QUANTUM ALGEBRA AND CYLINDER TOPOLOGY

by Tammo Tom Dieck[1]

Notes prepared by Stephane Baseilhac[2]

1. Introduction

These lectures concern the topology of braids and knots in the cylinder: we construct polynomial invariants generalizing the Jones invariant of links in S^3, using (some explicitly given) tensor representations of the *generalized (cylinder) braid group* ZB_n (see section 3 for a definition). This approach is also formalized into a categorical framework, thus it parallels the combinatorial realizations of representation categories of quantum groups by *colored and oriented* tangle categories, as presented in [Tu].

We use in an essential manner quantum algebra, so the reader is assumed to be familiar with the basic objects and constructions in this domain. We refer to [Kas] or [CP] for more information.

The details of the construction are to be found at the beginning of each section. We sometimes quote results without proofs, but we give some references where full details may be found.

2. Quantum algebra

First we introduce quantum objects, such as q-exponentials and q-deformed algebras (q being usually an invertible element in a commutative ring). We deduce from their very definition a "twist identity" in the quantum group $U_q(sl_2)$, that will give in section 5 a fundamental property of the category of integrable modules over $U_q(sl_2)$, denoted by $U - Int$, by mean of the existence of a "twist morphism" inside $Hom(U_q(sl_2) - Int)$.
In this way, we follow the philosophy behind the use of quantum algebra in low dimensional topology: we extract and realize abstract rules from geometric categories (here we consider the category of braids in the cylinder), such as tensor product or gluing in cobordism categories, into linear categories: roughly speaking, (generic) one-parameter deformations of representation categories of Lie algebras are given by representation categories of quantum groups, and the rigidity properties of the spaces of morphisms of these linear categories provide a powerful tool to obtain non-trivial functors from cobordism categories to them.
Moreover, quantum algebra presents itself as a very natural algebraic approach to non commutative geometry, where spaces have to be interpreted in terms of ring of operators (see 2.4 and 6.2), and geometrical assertions into ring theoretical assertions.

Let k be a commutative ring, $k^\times$ the set of invertible elements in k and $q \in k^\times$.

2.1. **the q-exponential function.** Let $x,\ y$ be q-commuting variables: $xy = qyx$, q being of a sufficiently large order (e.g. of infinite order) in $k^\times$, and set:

$$(n;q) = \frac{1-q^n}{1-q},\ (n;q)! = (1;q)(2;q)\dots(n;q),$$
$$\begin{pmatrix} n \\ k;q \end{pmatrix} = \frac{(n;q)!}{(k;q)!(n-k;q)!}\ .$$

Then, one can show by induction [Kas, ch. 4] that:

$$(x+y)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k;q \end{pmatrix} y^k x^{n-k} \tag{1}$$

Date: July 1997.

[1]Mathematisches Institut, Bunsenstraße 3/5, D-37073 Göttingen, Germany; e-mail: tammo@cfgauss.uni-math.gwdg.de

[2]Laboratoire Emile Picard, Universit Paul Sabatier, 118 route de Narbonne, 31000 Toulouse; e-mail: baseilha@picard.ups-tlse.fr

$$\left(\begin{array}{c} n \\ k;q \end{array} \right) = \left(\begin{array}{c} n-1 \\ k-1;q \end{array} \right) + q^k \left(\begin{array}{c} n-1 \\ k;q \end{array} \right)$$

This gives clearly a q-deformed Pascal formula; note that the q-binomial coefficients $\left(\begin{array}{c} n \\ k;q \end{array} \right)$ are polynomials in $\mathbb{Z}[q]$.
We shall also need, in section 5, the symbols:

$$[n] = [n;q] = \frac{q^n - q^{-n}}{q - q^{-1}},\ n \in \mathbb{Z}$$
$$[n]! = [n;q]! = [1][2]\ldots[n],\ n \in \mathbb{N},\ [0]! = 1$$

Now define the following formal power series in $\mathbb{Z}[q][[z]]$, where q is a "generic" parameter, i.e. $(q|q)_n \in k^\times$ (z does not necessarily lives in $\mathbb{C}$):

$$(x|q)_n = \prod_{j=0}^{n-1}(1 - q^j x)$$

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q|q)_n},\ E_q(z) = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{z^n}{(q|q)_n} \tag{2}$$

Proposition 1. *: The following formal identities hold in $\mathbb{Z}[q][[z]]$:*

$$e_q(z) = \prod_{j=0}^{\infty}\frac{1}{(1-q^j z)},\ E_q(z) = \prod_{j=0}^{\infty}(1+q^j z)$$
$$e_q(z)E_q(-z) = 1,\ e_q(qz) = (1-z)e_q(z)$$

$$e_q(z)e_q(-z) = e_{qz}(z^2) \tag{3}$$

proof: the functional equation $f(qz) = (1-z)f(z)$ admits a unique formal power series in z as solution, and we verify immediately that both sides of the first identity satisfy this equation (the left one by induction).
The same proof is valid for the second identity, and these two equalities imply directly the others. Note that $e_q(z) = (z|q)_\infty^{-1}$. □

Consider now a pair $x,\ y$ of q-commuting variables and the associated polynomial ring $P_q[x,\ y] = k[x,\ y]/\{xy - qyx\}$ (which is also called the quantum plane).
We can choose a basis in the form $(y^i x^j)_{i,\ j \in \mathbb{N}}$ for it, and complete $P_q[x,\ y]$ with respect to this basis into a ring $P_q[[x,\ y]]$ of power series in $x,\ y$, whose coefficients may be calculated from the q-commuting relation.

Proposition 2. *: Let $xy = qyx$ be q-commuting variables. Then the following formal identity holds in $P_q[[x,\ y]]$:*

$$e_q(x+y) = e_q(y)e_q(x) \tag{4}$$

proof: identifying the coefficients in a basis $\{y^k x^l\}_{k,\ l \in \mathbb{N}}$ on both sides, we obtain the identity (1). □

2.2. **The Heisenberg algebra, its q-deformations and the exponential operator.** Consider the 3-dimensional nilpotent Lie algebra $\mathcal{H}$ over k generated by 3 variables $X,\ P,\ W$, with relations:

$$[X,\ W]=[P,\ W]=0,\ [X,\ P]=W$$

It is called the *Heisenberg algebra*, or the *oscillator algebra*. One can realize $\mathcal{H}$, over $k=\mathbb{R}$, as the Lie algebra of the Lie group T of real upper triangular $(3,\ 3)$-matrices with unit diagonal, the exponential map $\mathcal{H}\to T,\ A\mapsto expA$ being a diffeomorphism. Now, deform the commutator of $\mathcal{H}$ into a *q-commutator*:

$$[A,\ B]_q=AB-qBA$$

Then the defining relations of $\mathcal{H}$ may be turned into the followings:

$$xp-qpx=w,\ wx=q^{-1}xw,\ wp=qpw$$

and we define the *q-deformed Heisenberg* algebra (or *q-oscillator* algebra), $\mathcal{H}_q$, as the k-algebra with generators $x,\ p,\ w$ and the preceding relations.

In view to simplify the presentation of the next result, we now consider a version of $\mathcal{H}_q$, with generators $x,\ y$ and v and relations:

$$xy-qyx=(1-q)v,\ xv=q^2vx,\ yv=q^{-2}vy \tag{5}$$

Proposition 3. *:The generators of $\mathcal{H}_q$ satisfy the formal identity:*

$$e_q(x+y)=e_q(y)e_{q^2}(v)e_q(x) \tag{6}$$

proof: setting $v=z^2$, one can verify that $x-z$ and $y+z$ are q-commuting variables. Hence:

$$\begin{aligned}e_q(x+y)&=e_q(x-z+z+y)=e_q(z+y)e_q(x-z)\\&=e_q(y)e_q(z)e_q(-z)e_q(x)=e_q(y)e_{q^2}(v)e_q(x)\end{aligned}$$

using respectively the identities (4) and (3). □

2.3. **The deformed enveloping algebra and the exponential operator.** In view to get a more practical version of equation (6), we are going to embed it in an *algebraic* q-deformation of the universal enveloping algebra of the Lie algebra $sl(2)$.

Recall that the Lie algebra $sl(2,\ \mathbb{C})$ over $k=\mathbb{C}$ may be realized as a matrix algebra with $\mathbb{C}$-basis:

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ F=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

with the following bracket relations: $[H,\ F]=-2F,\ [H,\ E]=2E,\ [E,\ F]=H.$

The universal enveloping algebra U of $sl(2)$ is the associative unital k-algebra generated by $E,\ F,\ H$ with relations:

$$EF-FE=H,\ HF-FH=-2F,\ HE-EH=2E.$$

Its q-deformation $U_q=U_q(sl(2))$ may be presented with generators $K,\ K^{-1},\ E,\ F$ and relations:

$$KK^{-1}=K^{-1}K=1,\ KE=q^2EK,\ KF=q^{-2}FK,\ EF-FK=\frac{K-K^{-1}}{q-q^{-1}}\ ,$$

where the ring k contains q as a unit such that $q-q^{-1}$ is invertible. This presentation of U_q is not convenient in order to see that the "classical limit" of these relations when $q\to 1$ gives U, and we refer to [Kas, ch. 6] for the explicit proof of this fact.

There is the following Poincar-Birkhoff-Witt type k-basis for the algebra U_q: $\{E^aK^bF^c;\ a,\ c\in\mathbb{N}\setminus\{0\},\ b\in\mathbb{Z}\}$. The defining relations of U_q show that theses elements generate the algebra, and the linear independence follows, for instance, from the action on the quantum plane (see [Kas, ch. 4]).

We can endow U_q with the structure of a Hopf algebra with invertible antipode, which means that the following maps $\mu,\ \epsilon$ are homomorphisms of algebra and s is an antihomomorphism of algebra. This structure is defined by:

$$\mu(E) = E \otimes 1 + K \otimes E,\ \mu(F) = F \otimes K^{-1} + 1 \otimes F,\ \mu(K) = K^{-1},$$
$$\epsilon(K) = 1,\ \epsilon(F) = \epsilon(E) = 0,$$
$$s(E) = -K^{-1}E,\ s(F) = -FK,\ s(K) = K^{-1}.$$

Formula (6) may now be expressed as a formal identity in $U_q \otimes U_q$: indeed, the ideals B_q^+ and B_q^- of U_q, respectively generated by $E,\ K,\ K^{-1}$ and $F,\ K,\ K^{-1}$ (the q-Borel subalgebras of U_q), prove to be Hopf subalgebras of U_q, and the induced comultiplication gives:

$$\mu(E) = \underbrace{E \otimes 1}_{x} + \underbrace{K \otimes E}_{y} \tag{7}$$

Then, setting $v = -qKE \otimes E$, it is straightforward to verify the relations (5) for the elements $x,\ y,\ v \in U_q \otimes U_q$, with q^{-1} in place of q. Then equation (6) reads:

$$e_{q^{-1}}(\mu(E)) = e_{q^{-1}}(K \otimes E)e_{q^{-2}}(-qKE \otimes E)e_{q^{-1}}(E \otimes 1) \tag{8}$$

This is the *twist identity.* A similar identity holds if we set

$$x = F \otimes K^{-1},\ y = 1 \otimes F,\ v = -qF \otimes FK^{-1}$$

All this has been done in a formal setting, but no ambiguity will arise when considering the action of U_q on integrable modules (we shall define this representation category in 5.2).

Remark: Note that the formal limit of $e_q((1-q)z)$ when $q \to 1$ is the exponential power series expansion of $exp(z)$. In fact, it is shown in [Ba-Re] that when we set $q = exp(-\tau)$, the singular part in τ of the asymptotic of $(x|q)_\infty$ when $\tau \to 0$ is equal to $(1-x)^{-1/2}exp(-Li_2/\tau)$ (where Li_2 denotes Euler's dilogarithm power expansion series). One can then ask what is the asymptotic behaviour of all the preceeding identities when q tends to 1 or a primitive root of unity. In connection with the representation theory of $\mathcal{H}_q$ (see 2.2), this problem has been thoroughly studied in mathematical physics in view to understand a quantum analog to the dilogarithm function.

2.4. **Comodule structures.** Before giving some formal definitions, let us explain roughly a way to see how comodule structures arise in quantum algebra.
The category of comodules over a cobraided bialgebra A, denoted by $A - Comod$, is by definition endowed with a braiding (see 5.1). Moreover, for most of the usual quantum groups, such as the quantum *linear* groups $U_q(sl_n)$, there is a duality isomorphism between their modules and comodules over their q-deformed coordinate rings, which are cobraided bialgebra (see 6.2 for the example of $SL_q(2)$ and [Kas], chpt. 7, for more details). Since the FRT-construction (see 6.1) produces from any solution to the Yang-Baxter equation a (unique) cobraided bialgebra, these categories arise very naturally.

Fix a coalgebra $(C,\ \mu,\ \epsilon)$. A left C-comodule is a pair $(N,\ \mu_N)$ where N is a vector space and $\mu_N : N \to C \otimes N$ is a linear map, called the coaction of C on N, such that the following diagrams commute:

$$\begin{tikzcd} N \arrow[r, "\mu_N"] \arrow[d, "\mu_N"'] & C \otimes N \arrow[d, "id \otimes \mu_N"] \\ C \otimes N \arrow[r, "\mu \otimes id"] & C \otimes C \otimes N \end{tikzcd}$$

$$\begin{tikzcd} N \arrow[r, "\mu_N"] \arrow[dr, "\cong"'] & C \otimes N \arrow[d, "\epsilon \otimes id"] \\ & k \otimes N \end{tikzcd}$$

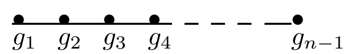
If two left C-comodules $(N,\ \mu_N)$ and $(N',\ \mu_{N'})$ are given, a linear map f from N to N' is a morphism of C-comodules if $(id \otimes f) \circ \mu_N = \mu_{N'} \circ f$.
Right C-comodules are defined in the same way, by twisting the factors in the target of μ_N.

3. Braids in the cylinder

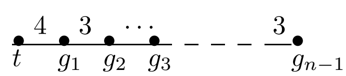
We define the generalized braid groups, and more specifically the generalized B-type braid group (respectively B-type Weyl-group) ZB_n (resp. CB_n) and quote some relations with the classical Artin braid group and the so-called affine Artin braid group. These relations yield a topological realization of ZB_n as the group of braids in the cylinder, and a graphical calculus for ZB_n is introduced. The latter makes clear, in a very simple way, the particular role of a certain "twist" element t in ZB_n.
We shall see at the end of this section that representations for knot algebras such as Hecke or Birman-Wenzl-Murakami type algebras, may be obtained from tensor representations of the braid group ZB_n.

3.1. **Braid groups.** Consider a finite set S with a symmetric map $m : S \times S \to \mathbb{N} \cup \infty$ satisfying $m(s,\ s) = 1,\ m(s,\ t) \geq 2$ for $s \neq t$. Then one can define the *weighted Coxeter graph* $\Gamma(S,\ m)$ as the graph with S as its set of vertices and with an edge weighted by $m(s,\ t)$ between vertices $s,\ t$ if and only if $m(s,\ t) \geq 3$ (often omitted in the notation in case of equality). Here are three important examples (which are respectively the Dynkin diagrams of the classical Lie algebras $su(n)$, $so(2n{+}1)$ and the affine Lie algebra $\tilde{A}_{n-1}$):

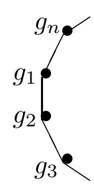
A_{n-1}



B_n



$\tilde{A}_{n-1},\ n \geq 3$



The generalized braid groups are denoted $Z(\Gamma(S,\ m))$, or also

$$Z(S,\ m) := \langle g_s,\ s \in S \mid \underbrace{g_s \quad g_t \quad g_s}_{m(s,\ t)<\infty\ alternative\ factors} \quad = \underbrace{g_t \quad g_s \quad g_t}_{idem} \ \rangle \tag{9}$$

The relations in this presentation of $Z(S,\ m)$ are called the *generalized braid relations*. They show that non connected vertices in the weighted Coxeter graph of $Z(S,\ m)$ are commuting (since $m(s,\ t) = 2$).
The n-th Artin braid group ZA_{n-1} is obtained from the graph A_{n-1} (with $n-1$ vertices) using the presentation (9) for $Z(S,\ m)$, and in the same way one may define the (n string-) cylinder braids from the graph B_n, as the group generated by t and $g_s,\ s \in S$, with $\sharp S = n-1$ and with the relations:

$$\begin{cases} g_i g_j g_i = g_j g_i g_j,\ |i-j| = 1, \\ g_i g_j = g_j g_i,\ |i-j| \geq 2, \\ t g_j = g_j t,\ j \geq 2, \\ t g_1 t g_1 = g_1 t g_1 t. \end{cases} \tag{10}$$

Note that the first two relations imply that $ZA_{n-1} \subset ZB_n$; the fourth relation is the *four braid relation.* Conversely, the map $\lambda : ZB_n \to ZA_{n-1},\ \lambda(g_i) = g_i,\ \lambda(t) = 1$ splits by $g_j \mapsto g_j$, which gives a semi-direct product by $K_n = kernel(\lambda)$:

$$K_n \to ZB_n \to ZA_{n-1} \tag{11}$$

In fact, one may prove that (see [TD3]):

Theorem 1. *: The kernel $k = kernel(\lambda)$ of λ is a free group generated by the following elements of ZB_n:*

$$y_0 = t,\ y_1 = g_1 t g_1^{-1},\ \dots,\ y_{n-1} = g_{n-1} \dots g_1 t g_1^{-1} \dots g_{n-1}^{-1}$$

It is straightforward from the picture of the affine Coxeter graph $\tilde{A}_{n-1}$ that the n-th *affine braid group* $Z\tilde{A}_{n-1}$ generated by $g_1,\ g_2,\ \dots,\ g_n$ may be defined by the relations:

$$g_i g_j g_i = g_j g_i g_j,\ i \equiv j \pm 1 \pmod n,$$
$$g_i g_j = g_j g_i \ \textit{otherwise}.$$

Indeed, for $n > 2$, the vertices may be denoted by $i \in \mathbb{Z}_n$ and we have $m(i,\ j) = 3$ if $i \equiv j \pm 1 \pmod n$ and $m(i,\ j) = 2$ otherwise (if $n = 2$ the single edge has weight ∞, i.e. $\tilde{A}_1$ is the free group generated by g_1 and g_2).
Consider the automorphism of $\tilde{A}_{n-1}$ which permutes cyclically the vertices, and the induced automorphism s of $Z\tilde{A}_{n-1}$: $s(g_i) = g_{i-1}, \forall\ i \pmod n$. Form the semi-direct product $Z\tilde{A}_{n-1} \rtimes_s \mathbb{Z}$ where the group structure on the set $Z\tilde{A}_{n-1} \times \mathbb{Z}$ is given by $(x,\ m) \cdot (y,\ n) = (x \cdot s^m(y),\ m + n)$; beware that s^n is the identity.

Theorem 2. *: ZB_n is the semi direct product of $Z\tilde{A}_{n-1}$:*

$$Z\tilde{A}_{n-1} \to ZB_n \to \mathbb{Z} \tag{12}$$

$$\begin{array}{lcl} g_i & \mapsto & g_i \mapsto 0,\ 1 \le i \le n-1 \\ g_n & \mapsto & g t g_1 t^{-1} g^{-1} \\ t & \mapsto & 1 \end{array}$$

where $g = g_{n-1} g_{n-2} \dots g_1$.

3.2. **Weyl groups of type A and B.** A *Coxeter group* $C(S,\ m)$ may be presented with generators $s,\ s \in S$, with the generalized braid relations and $s^2 = 1$. Then we have a surjective group homomorphism

$$Z(S,\ m) \xrightarrow{p} C(S,\ m),\ g_s \mapsto s$$

Define the *length* $l(x)$ of a word $x = s_1 s_2 \dots s_r := (s_1, \dots,\ s_r) \in C(S,\ m)$, as the minimum of r for all such expressions of x; when $r = l(x)$, then x is called a *reduced* word. This combinatorial invariant $l : C(S,\ m) \to \mathbb{N} \setminus \{0\}$ allows to construct a partial section to the map p:

Proposition 4. *: Suppose $(s_1,\ s_2,\ \dots,\ s_r)$ is a reduced expression of $x \in C(S,\ m)$. Then the product $g_x := g_{s_1} g_{s_2} \dots g_{s_r} \in Z(S,\ m)$ is independent of the reduced expression and only depends on x.*

This is a fundamental fact: in particular, taking $Z(S,\ m) = ZB_n$, define the elements

$$t_0 = t,\ t_1 = g_1 t g_1,\ \dots,\ t_{n-1} = g_{n-1} \dots g_2 g_1 t g_1 g_2 \dots g_{n-1}$$

These elements of ZB_n pairwise commute; see in 3.4 how we will use this property.

Examples of Coxeter groups are the symmetric groups $\mathcal{S}_n = CA_{n-1}$, associated with the graphs A_{n-1}, where the length function on a permutation π is $l(\pi) = \sharp\{(i,\ j) | i < j,\ \pi(i) > \pi(j)\}$; the longest element is of length $\dfrac{n(n-1)}{2}$.

Let us introduce a graphical calculus for permutations: to any $\pi \in \mathcal{S}$, associate n arcs in a strip $\mathbb{R} \times [1,\ 0]$, such that the j-th arc connects $(i,\ 0)$ to $(\pi(i),\ 1)$, two arcs have at most one transverse intersection point and the whole figure has at most double points. Assume also that the

intersection points have different heights.Notice that the number of double points is the length l of π, which shows the lost in topology when looking at the length only.
A reduced expression can be read off such a diagram, by composing in an ascending order the transpositions represented by horizontal substrips with at most one double point in the interior and no double point on the boudary of each subtrip; every reduced expression arises in this way. Then, the preceding theorem has the following geometrical meaning: take a reduced word in $\mathcal{S}_n$, and desingularize each double point of its diagram by an overcrossing. Do it so that the first arc is over the others, etc ...You get an "ascending" braid, independent of the reduced expression!
A picture for a transposition $\pi \in CA_3$ is as follows:

The next example is the Coxeter group for the Coxeter graph B_n. It is the group of permutations π of $\{\pm n\} = \{-n,\ -n+1,\ \dots,\ -1,\ 1,\ \dots,\ n\}$ such that $\pi(-i) = -\pi(i),\ i \in [\pm n]$. Define the generator t by the transposition $(-1,\ 1)$ and the generators g_i by the products $(i,\ i+1)(-i,\ -i-1)$. There is a symmetry with respect to the axis $\{0\} \times [1,\ 0]$ of the strip $\mathbb{R} \times [1,\ 0]$ in the corresponding pictures.
Again, one can read off a reduced expression from a diagram in which two arcs have at most one intersection point.

Now, the sequence (11) implies that CB_n is isomorphic to a semi-direct product:

$$1 \to (\mathbb{Z}_2)^n \to CB_n \to CA_{n-1} \to 1 \tag{13}$$

Finally, we quote that there exists, aside of the usual length function, a graded length for these equivariant permutations, given by a pair $(a,\ b)$: b is the number of double points along the axis $\{0\} \times [0,\ 1]$ and a is half the number of the remaining double points. The longest element has length $n^2 = 2 \times \dfrac{n(n-1)}{2} + n = max_{CB_n} a + max_{CB_n} b$.

Remark: the last two Coxeter groups CA_{n-1} and CB_n are respectively the Weyl groups of the compact Lie groups $U(n)$, and $SO(2n+1)$ (or $Sp(n)$). In particular, one may think of CB_n as the Coxeter group with generators $t,\ g_1,\ g_2,\ \dots,\ g_{n-1}$, where g_j acts on the complex n-space $\mathbb{C}^n$ by the transposition $(j,\ j+1)$ between coordinates and t acts by $z_1 \to -z_1$. This interpretation follows from the sequence (13), with $\mathcal{S}_n = CA_{n-1}$ acting on $\mathbb{C}^n$ by permutation of the coordinates and $(\mathbb{Z}_2)^n$ acting by sign changes

$$(z_1,\ z_2,\ \dots,\ z_n) \to (\epsilon_1 z_1,\ \epsilon_2 z_2,\ \dots,\ \epsilon_n z_n),\ \epsilon_i \in \{\pm 1\}$$

This representation of CB_n is then generated by the reflections in the hyperplanes $z_i = \pm z_j,\ i \neq j$ and $z_i = 0,\ i = 1,\ \dots,\ n$.

3.3. **Topology of braids.** We use the reflection representation of the Weyl group CB_n to derive a geometric interpretation of the braid group ZB_n. Let X be the complement of the previous hyperplanes: $z_i = \pm z_j,\ i \neq j,\ z_i = 0,\ i = 1,\ \dots,\ n$; then, CB_n acts freely on X, and

Theorem 3. *: There is an isomorphism of groups between the fundamental group $\pi_1(X/CB_n)$ of the space of regular orbits of CB_n and the braid group ZB_n.*

Let us proceed now with a geometric construction of this isomorphism.
Removing the hyperplanes $z_i = 0,\ i = 1,\ \dots,\ n$ from $\mathbb{C}^n$, we get the n-fold product of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, and removing the remaining reflection hyperplanes yields the space X of n-tuples in ${\mathbb{C}^*}^n$ with pairwise different squares z_j^2. Define the *configuration space* $C^n(\mathbb{C}^*)$ as the space of subsets of cardinality n in $\mathbb{C}^*$; it is obviously the orbit space $Y/\mathcal{S}_n$, where $Y \subset {\mathbb{C}^*}^n$ is the set of n-tuples with pairwise distinct components and $\mathcal{S}_n$ acts by permutations.

Proposition 5. *: The orbit space X/CB_n is homeomorphic to $C^n(\mathbb{C}^*)$, hence we have an isomorphism of groups $\pi_1(C^n(\mathbb{C}^*)) \cong ZB_n$*

proof: note that the quotient of X by the maps $z_j \mapsto z_j^2,\ j = 1,\ \dots,\ n$ is homeomorphic to $X/(\mathbb{Z}_2)^n \cong Y$, as an $\mathcal{S}_n$ equivariant homeomorphism. □

Here is a geometric interpretation: a loop in $C^n(\mathbb{C}^*)$ lifts to a path

$$w : [0,\ 1] \to Y,\ t \mapsto (\omega_1(t),\ \omega_2(t), \dots,\ \omega_n(t))$$

about the "axis" $(0,\ 0,\ \dots,\ 0)$, starting for example from the base point

$$(1,\ \omega,\ \omega^2,\ \dots,\ \omega^{n-1}),\ \omega = exp(2i\pi/n).$$

This path ends in $(\sigma(1),\ \sigma(2),\ \dots,\ \sigma(n))$, where σ is a permutation of the set of points $\{1,\ \omega,\ \dots,\ \omega^{n-1}\}$, and there is no self intersection points on the strings $Image(\omega_i)$ and between them.
Passing to the quotient by $\mathbb{Z}_n = \{1,\ \omega,\ \dots,\ \omega^{n-1}\}$, this gives a well defined cylinder braid z_ω from $\mathbb{Z}_n \times \{0\}$ to $\mathbb{Z}_n \times \{1\}$, with n strings, in $\mathbb{C}^* \times [0,\ 1]$. Since homotopy classes of loops correspond to isotopy classes of such braids and the multiplication of loops lifts to concatenation of braids, we get:

Theorem 4. *: The braid group ZB_n is the group of n-string braids in the cylinder $\mathbb{C}^* \times [0,\ 1]$.*

The same proof implies that the Artin braid group is the group of braids in the strip $\mathbb{C} \times [0,\ 1]$. Alternatively, the map $ZB_n \to ZA_n$ corresponds to forgetting the axis: $g_1,\ g_2,\ \dots, g_{n-1} \mapsto g_1,\ g_2,\ \dots, g_{n-1}$ respectively and $t \mapsto g_0^2$.

By lifting a loop in X/CB_n to a path $w : [0,\ 1] \to X$, we obtain in the same manner an isomorphism of groups between ZB_n and the group of *symmetric* braids (with respect to an axis) with $2n$ strings in $\mathbb{C} \times [0,\ 1]$, i.e. braids that are $\mathbb{Z}_2$-equivariant under $\mathbb{C}^* \times [0,\ 1] \to \mathbb{C}^* \times [0,\ 1],\ (z,\ t) \mapsto (-z,\ t)$.

The previous geometric realizations of ZB_n allow a graphical calculus for ZB_n: the generators t and g_i of the group of cylinder braids are respectively represented by a strand spinning around an axis and the usual braid pictures on an arbitrary side of this axis. The group of symmetric braids is obtained by substituing to the generator t two segments, which cross in a single point along the axis, and to double the generators g_i in a symmetric way with respect to the axis. Then the geometric realization of t is a two fold ramified covering along the axis of this *singular* braiding. Notice that we have a kind of generalized third Reidemeister move: consider the second figure in 19, and forget the notations inside. Slide the lower strand along the axis upward, so that it becomes the higher strand: this move is a geometric realization of the equation $tg_1tg_1 = g_1tg_1t$. Beware that the symmetry is *different* from the reflection in the axis: the former corresponds to a spatial rotation about the axis.

Here is finally an homotopy theoretic interpretation of the sequence (12).

The map ${\mathbb{C}^*}^n \to \mathbb{C}^*,\ (z_1,\ z_2,\ \dots,\ z_n) \mapsto z_1 \cdot z_2 \cdot \dots \cdot z_n$ is $\mathcal{S}_n$-equivariant, so it induces a map $\alpha : C^n(\mathbb{C}^*) \to \mathbb{C}^*$, and a homomorphism α_* between fundamental groups.
Let us prove that α is a fibre bundle.
The set $H = \{(z_1,\ \dots,\ z_n) \in \mathbb{C}^* | \prod_j x_j = 1\}$ is $\mathcal{S}_n$-equivariant and the map $\gamma : \mathbb{C}^* \times_{\mathbb{Z}_n} H \to {\mathbb{C}^*}^n,\ (z,\ z_1,\ \dots,\ z_n) \mapsto (zz_1,\ \dots,\ zz_n)$ from the semi-direct product $\mathbb{C}^* \times_{\mathbb{Z}_n} H$ is an $\mathcal{S}_n$-equivariant homeomorphism. Hence it is a fibre bundle with fiber H, associated to the $\mathbb{Z}_n$-principal bundle $\mathbb{C}^* \to \mathbb{C}^*,\ z \mapsto z^n$.

Removing from ${\mathbb{C}^*}^n$ the set C of n-tuples where at least two components are equal, and setting $D = H \cap C$, the map γ induces an $\mathcal{S}_n$-equivariant homeomorphism $\gamma : \mathbb{C}^* \times_{\mathbb{Z}_n} (H \backslash D) \to {\mathbb{C}^*}^n \backslash C = Y$. Then the following fibre bundle description of the configuration space is correct:

$$\alpha : C^n(\mathbb{C}^*) \cong Y/S_n \overset{\gamma^{-1}}{\cong} \mathbb{C}^* \times_{\mathbb{Z}_n} (H \setminus D)/S_n \to \mathbb{C}^*$$

The homotopy exact sequence for this fibration reduces to a sequence for the fundamental groups:

$$1 \to kernel(\alpha_*) \to ZB_n \to \mathbb{Z} \to 0$$

The 5-lemma applied to this sequence and to the sequence (12) proves that $Z\tilde{A}_{n-1}$ is the fundamental group of the fibre of α.

3.4. **Tensor representations of ZB_n.** To conclude this section, let us see how to construct representations of the braid group ZB_n on tensor powers of a k-module V, with k an integral domain.

Define a *four braid pair* $(X,\ F)$ as a pair of automorphisms of the modules $V \otimes V$ and V, respectively, such that:

$$\begin{array}{ll} (X \otimes 1)(1 \otimes X)(X \otimes 1) = (1 \otimes X)(X \otimes 1)(1 \otimes X) & \qquad (YB) \\ X(F \otimes 1)X(F \otimes 1) = (F \otimes 1)X(F \otimes 1)X & \qquad (FBP) \end{array}$$

The automorphism X is called an *R-matrix*, or *Yang-Baxter Operator*. The FBP relation shows that we consider here, with a view towards the representation theory of braid groups, the solutions of the more general four braid relation $XYXY = YXYX$ on $V \otimes V$ *only* when Y can be written in the form $F \otimes 1$ (see the proposition below).
Tensor representations of ZB_n on $V^{\otimes n}$ are obtained by setting:

$$\begin{array}{lll} t & \mapsto & F \otimes 1 \otimes \ldots \otimes 1 \\ g_i & \mapsto & X_i = 1 \otimes \ldots \otimes X \otimes \ldots \otimes 1 \end{array}$$

where X_i acts by X on the factors i and $i+1$. The main task of the remaining part of these lectures is to investigate the algebraic and categorical structures in which the (FBP) relation may live.

Let us display already at this point some identities to be used in the next sections. Set

$$t(1) = t,\ t(j) = g_{j-1}g_{j-2}\ldots g_1 t g_1 \ldots g_{j-1},\ t_n = t(1)t(2)\ldots t(n)$$
$$g(j) = g_j g_{j+1} \ldots g_{j+n-1},$$
$$x_{m,\ n} = g(m)g(m-1)\ldots g(1).$$

By a remark in 3.2, the elements $t(j)$ pairwise commute. If we denote by $T_n : V^{\otimes n} \to V^{\otimes n}$ and $X_{m,\ n} : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ the operators induced by t_n and by $x_{m,\ n}$ respectively, under the preceding tensor representation of ZB_n, we have

Proposition 6. *: The following identities hold*

$$T_{m+n} = X_{n,\ m}(T_n \otimes 1)X_{m,\ n}(T_m \otimes 1) = (T_m \otimes 1)X_{n,\ m}(T_n \otimes 1)X_{m,\ n}$$

proof: the element $t_n \in ZB_n$ is sent in the Coxeter group CB_n to a product of n^2 generators $t,\ g_j$; but we have shown in 3.2 that the longest element of CB_n, which is uniquely determined, also have length n^2: then it is equal to t_n. Since the element $x_{n,\ m}t_n x_{m,\ n}t_m$ of CB_{m+n} has length $(m+n)^2$, it is equal to t_{m+n} in CB_{m+n}. Now, these identities lift to the corresponding elements in ZB_{m+n} by proposition 4. Applying the tensor representation to both sides, we obtain the first equality. The second one follows from the symmetric procedure. □

Proposition 7. *: The element t_n is contained in the center of ZB_n.*

See 4.3 for a categorical interpretation.

proof: we know that t commutes with t_n; the relation

$$g_j t(j-1)t(j) = t(j-1)t(j)g_j$$

comes directly from the definition of the elements $t(j)$. Finally, the element g_j commutes with $t(k)$ for $k \neq j-1,\ j$. Then t_n commutes with any element of ZB_n. □

Here are two examples of four braid pairs: fix a ground ring k and parameters $\alpha,\ \beta,\ \rho,\ p \in k$; define

$$\theta = \rho - \rho^{-1},\ p^2 = q,\ \delta = q - q^{-1},\ \delta^* = q^2 - q^{-2}$$
$$\mu = \delta^*(1-q^{-2}),\ \lambda = q^{-1}\delta^*,\ \omega = \sqrt{q+q^{-1}}$$

The following matrices F_j (where an empty place carries a 0) act on a free module V_j over k, which has to be thought of as an irreducible module for the quantum group $U_q(sl_2)$. With this in mind, the R-matrices X_j are then the specializations onto $V_j \otimes V_j$ of the universal R-matrix of $U_q(sl_2)$. Besides explicit calculations may show that the pairs $(X_j,\ F_j)$ verify the four braid relation, it is not yet clear what is the four braid pair analog of the FRT construction (see 6.2).

$$F_2 = t(\alpha,\ \beta,\ \theta) = \begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix},\ X_2 = g(p) = \begin{pmatrix} p & & & \\ & 0 & p^{-1} & \\ & p^{-1} & p-p^3 & \\ & & & p \end{pmatrix}$$

$$X_3 = \left(\begin{array}{ccc|ccc|ccc} q^2 & & & & & & & & \\ & & & 1 & & & & & \\ & & & & & & q^{-2} & & \\ \hline & 1 & & \delta^* & & & & & \\ & & & & 1 & & \lambda & & \\ & & & & & & & 1 & \\ \hline & & q^{-2} & & \lambda & & \mu & & \\ & & & & & 1 & & \delta^* & \\ & & & & & & & & q^2 \end{array}\right)$$

$$F_3 = \begin{pmatrix} 0 & 0 & -q \\ 0 & -q^2 & -p^3\omega\theta \\ -q & -p^3\omega\theta & 1-q^2-q^2\theta^2 \end{pmatrix}$$

Remark: We can factor the tensor representation of ZB_n associated to the pair $(X_2,\ F_2)$, to a representation of a certain Hecke algebra associated to B_n (see 6.1.2); this is due to a quadratic relation that is verified by $(X_2,\ F_2)$. In the second case, the pair satisfies a cubic equation, so that we can factor the representation to a representation of a B-type generalization of the algebras of Birman-Wenzl [BW] and Murakami [M]; the cubic relations correspond to the algebraic counterpart, in these respective knot algebras, to skein relations (see [TD1] and [TD2]).

4. Tensor categories with cylinder braiding

We now investigate categories that are related to the preceding algebra. A global point of view for the understanding of representations of categories with a cylinder braiding emerges through tensor representations of ZB_n, with the introduction of a (oriented and colored) graphical calculus in remark 4.5. But we first give a definition of tensor categories with cylinder braiding, which clarify the properties of t in the representation category $ZB_n - Mod$ of ZB_n, and we interpret the resulting *cylinder twist* morphisms in the case of categories of modules over bialgebras.

4.1. **Tensor module categories.** Consider a tensor category $(\mathcal{A},\ \otimes,\ I,\ a,\ r,\ l)$: $\mathcal{A}$ is a category endowed with a (tensor product) functor $\otimes:\ \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and with a neutral object I for $\otimes$, and natural isomorphisms

$$\forall\ X,\ Y,\ Z\in\mathcal{A}\qquad \begin{array}{rccc} a_{X,\ Y,\ Z}: & (X\otimes Y)\otimes Z & \to & X\otimes(Y\otimes Z)\\ r_X: & X\otimes I & \to & X\\ l_X: & I\otimes X & \to & X \end{array}$$

are given, satisfying respectively the *Pentagon* and the *Triangle* axioms. These axioms are constraints imposed on $\otimes$; one can present them by commutative diagrams, as shown below, replacing all $\cdot$ symbols by $\otimes$ and α by a at every place they appear. We call a the *associator of* $\mathcal{A}$, and r (resp. l) the right unit (resp. the left unit).

Given a tensor category $\mathcal{A}$ as above and an arbitrary category $\mathcal{B}$, we define a right action $(*,\ \alpha,\ \rho)$ of $\mathcal{A}$ on $\mathcal{B}$ as the 3-tuplet of maps

- a functor $\mathcal{B}\times\mathcal{A}\to\mathcal{B},\ (Y,\ X)\mapsto Y*X$
- a natural isomorphism $\alpha,\ \alpha_{U,\ V,\ W}:(U*V)*W\to U*(V\otimes W)$, where $U\in Ob(\mathcal{B})$, V and $W\in Ob(\mathcal{A})$ (this map corresponds clearly to the above associator a, see the diagrams below).
- a natural isomorphism $\rho,\ \rho_X: X*I\to X,\ X\in Ob(\mathcal{B})$.

The last two maps satisfy the Pentagon and the Triangle axioms:

$$\begin{tikzcd}
((U*V)*W)*X \arrow[r, "\alpha_{U*V,W,X}"] \arrow[d, "\alpha_{U,\ V,\ W}\otimes id_X"] & (U*V)*(W\otimes X) \arrow[r, "\alpha_{U,V,W\otimes X}"] & U*(V\otimes(W\otimes X)) \\
(U*(V\otimes W))*X \arrow[rr, "\alpha_{U,V\otimes W,X}"] & & U*((V\otimes W)\otimes X) \arrow[u, "id_U\otimes a_{V,\ W,\ X}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(U*I)*V \arrow[r, "\alpha_{U,I,V}"] \arrow[dr, "\rho_U*id"'] & U*(I\otimes V) \arrow[d, "id*l_V"] \\
& U*V
\end{tikzcd}$$

Such a pair $(\mathcal{B},\ \mathcal{A})$ is called a right tensor $\mathcal{A}$-module category $\mathcal{B}$. Left actions are defined similarly. Equivariant morphisms are easily defined between tensor module categories (see [TD2]).

4.2. **Categories with cylinder braiding.** Let us now be given two categories $\mathcal{A}$ and $\mathcal{B}$ such that:

- $(A,\ Z)$ is a braided tensor category, where
$$\forall\ M,\ N\in Ob(\mathcal{A}),\ Z_{M,\ N}: M\otimes N\to N\otimes M$$
are the braiding morphisms for $\mathcal{A}$ (see [Kas, ch. 13]).
- $(\mathcal{B},*,\ \alpha,\ \rho)$ is a right $\mathcal{A}$-module,
- $\mathcal{A}$ is a subcategory of $\mathcal{B}$, with $Ob(\mathcal{A})=Ob(\mathcal{B})$ (for example, extend $\mathcal{A}$ to $\mathcal{B}$ by adding k-linear morphisms to the set of $\mathcal{A}$-morphisms $Hom_{\mathcal{A}}$, k being the ground ring).
- $(\alpha,*,\ \rho)$ restricts to $(a,\ \otimes,\ r)$ on $\mathcal{A}\times\mathcal{A}$.

A $\mathcal{B}$-*endomorphism* of $\mathcal{A}$ consists of a family $t_X\in Hom_{\mathcal{B}}(X,\ X)$ such that $\forall\ f\in Hom_{\mathcal{A}}(X,\ Y)$, the following commutative diagram is valid:

$$\begin{array}{ccc} X & \xrightarrow{t_X} & X\\ \Big\downarrow f & & \Big\downarrow f\\ Y & \xrightarrow{t_Y} & Y \end{array}$$

Finally, a *cylinder twist* for $(\mathcal{B},\ \mathcal{A})$ is defined as a $\mathcal{B}$-endomorphism t of $\mathcal{A}$ such that

$$t_{X\otimes Y}=(t_X\otimes id_Y)Z_{Y,\ X}(t_Y\otimes id_X)Z_{X,\ Y}$$
$$Z_{Y,\ X}(t_Y\otimes id_X)Z_{X,\ Y}(t_X\otimes id_Y)=(t_X\otimes id_Y)Z_{Y,\ X}(t_Y\otimes id_X)Z_{X,\ Y}$$

The second identity is a categorical version of the FBP relation: the existence of a cylinder twist specifies a strong property of the action of $\mathcal{A}$ on $\mathcal{B}$.
A pair $(\mathcal{B},\ \mathcal{A})$ with a cylinder twist is called a tensor module pair with *cylinder braiding*.

Let us anticipate on 4.5 with the following result:

Proposition 8. *: The first identity, which we call T, implies the second.*

proof: Naturality gives

$$Z_{X,\ Y} t_{X\otimes Y} Z_{X,\ Y}^{-1} = t_{Y\otimes X}$$

hence $t_{Y\otimes X} = Z_{X,\ Y}(t_X \otimes id_Y) Z_{Y,\ X}(t_Y \otimes id_X)$ by the first identity. Now interchange X and Y to find the result. □

One may also show that a cylinder twist is compatible with the neutral object: $T_I = id_I$.

4.3. **Categories of braids and cylinder braids.** Define the categories of braids $\mathcal{A}$ and cylinder braids $\mathcal{B}$ as follows:

- $Ob(\mathcal{A}) = Ob(\mathcal{B}) = \mathbb{N} \setminus \{0\}$
- $Hom_{\mathcal{A}}(m,\ n) = \begin{cases} \emptyset\ if\ m \neq n \\ ZA_{n-1}\ if\ m = n \end{cases}$
- $Hom_{\mathcal{B}}(m,\ n) = \begin{cases} \emptyset\ if\ m \neq n \\ ZB_n\ if\ m = n \end{cases}$

Tensor product between objects in the respective categories is given by the addition in $\mathbb{N} \setminus \{0\}$: $m \otimes n = m + n$.
It is easy to verify that, using the graphical calculus introduced in section 3.3, the braiding $x_{m,\ n}$ in $\mathcal{A}$ may be represented in ZA_{m+n-1} by:

In view to find a cylinder twist in the pair $(\mathcal{B},\ \mathcal{A})$, let us recall the definition (see 3.4) of the family $t(j),\ j = 1,\ \dots,\ n$ of elements of ZB_n:

$$t(1) := t,\ t(2) := g_1 t g_1,\ \dots,\ t(j) := g_{j-1}g_{j-2}\dots g_1 t g_1 \dots g_{j_1},\ \dots$$
$$t_n = t(1)t(2)\dots t(n)$$

The elements $t(j)$ are pictorially represented by

The figures make clear the following proposition (recall the concrete application in 3.4 and its proof):

Proposition 9. *: The following identity holds in $Hom_{\mathcal{B}}(m+n,\ m+n)$:*

$$t_{m+n} = x_{n,\ m}(t_n \otimes 1)x_{m,\ n}(t_m \otimes 1)$$

Moreover, the element t_n is in the center of ZB_n.

proof: In the symmetric picture for B-braids, we have for $t_{m+n} \in ZB_{m+n}$:

And its image in CB_{m+n} is

Since there is one intersection point between any pair of arcs in the previous picture (this is a reduced word for the longest element in CB_n), we easily see that it may also be presented as the image of $x_{n,\ m}(t_n \otimes 1)x_{m,\ n}(t_m \otimes 1)$ in CB_n. Then use proposition 4.
Moreover, an half twist gives (with a symmetric bloc at the middle of the picture):

which is also

$\square$

Corollary: *The element t_n yields a cylinder braiding on the pair $(\mathcal{B},\ \mathcal{A})$.*

Remark: The wellknown universality of $\mathcal{A}$ for the braiding property extends naturally to the category $\mathcal{B}$ for the cylinder braiding property (see 4.5).

4.4. **Example.** Consider a braided bialgebra $(A,\ \mu,\ \epsilon)$ over a commutative ring k, with universal R-matrix $R = \sum_i a_i \otimes b_i \in A \otimes A$. Define $\mathcal{A}$ as the category of left A-modules and A-linear maps, with braiding $Z_{M,\ N} : M \otimes N \to N \otimes M,\ x \otimes y \mapsto \sum_i b_i y \otimes a_i x$, and $\mathcal{B}$ as the category of left $\mathcal{A}$-modules with k-linear maps.

Proposition 10. *: The cylinder twists for $(\mathcal{B},\ \mathcal{A},\ R)$ correspond bijectively to elements $v \in A$ such that*

$$\mu(v) = (v \otimes 1)\hat{R}(1 \otimes v)R \tag{14}$$

where $\tau(x \otimes y) = y \otimes x,\ \tau(R) = \hat{R}$. Moreover we have $\epsilon(v) = 1$.

sketch of proof: Given an element $v \in A$ satisfying equation (14), define a k-linear morphism in $\mathcal{B}$ by $t_X : X \to X,\ x \mapsto vx$. It is a straightforward calculation to show that t verifies the FBP relation; note that t is not in general $\mathcal{A}$-linear (since v is not assumed to be central in A), hence v is not a ribbon element in A (see [Tu] or [Kas, ch. 14] for a definition) .
Conversally, given a cylinder twist $t \in Hom_{\mathcal{B}}$, the element $v = t_A(1) \in A$ satisfies the identity

(14).
Finally, applying $m(\epsilon\otimes 1)$ on the left of both sides of (14) (m denotes the multiplication in A), we obtain

$$v = \epsilon(v)\ m(\epsilon\otimes 1)\hat{R}\ v\ m(\epsilon\otimes 1)R$$

which gives $v = \epsilon(v)\ v$, since (see [Kas., ch. 8]) we have

$$m(\epsilon\otimes 1)\hat{R} = m(\epsilon\otimes 1)R = 1$$

□

4.5. **Remark on a graphical calculus and the duality.** We want to introduce in this section an orientation and a coloration on cylindrical braids, by elements in $Ob(\mathcal{B})$; the pair $(\mathcal{B},\ \mathcal{A})$ is only supposed to have a cylinder braiding and an extended *duality* (see below). Let us present briefly how we do this.
Define the (geometric) category ORR of *oriented rooted ribbons* with generators all the possible elementary cylindrical tangles that may be used to build (by concatenation) cylindrical braids. The relations are the "obvious" ones, that is they are all the relations that are induced by ambiant isotopy of braids in the cylinder. It is clear that the category ORR is a natural extention of the category $\mathcal{T}$ of oriented tangles.
It is shown in [TD2] how to construct a functor from ORR into any tensor module pair $(\mathcal{B},\ \mathcal{A})$ with cylinder braiding endowed with an extended notion of duality: in particular, the cylindrical (upward oriented) tangles made of a segment with an extremity on the top or the bottom line of the strip $\mathbb{R}\times[1,\ 0]$ and the other extremity on the axis of the cylinder (in the graphical calculus) are sent onto the so-called *rooting* and *corooting* morphisms of $\mathcal{B}$ (see below for their definition).
This rigid representation (i.e. rigid tensor functor) from the oriented category ORR to $\mathcal{B}$ produces a graphical calculus for $\mathcal{B}$ based on the presentation of ORR by generators and relations. In other words, a tensor module category with extended duality *and* with cylinder braiding can be presented by generators and relations as a categorical quotient of ORR.
Since everything above is classical (except the definition of the (co-)rooting morphisms) we present here only a few examples and we refer to [Tu, ch. 12] and [TD2] for more details.

Suppose that there exists a functor $*$ in $\mathcal{A}$ such that for any object V in $Ob(\mathcal{A})$, $*_V : V \to V^*$ is an involution and $1^* = 1$. Furthermore, we demand that $*$ is compatible with the monoidale structure of $\mathcal{A}$, that is $\forall\ a,\ b \in Ob(\mathcal{A})$ (resp. $Hom_{\mathcal{A}}$) we have an isomorphism between $(a\otimes b)^*$ and $b^*\otimes a^*$. Notice in the diagrams below that an ascending arrow at any point on the boundary of the strips indicates it has a non dual object as label. The transposed morphisms in $\mathcal{B}$ are obtained by inversing the orientation.

Let $X,\ Y$ be some objects in a tensor module category $\mathcal{B}$ endowed with a cylinder braiding. The above discussion implies that we have for t_X:

and $(t_X \otimes id_Y)Z_{Y,\ X}(T_Y \otimes id_X)Z_{X,\ Y}$ *is*

Tensor product for t, over X and Y, implies juxtaposition (with proposition 8):

Indeed, a braid in the cylinder may be considered as a band, always looking to the axis, and proposition 8 shows that the verification of the algebraic counterpart of this geometric property is a sufficient condition to have a tensor module category category with cylinder braiding.

We define a left *duality* in $\mathcal{A}$ by the existence of a pair of morphisms $b,\ d \in Hom_{\mathcal{A}}$, with $b_V : I \to V \otimes V^*,\ d_V : V^* \otimes V \to I$. We demand that these morphisms satisfy the following composition rules:

$$V \stackrel{b_V \otimes id_V}{\to} V \otimes V^* \otimes V \stackrel{id_V \otimes d_V}{\to} V = id_V \qquad (1)$$

$$V^* \stackrel{id_{V^*} \otimes b_V}{\to} V^* \otimes V \otimes V^* \stackrel{d_V \otimes id_{V^*}}{\to} V^* = id_{V^*} \qquad (2)$$

and are pictorially represented by the generators of the Temperley-Lieb category TA (see 6.1.1) with the obvious orientation (from left to right).
A right duality is defined similarly with morphisms $a,\ e$ with $a_V : I \to V^* \otimes V,\ e_V : V \otimes V^* \to I$, satisfying versions (with the obvious modifications) of the identities (1) and (2) above.

Proposition 11. *: A cylinder twist t ($\in Hom_{\mathcal{B}}$) is always compatible with a duality (in $\mathcal{A}$), i.e. we have*

which is formally

$$d_X(t_{X^*} \otimes id)Z_{X,\ X^*}(t_X \otimes id)Z_{X^*,\ X} = d_X$$

and the corresponding formal "upside-down" version:

$$Z_{X^*,\ X}(t_{X^*} \otimes id_X)Z_{X,\ X^*}(t_X \otimes id_{X^*})b_X = b_X$$

proof: we restrict to the case of a left duality. Since $b,\ d$ are morphisms in $\mathcal{A}$ and t is a $\mathcal{B}$-endomorphism, we have

$$t_{X \otimes X^*} b_X = b_X t_I = b_X,\ d_X t_{X^* \otimes X} = d_X$$

But these identities are equivalent to the conditions of compatibility, in the statement of the proposition. □

The above identities allow, in particular, to write down an expression for t_{X^*} in terms of t_X:

$$t_{X^*} = ((d_X Z^{-1}_{X^*,\ X}(t_X^{-1} \otimes id)Z^{-1}_{X,\ X^*}) \otimes id) \circ (id \otimes b_X)$$

We now define an extended notion of duality for a pair $\mathcal{C} = (\mathcal{B},\ \mathcal{A})$ with cylinder braiding.

Let $(b,\ d)$ be a left duality for $\mathcal{A}$, and consider two morphisms (the *rooting* and *corooting* morphisms)

$$\beta_X : I \to X^*,\ \delta_X : X \to I$$

in $Hom_{\mathcal{B}}$, such that

$$\begin{array}{ll} d_X(\beta_X \otimes id_X) = \delta_X,\ (\delta_X \otimes id_{X^*})b_X = \beta_X & (1) \\ \beta_{X \otimes Y} = (\beta_X \otimes id_Y)\beta_Y,\ \delta_{X \otimes Y} = \delta_Y(\delta_X \otimes id_Y) & (2) \end{array}$$

A left duality for $(\mathcal{B},\ \mathcal{A})$, also called a rooted structure for the pair, is a left duality for $\mathcal{A}$ and a pair of morphisms $\beta,\ \delta \in Hom_{\mathcal{B}}$ as above.

It is graphically clear to see how a left duality for a pair $(\mathcal{B},\ \mathcal{A})$ is compatible with a cylinder braiding: recall that the rooting maps are represented by a segment with an extremity on the top

or the bottom line of the strip, and the other extremity on the axis of the cylinder (in our graphical calculus), oriented from left to right . Then the compatibility of the morphism t with the rooting maps is graphically represented by spinning the free extremity of the segment once around the axis.
The formula is $\delta_X t_X = \delta_X$ (and $t_X \beta_X = \beta_X$ for the upside down version) and $t_Y(\delta_X \otimes id_Y) = (\delta_X \otimes id_Y) Z_{Y,\ X}(t_Y \otimes id_X) Z_{X,\ Y}$ (and $Z_{Y,\ X}(t_Y \otimes id_X) Z_{X,\ Y}(\beta_X \otimes id_Y) = (\beta_X \otimes id_Y) t_Y$ for the upside down version, for the rooting).

Similar axioms hold with a right duality.

5. Universal cylinder twist

In this section we describe in concrete terms some pairs of categories with a cylinder braiding. They are all representation categories for a quantum group (in each pair, one category is extended by k-linear maps); in particular, the category of $U_q(sl(2))$-modules provides the *highly non trivial* examples of four braid pairs we gave in 3.4.

5.1. **Cylinder twist for comodules.** Here is the dual formalism to the cylinder braiding property in a braided category of modules.

Let A be a bialgebra over a ring k, with comultiplication μ and counit ϵ. Given a vector space M, define a left A-comodule structure on M by the map

$$\mu_M \to A \otimes M,\ x \mapsto \sum x^1 \otimes x^2$$

Given a k-linear form

$$r : A \otimes A \to k$$

consider the deformed *flip* morphism

$$Z_{M,\ N} : M \otimes N \to N \otimes M$$
$$x \otimes y \mapsto \sum r(y^1 \otimes x^1) y^2 \otimes x^2$$

The map r is called *a braid form* if and only if $\{Z_{M,\ N}\}_{M,\ N}$ is a braiding in the category of left A-comodules. When such an r exists, we say that $(A,\ r)$ is a *cobraided* bialgebra.
¿From now on, we shall only refer to finite dimensional bialgebras. The multiplication in the dual algebra A^* (denoted by a convolution symbol) is defined by

$$\forall\ f,\ g : A \to k,\ (f * g)(a) = \sum f(a^1) g(a^2)$$

where $\mu(a) = \sum a^1 \otimes a^2$.
Moreover, we denote by m the multiplication in A and we define a tensor product for elements in A^* by

$$\forall\ f,\ g : A \to k,\ f \hat{\otimes} g : A \otimes A \to k,\ (f \hat{\otimes} g)(a \otimes b) = f(a) g(b)$$

Let $(A,\ r)$ be a cobraided bialgebra. A map $f : A \to k \in A^*$ is called *a cylinder form* if and only if f is convolution invertible and

$$f \circ m = (f \hat{\otimes} \epsilon) * r\tau * (\epsilon \hat{\otimes} f) * r$$

where τ is the canonical flip map; in formal notations with elements:

$$\forall\ a,\ b \in A,\ f \circ m(a,\ b) = \sum f(a^1) r(b^1 \otimes a^2) f(b^2) r(a^3 \otimes b^3)$$

where $(\mu \otimes id)\mu(a) = \sum a^1 \otimes a^2 \otimes a^3$.

One may verify immediately that

Proposition 12. *: Let f be a cylinder form. Define*

$$t_M : M \to M,\ x \mapsto \sum f(x^1) x^2$$

where M is a left A-comodule. Then $\{t_M\}_M$ defines a cylinder braiding on the braided category of left A-comodules. In particular, for each left A-comodule M, the pair $(Z_{M,\ M},\ t_M)$ is a four braid pair on M.

Here is an example of braid form: let $A = kG$ be the group algebra of an abelian group G, over the ring k. Then, the explicit description of the cobraided structure of A (see [Kas, ch.4]) shows that braid forms in A are in one-to-one correspondence with the bicharacters of G, i.e. with maps $r : G \times G \to k^\times$ such that

$$r(gh,\ k) = r(g,\ k)r(h,\ k)$$
$$r(k,\ gh) = r(k,\ g)r(k,\ h)$$

Then a k-linear map $f : G \to k$ yields a cylinder form if and only if

$$\forall\ a,\ b \in G\ f(ab) = f(a)f(b)r(a,\ b)r(b,\ a)$$

This last identity means that f is a quadratic form with associated bilinear form r (write it with additive notations)!

5.2. **The braided category of $U_q(sl(2))$-modules.** Let us denote $\mathcal{U} = U_q(sl(2))$, and recall that in section 2 we used generators $E,\ F,\ K,\ K^{-1} \in \mathcal{U}$. Define $\mathcal{U} - Int$ as the category of integrable $\mathcal{U}$-modules, i.e. vector spaces M such that

1) $M = \oplus_{n\in\mathbb{Z}} M^n$, where M^n is an eigenspace of K with eigenvalue q^n.
2) E and F act locally nilpotently, i.e. $\forall\ x \in M,\ \exists\ N \in \mathbb{N} :\ E^N x = F^N x = 0$.

Remark: consider operators on $\mathcal{U} - Int$ (which commutes with $\mathcal{U}$-linear maps). Each $x \in \mathcal{U}$ gives an operator by left multiplication, but note that in this category, *infinite sums of elements of $\mathcal{U}$* are also interpretable in terms of operators.

Then, assume that k is a field, $q^{1/2} \in k$ and q is not a root of unity. If H is the operator $M^m \to M^m,\ x \mapsto mx$, then $x = q^{H\otimes H/2}$ acts on $M^m \otimes N^n$ by multiplication by $q^{mn/2}$.
The infinite sum

$$\Psi = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{(q - q^{-1})^n}{[n; q]!} F^n \otimes E^n \tag{15}$$

is a well defined operator on $\mathcal{U} - Int \times \mathcal{U} - Int$ called a quasi-R-matrix. Indeed, the universal R-matrix R for $\mathcal{U}$ is $R = x \circ \Psi$ (see [Lus]).

Proposition 13. *: With $\delta = q - q^{-1}$, we have the equality $\Psi = e_{q^{-2}}(\frac{\delta^2}{q} F \otimes E)$.*

proof: Rewrite the coefficients of $F^n \otimes E^n$:

$$q^{n(n-1)/2} \frac{\delta^n}{[n]!} = q^{n(n-1)/2} \frac{\delta^{2n}}{(q - q^{-1}) \dots (q - q^{-n})} =$$
$$q^{n(n-1)/2} \frac{\delta^{2n}}{q^{n(n+1)/2}(1 - q^{-2}) \dots (1 - q^{-2n})}$$

The definitions of e_q and $(q^{-2}|q^{-2})_n$ in (2) gives the result. □

5.2.1. *The quantum Weyl group for $\mathcal{U} - Int$.* We shall use in this section some material from [Lus., ch. 1] without proofs, and we refer to this book for more details. Roughly speaking, there is an action of the braid group associated to any given Lie algebra on the category of its integrable modules. This action is generated by some *symmetries* (in Lusztig's book, they are refered as $T'_{i,\pm}$ and $T"_{i,\pm}$), defined by automorphisms in this category, and there is an operator L on $\mathcal{U} - Int$ that intertwines the action of these symmetries with the coproduct. We shall take a look at the action of this operator on integrable modules, and see how it generates a cylinder twist.

The category $\mathcal{U} - Int$ is semi-simple, since any of its object M is the direct sum of simple modules $M^m,\ m \in \mathbb{N}$. The simple modules V_n may be presented with basis $x_0,\ \dots,\ x_n$ and with the action ($x_{-1} = x_{n+1} = 0$):

$$F(x_i) = [i + 1]x_{i+1},\ E(x_i) = [n + 1 - i]x_{i-1},\ K(x_i) = q^{n-2i}x_i$$

Proposition 14. *: There exists an operator L on $\mathcal{U} - Int$ which ver ifies on V_n:*

$$x_j \mapsto (-1)^j q^{j(n+j)} x_{n-j}$$

In Lusztig's book (chapter 1), the operator L is refered as $T^{'}_{i,\ \pm 1},\ T^{"}_{i,\ \pm 1}$.

Let us denote, as usual, by $\mu(L)$ the operator that L induces when acting on tensor products of modules (where μ is the comultiplication in $\mathcal{U}$). We list in the next proposition a few essential properties of L.

Proposition 15. *: The above operator L has the following properties (as identities between operators):*

$$LEL^{-1} = -KF,\ LFL^{-1} = -EK^{-1},\ LKL^{-1} = K^{-1} \tag{16}$$

$$\mu(L) = (L \otimes L) \circ \Psi = \tau R \circ (L \otimes L) \circ x^{-1} \tag{17}$$

$$x(L \otimes 1) = (L \otimes 1)x^{-1},\ x(1 \otimes L) = (1 \otimes L)x^{-1} \tag{18}$$

$$(L \otimes L)\Psi(L \otimes L)^{-1} = x \circ \tau\Psi \circ x^{-1} \tag{19}$$

sketch of proof: (16) and (18) follow from a direct verification in V_n, and (19) by applying (16) on each summand of the expansion of Ψ. Let us turn to (17): the second equality follows from the first one by using (18) and (19). Now, by the quantum Clebsch-Gordan decomposition (see [Kas], chpt. 7)

$$V_m \otimes V_n = V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{|m-n|}$$

one can determine $\mu(L)$ on each module in the form $V_m \otimes V_n$; the action of Ψ on $V_m \otimes V_n$ may also be computed explicitly using the same fact. Applying finally $L \otimes L$, we get the result ... after all these long calculations. $\square$

5.2.2. *The universal twist for $\mathcal{U} - Int$.*

5.2.3. *Generalities.* A universal twist for $\mathcal{U} - Int$ is, by definition, a universal operator t such that (as an identity between operators)

$$\mu(t) = \tau R \circ (1 \otimes t) \circ R \circ (t \otimes 1) \tag{20}$$

When applied on the module M, we denote t as t_M.
We shall now look explicitly at the form of t_M (supposing that it exists).

1) If $M \xrightarrow{i} N \xrightarrow{p} M$ and $p \circ i = id$, then $t_M = pt_N i$ (by the naturality of the universal cylinder twist). Hence t_N determines t_M for each direct summand M of N.
2) If $M = M_1 \oplus M_2$, then obviously $t_M = t_{M_1} \oplus t_{M_2}$.
3) Let $V = V_1$ be the fundamental 2-dimensional $\mathcal{U}$-module. Then each finite dimensional $\mathcal{U}$-module is a direct summand of some tensor power $V^{\otimes N}$ (by the Clebsch-Gordan formula). For example, V_n appears with multiplicity 1 in $V^{\otimes n}$. Hence the operator t is determined by the endomorphisms $t_{V^{\otimes n}}$.
4) The twist identity

$$t_{V^{\otimes (m+n)}} = Z_{V^{\otimes n},\ V^{\otimes m}}(t_{V^{\otimes n}} \otimes 1)Z_{V^{\otimes m},\ V^{\otimes n}}(t_{V^{\otimes m}} \otimes 1)$$

implies that $t_{V^{\otimes n}}$ is determined inductively by t_V.

In conclusion t is uniquely determined by the fundamental four braid pair on V, which has necessarily the form already seen in section 3.4 (unless F_2 is a multiple of the identity):

$$F_2 = t(\alpha,\ \beta,\ \theta) = \begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix},\ X_2 = g(p) = \begin{pmatrix} p & & & \\ & 0 & p^{-1} & \\ & p^{-1} & p - p^3 & \\ & & & p \end{pmatrix}$$

Let us precise this result:

5) Denoting by V_0 the trivial module, the decomposition $V^{\otimes 2} = V_2 \oplus V_0$ and the computation of $t_{V^{\otimes 2}}$ implies that $t_{V_0} = id \Leftrightarrow \alpha\beta = -q$.
6) An inductive computation of t_{V_n} using all the above remarks shows that its matrix has a bottom-right triangular form (i.e. with 0 coefficients *above* the codiagonal).
7) In case where $(\alpha,\ \beta) = (1,\ -q)$, the codiagonal is given by Lusztig's operator L.

Now observe that some parameter transformations on a general F suffice to put it in the form $\begin{pmatrix} 0 & -q \\ 1 & \theta \end{pmatrix}$. So, up to these transformations on F (this is the *only* restriction we put on the general form of t for $\mathcal{U}-Int$) we can set

$$t = L \circ T$$

where T acts on V_n as an upper triangular matrix with unit diagonal.

Therefore, if there is such a universal twist operator t for $\mathcal{U}-Int$, then T has the form $\sum_{n=0}^{\infty} \alpha_n E^n$, i.e. it lives in the Borel subalgebra $\mathcal{B}_q^+$ of $\mathcal{U}$.

5.2.4. *Construction of the twist.* The properties of L listed above allow us to simplify the conditions on T under which t is indeed a universal cylinder twist.

Proposition 16. : *T induces a universal twist if and only if:*

$$\mu(T) = x(1 \otimes T)x^{-1} \circ (L^{-1} \otimes 1)\Psi(L \otimes 1) \circ (T \otimes 1) \tag{21}$$

proof: Rewriting

$$\mu(t) = \tau R \circ (1 \otimes t) \circ R \circ (t \otimes 1)$$

with $t = LT$ and $R = x \circ \Psi$, we get

$$\mu(T) = \mu(L^{-1})\tau R \circ (1 \otimes LT) \circ x\Psi \circ (LT \otimes 1)$$

But $\mu(L^{-1}) = x(L^{-1} \otimes L^{-1})(\tau R)^{-1}$, hence the latter identity is equal to

$$x(1 \otimes T)(L^{-1} \otimes 1)x\Psi(L \otimes 1)(T \otimes 1)$$

which is the result (using (18)). □

Continuing to assume that $T = \sum_k \alpha_k E^k$, we get directly from the definitions the following facts:

$$x(1 \otimes T)x^{-1} = \sum_k \alpha_k (K \otimes E)^k, \ (L^{-1} \otimes 1)\Psi(L \otimes 1) = e_{q^{-2}}(-\frac{\delta^2}{q} KE \otimes E)$$

where $\delta = q - q^{-1}$.
Define a formal power series in z with coefficients in $\mathbb{Z}[\theta, \ q^{1/2}, \ q^{-1/2}]$ by the identity

$$\tau_{q,\ \theta}(z) = \prod_{j=0}^{\infty} \frac{1}{1 - \frac{\theta}{\sqrt{q}} z q^{-2j} + z^2 q^{-4j}} \tag{22}$$

Theorem 5. : *The operator $T = \tau_{q,\ \theta}(\frac{\delta}{\sqrt{q}} E)$ on $\mathcal{U} - Int$ yields a universal cylinder twist.*

Remark: Consider a quadratic extension of $\mathbb{Z}[\theta, \ q^{1/2}, \ q^{-1/2}]$ in λ, such that $\frac{\theta}{\sqrt{q}} = \lambda + \lambda^{-1}$. The denominators in (22) factor into linear factors, giving

$$\tau_{q,\ \theta}(z) = e_{q^{-2}}(\lambda z) e_{q^{-2}}(\lambda^{-1} z) \tag{23}$$

proof of the theorem: set $x = \frac{\delta}{\sqrt{q}} E \otimes 1$ and $y = \frac{\delta}{\sqrt{q}} K \otimes E$; then $yx = \frac{\delta^2}{q} KE \otimes E$ and $x, \ y$ are q^{-2}-commuting variables. If we change q^{-2} to q, the verification of the twist identity (21) is a particular case of the following lemma:

Lemma: *Let $x, \ y$ be q-commuting variables. Then the following formal identity holds:*

$$e_q(\lambda(x+y)) e_q(\lambda^{-1}(x+y)) = e_q(\lambda y) e_q(\lambda^{-1} y) e_q(-yx) e_q(\lambda x) e_q(\lambda^{-1} x)$$

proof of the lemma: Since $x, \ y$ are q-commuting, we have

$$e_q(\lambda^{\pm 1} x + \lambda^{\pm 1} y) = e_q(\lambda^{\pm 1} y) e_q(\lambda^{\pm 1} x)$$

See section 2.1 for details. Hence the identity reads

$$e_q(\lambda^{-1}y)e_q(-yx)e_q(\lambda x) = e_q(\lambda x)e_q(\lambda^{-1}y)$$

Now, choosing as new variables $\lambda x,\ \lambda^{-1}y$, we have to prove that

$$e_q(x)e_q(y) = e_q(y)e_q(-yx)e_q(x)$$

Recall that in 2.3 we found that (up to $\dfrac{\delta}{\sqrt{q}}$), our q^{-2}-commuting variables $x,\ y$ satisfy the identity $e_q(x+y) = e_q(y)e_{q^2}(-qyx)e_q(x)$. Inserting the (always verified) relation $e_{q^2}(z)e_{q^2}(qz) = e_q(z)$ between formal power series on both sides, this gives

$$e_{q^2}(x+y)e_{q^2}(q(x+y)) = e_{q^2}(y)e_{q^2}(qy)e_{q^2}(-qyx)e_{q^2}(x)e_{q^2}(qx)$$

The change of variables $qy \mapsto y,\ q^2 \mapsto q$ finally yields the lemma. □

In conclusion, we see that the only possible obtruction to find a cylinder twist morphism in the braided category $\mathcal{U} - Int$ was to verify twist identity in $U_q(sl(2))$.

6. Applications and further results

We first describe some facts in the representation theory of Temperley Lieb categories, and then we discuss the integrability of the defining equation of a Four Braid Pair in a given representation category.

6.1. **Temperley-Lieb categories of type A and B, representations of knot algebras.**

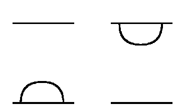
This section presents algebraic models for two linear categories, the Temperley Lieb categories of type A and B, through the braiding and the cylinder braiding properties in the representation category of $U_q(sl(2))$. Moreover, the Kauffman functor from the category of unoriented tangles to both categories may be used to induce representations of some well known knot algebras, and an extension of the Jones polynomial to links in the cylinder is finally obtained. Details may be found in [TD1] and [TD2].

6.1.1. *Temperley Lieb category of type A.* An $(m,\ n)$ bridge $(m+n = 2k,\ k \in \mathbb{N})$ is a collection of k arcs in a strip $\mathbb{R} \times [0,\ 1]$ without double points and with end points in $\{1,\ 2,\ \ldots,\ n\} \times \{0\} \cup \{1,\ 2,\ \ldots,\ n\} \times \{1\}$.
Given a $(m,\ n)$-bridge S and a $(n,\ p)$-bridge P, one may define a new $(m,\ p)$-bridge $T \wedge S$ by concatenation and elimination of the circles produced (T is over S). The number of such circles is denoted by $k(T,\ S)$.

Let k be a commutative ring, $d \in k$ an invertible parameter. We define the Temperley Lieb category of type A as the k linear category with objects the set $\mathbb{N} \setminus \{0\}$, and with set of morphisms $Hom_{TA}(m,\ n)$ the free k-module on the set of $(m,\ n)$-bridges.
The following composition rule on bridges

$$T \circ S = d^{k(T,\ S)} T \wedge S$$

is a k-bilinear map, with which the juxtaposition in the plane endow TA with the structure of a tensor category. The generators of TA as a tensor category are respectively denoted by p and i:



Their relations are $(id \otimes p) \circ (i \otimes id) = (p \otimes id) \circ (id \otimes i) = id$ which correspond diagrammatically to the torsion of a single free strand in the plane. Notice that, as suggested in section 4.5, these relations between morphisms in any tensor category endow it with a duality.

Clearly, the Temperley Lieb algebra T_nA with n generators $g_1,\ \ldots,\ g_n$ and relations $g_i g_{i\pm 1} g_i = g_i,\ g_i^2 = dg_i$ and $g_i g_j = g_j g_i\ if|i-j| > 1$, may be identified with $Hom_{TA}(n,\ n)$.
The interest in Temperley Lieb categories for topology is that their representations (functors into

modules) can be viewed as categorical quotients of the representations of the category $\mathcal{T}'$ of unoriented tangles, by mean of functors providing powerful families of link invariants. For example, the Kauffman tensor functor K from $\mathcal{T}'$ to TA is obtained by (we denote by $X \in \mathcal{T}$ the elementary geometric braiding):

with $d = -A^2 - A^{-2}$ (K being the identity on the other generators). It may be extended to colored tangles endowed with an orientation: one may then verify that K still gives a tensor functor between categories with duality.

Before stating the main result of this section, let us now present an interpretation of $Hom_{TA}(m,\, n)$ through representation theory. Let, as before, the basic 2-dimensional $U_q(sl(2))$-module be denoted by V, and define a tensor functor by

$$\begin{aligned} \Phi : n \in Ob(TA) &\mapsto V^{\otimes n} \\ \Phi(p) : V \otimes V &\to k, \\ \Phi(i) : k &\to V \otimes V. \end{aligned}$$

The generators of Hom_{TA} are sent by Φ respectively to the multiple of the projection to the trivial summand $V \otimes V \to k$ and to the multiple of the injection of the trivial summand $k \to V \otimes V$, in the Clebsch-Gordan decomposition of $V \otimes V = V_4 \oplus V_2 \oplus V_0$ (with $V_0 \cong k$ as $U_q(sl(2))$ modules). Note that the projection and the injection of the only one-dimensional summand in a block diagonalization of the Yang-Baxter operator yield these maps. Now extend Φ by K:

$$\Phi \circ K(X) = A\Phi(id) + A^{-1}\Phi(p \circ i)$$

For example, setting $A = q$, the defining relation of Φ for the braiding X on $V \otimes V$ gives:

$$X := \Phi \circ K(X) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} =$$

$$q\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+q^{-1}\begin{pmatrix}0&0&0&0\\0&-q&1-q&0\\0&1-q&-q^{-1}&0\\0&0&0&0\end{pmatrix}$$

We state the main result of this section without proof:

Theorem 6. *: The tensor functor* Φ*:*

$$\Phi : Hom_{TA}(m,\ n) \to Hom_{U_q(sl(2))}(V^{\otimes m},\ V^{\otimes n}) \tag{24}$$

is an isomorphism between categories with duality.

Remarks:

1) The theorem gives a *combinatorial* description of the representation category $U_q(sl(2)) - Mod$, since the Clebsch-Gordan injection and projection homomorphisms onto a summand allow to reconstruct any module over $U_q(sl(2))$.
2) The "representation" version of the quantum trace (evaluation of any oriented link, decomposed into oriented tangles, under the functor $\Phi \circ K$) in the second set is the Jones polynomial of oriented links (which is an invariant of ambiant isotopy of oriented links in the three-sphere).

6.1.2. *Temperley Lieb category of type B.* All is similar to the construction of the previous section: the objects of TB is again $\mathbb{N} \setminus \{0\}$, but the set of morphisms $Hom_{TB}(m,\ n)$ is the symmetric $(m,\ n)$-bridges, that is bridges with a symmetry with respect to the axis $\{0\} \times [0,\ 1]$ in the strip $\mathbb{R} \times [0,\ 1]$.
Composition and tensor product between objects and morphisms are as before: they are obtained by concatenation and juxtaposition of diagrams. We add a parameter D to d: the latter evaluates two symmetric circles while the former evaluates a circle which intersect the axis, in the graphical calculus, in two points.
A symmetric concatenation of diagrams show that TB is a tensor module over TA, with generators (the symmetry in the axis allows to consider only one side in each diagram) those of TA plus a segment with both extremities on the top or the bottom line, but in a *symmetric* position with respect to the axis. Thus, we obtain the diagrammatics we used for the rooting maps in section 4.5.

The set $Hom_{TB}(n,\ n)$ is called the n^{th} *Temperley Lieb algebra of type B*, and it is denoted by TB_n: it is easily seen that it may be realized with generators $e_0,\ e_1,\ \ldots,\ e_{n-1}$ and relations

$$\begin{aligned}
&e_0^2 = De_0\\
&e_1e_0e_1 = Fe_1\\
&e_j^2 = de_j,\ j \geq 1\\
&e_ie_je_i = e_i,\ |i-j| = 1,\ i,\ j \geq 1\\
&e_ie_j = e_je_i,\ |i-j| \geq 2
\end{aligned}$$

With generic parameters, TB_n is a semi-simple algebra, with $n{+}1$ irreducible modules $M_0,\ \ldots,\ M_n$ of dimension $dim(M_j) = \begin{pmatrix} n\\ j\end{pmatrix}$ (see [TD2]). The total dimension of TB_n is

$$\begin{pmatrix} 2n\\ n\end{pmatrix} = \sum_{j=0}^{n}\begin{pmatrix} n\\ j\end{pmatrix}^2 \tag{25}$$

We see from the procedure explained in 3.4 and the above defining relations of TB_n that the representation theory of TB is based on R-matrices satisfying quadratic relations.

As an interesting example, consider the *Hecke algebra* $H_nB(q,\ Q))$ of type B:

$$H_nB(q,\ Q) = \langle t,\ g_1,\ \ldots,\ g_{n-1}|\ B - type\ braid\ relations\ (11),$$
$$t^2 = (Q-1)t + Q,\ g_j^2 = (q-1)g_j + q,\ \forall\ j = 1,\ \ldots,\ n-1\rangle$$

It appears, in particular, in the skein theory of polynomial knots invariants.

One may show that (see [TD1] and [TD2])

Theorem 7. *: The B-type Temperley Lieb algebra $TB_n(d,\ D)$ with parameters $d,\ D$ is a quotient of the B-type Hecke algebra $H_nB(q,\ Q)$, with $d = q^{1/2} + q^{-1/2}$ and $D = Q^{1/2} + Q^{-1/2}$.*
More precisely, setting $q = p^2,\ d = p + p^{-1},\ D = a(1 + Q), F = a(p + p^{-1}Q)$, there is a surjective homomorphism

$$\begin{array}{ccc} H_nB(q,\ Q) & \to & TB_n(d,\ D,\ F) \\ t & \mapsto & a^{-1}e_0 - 1 \\ g_j & \mapsto & pe_j - 1 \end{array}$$

We have seen in 3.4 that a tensor representation of the braid group ZB_n on a tensor power $V^{\otimes n}$ gives a four braid pair $(X,\ F)$. If $(X,\ F)$ satisfies the relations

$$X^2 = (q-1)X + q,\ F^2 = (Q-1)F + Q$$

we then obtain a tensor representation of $H_nB(q,\ Q)$. Such a pair is provided by the standard example $(X_2,\ F_2)$ on the fundamental $U_q(sl(2))$-module V (see 3.4), after a suitable normalization of F_2. Furthermore,

Proposition 17. *: This tensor representation of $H_nB(q,\ Q)$ factors over $T_nB(d,\ D,\ F)$, by setting $d = p + p^{-1},\ D = a + y,\ F = pa + p^{-1}y$ and*

$$\begin{array}{ccc} e_0 & \mapsto & E_0 \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-1 times} \\ e_j & \mapsto & 1 \otimes \ldots \otimes \underbrace{E}_{i^{th}} \otimes \ldots \otimes 1 \end{array}$$

where

$$E_0 = \begin{pmatrix} a & b \\ x & y \end{pmatrix},\ E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & -1 & 0 \\ 0 & -1 & p^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Conclusion: the $U_q(sl(2))$-decomposition of the module $V^{\otimes n}$ into irreducible direct summands is an irreducible $H_nB(q,\ Q)$-decomposition *and* a direct sum of irreducible $T_nB(d,\ D,\ F)$-modules, for suitable parameters $d,\ D,\ F$. One can consequently ask what is the analog in TB of the isomorphism Φ.

Here is an algebraic model for TB, answering the previous question.

The functor Φ is an isomorphism from TA onto the category of $U_q(sl(2))$-linear maps over tensor powers of the fundamental 2-dimensional $U_q(sl(2))$-module V. Since the category TB is an extension of TA in the set of morphisms, the set of intertwinners would have to become smaller in any extension of Φ to TB. Indeed, let $Hom_t(V^{\otimes n},\ V^{\otimes m})$ denote the set of k-linear maps from $V^{\otimes n}$ to $V^{\otimes m}$ which commute with the cylinder twists $t_{\otimes n}$ on $V^{\otimes n}$ and $t_{\otimes m}$ on $V^{\otimes m}$, obtained from the fundamental four braid pair $(X_2,\ F_2)$ on V. Then we have

Theorem 8 (TD2)**.** *: There is an isomorphism, which extends Φ from TA to TB, and compatible with the composition of morphisms:*

$$\mathcal{L} : Hom_{TB}(n,\ m) \to Hom_t(V^{\otimes n},\ V^{\otimes m})$$

This isomorphism yields also a representation of the (unoriented) category of rooted cylinder ribbons RRB (see 4.5), which factor over the extension of the Kauffman functor from $\mathcal{T}$ to the (unoriented) category RRB.

elements for the proof: One may prove (although it is rather long and difficult) that the eigenspaces of $t_{\otimes n}$ on $V^{\otimes n}$ are the irreducible modules of the tensor representation of TB induced by the tensor representation of the braid group ZB_n (see the conclusion above). Since t comes from a central element in the braid group, the eigenspaces are invariant under elements in $Hom_t(V^{\otimes n},\ V^{\otimes m})$. But there are $n+1$ different eigenspaces with multiplicities $\begin{pmatrix} n \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} n \\ n \end{pmatrix}$.
This makes the dimension of the target of $\mathcal{L}$ equal to $\begin{pmatrix} 2n \\ n \end{pmatrix}$, with the help of (25).
The functoriality between tensor module categories with rigidity comes from an analysis of the quantum trace in T_nB. □

Let us now turn to the Kauffman functor K. As suggested in Theorem 8, it may be extended naturally to a functor $\bar{K}$ from the symmetric (unoriented) tangles in the cylinder to the category TB, defined with an additional rule (depending on parameters a and b) of the type

For example, the choice of the pair $(a,\ b) = (A^2,\ D^{-1}(1-A^2))$ may be shown to turn $\bar{K}$ into an extension of K that gives, as in remark 6.1.1, 2), the extension of the Jones polynomial to knots and links in the cylinder. Furthermore, the rooting maps defined in 4.5 are compatible with the preceding functors: hence $\bar{K}$ extends over the unoriented version of the category ORR (which itself extends $\mathcal{T}$).
Here is an example, where we use the fact that one may equally view links in the cylinder as framed closed braids, or *ribbons* always looking to the axis: an unknotted component linked with the axis may be distinguished from the unknot, since, when taking $(a,\ b) = (A^2,\ D^{-1}(1-A^2))$, its value under $\Phi \circ \bar{K}$ is $-A^3(A+A^{-1})$ or $-A^{-3}(A+A^{-1})$, depending on the sign of the linking number with the axis. With the same parameters for $\hat{K}$, the root map is $(\frac{D}{1-a^2})^{1/2}(ia,\ 1)$ and the coroot map is $(\frac{D}{1-a^2})^{1/2}\begin{pmatrix} ia \\ 1 \end{pmatrix}$.

Then, one may show that the extension of the Jones polynomial to links in the cylinder is given by the following proposition:

Theorem 9. *: Let L be a link obtained as the closure of an $(n,\ n)$-ribbon with value $\alpha_L : V^{\otimes n} \to V^{\otimes n}$ under $\mathcal{L}$. Then the Kauffman functor $\bar{K}$ maps L to the linear algebra trace of $\alpha_L \circ u^{\otimes n}$, where u is the diagonal matrix $Diag(-A^{-2},\ A^2)$.*

6.2. **The FRT-construction.** Let V be a free module over the ring k, with basis $v_1,\ v_2,\ \dots,\ v_n$, and

$$X : V \otimes V \to V \otimes V$$

a Yang Baxter operator on V. The following construction, called the *FRT-construction*, associates to the pair $(X,\ V)$ a unique cobraided bialgebra $(A(X),\ r)$; for more details, see [Kas, ch. 8].

Let $E_n = Hom(V^{\otimes n},\ V^{\otimes n})$, and $\tilde{A} = \oplus_{n=0}^{\infty} E_n$ be the graduate sum of homogeneous k-intertwinners; define the endomorphisms

$$T_i^j : v_m \mapsto \delta_i^m v_j \in E$$

The canonical isomorphism $E_m \otimes_k E_n \cong E_{m+n}$ shows that the family $\{T_i^j\}_{i,\ j}$ induces a basis $T_i^j = T_{i_1}^{j_1} \otimes \dots \otimes T_{i_n}^{j_n}$ for E_n, $i = (i_1,\ \dots,\ i_n)$, $j = (j_1, \dots,\ j_n)$ and the braiding morphisms $X = X_{m,\ n}$ on $V^{\otimes m} \otimes V^{\otimes n}$ may be written in multi-index notation as:

$$X(v_i \otimes v_j) = \sum_{a,\ b} X_{i\ j}^{a\ b} v_a \otimes v_b$$

It is then clear that we may define a braid form over $\tilde{A}$ by setting $r(T_i^a \otimes T_j^b) = X_{i\ j}^{a\ b}$.

One may show that the quotient of $\tilde{A}$ by the relations

$$C_{i\ j}^{k\ l} = \sum_{\alpha,\ \beta} X_{i\ j}^{\alpha\ \beta} T_{\alpha\ \beta}^{k\ l} - \sum_{\alpha,\ \beta} T_{i\ j}^{\alpha\ \beta} X_{\alpha\ \beta}^{k\ l}$$

is a cobraided bialgebra $A(X)$, in which r appears as a canonical element in A^*.

Suppose now that we have a four braid pair $(X,\ F)$ on V; the twist operator $t_{\otimes n} \in E_n$, expressed in the basis $\{v_i\}$ (in multi-index notation) as $t_n(v_i) = \sum_j F_i^j v_j$ (in multi-index notation), induces a linear form $T_i^j \to F_i^j$. It is clear that, as we saw in section 5, this linear form yields a cylinder form on the cobraided bialgebra $(A(X),\ r)$ obtained from the FRT-construction. Therefore, given a four braid pair $(X,\ F)$ on V, the category of $A(X)$-comodules has a cylinder braiding.
Since the pair $(X,\ id)$ is nothing more than a solution of the Yang Baxter equation, we may expect that the FRT-construction is a particular case of a more general construction, which would produce *universal* four braid pairs as special elements in a more general structure than cobraided bialgebra.

Question: What is the simplest algebraic structure supporting universal solutions to the (FBP) relation ?

Remark: The FRT-construction is dual to the approach of the preceding sections. In fact, the cobraided biagebra $A(X)$ corresponds merely to a q-deformation of the coordinate ring $\mathcal{O}_q$ over a Lie algebra (see [CP] or [Kas.]), rather than to a q-deformation of its universal algebra.
For example, in the case of sl_2, the FRT-construction produces directly a cylinder form on the category of $SL_q(2)$-comodules (see below the definition of $SL_q(2)$), starting with the fundamental automorphism F_2 of the pair $(X_2,\ F_2)$ on V. To get the existence of the cylinder twist in $U_q(sl_2) - Int$, in section 5, we worked with the braiding and the cylinder braiding characterizations for $U_q(sl(2))$. The duality between these approachs is *hidden* in the isomorphism between $SL_q(2) - comodules$ and $U_q(sl(2)) - modules$ (see 2.4). This remark is the heart of what is called the quantum duality principle, and it shows by the FRT-construction the usefulness of comodules structures, which behaves better than modules over quantum enveloping algebras.

We finish this lecture by an application of the FRT-construction to the $sl(n)$-theory, $n \geq 2$.
Given a matrix

$$P = (\{p_{i,\ j}\},\ 1 \leq i,\ j \leq n|\ p_{i,\ i} = q;\ p_{i,\ j}p_{j,\ i} = 1,\ i \neq j)$$

we can construct a Yang Baxter operator

$$X(v_i \otimes v_j) = \begin{cases} p_{i,\ j} v_j \otimes v_i,\ i \geq j \\ p_{i,\ j} v_j \otimes v_i + \delta v_i \otimes v_j,\ i < j \end{cases}$$

The FRT-construction associates to X a *multi-parameter version* $\mathcal{O}_q(\mathcal{M}_n)$ of the coordinate algebra of the group of $n \times n$-matrices $\mathcal{M}_n$ (obtained in the special case $p_{i,\ j} = 1,\ i \neq j$).
There is a special element in $\mathcal{O}_q(\mathcal{M}_n)$, the *quantum determinant* Det_q, which is a q-deformation of the classical determinant for operators. Then, the q-deformed coordinate algebra $\mathcal{O}_q(sl_n)$ is obtained by setting Det_q equal to 1 (we denoted $\mathcal{O}_q(sl_2)$ in the previous paragraph by $SL_q(2)$).
Finally, consider an automorphism

$$F : V \to V,\ F(v_j) = \begin{cases} \beta_j v_{n+1-j}\ if\ 2j \leq n+1 \\ \beta_j v_{n+1-j} + w v_j\ if\ 2j > n+1 \end{cases}$$

where w is an arbitrary parameter in k.

Lemma: *One may find a family of elements $\{\beta_j\}_{1 \leq j \leq n}$ in k, depending on w and a further parameter $z \in k$, such that the pair $(X,\ F)$ is a four braid pair over V.*

Then, verifying that F defines a cylinder form f over $\mathcal{O}_q(sl_n)$, i. e. that f(Det_q)=1, one proves that

Theorem 10. *: There is a cylinder braiding on the category of $\mathcal{O}_q(sl_n)$-comodules.*

References

[Ba-Re] V. V. Bazhanov & N. Yu. Reshetikhin, *Remarks on the quantum dilogarithm*, J. Phys. A: Math. Gen. 28 (1995), 2217-2226;

[BW] S. J. Birman & H. Wenzl, *Braids, link polynomials and a new algebra*, Trans. Amer. Math. Soc. 313 (1989), 249-273;

[CP] Chari & Pressley, *A guide to quantum groups*, Cambridge University Press (1994);

[Kas] C. Kassel, *Quantum groups*, Springer, GTM 155 (1994);

[Lus] G. Lusztig, *An introduction to quantum groups*, Birkhäuser, PM 110 (1993);

[M] J. Murakami, *The representations of the q-analogue of Brauer's centralizer algebras and the Kauffman polynomial of links*, Publ. Res. Inst. Math. Sci. 26 (1990), 935-945;

[TD1] T. Tom Dieck, *On tensor representations of knot algebras*, Manuscr. Math. 93 (1997), 163-176;

[TD2] T. Tom Dieck, *Categories of rooted cylinder ribbons*, J. reine angew. Math. 494 (1998), 35-63;

[TD3] T. Tom Dieck, *Private communication*;

[Tu] V. Turaev, *Quantum invariants of knots and 3-manifolds*, W. de Gruyter (1994).