n acts as zero and E^n does not is isomorphic to some $V(\lambda,0,b)$ by applying the Cartan automorphism (2.13). In particular, the cyclic U_ε-modules map under φ to a dense subset of $PSL_2\mathbb{C}$, and together with the diagonal modules they cover a neighborhood of the identity.

4.2. **Quantum coadjoint action and the bundle Ξ_M.** We are going to show that the central character map $\Xi\colon \mathrm{Rep}(U_\varepsilon)\to\mathrm{Spec}(Z_\varepsilon)$ can be used to define a bundle Ξ_M of U_ε-modules over $\mathrm{Spec}(Z_\varepsilon)$, endowed with an action of the subgroup $\mathcal{G}$ of $\mathrm{Aut}(\hat{U}_\varepsilon)$.
Let $V\in\mathrm{Rep}(U_\varepsilon)$, and

$$\rho_V : U_\varepsilon\to\mathrm{End}(V)$$

be the corresponding linear representation of U_ε. Recall (3.13) and Definition (3.7). Since Z_0 acts by scalar operators on V, the same is true for $\hat{Z}_0$, which consists of holomorphic functions of $x,\ y,z^{\pm1}\in Z_0$. Hence V is naturally a $\hat{U}_\varepsilon$-module. Since any $\hat{U}_\varepsilon$-module is uniquely determined by the action of U_ε on it, we will henceforth identify U_ε-modules and $\hat{U}_\varepsilon$-modules.

Definition 4.8. Given $g\in\mathcal{G}$, the *twisted* U_ε-module ${}^gV$ is defined by

$$\rho_{{}^gV}(u)=\rho(gu),\quad u\in U_\varepsilon.$$

Note that $\chi_{{}^gV}=\chi_V\circ g$ for all $g\in\mathcal{G}$ and $\chi\in\mathrm{Spec}(Z_\varepsilon)$, so the action of $\mathcal{G}$ on simple U_ε-modules lifts the opposite of the action of $\mathcal{G}$ on $\mathrm{Spec}(Z_\varepsilon)$.
The basic properties of the central character map Ξ are given by the following result. It is a special case of [DCK, §3.7-3.8 & Th. 4.2] (see also [DCP, §20]), which applies to the quantum groups of an arbitrary complex semi-simple Lie algebra. Recall the algebras U_ε^χ in (4.2).

Theorem 4.9. *(i) If $\chi\in\mathrm{Spec}(Z_\varepsilon)\setminus\mathcal{D}$, then $U_\varepsilon^\chi\cong M_n(\mathbb{C})$. Hence there is up to isomorphism a unique simple U_ε-module V_χ with central character χ; we have $V_\chi\cong V(\lambda,a,b)$ for any $(\lambda,a,b)\in\mathbb{C}^*\times\mathbb{C}^2$ satisfying (4.8)-(4.9).*
(ii) If $\chi\in\mathcal{D}$, there are exactly two simple U_ε-modules V_χ with central character χ: if $(x_\chi,y_\chi,z_\chi,c_\chi)=(0,0,\pm1,c_r^\pm)$ we have $\Xi^{-1}(\chi)=\{V_r^\pm,V_{n-r-2}^\pm\}$.
(iii) The restriction map $\tau\colon\mathrm{Spec}(Z_\varepsilon)\to\mathrm{Spec}(Z_0)$ has degree n, and the coordinate ring Z_ε is generated by Z_0 and Ω subject to any one of the equivalent relations $(R_\pm)$ given by

$$\prod_{j=0}^{n-1}\left(\Omega-c_j^\pm\right)=E^nF^n+\frac{K^n+K^{-n}\mp2}{(\varepsilon-\varepsilon^{-1})^{2n}}. \tag{4.13}$$

where, as usual, $c_j^\pm=\pm\frac{\varepsilon^{j+1}+\varepsilon^{-1-j}}{(\varepsilon-\varepsilon^{-1})^2}$.

Proof. (i) Consider a cyclic central character χ (so $\chi \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$). For any $V \in \Xi^{-1}(\chi)$, ρ_V induces an irreducible representation

$$\bar\rho_V : U_\varepsilon^\chi \to \mathrm{End}(V).$$

Because χ is cyclic, $\dim(V) = n$, and U_ε^χ has no non trivial proper ideal (by Theorem 2.4 the unit necessarily belongs to any non zero ideal). Hence $\bar\rho_V$ is faithful. Then, Wedderburn's theorem implies that $\bar\rho_V$ is an isomorphism, that is, $U_\varepsilon^\chi \cong M_n(\mathbb{C})$ [L, §XVII.3]. Conjugation with $\mathcal{G}$ yields a similar isomorphism for any module in the $\mathcal{G}$-orbit of V. Since any non trivial conjugacy class of $PSL_2\mathbb{C}$ contains an element whose entries are all non zero, by Theorem 3.8 the $\mathcal{G}$-orbit of any character $\chi \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ contains one which is cyclic. Hence $U_\varepsilon^\chi \cong M_n(\mathbb{C})$. Since $M_n(\mathbb{C})$ is a simple ring [L, §XVII.4 & 5], this implies the uniqueness of $V_\chi \in \Xi^{-1}(\chi)$ when $\chi \notin \mathcal{D}$.
(ii) is the content of (4.12).
(iii) Because of (i), in order to prove that $\deg(\tau) = n$ it is enough to find an open subset $\mathcal{O}$ of $\mathrm{Spec}(Z_0)$ (in the complex topology) such that for all $\tau(\chi) \in \mathcal{O}$, $(\tau \circ \Xi)^{-1}(\tau(\chi))$ consists of n simple U_ε-modules. In fact, this is true for any regular diagonal Z_0-character $\tau(\chi)$, for $(\tau \circ \Xi)^{-1}(\tau(\chi))$ consists of the $V(\lambda, 0, 0)$ with $\lambda^n = z_\chi$ ($\neq \pm 1$). Since the $\mathcal{G}$-action on $\mathrm{Rep}(U_\varepsilon)$ preserves the isomorphism type of the fibers of Ξ, and in particular their dimension and number of components, it is also true for any Z_0-character in the $\mathcal{G}$-orbit of a regular diagonal one. Hence we can take $\mathcal{O}$ to be the set of regular semisimple Z_0-characters. Theorem 3.8 implies that $\mathcal{O}$ is Zariski open and dense.
By Proposition 4.4 we know that Z_ε is generated by E^n, F^n, $K^{\pm n}$ and Ω. On another hand, (2.11) gives by an easy induction on r the relation

$$\prod_{j=0}^{r-1} \left(\Omega - \frac{\varepsilon^{2j+1}K + \varepsilon^{-2j-1}K^{-1}}{(\varepsilon - \varepsilon^{-1})^2} \right) = F^r E^r. \tag{4.14}$$

When $r = n$ this relation makes sense in Z_ε. To see this, let us expand the left-hand side as $\sum_{k=0}^n (-1)^k \sigma_k \Omega^{n-k}$. The coefficients σ_k are the elementary symmetric functions of the variables $x_j = \frac{\varepsilon^{2j+1}K+\varepsilon^{-2j-1}K^{-1}}{(\varepsilon-\varepsilon^{-1})^2}$, $0 \leq j \leq n-1$. It is classical that σ_k can be expressed as a polynomial with rational coefficient of the power sum functions $t_i = \sum_{j=0}^{n-1} x_j^i$, $0 \leq i \leq k$. Then, by using $\sum_{i=0}^{n-1} \varepsilon^{2i} = 0$ it is immediate to check that all σ_k are complex numbers not involving K, except $\sigma_n = -\frac{K^n+K^{-n}}{(\varepsilon-\varepsilon^{-1})^{2n}}$. Hence, for $r = n$ we can rewrite (4.14) as

$$\prod_{j=0}^{n-1} (\Omega - \alpha_j) = F^n E^n + \frac{K^n + K^{-n}}{(\varepsilon - \varepsilon^{-1})^{2n}}.$$

for some $\alpha_j \in \mathbb{C}$. The relation (4.13) follows from this by noting that the left-hand side (resp. right-hand side) acts as 0 (resp. $\pm 2(\varepsilon - \varepsilon^{-1})^{-2n}$) on all the simple U_ε-modules $V_r^\pm$ if and only if $\alpha_j = c_j^\pm$ for all $0 \leq j \leq n-1$. □

Remark 4.10. (i) We have seen in Theorem 2.7 that $\mathrm{rk}_{Z_0} U_\varepsilon = n^3$. From this one deduces that $\deg(\tau) = n$ and $V \in \Xi^{-1}(\chi)$ is unique for generic χ, by using that $\dim(V) = n$ for regular diagonal characters χ, as in the first part of the proof of Theorem 4.9 (iii).
(ii) It follows directly from (4.8)-(4.9) and Remark 4.6 that $\chi \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ determines uniquely $V(\lambda, a, b)$ up to isomorphism. This implies the uniqueness of $V_\chi \in \Xi^{-1}(\chi)$, but depends on the classification of the simple U_ε-modules, which is provided by Theorem 4.5.

Following [DCKP, §6] and [DCP, §11 & 21], we use Theorem 4.9 to provide the collection of simple regular U_ε-modules a structure of vector bundle over the smooth part of $\mathrm{Spec}(Z_\varepsilon)$.

To simplify notations, let

$$G = PSL_2\mathbb{C}.$$

It will be useful to distinguish between the Cartan subgroup $T = \mathbb{C}^*$ of H, and its image $\bar{T} = T/(\pm 1)$ under the 2-fold covering $\sigma\colon H \to G^0$.

Recall that G is an *affine* algebraic group, since the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ identifies G with a subgroup of the complex orthogonal group $SO_3\mathbb{C}$ for the Killing form of $\mathfrak{g}$. Denote by $G//G$ the affine variety with coordinate ring $\mathbb{C}[G]^G$, the ring of regular functions on G invariant under conjugation. The points of $G//G$ are in one to one correspondence with the conjugacy classes of elements of G having distinct traces up to sign, and we have isomorphisms

$$\mathbb{C}[G]^G \cong \mathbb{C}[\bar{T}]^W \ , \ G//G \cong \mathbb{C}^*/(z \sim \pm z^{-1}). \tag{4.15}$$

Here $W \cong \mathbb{Z}/2$ is the Weyl group of G acting on the torus $\bar{T} \subset G$ by inversion, and z is the coordinate function of $T \subset H$. Consider the maps

$$p_1 = p \circ \sigma\colon H \longrightarrow G//G \ , \ p_2 : G//G \longrightarrow G//G, \tag{4.16}$$

where $p\colon G \to G//G$ is the quotient map, and p_2 is induced by the n-th power map $g \mapsto g^n$, $g \in G$. The *fibered product* of p_1 and p_2 is the variety

$$H \times_{G//G} G//G = \{(h, \bar{g}) \in H \times G//G \mid p_1(h) = p_2(\bar{g})\}. \tag{4.17}$$

Theorem 4.11. *(i) The action of $\mathcal{G}$ on Spec(Z_0) extend to Spec(Z_ε) by a trivial action on Spec$(\mathbb{C}[\Omega])$. Hence, if $\mathcal{O}$ is an orbit of $\mathcal{G}$ in Spec(Z_0), the connected components of $\tau^{-1}\mathcal{O}$ are orbits of $\mathcal{G}$ in Spec(Z_ε).*

(ii) Spec(Z_ε) *is a 3-dimensional affine algebraic variety with singular set $\mathcal{D}$, and is isomorphic to $H \times_{G//G} G//G$.*

Proof. (i) The claim follows from Proposition 4.4 (ii) and the fact that for all q we have $\Omega \in Z_q$, so that $D_a(\Omega) = 0$ for all $a \in Z_0$.

(ii) According to Theorem 4.9 (iii), Spec(Z_ε) is the hypersurface in $\mathbb{C}^4$ given in coordinates (e, y, z, c) by (see (2.16)-(2.17))

$$(\varepsilon - \varepsilon^{-1})^{2n} \prod_{j=0}^{n-1} \left(c - c_j^{\pm}\right) = ey + z + z^{-1} \mp 2. \tag{4.18}$$

It follows from (4.10) that the singularities of Spec(Z_ε) are quadratic and coincide with $\mathcal{D}$ (note, in particular, that $(0, 0, \pm 1, c_{n-1}^{\pm})$ is a smooth point).

Let us prove now the isomorphism with (4.17). By Proposition 4.4 (ii) we have

$$Z_\varepsilon = Z_0 \otimes_{Z_0 \cap \mathbb{C}[\Omega]} \mathbb{C}[\Omega],$$

and $\mathbb{C}[\Omega] \cong \mathbb{C}[K + K^{-1}]$. Let us identify $\mathbb{C}[K^{\pm 1}]$ with $\mathbb{C}[T]$, the coordinate ring of the torus T of H. Then $\mathbb{C}[\Omega] \cong \mathbb{C}[T]^W$. By Proposition 3.1, $Z_0 \cong \mathbb{C}[H]$. The Harish-Chandra homomorphism applied to (4.13) gives $Z_0 \cap \mathbb{C}[\Omega] \cong \mathbb{C}[K^n + K^{-n}]$. On another hand, $\mathbb{C}[K^n + K^{-n}] \cong \mathbb{C}[T/\mu_n]^W$, where μ_n is the subgroup of T of n-th roots of unity. (Alternatively, Theorem 3.8 gives $Z_0^{\mathcal{G}} = \mathbb{C}[T/\mu_n]^W$, where $\mathbb{C}[T/\mu_n]^W$ is identified with the subalgebra of Z_0 generated by $z + z^{-1}$, and so $Z_0 \cap \mathbb{C}[\Omega] = \mathbb{C}[T/\mu_n]^W$ by (i) above). Hence

$$Z_\varepsilon \cong \mathbb{C}[H] \otimes_{\mathbb{C}[T/\mu_n]^W} \mathbb{C}[T]^W.$$

Since n is odd, this is equivalent to

$$Z_\varepsilon \cong \mathbb{C}[H] \otimes_{\mathbb{C}[\bar{T}/\mu_n]^W} \mathbb{C}[\bar{T}]^W.$$

The isomorphism of Spec(Z_ε) with $H \times_{G//G} G//G$ follows by duality. $\square$

Denote by $X = \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ the subset of regular central characters. By Theorem 4.11, it coincides with the smooth part of $\mathrm{Spec}(Z_\varepsilon)$.
Take a point $\chi \in X$. Any choice of parameters (λ, a, b) as in (4.8)-(4.9) determines an isomorphism of U_ε^χ-modules $V_\chi \cong V(\lambda, a, b)$, and an isomorphism of algebras $U_\varepsilon^\chi \cong M_n(\mathbb{C})$ by identifying $V(\lambda, a, b)$ with $\mathbb{C}^n$. Both isomorphisms are varying smoothly with respect to (λ, a, b) over a sufficiently small open neighborhood O_χ of χ in X. Hence we have coordinate charts $g_{O_\chi}\colon \coprod_{\rho \in O_\chi} U_\varepsilon^\rho \to O_\chi \times M_n(\mathbb{C})$ with smooth transition functions

$$g_{O_\chi} \circ g_{O_{\chi'}}^{-1} \colon (O_\chi \cap O_{\chi'}) \times M_n(\mathbb{C}) \to (O_\chi \cap O_{\chi'}) \times M_n(\mathbb{C})$$

that restrict to algebra automorphisms on the second component. By taking a locally finite covering of X made of neighborhoods O_χ we thus obtain a trivializing atlas for a smooth vector bundle of matrix algebras

$$\Xi_A \colon A_\varepsilon \to X, \tag{4.19}$$

where $A_\varepsilon = \coprod_{\chi \in X} U_\varepsilon^\chi$ as a set. By a "smooth bundle of algebras with unit" we mean that a smooth product with unit is defined on sections. For such bundles the functions on the base can be identified with multiples of the unit section.
Similarly, we have a rank n vector bundle

$$\Xi_M \colon M_\varepsilon \to X$$

with fibers the U_ε-modules $\Xi_M^{-1}(\chi) = V_\chi$. Clearly, Ξ_A and Ξ_M are associated bundles via the action of U_ε^χ on V_χ. They are topologically non trivial, since by using Remark 4.6 we can find a loop in $\mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ with non trivial holonomy. Namely, put $\varepsilon := e^{\frac{2\sqrt{-1}\pi d}{n}}$. For $b \neq 0$, the path in $\mathbb{C}^* \times \mathbb{C}^2$ given by

$$\gamma \colon t \longmapsto \left(\lambda e^{-\frac{4\sqrt{-1}\pi dt}{n}}, a + t\frac{\lambda - \lambda^{-1}}{b(\varepsilon - \varepsilon^{-1})}, b \right), \quad t \in [0,1],$$

projects to a loop in $\mathrm{Spec}(Z_\varepsilon)$ with holonomy in Ξ_M the permutation matrix $(\delta_{i+1,j})$ (indices mod n).

Theorem 4.12. *The group $\mathcal{G} \subset \mathrm{Aut}(\hat{U}_\varepsilon)$ acts on Ξ_A and Ξ_M by bundle morphisms.*

In particular, the orbits of $\mathcal{G}$ correspond to symplectic leaves (resp. conjugacy classes) of $\mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ (resp. $PSL_2\mathbb{C}^0$).

Proof. Since by Proposition 3.6 the group $\mathcal{G}$ maps the algebra of functions on $\mathrm{Spec}(Z_\varepsilon)$ into itself, its acts linearly on the fibers of Ξ_A and Ξ_M by automorphisms of algebras and U_ε-modules, respectively. More precisely, for any $g \in \mathcal{G}$ and $\chi \in \mathrm{Spec}(Z_\varepsilon)$, the action of g^{-1} on $\hat{U}_\varepsilon$ maps isomorphically the ideal $\mathcal{I}^\chi$ to $\mathcal{I}^{g \cdot \chi}$, and hence the algebra U_ε^χ to $U_\varepsilon^{g\cdot\chi}$ and its simple module V_χ to $V_{g \cdot \chi}$. □

Remark 4.13. (a) In the terminology of [RVW], Theorem 4.12 reflects the fact that the pair (U_ε, Z_0) is a *Poisson fibered algebra.* The infinitesimal action of $\mathcal{G}$ on Ξ_A defines a morphism of vector bundles $D\colon \Omega^1 X \to \mathcal{A}_\varepsilon$ with non trivial curvature, where $\mathcal{A}_\varepsilon$ is the bundle of first order differential operators on Ξ_A with symbols $\mathrm{Id} \otimes \xi$, $\xi \in TX$.
(b) The bundle Ξ_M extends to the whole of $\mathrm{Spec}(Z_\varepsilon)$, with singular (non simple) fibers $V(\pm\varepsilon^r, 0, 0)$, $0 \leq r \leq (n-3)/2$, over $\mathcal{D}$.

5. Intertwinners: the category U_ε-Mod

The Hopf algebra structure of U_ε endows the category U_ε-Mod of finite dimensional left U_ε-modules with a tensor product and a duality. We are going to describe them in geometric terms when applied to *regular* U_ε-modules, by using the theorems 4.11 and 4.12. To clarify the picture, after some preliminaries we recall well-known results on a subcategory based on singular modules and generating the Reshetikhin-Turaev TQFT.

5.1. **A few basic definitions** [Ks]**.** Unless stated otherwise all the modules we are going to consider will be left modules, finite dimensional over $\mathbb{C}$.
The tensor product vector space $V \otimes W$ of two U_ε-modules is naturally a $U_\varepsilon \otimes U_\varepsilon$-module. It is made into a U_ε-module by setting

$$a.(v \otimes w) = \Delta(a).(v \otimes w) \tag{5.1}$$

for all $a \in U_\varepsilon$, $v \in V$, $w \in W$, where $\Delta(a) \in U_\varepsilon \otimes U_\varepsilon$ is the coproduct of a. This action is compatible with the natural tensor product of linear maps of U_ε-modules, so it defines a bifunctor $\otimes\colon\ U_\varepsilon\text{-Mod} \times U_\varepsilon\text{-Mod} \to U_\varepsilon\text{-Mod}$.
Consider the trivial action of U_ε on $\mathbb{C}$ given by the counit, $a.z = \eta(a)z$. By (2.3), for all U_ε-modules U, V, W the formula (5.1) turns the canonical isomorphisms of vector spaces $l_V\colon \mathbb{C} \otimes V \cong V$, $r_V\colon V \otimes \mathbb{C} \cong V$ and

$$a_{U,V,W}\colon (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W) \tag{5.2}$$

into isomorphisms of U_ε-modules which are natural with respect to morphisms of U_ε-modules. Hence they define *natural isomorphisms* of functors l,r and a, with obvious compatibility relations. In particular

$$a\colon\ \otimes(\otimes \times \mathrm{id}) \longrightarrow \otimes(\mathrm{id} \times \otimes) \tag{5.3}$$

makes the following *Pentagonal Diagram* commutative:

$$\begin{array}{ccc}
(U \otimes (V \otimes W)) \otimes X & \xleftarrow{a_{U,V,W} \otimes \mathrm{id}_X} & ((U \otimes V) \otimes W) \otimes X \\
& & \downarrow a_{U\otimes V,W,X} \\
\downarrow a_{U,V\otimes W,X} & & (U \otimes V) \otimes (W \otimes X) \\
& & \downarrow a_{U,V,W\otimes X} \\
U \otimes ((V \otimes W) \otimes X) & \xrightarrow{\mathrm{id}_U \otimes a_{V,W,X}} & U \otimes (V \otimes (W \otimes X)).
\end{array} \tag{5.4}$$

The category U_ε-Mod endowed with $(\otimes, a, l, r)$ is an exemple of *tensor category*, with *associativity constraint* a and *unit* $\mathbb{C}$. Note that a explicits the different module structures at both sides.
The category U_ε-Mod has also a *(left) duality*, that is a pair (b, d) of natural transformations given on any U_ε-module V by morphisms

$$\begin{array}{llcl}
d_V\colon & V^* \otimes V & \longrightarrow & \mathbb{C} \\
b_V\colon & \mathbb{C} & \to & V \otimes V^*,
\end{array} \tag{5.5}$$

where V^* is an U_ε-module to be specified, satisfying

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V*} \otimes b_V) = \mathrm{id}_{V^*}.$$

Naturally we put $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, the dual linear space, and define d_V and b_V as the canonical pairing of vector spaces between V and V^*, and the map taking $1 \in \mathbb{C}$ to $\sum_i v_i \otimes v^i$, where $\{v_i\}$ and $\{v^i\}$ are dual basis of V and V^*. Then a (*left*) U_ε-module structure is defined on V^* by

$$d_V((a.\xi)\otimes v) = d_V(\xi\otimes(S(a).v)) \tag{5.6}$$

for all $a\in U_\varepsilon$, $v\in V$ and $\xi\in V^*$ (recall that the antipode $S\colon U_\varepsilon\to U_\varepsilon$ is an *anti*automorphism). By using (2.4) one can check that d_V and b_V are U_ε-linear maps. Any U_ε-linear map $f\colon V\to W$ has a *transpose* $f^*\colon W^*\to V^*$ given by

$$f^* = (d_W\otimes \mathrm{id}_{V^*})(\mathrm{id}_{W^*}\otimes f\otimes \mathrm{id}_{V^*})(\mathrm{id}_{W^*}\otimes b_V),$$

so that * defines a contravariant functor U_ε-Mod $\to U_\varepsilon$-Mod. As in the case of linear spaces there are natural bijections $\mathrm{Hom}_{U_\varepsilon}(U\otimes V,W)\cong \mathrm{Hom}_{U_\varepsilon}(U,W\otimes V^*)$ and $\mathrm{Hom}_{U_\varepsilon}(U^*\otimes V,W)\cong \mathrm{Hom}_{U_\varepsilon}(V,U\otimes W)$. Moreover, we get from (2.5) an isomorphism of U_ε-modules

$$V^*\otimes W^*\cong (W\otimes V)^*. \tag{5.7}$$

More generally, the vector space $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is an U_ε-module by setting

$$a.f(v) = \sum_i a_i'.f(S(a_i'').v) \tag{5.8}$$

for all $f\in \mathrm{Hom}_{\mathbb{C}}(V,W)$ and $v\in V$, where we put $\Delta(a)=\sum_i a_i'\otimes a_i''$. Note that by counitality in (2.3), the action (5.8) reduces to (5.6) when $W=\mathbb{C}$, and (5.1) and (5.6) imply that the canonical $\mathbb{C}$-linear isomorphism

$$\begin{array}{rccc}\lambda_{V,W}: & W\otimes V^* & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(V,W)\\ & w\otimes\xi & \longmapsto & (v\mapsto \xi(v)w)\end{array} \tag{5.9}$$

is an isomorphism of U_ε-modules. We have $\lambda^{-1}_{V,W}(f) = (f\otimes \mathrm{id}_{V^*})b_V$.

Duality allows one to define a trace in the category. For that, we use the remarkable fact that the square of the antipode S is an inner automorphism: for all $u\in U_\varepsilon$ we have $S^2(u) = KuK^{-1}$. Then, the linear isomorphism

$$\begin{array}{rccc}\Phi_V: & V & \longrightarrow & V^{**}\\ & v & \longmapsto & d_V(\,\cdot\otimes K.v)\end{array}$$

is also U_ε-linear. The *quantum trace* of V is the U_ε-linear map

$$\mathrm{tr}_q\colon \mathrm{End}_{\mathbb{C}}(V)\xrightarrow{\lambda^{-1}_{V,V}} V\otimes V^*\xrightarrow{\Phi_V\otimes \mathrm{id}_{V^*}} V^{**}\otimes V^*\xrightarrow{d_{V^*}}\mathbb{C}.$$

Explicitly, for all $f\in \mathrm{End}_{\mathbb{C}}(V)$ we have

$$\mathrm{tr}_q(f) = \mathrm{tr}\left(v\mapsto K.f(v)\right), \tag{5.10}$$

where tr is the usual trace map of linear spaces. The *quantum dimension* of V is $\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V)$. We say that V has *trace zero* if for all U_ε-linear endomorphisms f of V we have

$$\mathrm{tr}_q(f)=0.$$

Since $\Delta(K)=K\otimes K$ (K is said to be *group like*), the quantum trace is multiplicative: for all $f\in \mathrm{End}_{\mathbb{C}}(V)$, $g\in \mathrm{End}_{\mathbb{C}}(W)$ the quantum trace of $f\otimes g\in \mathrm{End}_{\mathbb{C}}(V\otimes W)$ is

$$\mathrm{tr}_q(f\otimes g) = \mathrm{tr}_q(f)\mathrm{tr}_q(g). \tag{5.11}$$

Exercise 5.1. (a) Check that all the U_ε-linear maps above are indeed U_ε-linear, and that the isomorphism $\lambda_{(V,W)}: W^*\otimes V^*\to (V\otimes W)^*$ in (5.7) can be decomposed as

$$\lambda_{(V,W)} = (d_W\otimes \mathrm{id}_{(V\otimes W)^*})(\mathrm{id}_{W^*}\otimes d_V\otimes \mathrm{id}_W\otimes \mathrm{id}_{(V\otimes W)^*})(\mathrm{id}_{W^*}\otimes \mathrm{id}_{V^*}\otimes b_{V\otimes W}).$$

(b) *(Right duality)* Show that we have U_ε-linear maps $d_V'\colon V\otimes {}^*V\to\mathbb{C}$ and $b_V'\colon \mathbb{C}\to {}^*V\otimes V$ analogous to (5.5) but with tensorands permuted, where ${}^*V$ is defined by (5.6) with d_V and

S replaced by d'_V and S^{-1}. Show that $\xi \mapsto K^{-1}.\xi$ defines an isomorphism of U_ε-modules $V^* \longrightarrow {}^*V$, and that $\mathrm{tr}_q(f) = d'_V(f \otimes \mathrm{id}_{V^*})b_V$.

5.2. **The modular category $\bar{U}_q$-Mod.** The *singular* simple U_ε-modules $V_r^\pm$ generate a remarkable subcategory of U_ε-Mod which has been described in [RT1] (in the simply-connected version of U_ε, see Remark 2.2).

For all $0 \leq r \leq n-1$, the action of U_ε on $V_r^\pm$ factors through the *restricted quantum group*

$$\bar{U}_\varepsilon = U_\varepsilon/(E^n = F^n = 0, K^{2n} = 1). \tag{5.12}$$

The algebra $\bar{U}_\varepsilon$ is *finite dimensional*, *not semisimple*, and has for simple modules the *finite set* $\{V_r^\pm, r = 0, \ldots, n-1\}$. Note that $V_0^+ = \mathbb{C}$ (the trivial module), and

$$\text{if } r \leq n-2, \quad \dim_q(V_r^\pm) = \pm[r+1] \neq 0. \tag{5.13}$$

On another hand, $\dim_q(V_{n-1}^\pm) = 0$. Hence, by Schur's lemma, $\mathrm{tr}_q(f) = 0$ for all U_ε-linear endomorphisms f of $V_{n-1}^\pm$, which has thus a special status among singular modules, reminiscent of the fact that $V_{n-1}^\pm = V(\pm\varepsilon^{-1}, 0, 0)$.

Definition 5.2. A color is an integer r such that $0 \leq r \leq n-2$.

There is a unique Hopf algebra structure on $\bar{U}_\varepsilon$ such that the quotient map $U_\varepsilon \to \bar{U}_\varepsilon$ is a morphism of Hopf algebras (see eg. [Ks, Prop. IX.6.1]). Since $V_r^- \cong V_0^- \otimes V_r^+$, we can concentrate on the modules $V_r := V_r^+$. They are *self dual*, and satisfy:

Theorem 5.3. [RT1, Th. 8.4.3] *For any colors i, j there is a unique trace zero submodule $Z_{i,j}$ of $V_i \otimes V_j$ such that*

$$V_i \otimes V_j \cong (\oplus_k V_k) \oplus Z_{i,j}, \tag{5.14}$$

where the sum is over all colors k such that the triple (i, j, k) is ε-admissible*, that is:*

- $i + j + k \in 2\mathbb{Z}$ *and* $i + j + k \leq 2(n-2)$*;*
- $|i - j| \leq k \leq i + j$ *(the* triangle inequalities*).*

When $i + j \leq n-2$ we have $Z_{i,j} = \emptyset$.

The modules $Z_{i,j}$ are built on the highest weight modules $V(\varepsilon^i, 0, 0)$ and some $2n$-dimensional extensions thereof. Since their Z_0-characters are trivial, like the color modules V_r we can consider them as singular U_ε-modules. Their duals and tensor products with color modules V_r decompose into summands of the same form, so the V_r generate a subcategory of U_ε-Mod which is closed under tensor product and duality.

Remark 5.4. The category $\bar{U}_\varepsilon$-Mod gives rise to the so called *fusion rules* of Wess-Zumino-Witten conformal field theories with gauge group $SU(2)$. In this context, the admissibility conditions of Theorem 5.3 give a method for counting dimensions of the space of conformal blocks [Ko, Prop. 1.19] (compare also (5.19) below and [Ko, Lemma 2.6]).

The associativity constraint (5.2), when applied to color modules and computed *modulo trace zero $\bar{U}_\varepsilon$-modules*, defines the celebrated *ε-6j-symbols*. Let us recall how this goes (see [CFS, §4] for details).

Theorem 5.3 implies that for each ε-admissible triple (i, j, k) the space of U_ε-linear embeddings $V_k \to (V_i \otimes V_j)/Z_{i,j}$ has dimension one. There is a natural basis of lifts, the *Clebsch-Gordan operator*

$$Y_{i,j}^k \colon V_k \longrightarrow V_i \otimes V_j, \tag{5.15}$$

defined in terms of the *Jones-Wentzl idempotents* e_l $(l = i, j, k)$ of the Temperley-Lieb algebra; these can be realized as projectors

$$e_l \colon V_1^{\otimes l} \to V_1^{\otimes l} \tag{5.16}$$

in the algebra of U_ε-linear transformations of $V_1^{\otimes l}$, whose image is isomorphic to V_l [CFS, Prop. 4.3.8].

By using (5.2) and (5.11), we get for all colors a, b, c a U_ε-linear isomorphism relating two splittings

$$\begin{aligned}(V_a \otimes V_b) \otimes V_c &= \oplus_{(l,k)} (Y^l_{a,b} \otimes \mathrm{id}_{V_c}) Y^k_{l,c}(V_k) \oplus Z_1\\ V_a \otimes (V_b \otimes V_c) &= \oplus_{(j,k)} (\mathrm{id}_{V_a} \otimes Y^j_{b,c}) Y^k_{a,j}(V_k) \oplus Z_2\end{aligned} \tag{5.17}$$

where the sums are over all pairs of colors (l, k) (resp. (j, k)), such that (a, b, l) and (l, c, k) (resp. (b, c, j) and (a, j, k)) are ε-admissible, and Z_1, Z_2 are maximal trace zero submodules of $V_a \otimes V_b \otimes V_c$. It follows from Exercise 5.6 (c) below that any U_ε-linear map $V_k \to V_a \otimes V_b \otimes V_c$ can be written *uniquely* as a linear combination of the maps $(Y^l_{a,b} \otimes \mathrm{id}_{V_c})Y^k_{l,c}$, plus a map whose image is contained in a trace zero summand of $V_a \otimes V_b \otimes V_c$. The *$\varepsilon$-$6j$-symbols*

$$\left\{ \begin{matrix} a & b & l \\ c & k & j \end{matrix} \right\}_\varepsilon \in \mathbb{C} \tag{5.18}$$

are thus defined by

$$(\mathrm{id}_{V_a} \otimes Y^j_{b,c})Y^k_{a,j} = \sum_l \left\{ \begin{matrix} a & b & l \\ c & k & j \end{matrix} \right\}_\varepsilon (Y^l_{a,b} \otimes \mathrm{id}_{V_c})Y^k_{l,c} + S \tag{5.19}$$

where S maps into a summand of trace zero and the sum is taken over all colors l such that (a, b, l) and (l, c, k) are ε-admissible. Consider the *normalized* ε-$6j$-symbols

$$\left[\begin{matrix} a & b & f \\ e & d & c \end{matrix} \right]_\varepsilon = \frac{(-1)^f}{[f+1]} \sqrt{\frac{\Theta(a,b,f)\Theta(d,e,f)}{\Theta(a,c,d)\Theta(b,c,e)}} \left\{ \begin{matrix} a & b & f \\ e & d & c \end{matrix} \right\}_\varepsilon \tag{5.20}$$

where we fix once and for all a square root, and (see (2.9))

$$\Theta(a,b,k) = (-1)^{\frac{a+b+k}{2}} \frac{[\frac{a+b-k}{2}]![\frac{a-b+k}{2}]![\frac{-a+b+k}{2}][\frac{a+b+k}{2}+1]!}{[a]![b]![k]!}.$$

To an abstract tetrahedron with edges colored by a, b, c, d, e, f, let us associate the scalar (5.20). We have:

Theorem 5.5. ([KR], [RT1]; see [CFS, Th. 4.4.6]) *The normalized ε-$6j$-symbols* (5.20) *are well-defined, and satisfy:*

- *Invariance under full tetrahedral symmetries.*
- *The Elliot-Biedenharn identity:*

$$\left[\begin{matrix} c & d & h \\ g & e & f \end{matrix} \right]_\varepsilon \cdot \left[\begin{matrix} b & h & k \\ g & a & e \end{matrix} \right]_\varepsilon = \sum_j (-1)^j [j+1] \left[\begin{matrix} b & c & j \\ f & a & e \end{matrix} \right]_\varepsilon \cdot \left[\begin{matrix} j & d & k \\ g & a & f \end{matrix} \right]_\varepsilon \cdot \left[\begin{matrix} c & d & h \\ k & b & j \end{matrix} \right]_\varepsilon . \tag{5.21}$$

Note that the ε-$6j$-symbols (5.18) are only partially symmetric. The commutativity of the Pentagonal Diagram (5.4) for color modules shows up in the Elliot-Biedenharn identity.

Exercise 5.6. (a) Why is the algebra $\bar{U}_\varepsilon$ not semisimple ? (Hint: otherwise every $\bar{U}_\varepsilon$-module would be semisimple, that is, a sum of simple modules.)
(b) We have claimed the self duality of the modules V_r: determine explicitly an isomorphism $V_r^* \to V_r$ (use (5.6) and the formulas in Section 4.1 !).
(c) By Theorem 5.3, for any maximal trace zero submodule U of $V_a \otimes V_b \otimes V_c$ which is a summand, the complementary submodule W such that $V_a \otimes V_b \otimes V_c = W \oplus U$ is completely reducible. Show that any simple submodule V of $V_a \otimes V_b \otimes V_c$ such that $V \not\subset U$ is a color module. Deduce that given any two maximal trace zero summands U_i $(i = 1, 2)$ of $V_a \otimes V_b \otimes V_c$, any simple submodule of U_1 is a submodule of U_2.

Hence, for every maximal trace zero submodule U of $V_a \otimes V_b \otimes V_c$, the matrix

$$\left(\left\{ \begin{array}{ccc} a & b & l \\ c & k & j \end{array} \right\}_\varepsilon \right)_{l,j} \tag{5.22}$$

of ε-$6j$-symbols relates the two basis of invariant maps $V_k \to V_a \otimes V_b \otimes V_c/U$ given by $\{(Y^l_{a,b} \otimes \mathrm{id}_{V_c})Y^k_{l,c}\}_l$ and $\{(\mathrm{id}_{V_a} \otimes Y^j_{b,c})Y^k_{a,j}\}_j$.

5.3. **Pure regular Clebsch-Gordan decomposition and $6j$-symbols.** Similarly to Theorem 5.3 and the change of basis matrix (5.22), the tensor products of simple regular U_ε-modules can be split in different ways, related by morphisms that we are going to define.

First we consider the duals of regular U_ε-modules. Recall the coordinates (4.6) of the set $\mathrm{Spec}(Z_\varepsilon)$ of central characters of U_ε, and the degree n covering map over the regular ones,

$$\tau\colon \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D} \to \mathrm{Spec}(Z_0) \setminus \{\pm\mathrm{id}\}.$$

Let $\chi \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$. Denote by V_χ the corresponding simple U_ε-module, and let χ^{-1} be given by

$$\tau(\chi^{-1}) = \tau(\chi)^{-1} \quad \text{and} \quad c_{\chi^{-1}} = c_\chi$$

where $\tau(\chi)^{-1}$ is the inverse of $\tau(\chi)$ in the group $\mathrm{Spec}(Z_0) \cong H$. From (3.6) and (5.6) we get:

Lemma 5.7. *The dual module V_χ^* coincides with $V_{\chi^{-1}}$. Hence the duality of U_ε-Mod induces an isomorphism of the bundle Ξ that lifts the inversion map on* $\mathrm{Spec}(Z_0) \setminus \{\pm\mathrm{id}\}$.

It is immediate to check that the regular simple U_ε-modules $V(\lambda, a, b)$ have vanishing quantum dimension, and so are trace zero modules. Thus, we are in some sense in a situation opposite to that of Section 5.2, where we dealt with the color modules V_r. What makes the tensor products of the regular $V(\lambda, a, b)$s easy to handle are the following facts, that we have proved in Lemma 2.8 and Theorem 4.9 (i):

- (a) Z_0 is a Hopf subalgebra of U_ε, and in particular $\Delta(Z_0) \subset Z_0 \otimes Z_0$;
- (b) for all $\chi \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$, the algebra U_ε^χ is simple, and isomorphic to $M_n(\mathbb{C})$.

Let $h \in \mathrm{Spec}(Z_0) \setminus \{\pm\mathrm{Id}\}$ be a regular Z_0-character, and $\mathcal{I}^h \subset U_\varepsilon$ the ideal generated by $\mathrm{Ker}(h)$. For any $\chi \in \tau^{-1}(h)$, the algebra $U_\varepsilon^h = U_\varepsilon/\mathcal{I}^h$ is isomorphic to $U_\varepsilon^\chi \otimes_h Z_\varepsilon$. Then, in virtue of (b) above, it is semisimple, with a direct product decomposition into complementary ideals

$$U_\varepsilon^h = \prod_{\chi \in \tau^{-1}(h)} U_\varepsilon^\chi. \tag{5.23}$$

Correspondingly, the unit $1 \in U_\varepsilon^h$ can be written as

$$1 = \sum_{\chi \in \tau^{-1}(h)} e_\chi, \tag{5.24}$$

where the e_χs are the units of the subalgebras $U_\varepsilon^\chi \subset U_\varepsilon^h$, and satisfy $U_\varepsilon^\chi = U_\varepsilon^h e_\chi$, and $e_\chi e_{\chi'} = 0$ for $\chi \neq \chi'$ [L, Prop. XVII.4.3].
Since U_ε^h is semisimple, every U_ε^h-module is semisimple, that is, a sum of simple submodules. On another hand, because of (a) above, Z_0 acts by scalars on any tensor product V of simple U_ε-modules, and so V is naturally a U_ε^h-module for some $h \in \mathrm{Spec}(Z_0)$. In fact, (3.6) shows that h is equal to the product of the Z_0-characters of the tensorands.
This applies in particular to the U_ε-modules $V_\rho \otimes V_\mu$ for all $\rho,\ \mu \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$. They are U_ε^h-modules, where $h = \tau(\rho)\tau(\mu)$. If $h \neq \pm\mathrm{id}$, U_ε^h is semisimple, and hence

$$V_\rho \otimes V_\mu = 1.(V_\rho \otimes V_\mu) = \oplus_{\chi \in \tau^{-1}(h)}\ e_\chi.(V_\rho \otimes V_\mu), \tag{5.25}$$

where the projectors e_χ map $V_\rho \otimes V_\mu$ onto a submodule isomorphic to V_χ. We deduce the following analog of Theorem 5.3 for regular U_ε-modules:

Proposition 5.8. *(i) Any tensor product of simple U_ε-modules which has a regular Z_0-character is a semisimple U_ε-module.*
(ii) If $\rho,\ \mu \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ are such that $h = \tau(\rho)\tau(\mu) \neq \pm\mathrm{id}$, then

$$V_\rho \otimes V_\mu \cong \oplus_{\chi \in \tau^{-1}(h)} V_\chi. \tag{5.26}$$

Remark 5.9. The idempotents e_χ of U_ε^h play in Proposition 5.8 the same role as the Jones-Wentzl idempotents (5.16) do in Theorem 5.3. In both cases, the simple summands are distinguished by the Casimir element. For color modules, its values are determined by the classical or quantum dimension. For regular modules, these are constantly equal to n and 0, respectively; the Casimir element selects a nth-root of z (see Theorem 4.11 (ii)).

Definition 5.10. Let $h \in \mathrm{Spec}(Z_0) \setminus \{\pm\mathrm{id}\}$ be a regular Z_0-character. The *multiplicity module* $M(h)$ is the n dimensional vector space spanned by the idempotents e_χ of U_ε^h, where $\chi \in \tau^{-1}(h)$.
A tuple $(\rho_1, \ldots, \rho_p)$ of regular central characters $\rho_i \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ is *regular* if for all $1 \leq k \leq k + l \leq p$, we have

$$\tau(\rho_k)\tau(\rho_{k+1}) \ldots \tau(\rho_{k+l}) \in \mathrm{Spec}(Z_0) \setminus \{\pm\mathrm{id}\}.$$

By Theorem 4.9 (i) we have a corresponding notion of regular tuples of U_ε-modules.

Remark 5.11. For any two $h,\ h' \in \mathrm{Spec}(Z_0) \setminus \{\pm\mathrm{id}\}$, the decomposition (5.24) provides canonical isomorphisms $M(h) \cong M(h')$. Also, Proposition 5.8 (ii) implies that any two regular pairs (ρ, μ) and (ν, κ) satisfying $h = \tau(\rho)\tau(\mu) = \tau(\nu)\tau(\kappa)$ give isomorphic U_ε-modules $V_\rho \otimes V_\mu$ and $V_\nu \otimes V_\kappa$.

We can reorganize the direct sum $\oplus_{\chi \in \tau^{-1}(h)} V_\chi$ into a tensor product as follows. Let V be the vector space underlying the modules V_χ. Define an action of U_ε on $V \otimes M(h)$ by extending linearly the formula

$$a.(v \otimes e_\chi) = (a_\chi.v) \otimes e_\chi \tag{5.27}$$

for all $a \in U_\varepsilon$ and $v \in V$, where $a_\chi \in U_\varepsilon^\chi$ is the coset of a, with its canonical action on V. We have a canonical isomorphism $\oplus_{\chi \in \tau^{-1}(h)} V_\chi \cong V \otimes M(h)$ of U_ε-modules, mapping $v \in V_\chi$ to $v \otimes e_\chi$. Then, for all regular pairs (ρ, μ) with $h = \tau(\rho)\tau(\mu)$, any isomorphism of the form (5.26) defines an isomorphism of U_ε-modules

$$\begin{array}{rccl} F(\rho,\mu): & V \otimes V & \longrightarrow & V \otimes M(h) \\ & v_1 \otimes v_2 & \longmapsto & \sum_{\chi \in \tau^{-1}(h)} e_\chi(v_1 \otimes v_2) \otimes e_\chi \end{array} \tag{5.28}$$

by putting the U_ε-module structure of $V_\rho \otimes V_\mu$ on $V \otimes V$; when this structure is clear from the context, we write F for $F(\rho,\mu)$, and similarly for the inverse *evaluation map*

$$K(\rho,\mu)\colon V \otimes M(h) \longrightarrow V \otimes V. \tag{5.29}$$

Let $\Delta_{(\rho,\mu)}\colon U_\varepsilon \to U_\varepsilon^\rho \otimes U_\varepsilon^\mu$, $a \mapsto \Delta(a) \bmod(\mathcal{I}^\rho \otimes \mathcal{I}^\mu)$. The formula (5.27) shows that $K(\rho,\mu)$ and $\Delta_{(\rho,\mu)}$ are equivalent data, related by

$$\Delta_{(\rho,\mu)}(a) = K(\rho,\mu)(a \otimes 1)K(\rho,\mu)^{-1} \tag{5.30}$$

as operators in $\mathrm{Aut}(V_\rho \otimes V_\mu)$.

Remark 5.12. Since the modules V_χ are simple, $\mathrm{End}_{U_\varepsilon}(V_\chi) \cong \mathbb{C}$, and so $K(\rho,\mu)$ is uniquely determined up to a scalar factor only. We can reduce this to an ambiguity modulo *nth roots of unity* by requiring that $\det(K(\rho,\mu)) = 1$. This produces a degree n polynomial in the coordinates of ρ, $\mu \in \mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$.

We are now ready to define the operators and functional relations describing the isomorphisms

$$(V_\rho \otimes V_\mu) \otimes V_\nu \cong V_\rho \otimes (V_\mu \otimes V_\nu) \cong \oplus_{\chi \in \tau^{-1}(h)}(\underbrace{V_\chi \oplus \ldots \oplus V_\chi}_{n \text{ times}})$$

for regular triples (ρ,μ,ν), where $h = \tau(\rho)\tau(\mu)\tau(\nu)$.

Definition 5.13. Let (ρ,μ,ν) be a regular triple of U_ε-modules. Put $f = \tau(\rho)$, $g = \tau(\mu)$, $h = \tau(\nu)$. The *regular 6j-symbol operator* of (ρ,μ,ν) is the linear isomorphism

$$\mathcal{R}(\rho,\mu,\nu)\colon M(fgh) \otimes M(gh) \longrightarrow M(fg) \otimes M(fgh) \tag{5.31}$$

that makes the following diagram commutative:

$$\begin{array}{ccc}
V \otimes M(fgh) \otimes M(gh) & \xrightarrow{\mathrm{id}\otimes\mathcal{R}(\rho,\mu,\nu)} & V \otimes M(fg) \otimes M(fgh) \\
\Big\downarrow{\scriptstyle (\mathrm{id}\otimes\Delta_0)(K)} & & \Big\downarrow{\scriptstyle (\Delta_0\otimes\mathrm{id})(K)} \\
V_\rho \otimes V \otimes M(gh) & & V \otimes M(fg) \otimes V_\nu \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes K} & & \Big\downarrow{\scriptstyle K\otimes\mathrm{id}} \\
V_\rho \otimes (V_\mu \otimes V_\nu) & \xleftarrow[a_{V_\rho,V_\mu,V_\nu}]{} & (V_\rho \otimes V_\mu) \otimes V_\nu
\end{array} \tag{5.32}$$

where $\Delta_0(x) = x \otimes 1$ (the *standard* coproduct). In operator form, we have

$$K_{12}(\rho,\mu)K_{13}(\chi_r,\nu)\mathcal{R}_{23}(\rho,\mu,\nu) = K_{23}(\mu,\nu)K_{12}(\rho,\chi_l) \tag{5.33}$$

for all $\chi_r \in \tau^{-1}(fg)$, $\chi_l \in \tau^{-1}(gh)$.

Thus, $\mathcal{R}$ determines via (5.33) the associativity constraint a of U_ε-Mod over regular modules, like K determines via (5.30) the tensor product of U_ε-Mod over regular modules. The Pentagonal Diagram (5.4) translates as follows:

Proposition 5.14. *Let* (κ, ρ, μ, ν) *be a regular* 4*-tuple of* U_ε*-modules. Set* $f = \tau(\kappa)$, $g = \tau(\rho)$, $h = \tau(\mu)$, $k = \tau(\nu)$. *The following diagram is commutative:*

$$\begin{array}{ccc}
M(fghk) \otimes M(gh) \otimes M(ghk) & \xleftarrow{\mathrm{id}\otimes\mathcal{R}(\rho,\mu,\nu)} & M(fghk) \otimes M(ghk) \otimes M(hk) \\
\Big\downarrow (\Delta_0 \otimes \mathrm{id})(\mathcal{R}) & & \Big\downarrow (\mathrm{id}\otimes\Delta_0)(\mathcal{R}) \\
 & & M(fg) \otimes M(fghk) \otimes M(hk) \\
 & & \Big\downarrow (\tau\Delta_0\otimes\mathrm{id})(\mathcal{R}) \\
M(fgh) \otimes M(gh) \otimes M(fghk) & \xrightarrow{\mathcal{R}(\kappa,\rho,\mu)\otimes\mathrm{id}} & M(fg) \otimes M(fgh) \otimes M(fghk)
\end{array}$$

where τ *is the flip map. In operator form, we have the* Pentagon Equation*:*

$$\mathcal{R}_{12}(\kappa, \rho, \mu)\mathcal{R}_{13}(\kappa, \chi_1, \nu)\mathcal{R}_{23}(\rho, \mu, \nu) = \mathcal{R}_{23}(\chi_2, \mu, \nu)\mathcal{R}_{12}(\kappa, \rho, \chi_3) \tag{5.34}$$

for all $\chi_1 \in \tau^{-1}(gh)$, $\chi_2 \in \tau^{-1}(fg)$, $\chi_3 \in \tau^{-1}(hk)$.

Proof. Commutativity of the diagram is equivalent to (5.34). The latter is proved by a straightforward computation using (5.33); details are left as an exercise (or, compare with the proof of Proposition 6.12 iii)). □

Remark 5.15. *(The Borel case)* The notion of regular module makes sense as well for any Borel subalgebra $U_\varepsilon b$ of U_ε, say the positive one, generated by $K^{\pm 1}$ and E. Since $U_\varepsilon b$ has no Casimir element, the isomorphism classes of simple regular $U_\varepsilon b$-modules correspond under the map φ of (4.11) to triangular matrices up to sign in $PSL_2\mathbb{C} \setminus \{\pm \mathrm{Id}\}$. Then, for regular pairs (ρ, μ) of simple $U_\varepsilon b$-modules, (5.26) simplifies to a splitting into n isomorphic copies of a single regular $U_\varepsilon b$-module $V_{\rho\mu}$. The multiplicity module $M(h)$ becomes the space of equivariant projections $\mathrm{Hom}_{U_\varepsilon b}(V_\rho \otimes V_\mu, V_{\rho\mu})$, and (5.29) is the map

$$\begin{array}{rccc}
K: & V_{\rho\mu} \otimes \mathrm{Hom}_{U_\varepsilon b}(V_{\rho\mu}, V_\rho \otimes V_\mu) & \longrightarrow & V_\rho \otimes V_\mu \\
 & v \otimes i & \longmapsto & i(v).
\end{array}$$

Exercise 5.16. Let V be a finite dimensional vector space and f, $g \in \mathrm{End}(V)$ such that $f^2 = f$, $g^2 = g$ and $fg = gf$. Show that $R := f \otimes g$ satisfies

$$R_{12}R_{13}R_{23} = R_{23}R_{12} \tag{5.35}$$

in $\mathrm{End}(V \otimes V \otimes V)$. Show that if $R := f \otimes \mathrm{id}$ or $R := \mathrm{id} \otimes f$ is a solution of (5.35), then $f^2 = f$.

6. The regular 6j-symbols as bundle morphisms

The FRT method associates a cobraided Hopf algebra to any finite dimensional invertible solution of the Quantum Yang-Baxter Equation (see eg. [Ks, Ch. VIII]). Similarly, any finite dimensional invertible solution of the *constant* Pentagon Equation (5.35) is the canonical element of the *Heisenberg double* of some Hopf algebra [BS, Mi, D].
We are going to see how this result can be adapted to equation (5.34), which has the form of a (non-Abelian) 3-cocycle identity over the group $PSL_2\mathbb{C}$ via the map $\sigma\tau\colon \mathrm{Spec}(Z_\varepsilon) \to PSL_2\mathbb{C}$. We proceed in several steps to identify both the evaluation map (5.29) and the regular 6j-symbol operator (5.31) as instances of a same bundle morphism $\mathcal{R}\colon \Xi^{(2)} \longrightarrow \Xi^{(2)}$.

6.1. **The QUE algebra U_h.** The *quantum universal envelopping (QUE) algebra* $U_h = U_h(sl_2)$ is the Hopf algebra over $\mathbb{C}[[h]]$ topologically generated by three variables X, Y and H with relations

$$[H,X]=2X\ ,\ [H,Y]=-2Y\ ,\ [X,Y]=\frac{\sinh(hH/2)}{\sinh(h/2)}=\frac{e^{hH/2}-e^{-hH/2}}{e^{h/2}-e^{-h/2}}.$$

(See eg. [CP, Ch. 6–8] and [Ks, Ch. XVII].) The comultiplication and counit are determined by

$$\Delta(H)=H\otimes 1+1\otimes H,$$
$$\Delta(X)=\ X\otimes e^{hH/4}+e^{-hH/4}\otimes X\ ,\ \Delta(Y)=Y\otimes e^{hH/4}+e^{-hH/4}\otimes Y$$

and

$$\eta(H)=\eta(X)=\eta(Y)=0.$$

The antipode is

$$S(H)=-H\ ,\ S(X)=-e^{hH/2}X\ ,\ S(Y)=-e^{-hH/2}Y.$$

The QUE algebra U_h is a *topological deformation* of the universal enveloping algebra Usl_2, in the sense that $U_h\cong Usl_2[[h]]$ as $\mathbb{C}[[h]]$-modules, and $U_h/hU_h\cong Usl_2$ as Hopf algebras, where Usl_2 has the Hopf algebra structure determined by

$$\Delta(x)=x\otimes 1+1\otimes x\ ,\ \eta(x)=0\ ,\ S(x)=-x \tag{6.1}$$

for all $x\in sl_2$. In particular, U_h is equipped with the h-adic topology, and is a *topologically* free $\mathbb{C}[[h]]$-module; the algebraic tensor product $U_h\otimes U_h$ is thus equally completed in h-adic topology. Then, the coproduct Δ is well-defined, and all the above structure maps are continuous.

Let us identify U_q with the algebra U_q' of Exercise 2.3. The latter is explicitly defined for all values of q and inherits from U_q a structure of Hopf algebra. There is an injective morphism of Hopf algebras $i\colon U_q'\to U_h$, given by $i(q)=e^{h/2}$ and

$$\begin{array}{lcl} i(E)=Xe^{hH/4} & , & i(F)=e^{-hH/4}Y\ , \\ i(K^{\pm1})=e^{\pm hH/2} & , & i(L)=XY-e^{-hH/4}YXe^{hH/4}. \end{array} \tag{6.2}$$

Hence, for what regards properties independent of the specific evaluation $q=\varepsilon$, we will consider U_q as a subHopf algebra of U_h.

6.1.1. *The QUE dual U_h°.* Duality is a delicate question for infinite dimensional algebras. We say that two Hopf algebras A and A' over a ground ring k are *dual* if there exists a bilinear pairing $\langle\ ,\ \rangle\colon A\otimes A'\to k$ which is non degenerate in the sense that for all $f\in A'$ (resp. $u\in A$), $\langle u,f\rangle=0$ for all $u\in A$ (resp. $f\in A'$) implies $f=0$ (resp. $u=0$), and

$$\begin{array}{c} \langle u,fg\rangle=\langle\Delta(u),f\otimes g\rangle\ ,\ \langle u\otimes v,\Delta(f)\rangle=\langle uv,f\rangle \\ \langle u,S(f)\rangle=\langle S(u),f\rangle\ ,\ \eta(f)=\langle 1,f\rangle\ ,\ \eta(u)=\langle u,1\rangle \end{array} \tag{6.3}$$

for all $u,\ v\in A,\ f,\ g\in A'$, where we denote by the same letters the structure maps of A and A'. When suitably interpreted, these formulas give indeed the linear dual $U_h^*=\mathrm{Hom}_{\mathbb{C}[[h]]}(U_h,\mathbb{C}[[h]])$ a structure of topological Hopf algebra dual to U_h for the natural pairing $\langle\ ,\ \rangle\colon U_h\otimes U_h^*\to\mathbb{C}[[h]]$ [CP, Def. 6.3.3]. However U_h^* is not a QUE algebra since there does not exist any Lie algebra $\mathfrak{g}$ such that $U_h^*/hU_h^*\cong U\mathfrak{g}$ as Hopf algebras (otherwise U_h^* should be cocommutative up to first order by (6.1)).

A way to repair this inconvenience is to consider the space

$$U_h^\circ=\sum_{l\geq 0}h^{-l}I^l,$$

where $I = \sum_{i,j,k}(X^iY^jH^k)^*U_h^*$ is the maximal ideal of U_h^*, the elements $(X^iY^jH^k)^*$ being dual to PBW basis vectors of Usl_2 and the sums completed in h-adic topology. Recall from Corollary 3.2 and Remark 3.4 that the Poisson-Lie structure of $H \cong \mathrm{Spec}(Z_0)$ given by the bracket $\psi_*\{\ ,\ \}$ is dual to the so called standard one on $PSL_2\mathbb{C}$. We have:

Proposition 6.1. [ES, Prop. 10.3] *U_h° is a Hopf algebra dual to U_h under the completion of the natural pairing $\langle\ ,\ \rangle\colon U_h \otimes U_h^* \to \mathbb{C}[[h]]$. It is isomorphic to the QUE algebra $U_h(\mathfrak{h})$, where $\mathfrak{h}$ is the Lie algebra of H with bialgebra structure tangent to $\psi_*\{\ ,\ \}$.*

We call $U_h^\circ = U_h\mathfrak{h}$ the *QUE dual* of U_h.

6.1.2. *The finite dual.* There is another notion of dual Hopf algebra that we will meet in the sequel. Its definition makes sense for arbitrary Hopf algebras (see [Mo, § 9.1] or [KS, Ch. 3, Prop. 1.1.3]):

Definition 6.2. The *finite dual* of a k-algebra A is the subspace $A^\bullet$ of the linear dual A^* defined by

$$A^\bullet = \{f \in A^* \mid f(I) = 0 \text{ for some ideal } I \text{ such that } \dim_{\mathbb{C}}(A/I) < \infty\}.$$

We have:

Proposition 6.3. *If A is a Hopf algebra, the finite dual $A^\bullet$ is a Hopf algebra, with structure maps defined by* (6.3) *under the natural pairing $\langle\ ,\ \rangle\colon A \otimes A^* \to k$. Moreover, $A^\bullet$ is the largest subspace V of A^* with coproduct in $V \otimes V$.*

Note that the coproduct of A always gives A^* an algebra structure; only the product may cause some trouble, as its adjoint for $\langle\ ,\ \rangle$ may not map into the subspace $A^*\otimes A^* \subset (A\otimes A)^*$. The finite dual $A^\bullet$ can be equivalently defined as the subalgebra of A^* generated by the *matrix elements* of all finite dimensional A-modules V, that is, by the linear functionals

$$c_{l,v}^V\colon \begin{array}{ccc} A & \longrightarrow & \mathbb{C} \\ a & \longmapsto & l(a.v) \end{array} \tag{6.4}$$

where $v \in V$, $l \in V^*$. In these terms, Proposition 6.3 follows from the fact that the category of finite-dimensional A-modules is closed under taking duals, direct sums and tensor products. A topological interpretation of $A^\bullet$ is as follows. Let

$$\mathcal{J} = \{\mathrm{Ker}(\rho) \mid \rho\colon A \to \mathrm{End}(V) \text{ is a finite dimensional representation}\}.$$

Define the *$\mathcal{J}$-adic topology* on A by taking as a base of neighborhoods of $a \in A$ the sets $\{a + J \mid J \in \mathcal{J}\}$. Similarly, define a topology on $A\otimes A$ by taking as a base of neighborhoods of $a\otimes b \in A\otimes A$ the sets $\{a\otimes b + L \mid L = A\otimes J + K \otimes A \text{ and } J, K \in \mathcal{J}\}$. Then A is a topological algebra (all the structure maps are continuous), and if the ground ring k of A is given the discrete topology, $A^\bullet$ is the set of continous k-linear maps $A \to k$. For $A = U_h$, by taking representations ρ on free $\mathbb{C}[[h]]$-modules of finite rank, the finite dual $U_h^\bullet$ is the set of $\mathbb{C}[[h]]$-linear maps $U_h \to \mathbb{C}[[h]]$ which are continuous for the $\mathcal{J}$-adic topology. It is usually called the *quantized function ring*, and denoted by $SL_h(2)$ [CP, Th. 7.1.4].

The finite dual $U_q^\bullet$ is defined also for all values of $q \in \mathbb{C}$, $q \neq -1,\ 0,\ 1$. At $q = \varepsilon$ it is very big, as it contains the representative functions on the bundle Ξ_A of Theorem 4.12 (compare eg. with Theorem 6.7 below). So $U_q^\bullet$ is usually defined in the litterature by restricting the matrix elements (6.4) to the U_q-modules $V_{r,q}^\pm$; then it coincides with the rational form $SL_q(2)$ of $SL_h(2)$ (see [CP, Ch. 13], [Ks, Ch. VII], [KS, Ch. 3]).

6.2. **The Heisenberg double $\mathcal{H}_h = \mathcal{H}(U_h)$.** The Heisenberg double of U_h is a topological $\mathbb{C}[[h]]$-module isomorphic to $U_h \otimes U_h^\circ$, with an algebra structure given by a *smash product*. Let us recall this notion (see [Mo]).

Definition 6.4. Let A be a Hopf k-algebra and V a left A-module which is simultaneously an algebra. We say that V is an *A-module algebra* if both its product $V \otimes V \to V$ and unit $k \to V$ are morphisms of A-modules. The *smash product* $V \sharp A$ of A and an A-algebra V is the algebra isomorphic to $V \otimes A$ as a vector space, with product given by

$$(u \sharp a)(v \sharp b) = u(a_{(1)}.v) \sharp a_{(2)}b$$

for all u, $v \in V$ and a, $b \in A$, where we put $\Delta(a) = a_{(1)} \otimes a_{(2)}$ (Sweedler's sigma notation).

The smash product $V \sharp A$ encodes the commutation relations between the linear operators induced by the left action of V on itself, and the linear operators induced by the action of A on V. Indeed, one checks without difficulty that:

Lemma 6.5. *The map*

$$\begin{array}{rccc} \lambda: & V \sharp A & \longrightarrow & \mathrm{End}(V) \\ & v \sharp a & \longmapsto & (u \mapsto v(a.u)) \end{array} \tag{6.5}$$

defines a representation of $V \sharp A$ *(the* Heisenberg representation*).*

Definition 6.6. Let $\langle\ ,\ \rangle\colon A \otimes A' \to k$ be a duality of Hopf k-algebras A and A'. The *Heisenberg double* $\mathcal{H}(A)$ is the smash product $A \sharp A'$, where A is made into an A'-module via the *left regular action*

$$\begin{array}{rccc} \rightharpoonup: & A' \otimes A & \longrightarrow & A \\ & x \otimes a & \longmapsto & x \rightharpoonup a = a_{(1)}\langle a_{(2)}, x\rangle. \end{array} \tag{6.6}$$

For Heisenberg doubles, the fact that (6.5) defines a representation follows from the commutation relation

$$(1 \sharp x)(a \sharp 1) = (a_{(1)} \sharp 1)(1 \sharp \langle a_{(2)}, x_{(1)}\rangle x_{(2)}) \tag{6.7}$$

for all $a \in A$, $x \in A'$. In particular, the Heisenberg double $\mathcal{H}_h = \mathcal{H}(U_h)$ is the topological algebra isomorphic to $U_h \otimes U_h^\circ$ as a $\mathbb{C}[[h]]$-module, with product given by

$$(u \sharp x)(v \sharp y) = u(x_{(1)} \rightharpoonup v) \sharp x_{(2)}y = \langle v_{(2)}, x_{(1)}\rangle uv_{(1)} \sharp x_{(2)}y$$

for all u, $v \in U_h$, x, $y \in U_h^\circ$.

6.2.1. *A classical example: the cotangent bundle T^*G.* The Heisenberg double $\mathcal{H}_h$ may be understood as a deformation of the following classical geometric situation. Let G be an affine algebraic group over $\mathbb{C}$. There are three "classical" Hopf algebras associated to G:

- The group algebra $\mathbb{C}G$, where $\Delta(g) = g \otimes g$, $\eta(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$.
- The coordinate ring $\mathbb{C}[G] \subset \mathbb{C}G^\bullet$ (as the subset of algebraic maps).
- The universal envelopping algebra $U\mathfrak{g}$, with structure maps (6.1).

Note that the adjoint action of G on $\mathfrak{g}$ extends linearly to give an action of $\mathbb{C}G$ on $U\mathfrak{g}$ by Hopf algebra automorphisms. Denote by e the identity element of G, and identify $U\mathfrak{g}$ with the space of all left invariant differential operators on G. For simplicity, assume that G is connected, simply connected, and semisimple. We have:

Theorem 6.7. (See [A, Th. 4.3.13]) *The pairing*

$$\begin{array}{rccc} \langle\ ,\ \rangle: & U\mathfrak{g} \times \mathbb{C}[G] & \longrightarrow & \mathbb{C} \\ & X \otimes f & \longmapsto & (X.f)(e) \end{array} \tag{6.8}$$

induces an injective morphism $U\mathfrak{g} \to \mathbb{C}[G]^\bullet$ *and an isomorphism* $\mathbb{C}[G] \to U\mathfrak{g}^\bullet$. *The Hopf algebra* $\mathbb{C}[G]^\bullet$ *is generated by* $U\mathfrak{g}$ *and the evaluation maps* $f \mapsto f(g)$, $g \in G$. *More precisely,* $\mathbb{C}[G]^\bullet \cong U\mathfrak{g} \otimes \mathbb{C}G$ *as a coalgebra, and its product and antipode are given by*

$$(X \otimes f)(Y \otimes g) = X(f_{(1)}.Y) \ \otimes \ f_{(2)}g, \tag{6.9}$$

$$S(X \otimes f) = S(f_{(1)}).S(X) \otimes S(f_{(2)}).$$

In the situation of Theorem 6.7, the action of $U\mathfrak{g}$ on $\mathbb{C}[G] \cong U\mathfrak{g}^\bullet$ by left invariant derivations takes a form dual to (6.6), since

$$(X.f)(a) = \frac{\mathrm{d}}{\mathrm{dt}} \left(f(ae^{tX}) \right)_{|t=0} = f_{(1)}(a)\langle X, f_{(2)} \rangle \tag{6.10}$$

for all $X \in \mathfrak{g}$, $f \in \mathbb{C}[G]$ and $a \in G$, where $\langle \ , \ \rangle$ is as in (6.8). Hence, by dualizing Definition 6.6 and considering $\mathbb{C}[G]$ as an $U\mathfrak{g}$-module via (6.10), we find that $\mathcal{H}(\mathbb{C}[G]) = \mathbb{C}[G] \sharp U\mathfrak{g}$ coincides with the algebra of all differential operators on G, with its usual action (6.5) on $\mathbb{C}[G]$. On another hand, the symmetrization map

$$\begin{array}{ccc} S\mathfrak{g} & \longrightarrow & U\mathfrak{g} \\ x_1 \ldots x_r & \longmapsto & \sum_{\sigma \in S_r} x_{\sigma(1)} \ldots x_{\sigma(r)} \end{array}$$

yields vector space isomorphisms $\mathbb{C}[\mathfrak{g}^*] \cong S\mathfrak{g} \cong U\mathfrak{g}$ that allow one to identify $\mathcal{H}(\mathbb{C}[G])$ with the space of functions on the cotangent bundle $T^*G \cong G \ltimes \mathfrak{g}^*$. The algebra structure is determined by the pairing (6.8), and hence by the canonical symplectic structure on T^*G via the map assigning to each function its Hamiltonian vector field.

6.2.2. $\mathcal{H}_h$ *as a deformation of* $\mathbb{C}[T^*G]$. The cotangent bundle T^*G is an example of Drinfeld double Lie group. It is associated to the *trivial* Poisson-Lie structure of G. By starting with the *standard* Poisson-Lie structure of $PSL_2\mathbb{C}$ the general theory provides us with a Drinfeld double $\mathcal{D}(PSL_2\mathbb{C})$ diffeomorphic to $PSL_2\mathbb{C} \times H$ in a neighborhood of the identity element e. It carries a Poisson bracket $\{ \ , \ \}_s$ which is non degenerate at e, and given by (see [KS, Prop. 6.1.16])

$$\{f_1, f_2\}_s = \sum_i (\partial_i f_1 \partial^i f_2 - \partial^i f_1 \partial_i f_2) + \sum_i (\partial'_i f_1 (\partial^i)' f_2 - (\partial^i)' f_1 \partial'_i f_2). \tag{6.11}$$

Here, ∂_i and ∂^i (resp. ∂'_i and $(\partial^i)'$) are dual basis of right (resp. left) invariant vector fields on $PSL_2\mathbb{C}$ and H, respectively, considered as vector fields on $\mathcal{D}(PSL_2\mathbb{C})$. The function space $\mathbb{C}[\mathcal{D}(PSL_2\mathbb{C})]$ inherits from $\{ \ , \ \}_s$ a structure of Heisenberg double algebra $\mathcal{H}(\mathbb{C}[PSL_2\mathbb{C}])$ that may be deformed to $\mathcal{H}(U_h^\bullet)$ ([STS, Prop. 3.3], [Lu1, Th. 3.10]). More precisely, by working dually and letting $\mathfrak{d}$ denote the Lie algebra of $\mathcal{D}(PSL_2\mathbb{C})$, one obtains:

Proposition 6.8. *The Heisenberg double* $\mathcal{H}_h = \mathcal{H}(U_h)$ *is a topological algebra deformation of* $U\mathfrak{d}$ *over* $\mathbb{C}[[h]]$, *and is a quantization of* $(\mathcal{D}(PSL_2\mathbb{C}), \{ \ , \ \}_s)$.

The last claim means that the Poisson bracket $\{ \ , \ \}_s$ determines a bivector $\pi_s \in \mathfrak{d} \otimes \mathfrak{d}$ such that $r = 1 + h\pi_s \bmod(h^2)$, where

$$r = \lambda^{-1}(\mathrm{id}_{U_h}) \tag{6.12}$$

and λ is the canonical map $\lambda\colon U_h \otimes U_h^\circ \longrightarrow \mathrm{Hom}_{\mathbb{C}[[h]]}(U_h)$.

Remark 6.9. As suggested by T^*G and $\mathcal{D}(G)$ above, the quantum Heisenberg doubles (resp. (6.12)) are closely related to Fourier duality and quantum Drinfeld doubles (resp. universal R-matrices). We refer to [STS] and [Lu2, CR] for results in this direction.

6.3. **The canonical morphism $\mathcal{R}\colon \Xi^{(2)} \to \Xi^{(2)}$.** Recall the bundle Ξ_M of Theorem 4.12. Let $\mathrm{Spec}(Z_\varepsilon)^2_{reg}$ denote the set of regular pairs of central characters. Define a new bundle

$$\Xi^{(2)}\colon M_\varepsilon^{(2)} \to \mathrm{Spec}(Z_\varepsilon)^2_{reg} \tag{6.13}$$

by restricting the base space of the product bundle $\Xi_M \times \Xi_M$ and regarding each fiber as a tensor product of U_ε-modules; hence, for any $(\rho,\mu) \in \mathrm{Spec}(Z_\varepsilon)^2_{reg}$, the fiber over (ρ,μ) has the form $V_\rho \otimes V_\mu$. By Remark 5.11, the fibers over any two pairs $(\rho,\mu),(\nu,\kappa) \in \mathrm{Spec}(Z_\varepsilon)^2_{reg}$ such that $\tau(\rho)\tau(\mu) = \tau(\nu)\tau(\kappa)$ are isomorphic U_ε-modules.

Proposition 6.10. The coproduct induces an action $\Delta_\mathcal{G}$ of $\mathcal{G}$ on $\Xi^{(2)}$ by bundle morphisms.

Proof. As in Theorem 4.12 it is enough to show that the coproduct induces a homomorphism $\mathcal{G} \to \mathrm{Aut}(\hat{U}_\varepsilon \otimes \hat{U}_\varepsilon)$ whose image preserves $Z_\varepsilon \otimes Z_\varepsilon$. For all $x \in U_\varepsilon$ one computes in U_q that

$$\begin{aligned}\Delta(\left[\frac{E^n}{[n]!}, u\right]) &= \left[\frac{E^n}{[n]!} \otimes 1 + K^n \otimes \frac{E^n}{[n]!} + \ldots, \Delta(u)\right]\\ &= \left[\frac{E^n}{[n]!}, u_{(1)}\right] \otimes u_{(2)} + \left[\frac{K^n}{[n]!}, u_{(1)}\right] \otimes E^n u_{(2)} + \\ &\qquad\qquad + u_{(1)}K^n \otimes \left[\frac{E^n}{[n]!}, u_{(2)}\right] + \ldots\end{aligned}$$

where we put $\Delta(u) = u_{(1)} \otimes u_{(2)}$, and the dots refer to elements which are vanishing at $q = \varepsilon$. By specializing at $q = \varepsilon$ and using (3.3) we get

$$\Delta(D_e(u)) = (D_e \otimes 1 + z \otimes D_e - D_z \otimes zx)\Delta(u). \tag{6.14}$$

The right hand side is a derivation of $\hat{U}_\varepsilon \otimes \hat{U}_\varepsilon$ preserving $Z_\varepsilon \otimes Z_\varepsilon$. As in the proof of Proposition 3.6, it can be integrated to a 1-parameter group of automorphisms $\Delta(\exp(tD_e))$. Similar facts hold true for $\Delta(D_f(u))$ as well. We let $\Delta_\mathcal{G}$ be generated by the actions of $\Delta(\exp(tD_e))$ and $\Delta(\exp(tD_f))$, $t \in \mathbb{C}$. □

We wish to let the element (6.12) act on the left on $\Xi^{(2)}$, or equivalently, by conjugation on the bundle of algebras obtained from $\Xi^{(2)}$ by replacing Ξ_M with Ξ_A. In order for this to make sense, we realize it as an element of $\mathcal{H}_h \otimes \mathcal{H}_h$:

Definition 6.11. The *canonical element of* $\mathcal{H}_h$ is $R = \tau \circ (i \otimes i')(r)_{21} \in \mathcal{H}_h \otimes \mathcal{H}_h$, where the subscript "$_{21}$" means that the tensorands are permutated, and $i\colon U_h \to \mathcal{H}_h$ and $i'\colon U_h^\circ \to \mathcal{H}_h$ are the natural linear embeddings in $\mathcal{H}_h \cong U_h \otimes U_h^\circ$.

By writing $r = \sum_i e_i \otimes e^i$ in dual topological basis $\{e_i\}$ and $\{e^i\}$ of U_h and U_h°, we have

$$R = \sum_i (1 \sharp e^i) \otimes (e_i \sharp 1). \tag{6.15}$$

The next results sums up the fundamental properties of R (compare with [Ks, Th. IX.4.4] and [BS, Prop. 1.2]). To simplify notations, we will often identify $u \in U_h$ with $i(u) = u \sharp 1$ and $x \in U_h^\circ$ with $i'(x) = 1 \sharp x$.

Proposition 6.12. *i) We have* $R^{-1} = (S \otimes \mathrm{id})(R)$ and $(\mathrm{id} \otimes \Delta)(R) = R_{12}R_{13}$.
ii) The following identities hold true:

$$R(u \otimes 1) = \Delta(u)R\ ,\ (1 \otimes x)R = R\Delta(x) \tag{6.16}$$

for all $u \in U_h$ *and* $x \in U_h^\circ$, *and*

$$R_{12}R_{13}R_{23} = R_{23}R_{12}. \tag{6.17}$$

iii) Denote by μ the product of U_h. The image $\mathcal{R} = (\lambda \otimes \lambda)(R) \in \mathrm{Aut}_{\mathbb{C}[[h]]}(U_h \otimes U_h)$ of the canonical element under the Heisenberg representation (6.5) is given by

$$\mathcal{R} = (\mathrm{id} \otimes \mu)(\Delta \otimes \mathrm{id}). \tag{6.18}$$

Proof. i) We have $R(S \otimes \mathrm{id})(R) = \sum_{i,j}(1 \sharp e^i(e^j \circ S)) \otimes (e_i e_j \sharp 1)$. Then, by using (6.3) we see that for all x, $y \in U_h^\circ$ and u, $v \in U_h$ we have

$$\begin{aligned}\langle R(S \otimes \mathrm{id})(R), x \otimes u \otimes y \otimes v\rangle &= x(1)\eta(v) \sum_{i,j,(u)} y(e_i e_j e^i(u_{(1)}) e^j(S(u_{(2)}))) \\ &= x(1)\eta(v) y(u_{(1)} S(u_{(2)})) \\ &= x(1)\eta(v) y(1)\eta(u) \\ &= \langle 1 \otimes 1 \otimes 1 \otimes 1, x \otimes u \otimes y \otimes v\rangle.\end{aligned}$$

Here, η is the counit of U_h, and $\langle\ ,\ \rangle$ the product of the natural duality pairings. It follows that $R^{-1} = (S \otimes \mathrm{id})(R)$. As for the second identity, note that we have $(\langle u,\ \rangle \otimes \mathrm{id})(R) = u$, $(\mathrm{id} \otimes \langle\ , x\rangle)(R) = x$, and similarly for y. Hence

$$\langle u \otimes x \otimes y, (\mathrm{id} \otimes \Delta)(R)\rangle = \langle x \otimes y, \Delta(u)\rangle.$$

On another hand $\langle u \otimes x \otimes y, R_{12}R_{13}\rangle = \langle(\mathrm{id} \otimes \langle\ , x\rangle)(R)(\mathrm{id} \otimes \langle\ , y\rangle)(R), u\rangle = \langle xy, u\rangle$. The result follows again from (6.3).
ii) For all $u \in U_h$ and $x \in U_h^\circ$, we have $(\mathrm{id} \otimes \langle\ , x\rangle)(R(u \otimes 1)) = xu$, and

$$\begin{aligned}(\mathrm{id} \otimes \langle\ , x\rangle)(\Delta(u)R) &= \sum_i u_{(1)} e^i \langle u_{(2)} e_i, x\rangle \\ &= \sum_i u_{(1)} e^i \langle u_{(2)}, x_{(1)}\rangle \langle e_i, x_{(2)}\rangle \\ &= u_{(1)} \langle u_{(2)}, x_{(1)}\rangle x_{(2)}.\end{aligned}$$

Together with (6.7) this proves the first identity in (6.16). The second one is similar. Finally, $R_{23}R_{12} = (\mathrm{id} \otimes \Delta)(R)R_{23} = R_{12}R_{13}R_{23}$ by (6.16) and i).
iii) One checks (6.18) by a straighforward computation, writing R as in (6.15). Details are left as an exercise. □

Like (6.2) we have an embedding of the rational form $U_q\mathfrak{h}$ of $U_h^\circ = U_h\mathfrak{h}$ into U_h°, and hence of the smash product $\mathcal{H}_q = U_q \sharp U_q\mathfrak{h}$ into $\mathcal{H}_h$. One can check that the conjugation action of R on $\mathcal{H}_h \otimes \mathcal{H}_h$ induces an (outer) automorphism $\mathcal{R}$ of the subalgebra $\mathcal{H}_q \otimes \mathcal{H}_q$. Moreover, in complete analogy with the semi-classical situation considered in [WX], $\mathcal{H}_q$ has a double structure of quantum groupoid through which $\mathcal{R}$ factors to define an automorphism of $U_q \otimes U_q$. Then, by specializing at $q = \varepsilon$ and considering the resulting left action on tensor products of simple U_ε-modules, using (6.16)-(6.17), (5.30) and (5.33) we get:

Theorem 6.13. *The canonical element R of $\mathcal{H}_h$ induces a bundle morphism $\mathcal{R}\colon \Xi^{(2)} \longrightarrow \Xi^{(2)}$ such that*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{12} \tag{6.19}$$

and $\mathcal{R}(u \otimes \mathrm{id}) = \Delta(u)\mathcal{R}$ for all $u \in U_\varepsilon$. Hence $\mathcal{R}$ coincides at $(\rho, \mu) \in \mathrm{Spec}(Z_\varepsilon)^2_{reg}$ with the evaluation map $K(\rho, \mu)$, and the regular 6j-symbol operator $\mathcal{R}(\rho, \mu, \nu)$ is also a value of $\mathcal{R}$.

Note that the Pentagon equation (5.34) is equivalent to (6.19). By identifying K with $\mathcal{R}$ each multiplicity module $M(h)$ gets a natural structure of U_ε-module, such that $\mathcal{R}$ is $U_\varepsilon \otimes U_\varepsilon$-linear at points (ρ, μ) where $\tau(\rho)\tau(\mu) = h$.

Finally, let us go back to the quantum coadjoint action. By passing the coproducts on the left in (6.16), we get two actions of U_ε and $U_\varepsilon\mathfrak{h}$ on the set of bundle morphisms of $\Xi^{(2)}$, which leave $\mathcal{R}$ invariant. By using these actions one can show that:

Theorem 6.14. *The $\mathcal{G}$-action on Ξ_M extends to* $\mathrm{End}(\Xi^{(2)})$ *by preserving $\mathcal{R}$.*

In particular, $\mathcal{R}$ is constant along the $\Delta_{\mathcal{G}}$-orbits in $\Xi^{(2)}$. Since the $\mathcal{G}$-orbits in $\mathrm{Spec}(Z_\varepsilon) \setminus \mathcal{D}$ cover the orbits of the adjoint action in $PSL_2\mathbb{C}^0$, Theorem 4.11 (ii) and Remark 5.11 show that $\mathcal{R}$ descends to a morphism of a vector bundle of rank n^2 over a covering of $PSL_2\mathbb{C}//PSL_2\mathbb{C}$ of degree $2n$.

6.4. **Matrix dilogarithms.** According to Theorem 6.14 one can compute $\mathcal{R}$ by restricting to pairs (ρ, μ) such that $\varphi(\rho)$, $\varphi(\mu) \in PB_+$. So we could have developed the whole theory by starting with the Heisenberg double $\mathcal{H}(U_\varepsilon b)$ of a Borel subalgebra $U_\varepsilon b \subset U_\varepsilon$. This has been done in [BB1, BB2]. Explicit formulas have shown that $\mathcal{R}$ satisfies tetrahedral symmetry relations, and produced an elementary form of (6.19) reminiscent of the five-term identities satisfied by the classical dilogarithm functions. In order to state it we need a few preparation.

Define a *QH tetrahedron* $\Delta(b, w, f, c)$ as an oriented abstract tetrahedron Δ endowed with:

- a *branching* b, defined as an orientation of the edges inducing an ordering of the vertices v_i by stipulating that an edge points towards the greater of its endpoints. The 2-faces δ_i are then ordered as the opposite vertices, and the edges of δ_3 are denoted by e_j, where the source vertices of e_0 and e_1 are v_0 and v_1, respectively.
- a triple $w = (w_0, w_1, w_2)$, where $w_j \in \mathbb{C} \setminus \{0, 1\}$ is associated to e_j and the opposite edge, and $w_{j+1} = 1/(1 - w_j)$; hence w corresponds to the *cross-ratio moduli* of an ideal hyperbolic tetrahedron.
- a *flattening* $f = (f_0, f_1, f_2)$ and a *charge* $c = (c_0, c_1, c_2)$, where $f_j, c_j \in \mathbb{Z}$ are associated to e_j and the opposite edge and satisfy:

  The flattening condition: $\mathrm{l}_0 + \mathrm{l}_1 + \mathrm{l}_2 = 0$, where

$$\mathrm{l}_j = \mathrm{l}_j(b, w, f) = \log(w_j) + \sqrt{-1}\pi f_j; \tag{6.20}$$

  The charge condition:

$$c_0 + c_1 + c_2 = 1. \tag{6.21}$$

The branching endows Δ with a *b-orientation*, positive and denoted by $*_b = 1$ when the 2-face δ_3 inherits from the orientation of its edges the opposite of the boundary orientation. In Figure 1 we show the two branched tetrahedra for $*_b = \pm 1$ (up to global symmetries), together with the 1-skeletons of the cell decompositions dual to the interiors.

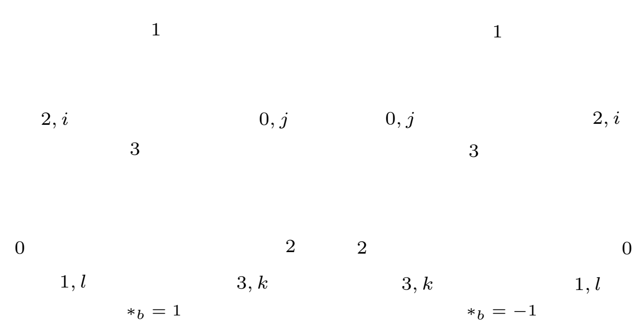

FIGURE 1. Branched tetrahedra.

Recall that we denote by n the order of ε. The *nth root cross-ratio moduli* of a QH tetrahedron $\Delta(b,w,f,c)$ are defined by

$$w'_j = \exp(\mathrm{l}_{j,n}), \quad \text{where} \quad \mathrm{l}_{j,n} = \frac{1}{n}\left(\log(w_j) + \pi\sqrt{-1}(n+1)(f_j - *_b c_j)\right). \tag{6.22}$$

The pairs $(w'_0, w'_1) \in \mathbb{C}^2$ define global coordinates of the Riemann surface $\widehat{\mathbb{C}}_n$ of the map

$$\begin{array}{ccc} \mathbb{C}\setminus\{0,1\} & \longrightarrow & \mathbb{C}^2 \\ w_0 & \longmapsto & (w_0^{\frac{1}{n}}, (1-w_0)^{-\frac{1}{n}}). \end{array} \tag{6.23}$$

Let us put $m := (n-1)/2$, and

$$[x] := n^{-1}\frac{1-x^n}{1-x}, \quad g(x) := \prod_{j=1}^{n-1}(1-x\zeta^{-j})^{j/n}, \quad h(x) := g(x)/g(1)$$

$$\omega(u',v'|n) := \prod_{j=1}^{n}\frac{v'}{1-u'\zeta^j}, \quad (u')^n + (v')^n = 1, \quad n \in \mathbb{N},$$

with $\omega(u',v'|0) := 1$ by convention, and $x^{1/n} := \exp(\log(x)/n)$ is extended to $0^{1/n} := 0$ by continuity. The function ω is n-periodic in its integer argument, and g is analytic over $\mathbb{C}\setminus\{r\zeta^k, r \geq 1, k = 1,\ldots,n-1\}$.

Definition 6.15. The (nth) *matrix dilogarithm* of a branched oriented tetrahedron $\Delta(b)$ is the regular map

$$\begin{array}{rccc} \mathcal{R}_n(\Delta,b): & \mathbb{Z}^2\times\widehat{\mathbb{C}}_n & \longrightarrow & \mathrm{Aut}(\mathbb{C}^n\otimes\mathbb{C}^n) \\ & (c_0,c_1,w'_0,w'_1) & \longmapsto & \left((w'_0)^{-c_1}(w'_1)^{c_0}\right)^{\frac{n-1}{2}}(\mathcal{L}_n)^{*_b}(w'_0,(w'_1)^{-1}) \end{array}$$

where

$$\mathcal{L}_n(u',v')^{i,j}_{k,l} = h(u')\ \zeta^{kj+(m+1)k^2}\ \omega(u',v'|i-k)\ \delta_n(i+j-l)$$

$$\left(\mathcal{L}_n(u',v')^{-1}\right)^{k,l}_{i,j} = \frac{[u']}{h(u')}\ \zeta^{-kj-(m+1)k^2}\ \frac{\delta_n(i+j-l)}{\omega(u'/\zeta,v'|i-k)},$$

We will write $\mathcal{R}_n(\Delta,b)(c_0,c_1,w'_0,w'_1) = \mathcal{R}_n(\Delta,b,w,f,c)$. Note that the branching b associates an index of $\mathcal{L}_n(w'_0,w'_1)^{\pm 1}$ to each 2-face of Δ by the rule indicated in Figure 1.

Consider a move $T \to T'$ between triangulated hexahedra T and T', as shown in Figure 2. Assume that we have QH tetrahedra on both sides, having branchings that coincide at every common edge; in Figure 2 we have fixed one such global branching, but there are five other possible choices up to global symmetries.

Define

$$W'_T(e) = \prod_{h\to e} w'(h)^{*_b}\ ,\ C_T(e) = \sum_{h\to e} c(h) \tag{6.24}$$

where:

- "$h \to e$" means that h is an edge of a QH tetrahedron of T that is identified with the edge e in T, and the products are over all such edges;
- $*_b = \pm 1$ according to the b-orientation of the QH tetrahedron that contains h;
- $w'(h)$ is the nth root cross-ratio modulus (6.22) at h, and $c(h)$ its charge.

The same notions are defined for T'. We say that $T \to T'$ is a $2 \to 3$ *transit* if at every common edge e we have

$$W'_T(e) = W'_{T'}(e)\ ,\ C_T(e) = C_{T'}(e). \tag{6.25}$$

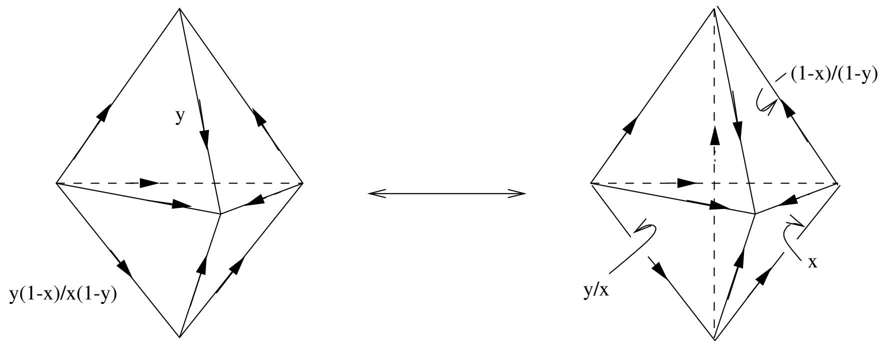

FIGURE 2. An instance of transit.

The transit conditions are very restrictive; for instance, in Figure 2 we have shown the relations between the cross-ratio moduli w_0 of the five QH tetrahedra.

To each QH tetrahedron of T or T', a matrix dilogarithm $\mathcal{R}_n(\Delta, b, w, f, c)$ is associated. Define an *n-state* of T or T' as a function that gives every 2-simplex an index, with values in $\{0, \ldots, n-1\}$. By the rule of Figure 1, every n-state determines a matrix dilogarithm entry. As two adjacent tetrahedra induce opposite orientations on a common face, an index is down for the matrix dilogarithm of one tetrahedron when it is up for the other. By summing over repeated indices we get a tensor

$$\mathcal{R}_n(T) = \textstyle\sum_s \prod_{\Delta \subset T} \mathcal{R}_n(\Delta, b, w, f, c)_s \tag{6.26}$$

where the sum is over all n-states of T, and $\mathcal{R}_n(\Delta, b, w, f, c)_s$ stands for the matrix dilogarithm entry determined by s.

We can now state the analog of Theorem 5.5 for regular U_ε-modules. By comparing formulas it makes explicit the relationship between their $6j$-symbols and the matrix dilogarithms. The first claim is essentially a consequence of Theorem 6.14. The rest is proved in [BB1, BB2]. Recall the map $\varphi\colon \mathrm{Rep}(U_\varepsilon) \to PSL_2\mathbb{C}^0$ in (4.11), and the isomorphism $PSL_2\mathbb{C} \cong \mathrm{Aut}(\mathbb{P}^1)$.

Theorem 6.16. *1) Let (ρ, μ, ν) be a triple of U_ε-modules which is regular and cyclic. Put $f = \varphi(\rho)$, $g = \varphi(\mu)$, $h = \varphi(\nu) \in PSL_2\mathbb{C}^0$. The $6j$-symbol operator $\mathcal{R}(\rho, \mu, \nu)$ coincides up to conjugacy with the map*

$$\begin{array}{rccl} \mathcal{R}: & \widehat{\mathbb{C}}^n & \longrightarrow & \mathrm{Aut}(\mathbb{C}^n \otimes \mathbb{C}^n) \\ & (w_0', w_1') & \longmapsto & \mathcal{L}_n(w_0', (w_1')^{-1}), \end{array}$$

where w_0 squared is the cross-ratio $[0 : f(0) : fg(0) : fgh(0)]$ of the points in $\mathbb{P}^1$, and the nth roots w_0' and w_1' are determined by the Casimir coordinates of $\Xi(\rho)$, $\Xi(\mu)$, $\Xi(\nu) \in \mathrm{Spec}(Z_\varepsilon)$.
2) The matrix dilogarithm $\mathcal{R}_n(\Delta, b)$ satisfies:

- *Invariance under full tetrahedral symmetries, up to a determined projective action of $SL_2\mathbb{Z}$ on the source and target spaces of* $\mathrm{Aut}(\mathbb{C}^n \otimes \mathbb{C}^n)$.
- *For any $2 \to 3$ transit $T \to T'$ we have a* five term identity

$$\mathcal{R}_n(T) = \mathcal{R}_n(T'). \tag{6.27}$$

Note that the tetrahedral symmetries of the matrix dilogarithms depend on the flattening and charge conditions (6.20)–(6.21). They are necessary to get the five term identities for all the $2 \to 3$ transits.

The cross-ratio w_0 is a complex number and is distinct from 0 and 1 because the triple (ρ, μ, ν) is regular and cyclic. The nth root modulus w_0' describes via Theorem 4.11 ii) a coset of $\mathrm{Spec}(Z_\varepsilon)$ mod $\mathcal{G}$. The 3-cocycloid identities (5.34) or (6.19) coincide with (6.27) exactly for the $2 \to 3$ transits $T \to T'$ with the branching of Figure 2. In this form, it is a non Abelian deformation of the celebrated five term identity

$$\mathrm{L}(x) - \mathrm{L}(y) + \mathrm{L}(y/x) - \mathrm{L}(\frac{1-x^{-1}}{1-y^{-1}}) + \mathrm{L}(\frac{1-x}{1-y}) = 0 \tag{6.28}$$

which x, y are real, $0 < y < x < 1$, and L is the dilogarithm function

$$\mathrm{L}(x) = -\frac{\pi^2}{6} - \frac{1}{2}\int_0^x \left(\frac{\log(t)}{1-t} + \frac{\log(1-t)}{t}\right) dt. \tag{6.29}$$

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